The Value of Public Information in Common-Value Tullock Contests*

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Abstract

We study how changes in the information available to the players of a symmetric common-value Tullock contest with incomplete information affect their payoffs and their incentives to exert effort. For the class of contests where players' state dependent cost of effort is multiplicative, we show that if the players' Arrow-Pratt measure of relative risk aversion is increasing (decreasing), then the value of Public information is positive (negative). Moreover, if players' cost of effort (value) is state independent, then players' effort decreases (increases) with the level of information.

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1 Introduction

We study how changes in the information available to the players of a symmetric common-value Tullock contest with incomplete information affect their incentives to exert effort and their payoffs. In a *Tullock contest* – see Tullock (1980) – a player's probability of winning the prize is the ratio of the effort he exerts and the total effort exerted by all players. In a symmetric common-value contest with incomplete information players have a common state dependent value for the object and a common state dependent cost of effort, and all players have the same information.

There are a variety of economic settings (rent-seeking, innovation tournaments, patent races) in which agents face a game strategically equivalent to a Tullock contest – see Baye and Hoppe (2003). Tullock contests may also arise by design, e.g., in sport competition, internal labor markets – see Konrad (2008) for a general survey; also, see Skaperdas (1996) and Clark and Riis (1998) for alternative axiomatizations of Tullock contests.

In our model, players' uncertainty about their value and cost is described by a probability space, and players' information is described by a subfield of the field on which players' common prior is defined. We show that if players' cost of effort is a twice differentiable, strictly increasing and convex function in every state, then the contest has a unique equilibrium, which is symmetric. Einy et al. (2013) have recently established a general existence theorem for Tullock contests with incomplete information when the probability space describing player information is finite, and have provided conditions for uniqueness of equilibrium – see also Wasser (2013) on the issue of existence of equilibrium, and Ewerhart and Quartieri (2013) on the issue of uniqueness of equilibrium. These results do not apply to our setting, in which both the set of the states of nature and the players' information field are unrestricted. Moreover, we show that the unique equilibrium is symmetric, a property that greatly simplifies our analysis.

For the class of contests on which the cost function has the properties required for our existence theorem to apply, in which equilibrium is unique and symmetric, the question "how the players' efforts and payoffs change when the information available to them changes" is well posed. We are able to provide an answer to this question when the players' cost of effort is a multiplicative function, that is, when the players cost of effort is the product of a random variable and a real valued function d on the player's effort. A function d identifies a class of symmetric common-value Tullock contest with incomplete information that differ on the two random variables identifying the players' value for the prize and the random component of their cost of effort, and on the subfield identifying the information available to the players. Changes in the level of information are represented as changes in the subfield describing the players' information.

Following Einy, Moreno and Shitovitz (2003), given a function d and any two random variables describing the players' common value (v) and common cost (w), which are the uncertain elements of the contest, we define a binary relation that ranks information fields according to the level of information they contain: an information field $\mathcal{H}$ is more informative than some other information field $\mathcal{G}$ if the predictions of the value and cost are the same whether players information is given by $\mathcal{H}$ or it is given by the aggregate information in $\mathcal{G}$ and $\mathcal{H}$.

We define two auxiliary real-value functions (S and U) that for any pair of random variables (v, w) provide the players' expected effort and payoff, respectively, in the unique equilibrium (which we show is symmetric) of the corresponding contest when players' information field is that where the common prior is defined. The restrictions of the functions S and U to the subset of positive constant random variables (which we denote by D and F) may be regarded as functions defined on $\mathbb{R}^2_{++}$. The curvature of these functions determine the effect on incentives and payoffs of the added flexibility (commitment) that more (less) information introduces in the game: if D (respectively, F) is convex, then the players' effort (payoff) increases with the level of information, whereas if D (F) is concave, then the players' effort (payoff) decreases with the level of information. Einy, Moreno and Shitovitz (2003)'s prove this result as an application of Jensen's inequality.

In our setting, the curvatures of the functions D and F are determined by the function d. Moreover, the function d also determines player's Arrow-Pratt relative measure of risk aversion. Using our results relating the curvature of the auxiliary functions mentioned above and the effect of changes of information in efforts and payoffs, we show that when the players' Arrow-Pratt measure of relative risk aversion implied by d is increasing (decreasing), then the value of public information is positive (negative) in every symmetric common-value Tullock contest with incomplete information in the class defined by the function d. Moreover, if the cost of effort is independent of the state of nature, then the players' effort decreases with the level of information when their Arrow-Pratt measure of relative risk aversion is increasing. And if the value of the prize is independent of the state of nature, then the players' effort increases with the level of information when their Arrow-Pratt measure of relative risk aversion is decreasing.

An interesting implication of our results is that when players' efforts are monetary, i.e., the function d is linear, the value of information is zero, and players' effort is invariant to changes in players' information. In contrast, when the players have a convex quadratic cost function, then their Arrow-Pratt measure of relative risk aversion is increasing, and the value of information is positive. Thus, if the cost of effort is independent of the state of nature, then players exert less effort the better informed they are. It is not difficult, however, to find examples in which the cost of effort is state-dependent, and players exert more effort the better informed they are – see Example 1 in Section 4.

The value of information in Tullock contests has been seldom studied in the literature. Denter, Morgan and Sisak (2011) study the effect of mandated transparency policy on lobbying, and identify conditions under which it leads to an increase in efforts. Their setting is a two-player Tullock contest with private values and one-sided incomplete information. Morath and Münster (2013), and Kovenock, Morath and Münster (2013) study the incentives for information acquisition and information sharing, respectively, in all-pay auction contests. Of course, there is a large literature studying the value of information and the incentives for information acquisition in auctions.

2 Common-Value Tullock Contests

A group of players $N = \{1, ..., n\}$, with $n \geq 2$, compete for a prize by choosing a level of *effort* in $\mathbb{R}_+$. There is uncertainty about the players' common value of the prize and each player's cost function. This uncertainty is described by a probability space $(\Omega, \mathcal{F}, p)$, where Ω is the set of states of nature, $\mathcal{F}$ is a σ-field of subsets of Ω, and p is σ-additive probability measure on $\mathcal{F}$. We interpret p as the players' common prior belief about the realized state of nature. The value of the prize is described by an integrable function $v : \Omega \to \mathbb{R}_{++}$. The cost function of each player $i \in N$ is described by a function $c_i : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for every integrable function $s_i : \Omega \to \mathbb{R}_+$ the function $c_i(\cdot, s_i(\cdot))$ is integrable. The private information about the state of nature of player $i \in N$ is described by a σ-subfield $\mathcal{F}_i$ of $\mathcal{F}$; that is, given an event $A \in \mathcal{F}_i$, player i knows whether the realized state of nature is a member of A. A *common-value Tullock contest with incomplete information* is thus a collection $T = (N, (\Omega, \mathcal{F}, p), v, c_1 ..., c_n, \mathcal{F}_1, ..., \mathcal{F}_n)$.

Associated with a common-value Tullock contest with incomplete information $T = (N, (\Omega, \mathcal{F}, p), v, c_1 ..., c_n, \mathcal{F}_1, ..., \mathcal{F}_n)$ is a Bayesian game $G(T) = (N, (\Omega, \mathcal{F}, p), \mathbb{R}^n_+,$ $u_1, ..., u_n, \mathcal{F}_1, ..., \mathcal{F}_n)$, to which we refer as the *Tullock Bayesian game associated with* T, in which the set of actions of each player is $\mathbb{R}_+$, and the payoff function of each player $i \in N$ is $u_i : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}$ given for every $\omega \in \Omega$ and $x \in \mathbb{R}^n_+ \backslash \{0\}$ by

$$u_i(\omega, x) = \frac{x_i}{\bar{x}} v(\omega) - c_i(\omega, x_i), \tag{1}$$

where $\bar{x} \equiv \sum_{k=1}^{N} x_k$, and by $u_i(\omega, 0) = \rho_i(\omega) v(\omega) - c_i(\omega, 0)$, where $\rho(\omega) \in \Delta^n$. (We assume that when players exert no effort the prize is allocated using some predetermined state-dependent probability vector $\rho(\omega) \in \Delta^n$. As we show in the Appendix, the values of ρ are inconsequential.) In this game a pure strategy of player $i \in N$ is an integrable $\mathcal{F}_i$-measurable function $s_i : \Omega \to \mathbb{R}_+$ that specifies player i's effort in each state of nature following the observation of his private information. We denote by S_i the set of all pure strategies of player i, and by $S = S_1 \times ... \times S_n$ the set of pure strategy profiles.

Given a strategy profile $s = (s_1, ..., s_n) \in S$, we denote by s_{-i} the profile obtained from s by suppressing the strategy of player $i \in N$. Also, if X is an integrable random variable on $(\Omega, \mathcal{F}, p)$ and $\mathcal{G}$ is a σ-subfield of $\mathcal{F}$, we write $E[X \mid \mathcal{G}]$ for the conditional expectation of X with respect to $\mathcal{G}$. A profile of strategies $s^* = (s_1^*, ..., s_n^*) \in S$ is a *Bayesian Nash equilibrium* of $G(T)$ if for every $i \in N$, every $s_i \in S_i$, and almost all $\omega \in \Omega$,

$$E[u_i(\cdot, s^*(\cdot)) \mid \mathcal{F}_i](\omega) \geq E[u_i(\cdot, s_{-i}^*(\cdot), s_i(\cdot)) \mid \mathcal{F}_i](\omega). \quad (2)$$

It is easy to see, e.g., Remark 2.1 in Einy, Moreno and Shitovitz (2003), that $s^* \in S$ is a Bayesian Nash equilibrium if and only if for every $i \in N$ and every $s_i \in S_i$

$$E[u_i(\cdot, s^*(\cdot))] \geq E[u_i(\cdot, s_{-i}^*(\cdot), s_i(\cdot))]. \quad (3)$$

Throughout the paper we restrict attention to pure strategy Bayesian Nash equilibria.

3 Symmetric Common-Value Tullock Contests

A *symmetric* common-value Tullock contests with incomplete information, i.e., a contests in which all players have the same cost function and the same information, may be described by a collection $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$, where c is the cost function of every player, and $\mathcal{G}$ is a σ-subfield of $\mathcal{F}$ describing the information of every player. We refer to $G(T)$ as a *symmetric* Tullock Bayesian game associated with T. Our first result establishes conditions implying that a Tullock game with symmetric information has a unique Bayesian Nash equilibrium.

Theorem 3.1. *Let $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$ be a symmetric common-value Tullock contest with symmetric information. Assume that for all $\omega \in \Omega$, $c(\omega, \cdot)$ is twice differentiable, strictly increasing and convex, and satisfies $c(\omega, 0) = 0$. Then the game $G(T)$ has a unique (pure strategy) Bayesian Nash equilibrium s^*. Moreover, s^* is symmetric, i.e., $s_1^* = s_2^* = ... = s_n^*$.*

Proof: For every $\omega \in \Omega$ define the n-person complete information game $G(\omega, T)$ in which the set of pure strategies of every player is $\mathbb{R}_+$ and the payoff function of each player $i \in N$, $h_i(\omega, \cdot) : \mathbb{R}_+^n \to \mathbb{R}_+$, is given for $x \in \mathbb{R}_+^n$ by

$$h_i(\omega, x) = E[u_i(\cdot, x) \mid \mathcal{G}](\omega).$$

The game $G(\omega, T)$ has a unique Nash equilibrium $t^*(\omega) = (t_1^*(\omega), ..., t_n^*(\omega))$, which is symmetric, i.e., $t_1^*(\omega) = t_2^*(\omega) = ... = t_n^*(\omega)$. (We establish this result in the Appendix along the lines of Szidarovszky and Okuguchi (1997)'s Theorem 1.) We show that the strategy profile $s^* \in S$ given for $\omega \in \Omega$ by $s^*(\omega) = t^*(\omega)$ is a Bayesian Nash equilibrium of the symmetric Tullock Bayesian game $G(T)$. We first show that s^* is a $\mathcal{G}$-measurable function. Define the correspondence $H : \Omega \to 2^{\mathbb{R}^n_+}$ by

$$H(\omega) = \{t \in \mathbb{R}^n_+ \mid t \text{ is a Nash equilibrium of } G(\omega, T)\}.$$

We show that the graph of the correspondence H is measurable with respect to the product σ-field $\mathcal{G} \otimes \mathbb{B}(\mathbb{R}^n_+)$, where $\mathbb{B}(\mathbb{R}^n_+)$ is the σ-field of Borel subsets of $\mathbb{R}^n_+$. For all $r \in \mathbb{R}^n_+$ let

$$J(r) = \{(\omega, x) \in \Omega \times \mathbb{R}^n_+ \mid h_i(\omega, x) \geq h_i(\omega, x_{-i}, r_i) \text{ for all } i \in N\}.$$

Since for all $i \in N$ and all $x \in \mathbb{R}^n_+$ the function $h_i(\cdot, x)$ is $\mathcal{G}$-measurable, and since for all $\omega \in \Omega$ the profile $0 \in \mathbb{R}^n_+$ is not a Nash equilibrium of $G(\omega, T)$, then $h_i(\omega, \cdot)$ is continuous on the set $\{x \in \mathbb{R}^n_+ \mid (\omega, x) \in J(r)\}$. Therefore the set $J(r)$ is $\mathcal{G} \otimes \mathbb{B}(\mathbb{R}^n_+)$-measurable for all $r \in \mathbb{R}^n_+$. Now, the graph of the correspondence H is

$$graph(H) = \bigcap_{r \in \mathbb{R}^n_+} J(r) = \bigcap_{r \in \mathbb{Q}^n_+} J(r),$$

where $\mathbb{Q}^n_+$ is the set of n-tuples of rational numbers. Since $\mathbb{Q}^n_+$ is countable, then $graph(H)$ is $\mathcal{G} \otimes \mathbb{B}(\mathbb{R}^n_+)$-measurable. Thus, by the Measurable Selection Theorem – see Aumann (1969) and Hildenbrand (1974)'s Theorem 1 (page 54) – there exists a $\mathcal{G}$-measurable function $\phi : \Omega \to \mathbb{R}^n_+$ such that $\phi(\omega) \in H(\omega)$ for almost all $\omega \in \Omega$. Since for all $\omega \in \Omega$ the set $H(\omega)$ is a singleton (because $G(\omega, T)$ has a unique equilibrium), then $\phi(\omega) = t^*(\omega) = s^*(\omega)$ for almost all $\omega \in \Omega$, and therefore s^* is a $\mathcal{G}$-measurable function.

Now for $i \in N$ and $s_i \in S_i$ we have

$$E[u_i(\cdot, s^*(\cdot)) \mid \mathcal{G}](\omega) \geq E[u_i(\cdot, (s^*_{-i}(\cdot), s_i(\cdot))) \mid \mathcal{G}](\omega)$$

for almost every $\omega \in \Omega$, and therefore

$$E[u_i(\cdot, s^*(\cdot))] \geq E[u_i(\cdot, (s^*_{-i}(\cdot), s_i(\cdot)))].$$

Hence s^* is a Bayesian Nash equilibrium of the game $G(T)$.

Uniqueness and symmetry follows from the fact that for all $\omega \in \Omega$ the profile $t^*(\omega) \in \mathbb{R}^n_+$ is the unique Nash equilibrium of $G(\omega, T)$, and $t^*_1(\omega) = t^*_2(\omega) = ... = t^*_n(\omega)$. $\square$

4 The Effect of Information on Payoffs and Effort

In this section we study the value of public information, and the effect of information on players' effort, in the class of symmetric common-value Tullock contests with incomplete information in which the players' cost function is multiplicative. Let $d : \mathbb{R}_+ \to \mathbb{R}_+$ be a twice differentiable, strictly increasing and convex function. Denote by $\mathcal{T}(d)$ the set of all symmetric common-value Tullock contests with incomplete information $(N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G})$ such that the cost function c satisfies for all $(\omega, x) \in \Omega \times \mathbb{R}_+$

$$c(\omega, x) = w(\omega)d(x),$$

where w is some non-negative integrable real-valued function on $(\Omega, \mathcal{F}, p)$, and $\mathcal{G}$ is a σ-subfield of $\mathcal{F}$. Thus, a symmetric common-value Tullock contests with incomplete information $T \in \mathcal{T}(d)$ is defined by pair of non-negative integrable functions (v, w) and a σ-subfield of $\mathcal{F}$.

Any pair of non-negative integrable functions (v, w) induces a binary relation on the family of all σ-subfield of $\mathcal{F}$ as follows: If $\mathcal{G}$ and $\mathcal{H}$ are two σ-subfields of $\mathcal{F}$, then

$$\mathcal{H} \succsim \mathcal{G} \Leftrightarrow \{E(v \mid \mathcal{H}) = E(v \mid \mathcal{G} \vee \mathcal{H}) \text{ and } E(w \mid \mathcal{H}) = E(w \mid \mathcal{G} \vee \mathcal{H})\}.$$

(Here $\mathcal{G} \vee \mathcal{H}$ is the smallest σ-subfield of $\mathcal{F}$ that contains both $\mathcal{G}$ and $\mathcal{H}$.) The interpretation of the binary relation $\succsim$ is simple: $\mathcal{H} \succsim \mathcal{G}$ if and only if the predictions of the common value and the cost function (the uncertain elements of the contest) are the same whether players information is given by $\mathcal{H}$ or it is given by the aggregate information in $\mathcal{G}$ and $\mathcal{H}$ (i.e., by $\mathcal{G} \vee \mathcal{H}$).

Let $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$. By Theorem 3.1 the Bayesian game $G(T)$ has a unique equilibrium, which is symmetric. Denote by $s^*_{\mathcal{G}}$ the strategy played by every player in equilibrium, and by $u^*_{\mathcal{G}}$ the equilibrium payoff of every player. We say that the value of public information in $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G})$ is positive (negative) if for every contest $T' = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{H})$,

$$\mathcal{H} \succsim \mathcal{G} \Rightarrow E(u^*_{\mathcal{H}}) \geq E(u^*_{\mathcal{G}}) \ (E(u^*_{\mathcal{H}}) \leq E(u^*_{\mathcal{G}})).$$

Also, we say that players' effort increase (decrease) with the level of information in $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$ if for every $T' = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{H})$,

$$\mathcal{H} \succsim \mathcal{G} \Leftrightarrow E(s^*_{\mathcal{H}}) \geq E(s^*_{\mathcal{G}}) \ (E(s^*_{\mathcal{H}}) \leq E(s^*_{\mathcal{G}})).$$

Let the functions $S, U : L^1_+(\Omega, \mathcal{F}, p) \times L^1_+(\Omega, \mathcal{F}, p) \to \mathbb{R}$ be given for each $(v, w) \in L^1_+(\Omega, \mathcal{F}, p) \times L^1_+(\Omega, \mathcal{F}, p)$ by

$$S(v, w) = E(s^*_{\mathcal{F}}),$$

and

$$U(v, w) = E(u^*_{\mathcal{F}}).$$

The functions S and U identify the expected equilibrium effort and payoff, respectively, in every contest $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{F}) \in \mathcal{T}(d)$. Let the functions $D, F : \mathbb{R}^2_{++} \to \mathbb{R}$ be given for each $(a, b) \in \mathbb{R}^2_{++}$ by

$$D(a, b) = S(a1_\Omega, b1_\Omega), \tag{4}$$

and

$$F(a, b) = U(a1_\Omega, b1_\Omega). \tag{5}$$

The functions S and F identify the expected equilibrium effort and payoff, respectively, in every contest $T = (N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega)\, d, \mathcal{F})$.

Proposition 4.1 establishes conditions identifying the effect of information on players' effort and payoff for a contest $T \in \mathcal{T}(d)$. The proof of Proposition 4.1 is identical to that of Proposition 3.3 in Einy, Moreno and Shitovitz (2003), and therefore is omitted.

Proposition 4.1. *Let $d : \mathbb{R}_+ \to \mathbb{R}_+$ be a twice differentiable, strictly increasing and convex function, and let $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$ be a symmetric common-value Tullock contest with incomplete information.*

(4.1.1) If the function D defined in (4) is convex (concave) on $\mathbb{R}^2_{++}$, then effort increases (decreases) with the level of information in T.

(4.1.2) If the function F defined in (5) is convex (concave) on $\mathbb{R}^2_{++}$, then the value of public information in T is positive (negative).

Proposition 4.2 identifies a necessary and sufficient condition for the function F to be either concave or convex.

Proposition 4.2. *Let $d : \mathbb{R}_+ \to \mathbb{R}_+$ be a twice differentiable, strictly increasing and convex function. The function F defined in (5) is convex (concave) on $\mathbb{R}^2_{++}$ if and only if $F_{aa}(a,b) \geq 0$ ($F_{aa}(a,b) \leq 0$) for all $(a,b) \in \mathbb{R}^2_{++}$.*

Proof. Let $(a,b) \in \mathbb{R}^2_{++}$. Write $s(a,b)$ for the strategy of each player in the unique Bayesian Nash equilibrium of the game $G(T)$, where $T = (N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega)\, d, \mathcal{F})$. Then

$$F(a,b) = \frac{a}{n} - bE(d(s(a,b))),$$

It is easy to see that s is homogeneous of degree zero on $\mathbb{R}^2_{++}$, i.e., $s(\lambda a, \lambda b) = s(a,b)$ for all $\lambda \in \mathbb{R}_{++}$. Hence

$$F(\lambda a, \lambda b) = \lambda F(a,b),$$

i.e., F is homogeneous of degree one on $\mathbb{R}^2_{++}$. By Euler's Theorem

$$F(a,b) = aF_a(a,b) + bF_b(a,b).$$

Differentiating with respect to a on both sides on this equation and simplifying yields

$$aF_{aa}(a,b) + bF_{ba}(a,b) = 0. \tag{6}$$

Likewise

$$aF_{ab}(a,b) + bF_{bb}(a,b) = 0. \tag{7}$$

Hence

$$a^2 F_{aa}(a,b) = b^2 F_{bb}(a,b), \tag{8}$$

and therefore

$$F_{aa}(a,b) \geq 0 \Leftrightarrow F_{bb}(a,b) \geq 0.$$

Further, (6) and (7) imply

$$F_{aa}(a,b)F_{bb}(a,b) - F_{ab}(a,b)F_{ba}(a,b) = 0. \tag{9}$$

Thus, the eigenvalues of the Hessian matrix of F are non-negative (non-positive) when F_{aa} is a non-negative (non-positive) function on $\mathbb{R}^2_{++}$. □

Let $u : \mathbb{R}_+ \to \mathbb{R}$ be a twice differentiable increasing and concave function. If u is an individual's von Neumann-Morgenstern utility function, then his Arrow-Pratt measure of relative risk aversion is given for all $x \in \mathbb{R}_+$ by

$$R_u(x) = -\frac{xu''(x)}{u'(x)}.$$

In a asymmetric common-value Tullock contests with incomplete information $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$ a player's utility if he wins the prize is $u(\cdot, x_i) = v(\cdot) - w(\cdot)d(x_i)$, and it is $u(\cdot, x_i) = -w(\cdot)d(x_i)$ if he does not win the price. Hence players' Arrow-Pratt measure of relative risk aversion is determined by the function d independently of the state, and is given for all $x \in \mathbb{R}_+$ by $R_{-d}(x)$.

Our next result establishes a relationship between the behavior of the Arrow-Pratt measure of relative risk-aversion implied by the function d and the value of public information.

Theorem 4.3. *Let $d : \mathbb{R}_+ \to \mathbb{R}_+$ be a thrice differentiable, strictly increasing and convex function, and let $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$ be a symmetric common-value Tullock contest with incomplete information. If the players' Arrow-Pratt measure of relative risk aversion R_{-d} is increasing (decreasing), then the value of public information in T is positive (negative).*

Proof. Assume that $R_{-d}(x)$ is an increasing function, and let $T = (N, (\Omega, \mathcal{F}, p), v, c, \mathcal{G}) \in \mathcal{T}(d)$. By Proposition 4.2 it suffices to show that $F_{aa}(a,b) \geq 0$ for all $(a,b) \in \mathbb{R}^2_{++}$, where F is the function defined in (5). For $(a,b) \in \mathbb{R}^2_{++}$,

$$F(a,b) = \frac{a}{n} - bE(d(s(a,b))), \tag{10}$$

where $s(a,b)$ is the strategy of each player in the unique Bayesian Nash equilibrium of the game $G(T)$, with $T = (N, (\Omega, \mathcal{F}, p), a1_\Omega, (b1_\Omega)\, d, \mathcal{F})$. Let

$$g(a,b) = d(s(a,b)).$$

Then

$$F(a,b) = \frac{a}{n} - bE(g(a,b)),$$

and therefore

$$F_{aa}(a,b) = -bE(g_{aa}(a,b)).$$

We show that $E(g_{aa}(a,b)) \leq 0$. Differentiating g we get

$$g_a(a,b) = d'(s(a,b))s_a(a,b)$$

and

$$g_{aa}(a,b) = d''(s(a,b))\,(s_a(a,b))^2 + d'(s(a,b))s_{aa}(a,b). \tag{11}$$

Since $s(a,b)$ maximizes $E[u_i(\cdot, (s^*_{-i}(\cdot), s_i(\cdot))) \mid \mathcal{G}]$, it satisfies the first order condition

$$E[\frac{a\,(n-1)}{n^2 s(a,b)} \mid \mathcal{G}](\omega) = bE[d'(s(a,b)) \mid \mathcal{G}](\omega)$$

for all $\omega \in \Omega$. Since $s(a,b)$ is $\mathcal{G}$-measurable, then

$$s(a,b)d'(s(a,b)) = \frac{n-1}{n^2 b}a. \tag{12}$$

Therefore

$$(sd'(s))'\,(a,b)s_a(a,b) = \frac{n-1}{n^2 b},$$

i.e.,

$$s_a(a,b) = \frac{n-1}{n^2 b\,(sd'(s))'\,(a,b)} \tag{13}$$

Differentiating this expression we get

$$s_{aa}(a,b) = -\frac{n-1}{n^2 b}\frac{(sd'(s))''\,(a,b)s_a(a,b)}{\left((sd'(s))'\,(a,b)\right)^2}. \tag{14}$$

Therefore

$$s_{aa}(a,b) \leq 0 \Leftrightarrow (sd'(s))''\,(a,b)s_a(a,b) \geq 0. \tag{15}$$

By (11) and (14),

$$g_{aa}(a,b) = d''(s(a,b))\,(s_a(a,b))^2 - \frac{n-1}{n^2b}\frac{d'(s(a,b))\,(sd'(s))''\,(a,b)s_a(a,b)}{\left((sd'(s))'\,(a,b)\right)^2}.$$

Hence $g_{aa}(a,b) \leq 0$ if and only if

$$\begin{aligned} d''(s(a,b))\,(s_a(a,b))^2 &\leq \frac{n-1}{n^2b}\frac{d'(s(a,b))\,(sd'(x))''\,(a,b)s_a(a,b)}{\left((sd'(s))'\,(a,b)\right)^2} \\ &= \frac{d'(s(a,b))\,(sd'(x))''\,(a,b)}{(sd'(s))'\,(a,b)} s_a(a,b) \frac{n-1}{n^2b\,(sd'(s))'\,(a,b)} \\ &= \frac{d'(s(a,b))\,(sd'(s))''\,(a,b)}{(sd'(s))'\,(a,b)}\,(s_a(a,b))^2\,, \end{aligned}$$

where the last expression is obtained by replacing $s_a(a,b)$ from equation (13). Hence

$$\begin{aligned} g_{aa}(a,b) \leq 0 \quad &\Leftrightarrow \quad \frac{d''(s(a,b))}{d'(s(a,b))} - \frac{(sd'(x))''\,(a,b)}{(sd'(x))'\,(a,b)} \leq 0 \\ &\Leftrightarrow \quad (\ln d'(s(a,b)))' - \left(\ln\,(sd'(s))'\,(a,b)\right)' \leq 0 \\ &\Leftrightarrow \quad \left(\ln \frac{(sd'(s))'\,(a,b)}{d'(s(a,b))}\right)' \geq 0 \\ &\Leftrightarrow \quad \left(\frac{(sd'(s))'\,(a,b)}{d'(s(a,b))}\right)' \geq 0 \\ &\Leftrightarrow \quad \left(1 + \frac{s(a,b)d''(s(a,b))}{d'(s(a,b))}\right)' \geq 0 \\ &\Leftrightarrow \quad (R_{-d}(s(a,b)))' \geq 0. \ \square \end{aligned}$$

Our next result concerns the effect on players' effort of changes in the level of information.

Proposition 4.4. *Let $d : \mathbb{R}_+ \to \mathbb{R}_+$ be a thrice differentiable, strictly increasing and convex function, and let $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$ be a symmetric common-value Tullock contest with incomplete information.*

(4.4.1) If w is constant on Ω and $(xd'(x))'' \leq 0$ ($(xd'(x))'' \geq 0$) for all $x \in \mathbb{R}_+$, then players' effort increases (decreases) with the level of information in T.

(4.4.2) If v is constant on Ω and $(xd'(x))'' \leq 0$ for all $x \in \mathbb{R}_+$, then players' effort increases with the level of information in T.

Proof. We prove (4.4.1). W.l.o.g. assume that $w(\cdot) = 1$ on Ω. Let $a \in \mathbb{R}_{++}$ and let $s(a, 1)$ be the pure strategy played by every player in the unique equilibrium of $G(T)$, where $T = (N, (\Omega, \mathcal{F}, p), a1_\Omega, 1_\Omega d, \mathcal{F})$. Let $\bar{D}(a) := D(a, 1) = E(s(a, 1))$. Then $\bar{D}''(a) = E(s_{aa}(a, 1))$. Since $s_a(a, 1) > 0$ by (13), then $(xd'(x))'' \leq 0$ (respectively, $(xd'(x))'' \geq 0$) implies $s_{aa}(a, 1) \geq 0$ (respectively, $s_{aa}(a, 1) \leq 0$) by (15), and therefore $\bar{D}''(a) \geq 0$ (respectively, $\bar{D}''(a) \leq 0$). Hence players' effort increase (decrease) with the level of information in T by Proposition 4.1.1.

We prove (4.4.2). W.l.o.g. assume that $v(\cdot) = 1$ on Ω. Let $\hat{D}(b) := D(1, b) = E(s(1, b))$. Then $\hat{D}''(b) = E(s_{bb}(1, b))$. Differentiating equation (12) with respect to b and using $a = 1$ yields

$$s_b(1, b) = -\frac{n-1}{n^2} \frac{1}{(sd'(s))'(1, b)} \frac{1}{b^2} < 0.$$

Differentiating this expression we get

$$\begin{aligned} s_{bb}(1, b) &= -\frac{n-1}{n^2}\left(-\frac{2}{b^3}\frac{1}{(sd'(s))'(1, b)} - \frac{1}{b^2}\frac{(sd'(s))''(1, b)s_b(1, b)}{[(sd'(s))'(1, b)]^2}\right) \\ &= \frac{n-1}{n^2}\frac{1}{b^2}\frac{1}{(sd'(s))'(1, b)}\left(\frac{2}{b} + \frac{(sd'(s))''(1, b)s_b(1, b)}{(sd'(s))'(1, b)}\right). \end{aligned}$$

Therefore $(sd'(s))''(1, b) \leq 0$ implies $s_{bb}(1, b) \geq 0$ and hence $\hat{D}''(b) \geq 0$. Thus, players' effort increases with the level of information in T by Proposition 4.1.1. □

When the Arrow-Pratt measure of relative risk aversion is increasing (decreasing) and the players' cost of effort (value) is independent of the state of nature, we can evaluate the impact of changes in the level of information on players' effort.

Proposition 4.5. *Let $d : \mathbb{R}_+ \to \mathbb{R}_+$ be a thrice differentiable, strictly increasing and convex function, and let $T = (N, (\Omega, \mathcal{F}, p), v, wd, \mathcal{G}) \in \mathcal{T}(d)$ be a symmetric common-value Tullock contest with incomplete information.*

(4.5.1) If w is constant on Ω and the players' Arrow-Pratt measure of relative risk aversion R_{-d} is increasing, then players' effort decreases with the level of information in T.

(4.5.2) If v is constant on Ω and the players' Arrow-Pratt measure of relative risk aversion R_{-d} is decreasing, then players' effort increases with the level of information in T.

Proof. We prove Proposition (4.5.1). Assume that R_{-d} is increasing, i.e., for all $x \in \mathbb{R}_+$

$$\left(\frac{xd''(x)}{d'(x)}\right)' \geq 0.$$

Then

$$\left(1 + \frac{xd''(x)}{d'(x)}\right)' \geq 0,$$

which implies

$$\left(2d''(x) + xd'''(x)\right) d'(x) \geq d'(x)d''(x) + x\left(d''(x)\right)^2 \geq 0,$$

and therefore

$$2d''(x) + xd'''(x) = (xd'(x))'' \geq 0$$

for all $x \in \mathbb{R}_+$. Hence Proposition 4.5.1 follows from Proposition 4.4.1.

We prove Proposition (4.5.2). Assume that R_{-d} is decreasing. Hence

$$\left(1 + \frac{xd''(x)}{d'(x)}\right)' \leq 0.$$

Taking log in equation (12) we may write

$$\ln s(a,b) + \ln d'(s(a,b)) = \ln \frac{n-1}{n^2} + \ln a - \ln b.$$

Setting $a = 1$ and differentiating with respect b we get

$$-\frac{1}{b} = \frac{s_b(1,b)}{s(1,b)} \left(1 + \frac{s(1,b)d''(s(1,b))}{d'(s(1,b))}\right).$$

Hence $s_b(1,b) < 0$. Differentiating this equation again with respect to b yields

$$\frac{1}{b^2} = \frac{s_{bb}(1,b)s(1,b) - s_b(1,b)^2}{s(1,b)^2} \left(1 + \frac{s(1,b)d''(s(1,b))}{d'(s(1,b))}\right)$$

$$+\frac{s_b(1,b)^2}{s(1,b)} \left(1 + \frac{s(1,b)d''(s(1,b))}{d'(s(1,b))}\right)'.$$

Since R_{-d} is decreasing, the second term in the right hand side is non-positive, and therefore the first term must be non-negative, i.e.,

$$s_{bb}(1,b)s(1,b) - s_b(1,b)^2 \geq 0.$$

Hence $s_{bb}(1,b) \geq 0$. Thus $\hat{D}(b) = D(1,b) = E(s(1,b))$ is convex, since $\hat{D}''(b) = E(s_{bb}(1,b)) \geq 0$. Therefore players' effort increases with the level of information in T by Proposition 4.1.1. □

Finally, we present some applications and examples. In a symmetric common-value *classic* Tullock contests with incomplete information the cost function of every player $i \in N$ is $c_i(\omega, x) = x$. Therefore the players' Arrow-Pratt measure of relative risk aversion R_{-d} is constant. Our next proposition is a direct consequence of Theorem 4.3 and Proposition 4.5.

Proposition 4.6. *In every symmetric common-value classic Tullock contest with incomplete information the value of public information is zero, and the players' effort is invariant to changes in the players' information.*

The following examples provide other applications of our results.

Example 1. Assume that $n = 2$, $\Omega = \{\omega_1, \omega_2\}$, $p(\omega_1) = p(\omega_2) = 1/2$, and let $d(x) = x^2/2 + 2x$. Then for $x \in \mathbb{R}_+$,

$$R_{-d}(x) = \frac{x}{x+1}.$$

Therefore $R'_{-d}(x) > 0$. Hence in every contest $T \in \mathcal{T}(d)$ the value of public information is positive. Moreover, if the players' cost of effort is independent of the state of nature, then their effort decrease with the level of public information. However, if the the cost of effort depends on the state, then players' effort may increase with the level of information. An example with this feature is as follows: let the value be $v(\omega_1) = v(\omega_2) = 2$, and let the multiplicative component of the cost be $w(\omega_1) = 1/6$, $w(\omega_2) = 1/48$. Consider the contest $T \in \mathcal{T}(d)$ in which the players have no information about the state. Then equilibrium efforts are $s^*(\omega_1) = s^*(\omega_2) = \sqrt{57}/3 - 1$.

Consider the contest $T' \in \mathcal{T}(d)$ in which the players have complete information about the state. Then equilibrium efforts are $s^*(\omega_1) = 1$ and $s^*(\omega_2) = 4$, and therefore the expected effort is $E(s^*) = 5/2 > \sqrt{57}/3 - 1$. Hence the expected effort increases with the level of information in T.

Example 2. Assume that $d(x) = x^\lambda$ for all $x \in \mathbb{R}_+$, where $\lambda > 1$. Then for $x \in \mathbb{R}_+$, $R_{-d}(x) = \lambda - 1$, and therefore $R'_{-d}(x) = 0$. Hence in every contest $T \in \mathcal{T}(d)$ the value of public information in T is zero. However, since

$$(xd'(x))'' = \lambda^2(\lambda - 1)x^{\lambda-2} > 0,$$

in every contest $T \in \mathcal{T}(d)$ in which the cost of effort is independent of the state of nature players' effort decrease with the level of public information by Proposition 4.4.1.

5 Extensions

The results of sections 3 and 4 apply to a broader class of *generalized* symmetric common-value Tullock contests with incomplete information in which the contests success function $\rho : \Omega \times \mathbb{R}^n_+ \to \gneqq^n$ is given by for $i \in N$ and $(\omega, x) \in \Omega \times \mathbb{R}^n_+ \backslash \{0\}$ by

$$\rho_i(\omega, x) = \frac{g(\omega, x_i)}{\sum_{j=1}^n g(\omega, x_j)},$$

where $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a *score* function such that for all $\omega \in \Omega$, $g(\omega, \cdot)$ is twice differentiable, strictly increasing and concave, and satisfies $g(\omega, 0) = 0$. In the Bayesian game associated with a contest T_g in this class, $G(T_g)$, the payoff function of each player $i \in N$ is given for all $(\omega, x) \in \Omega \times \mathbb{R}^n_+$ by

$$u_i(\omega, x) = \frac{g(\omega, x_i)}{\sum_{j=1}^n g(\omega, x_j)} v(\omega) - c(\omega, x_i).$$

Since there is a bijection between the equilibria of the game $G(T_g)$ and the game $G(T)$, where $T = (N, (\Omega, \mathcal{F}, p), v, \hat{c}, \mathcal{G})$, with $\hat{c}(\omega, \cdot) = g^{-1}(\omega, \cdot) \circ c(\omega, \cdot)$ for all $\omega \in \Omega$, then changes in information in T_g have the same impact as in T.

6 Appendix

Lemma. *A symmetric common-value Tullock contest with complete information in which the contenders' cost of effort is a twice differentiable strictly increasing and convex function* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $c(0) = 0$ *has a unique (pure strategy) Nash equilibrium.*

Proof. Denote by $v > 0$ the contenders common value. We show that the n-person complete information game in which the set of pure strategies of every player is $\mathbb{R}_+$ and the payoff function of each player $i \in N$ is $h_i : \mathbb{R}_+^n \to \mathbb{R}_+$ given for $x \in \mathbb{R}_+^n \backslash \{0\}$ by

$$h_i(x) = \frac{x_i}{\bar{x}} v - c(x_i),$$

where $\bar{x} = \sum_{j=1}^n x_j$, and

$$h_i(0) = \rho_i v - c(0) = \rho_i v,$$

where $\rho \in \Delta^n$ is predetermined, has a unique Nash equilibrium, which is symmetric.

Note that $h_i(\cdot, x_{-i})$ is twice differentiable and concave on $\mathbb{R}_+^n \backslash \{0\}$. For $x_{-i} \in \mathbb{R}_+^{n-1} \backslash \{0\}$ player i's optimal effort solves the problem

$$\max_{x_i \in \mathbb{R}_+} h_i(x_i, x_{-i}).$$

Differentiating h_i we get

$$\frac{\partial h_i}{\partial x_i} = \frac{\bar{x} - x_i}{\bar{x}^2} v - c'(x_i).$$

Write

$$\varphi(\bar{x}, x_i) := (\bar{x} - x_i) v - \bar{x}^2 c'(x_i).$$

Note that

$$\frac{\partial \varphi}{\partial x_i} = -\bar{x}\left(2c'(x_i) + \bar{x}c''(x_i)\right) < 0$$

for $(x_i, x_{-i}) \in \mathbb{R}_+^n \backslash \{0\}$. If $\varphi(\bar{x}, 0) = \bar{x}\left(v - \bar{x}c'(0)\right) \leq 0$, then $\partial h_i / \partial x_i \leq 0$ on $\mathbb{R}_+$, and therefore player i's optimal effort is $x_i = 0$. If $\varphi(\bar{x}, 0) > 0$, since $\varphi(\bar{x}, \bar{x}) = -\bar{x}^2 c'(\bar{x}) < 0$ and $\partial \varphi / \partial x_i < 0$, then player i's optimal effort is the unique solution to the equation $\varphi(\bar{x}, x_i) = 0$.

Clearly $x = 0 \in \mathbb{R}^n_+$ is not a Nash equilibrium: since $n \geq 2$, then $\rho_i < 1$ for some $i \in N$, and therefore

$$h_i(x) = \rho_i v < v - c(\varepsilon) = h_i(x_{-i}, \varepsilon).$$

for $\varepsilon > 0$ sufficiently small.

Therefore a Nash equilibrium $x \in \mathbb{R}^n_+ \setminus \{0\}$ satisfies $\varphi(\bar{x}, x_i) = 0$ for all $i \in N$, where $\bar{x} = \sum_{j=1}^n x_j$. Hence $x_i = x_j$ for all $i, j \in N$, and therefore $x = (t, ..., t)$, where $t \in \mathbb{R}_{++}$ solves $\varphi(nt, t) = 0$. Equivalently, t must solve the equation $\phi(t) = 0$, where

$$\phi(t) := \frac{\varphi(nt, t)}{t} = (n - 1)\, v - n^2 t c'(t).$$

Note that $\phi(0) > 0$ and $\phi'(t) = -n^2\,(c'(t)) + tc''(t)) < 0$. Moreover, since c is convex, there is $Q > 0$ sufficiently large that $Qc'(Q) > v$, and therefore

$$\phi(Q) = (n - 1)\, v - n^2 Q c'(Q) < nv\,(1 - n) < 0.$$

Hence the equation $\phi(t) = 0$ has a unique solution $t^* \in \mathbb{R}_{++}$, and therefore the unique (pure strategy) Nash equilibrium is $x = (t^*, ..., t^*)$.

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