Approximate Revenue Maximization with Multiple Items*

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Abstract

Myerson's classic result provides a full description of how a seller can maximize revenue when selling a single item. We address the question of revenue maximization in the simplest possible multi-item setting: *two items* and a *single buyer* who has *independently distributed* values for the items, and an additive valuation. In general, the revenue achievable from selling two independent items may be strictly higher than the sum of the revenues obtainable by selling each of them separately. In fact, the structure of optimal (i.e., revenue-maximizing) mechanisms for two items even in this simple setting is not understood.

In this paper we obtain *approximate* revenue optimization results using two simple auctions: that of selling the items *separately*, and that of selling them as a single *bundle*. Our main results (which are of a "direct sum" variety, and apply to any distributions) are as follows. Selling the items separately guarantees at least half the revenue of the optimal auction; for identically distributed items, this becomes at least 73% of the optimal revenue.

For the case of $k > 2$ items, we show that selling separately guarantees at least a $c/\log^2 k$ fraction of the optimal revenue; for identically distributed items, the bundling auction yields at least a $c/\log k$ fraction of the optimal revenue.

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1 Introduction

Suppose that you have one item to sell to a single buyer whose willingness to pay is unknown to you but is distributed according to a known prior (given by a cumulative distribution F). If you offer to sell it for a price p then the probability that the buyer will buy is[1] $1 - F(p)$, and your revenue will be $p \cdot (1 - F(p))$. The seller will choose a price p^* that maximizes this expression.

This problem is exactly the classical monopolist pricing problem, but looking at it from an auction point of view, one may ask whether there are mechanisms for selling the item that yield a higher revenue. Such mechanisms could be indirect, could offer different prices for different probabilities of getting the item, and perhaps others. Yet, Myerson's characterization of optimal auctions (Myerson [1981]) concludes that the take-it-or-leave-it offer at the above price p^* yields the optimal revenue among *all* mechanisms. Even more, Myerson's result also applies when there are multiple buyers, in which case p^* would be the reserve price in a second price auction.

Now suppose that you have two (different) items that you want to sell to a single buyer. Furthermore, let us consider the simplest case where the buyer's values for the items are independently and identically distributed according to F ("i.i.d.-F" for short), and furthermore that his valuation is additive: if the value for the first item is x and for the second is y, then the value for the *bundle* – i.e., getting both items – is[2] $x + y$. It would seem that since the two items are completely independent from each other, then the best we should be able to do is to sell each of them separately in the optimal way, and thus extract exactly twice the revenue we would make from a single item. Yet this turns out to be false.

Example: Consider the distribution taking values 1 and 2, each with probability 1/2. Let us first look at selling a single item optimally: the seller can either choose to price it at 1, selling always[3] and getting a revenue of 1, or choose to price the item at 2, selling it with probability 1/2, still obtaining an expected revenue of 1, and so the optimal revenue for a single item is 1. Now consider the following mechanism for selling both items: bundle them together, and sell the bundle for price 3. The probability that the sum of the buyer's values for the two items is at least 3 is 3/4, and so the revenue is $3 \cdot 3/4 = 2.25$ – larger than 2, which is obtained by selling them separately.

However, that is not always so: bundling may sometimes be worse than selling the

[1] Assume for simplicity that the distribution is continuous.

[2] Our buyer's demand is *not* limited to one item (which is the case in some of the existing literature; see below).

[3] Since we want to maximize revenue we can always assume without loss of generality that ties are broken in a way that maximizes revenue; this can always be achieved by appropriate small perturbations.

items separately. For the distribution taking values 0 and 1, each with probability 1/2, selling the bundle can yield at most a revenue of 3/4, and this is less than twice the single-item revenue of 1/2. In some other cases neither selling separately nor bundling is optimal. For the distribution that takes values 0, 1 and 2, each with probability 1/3, the unique optimal auction turns out to offer to the buyer the choice between any single item at price 2, and the bundle of both items at a "discount" price of 3. This auction gets revenue of 13/9 revenue, which is larger than the revenue of 4/3 obtained from either selling the two items separately, or from selling them as a single bundle. A similar situation happens for the uniform distribution on $[0, 1]$, for which neither bundling nor selling separately is optimal (Manelli and Vincent [2006]). In yet other cases the optimal mechanism is not even deterministic and must offer lotteries for the items. This happens in the following example from Hart and Reny [2011][4]: Let F be the distribution which takes values 1, 2 and 4, with probabilities 1/6, 1/2, 1/3, respectively. It turns out that the unique optimal mechanism offers the buyer the choice between buying any one good *with probability 1/2* for a price of 1, and buying the bundle of both goods (surely) for a price of 4; any deterministic mechanism has a strictly lower revenue.

So, it is not clear what optimal mechanisms for selling two items look like, and indeed characterizations of optimal auctions even for this simple case are not known. We shorty describe some of the previous work on these type of issues. McAfee and McMillan [1988] identify cases where the optimal mechanism is deterministic. However, Thanassoulis [2004] and Manelli and Vincent [2006] found a technical error in the paper and exhibit counter-examples. These last two papers contain good surveys of the work within economic theory, with more recent analysis by Fang and Norman [2006], Jehiel et al. [2007], Hart and Reny [2010], Hart and Reny [2011], Lev [2011]. In the last few years algorithmic work on these types of topics was carried out. One line of work (e.g. Briest et al. [2010] and Cai et al. [2012]) shows that for discrete distributions the optimal auction can be found by linear programming in rather general settings. This is certainly true in our simple setting where the direct representation of the auction constraints provides a polynomial size linear program. Thus we emphasize that the difficulty in our case is *not computational*, but is rather that of characterization and understanding the results of the explicit computations: this is certainly so for continuous distributions, but also for discrete ones.[5] Another line of work in computer science (Chawla et al. [2007], Chawla et al.

[4]Previous examples where randomization helps appear in Manelli and Vincent [2006], Manelli and Vincent [2007] and Thanassoulis [2004], but these require *interdependent* distributions of values, rather than independent and identically distributed values.

[5]If we limit ourselves to *deterministic* auctions (and discrete distributions), finding the optimal one is easy computationally in the case of one buyer (just enumerate), in contrast to the general case of multiple buyers with correlated values for which computational complexity difficulty has been established by Papadimitriou and Pierrakos [2011].

[2010a], Chawla et al. [2010b], Daskalakis and Weinberg [2011]) attempts approximating the optimal revenue by simple mechanisms. This was done for various settings, especially unit-demand settings and some generalizations. One conclusion from this line of work is that for many subclasses of distributions (such as those with monotone hazard rate) various simple mechanisms can extract a constant fraction of the expected value of the items[6]. This is true in our simple setting, where for such distributions selling the items separately provides a constant fraction of the expected value and thus of the optimal revenue.

The current paper may be viewed as continuing this tradition of approximating the optimal revenue with simple auctions. It may also be viewed as studying the extent to which auctions can gain revenue by doing things that appear less "natural" (such as pricing lotteries whose outcomes are the items; of course, the better our understanding becomes, the more things we may consider as natural.) We study two very simple and natural auctions that we show do give good approximations: the first simple auction is to sell the items separately and independently, and the second simple auction is to sell all items together as a bundle. We emphasize that our results hold for *arbitrary* distributions and we do not make any assumptions (such as monotone hazard rate).[7] In particular, our approximations to the optimal revenue also hold when the expected value of the items is arbitrarily (even infinitely) larger than the optimal revenue.

We will denote by $\textsc{Rev}(\mathcal{F}) \equiv \textsc{Rev}_k(\mathcal{F})$ the optimal revenue obtainable from selling, to a single buyer (with an additive valuation), k items whose valuation is distributed according to a (k-dimensional joint) distribution $\mathcal{F}$. This revenue is well understood only for the special case of one item ($k = 1$), i.e., for a one-dimensional F, in which case it is obtained by selling at the Myerson price (i.e., $\textsc{Rev}_1(F) = sup_{p\geq 0} p \cdot (1 - F(p))$. The first three theorems below relate the revenue obtainable from selling multiple independent items optimally (which is not well understood) to the revenue obtainable by selling each of them separately (which is well understood).

Our first and main result shows that while selling two independent items separately need not be optimal, it is not far from optimal and always yields at least half of the optimal revenue. We do not know of any easier proof that provides *any* constant approximation bound.[8]

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[6] In our setting ths is true even more generally, for instance whenever the ratio between the median and the expectation is bounded, which happens in particular when the tail of the distribution is "thinner" than $x^{-\alpha}$ for $\alpha > 1$.

[7] One may argue that there is no need for *uniform* approximation results on the ground that the seller knows the distribution of the buyer's valuation. However, as we have shown above, that does not help finding the optimal auction (even for simple distributions) – whereas the approximations are always easy and simple (as they use only optimal prices for one-dimensional distributions).

[8] There is an easy proof for the special case of *deterministic* auctions, which we leave as an exercise to the reader. It does not seem that this type of easy proof can be extended to general auctions since it

The joint distribution of two items distributed independently according to F_1 and F_2, respectively, is denoted by $F_1 \times F_2$.

Theorem 1 *For every one-dimensional distributions F_1 and F_2,*

$$\textsc{Rev}_1(F_1) + \textsc{Rev}_1(F_2) \geq \frac{1}{2} \cdot \textsc{Rev}_2(F_1 \times F_2).$$

This result is quite robust and generalizes to auctions with multiple buyers, using either the Dominant-Strategy or the Bayes-Nash notions of implementation. It also generalizes to multi-dimensional distributions, i.e., to cases of selling two collections of items, and even to more general mechanism design settings (see Theorems 20 and 30).[9] However, as we show in a companion paper Hart and Nisan [2012], such a result does *not* hold when the values for the items are allowed to be *correlated*: there exists a joint distribution of item values such that the revenue obtainable from each item separately is finite, but selling the items optimally yields infinite revenue.

For the special case of two identically distributed items (one-dimensional and single buyer), i.e., $F_1 = F_2$, we get a tighter result.

Theorem 2 *For every one-dimensional distribution F,*

$$\textsc{Rev}_1(F) + \textsc{Rev}_1(F) \geq \frac{e}{e+1} \cdot \textsc{Rev}_2(F \times F).$$

Thus, for two independent items, each distributed according to F, taking the optimal Myerson price for a single item distributed according to F and offering the buyer to choose which items to buy at that price per item (none, either one, or both), is guaranteed to yield at least 73% of the optimal revenue for the two items. This holds for *any* distribution F (and recall that, in general, we do not know what that optimal revenue is; in contrast, the Myerson price is well-defined and immediate to determine).

There is a small gap between this bound of $e/(e+1) = 0.73...$ and the best separation that we have with a gap of of 0.78... (see Corollary 29). We conjecture that the latter is in fact the tight bound.

We next consider the case of more than two items. It turns out that, as the number of items grows, the ratio between the revenue obtainable from selling them optimally to that obtainable by selling them separately is unbounded. In fact, we present an example showing that the ratio may be as large as $O(\log k)$ (see Lemma 8). Our main positive

would apply also to interdependent item values in which case, as we show in a companion paper Hart and Nisan [2012], there is no finite bound relating the two-item revenue to that of selling them separately.

[9]However, we have not been able to generalize these decomposition results to multiple buyers and multiple items simultaneously.

result for the case of multiple items is a bound on this gap in terms of the number of items. When the k items are independent and distributed according to $F_1, \ldots, F_k$, we write $F_1 \times \cdots \times F_k$ for their product joint distribution.

Theorem 3 *There exists a constant $c > 0$ such that for every integer $k \geq 2$ and every one-dimensional distributions $F_1, \ldots, F_k$,*

$$\textsc{Rev}_1(F_1) + \cdots + \textsc{Rev}_1(F_k) \geq \frac{c}{\log^2 k} \cdot \textsc{Rev}_k(F_1 \times \cdots \times F_k).$$

We then consider the other simple single-dimensional auction, the *bundling* auction, which offers a single price for the bundle of all items.[10] We ask how well it can approximate the optimal revenue. We first observe that, in general, the bundling auction may do much worse and only yield a revenue that is a factor of almost k times lower than that of the optimal auction (see Example 15; moreover, we show in Lemma 14 that this is tight up to a constant factor). However, when the items are independent and *identically* distributed, then the bundling auction does much better. It is well known (Armstrong [1999], Bakos and Brynjolfsson [1999]) that for every fixed distribution F, as the number of items distributed independently according to F approaches infinity, the bundling auction approaches the optimal one (for completeness we provide a short proof in Appendix D.) This, however, requires k to grow as F remains fixed. On the other hand, we show that this is not true uniformly over F: for every large enough k, there are distributions where the bundling auction on k items extracts less than 57% of the optimal revenue (Example 19). Our main result for the bundling auction is that in this case it extracts a logarithmic (in the number of items k) fraction of the optimal revenue. We do not know whether the loss is in fact bounded by a constant fraction. Since the distribution of the sum of k independent and identically distributed according to F items is the k-times convolution $F * \cdots * F$, our result is:

Theorem 4 *There exists a constant $c > 0$ such that for every integer $k \geq 2$ and every one-dimensional distribution F,*

$$\textsc{Rev}_1(\underbrace{F * \cdots * F}_{k}) \geq \frac{c}{\log k} \cdot \textsc{Rev}_k(\underbrace{F \times \cdots \times F}_{k}).$$

Many problems are left open. From the general point of view, the characterization of the optimal auction is still mostly open, despite the many partial results in the cited papers. In particular, it is open to fully characterize when selling separately is optimal;

[10]By Myerson's result, this is indeed the optimal mechanism for selling the bundle.

when the bundling auction is optimal[11]; or when are deterministic auctions optimal. More specifically, regarding our approximation results, gaps remain between our lower bounds and upper bounds.

The structure of the paper is as follows. In Section 2 we present our notations and the preliminary setup. Section 3 studies the relations between the bundling auction and selling separately; these relations are not only interesting in their own right, but are then also used as part of the general analysis and provide us with most of the examples that we have for gaps in revenue. Section 4 studies the case of two items, gives the main decomposition theorem together with a few extensions; Section 5 gives our results for more than two items. Several proofs are postponed to appendices. Finally, Appendix E provides a table summarizing our bounds on the revenue gaps between the separate auction and the optimal auction, and between the bundling auction and the optimal auction.

2 Notation and Preliminaries

2.1 Mechanisms

A mechanism for selling k items specifies a (possibly randomized) protocol for interaction between a seller (who has no private information and commits to the mechanism) and a buyer who has a private valuation for the items. The outcome of the mechanism is an allocation specifying the probability of getting each of the k items and an (expected)[12] payment that the buyer gives to the seller. We will use the following notations:

- **Buyer valuation:** $x = (x_1, \ldots, x_k)$ where $x_i \geq 0$ denotes the value of the buyer for getting item i.
- **Allocation:** $q = (q_1, \ldots, q_k) \in [0,1]^k$, where $q_i = q_i(x)$ denotes the probability that item i is allocated to the buyer when his valuation is x (alternatively, one may interpret q_i as the fractional quantity of item i that the buyer gets).
- **Seller revenue:** $s = s(x)$ denotes the expected payment[13] that the seller receives from the buyer when the buyer's valuation is x.

[11] We do show that this is the case for a class of distributions that decrease not too slowly; see Theorem 28.

[12] We only consider risk-neutral agents.

[13] In the literature this is also called transfer, cost, price, revenue, and denoted by p, t, c, etc. We hope that using the mnemonic s for the ***S***eller's final payoff and b for the ***B***uyer's final payoff will avoid confusion.

- **Buyer utility:** $b = b(x)$ denotes the utility of the buyer when his valuation is x, i.e., $b(x) = \sum_i x_i q_i(x) - s(x) = x \cdot q(x) - s(x)$.

We will be discussing mechanisms that are:

- **IR – (Ex-post) Individually Rational**: $b(x) \geq 0$ for all x.
- **IC – Incentive Compatible**: For all x, x': $\sum_i x_i q_i(x) - s(x) \geq \sum_i x_i q_i(x') - s(x')$.

The IC requirement simply captures the notion that the buyer acts strategically in the mechanism. Since we are discussing a single buyer, this is in a simple decision-theoretic sense and in particular there is no distinction between the domainant strategy and the Bayes-Nash implementation notions.

The following lemma gives well known and easily proven equivalent conditions for incentive compatibility.

Lemma 5 *The following three definitions are equivalent for a mechanism with* $b(x) = x \cdot q(x) - s(x) = \sum_i x_i q_i(x) - s(x)$*:*

1. *The mechanism is IC.*
2. *The allocation q is weakly monotone, in the sense that for all x, x' we have $(x - x') \cdot (q(x) - q(x')) \geq 0$, and the payment to the seller satisfies $x' \cdot (q(x) - q(x')) \leq s(x) - s(x') \leq x \cdot (q(x) - q(x'))$ for all x, x'.*
3. *The buyer's utility b is a convex function of x and for all x the allocation $q(x)$ is a subgradient of b at x, i.e., for all x' we have $b(x') - b(x) \geq q(x) \cdot (x' - x)$. In particular b is differentiable almost everywhere and there $q_i(x) = \partial b(x)/\partial x_i$.*

Proof. • *1 implies 2:* The RHS of the second part is the IC constraint for x, the LHS is the IC constraint for x', and the whole second part directly implies the first part.

• *2 implies 1:* Conversely, the RHS of the second part is exactly the IC constraint for x.

• *1 implies 3:* By IC, $b(x) = \sup_{x'} x \cdot q(x') - s(x')$ is a supremum of linear functions of x and is thus convex. For the second part, $b(x') - b(x) - q(x) \cdot (x' - x) = x' \cdot q(x') + s(x) - s(x') - x' \cdot q(x) \geq 0$, where the inequality is exactly the IC constraint for x'.

• *3 implies 1:* Conversely, as in the previous line, the subgradient property at x is exactly equivalent to the IC constraint for x'. ∎

Note that this in particular implies that any convex function b with $0 \leq \partial b(x)/\partial x_i \leq 1$ for all i defines an incentive compatible mechanism by setting $q_i(x) = \partial b(x)/\partial x_i$ (at non-differentiability points take q to be an arbitrary subgradient of b) and $s(x) = x \cdot q(x) - b(x)$.

When $x_1, \ldots, x_k$ are distributed according to the joint cumulative distribution function $\mathcal{F}$ on[14] $\mathbb{R}_+^k$, the expected revenue of the mechanism given by b is

$$R(b; \mathcal{F}) = \mathbb{E}_{x \sim \mathcal{F}}(s(x)) = \int \cdots \int \left(\sum_{i=1}^{k} x_i \frac{\partial b(x)}{\partial x_i} - b(x) \right) \mathrm{d}\mathcal{F}(x_1, \ldots, x_k).$$

Thus we want to maximize this expression over all convex functions b with $0 \le \partial b(x)/\partial x_i \le 1$ for all i. We can also assume

- **NPT – No Positive Transfers**: $s(x) \ge 0$ for all x.

This is without loss of generality as any mechanism can be converted to an NPT one, with the revenue only increasing.[15] This in particular implies that $b(0) = 0$ without loss of generality (as it follows from IR+NPT).

2.2 Revenue

For a cumulative distribution $\mathcal{F}$ on $\mathbb{R}_+^k$ (for $k \ge 1$), we consider the optimal revenue obtainable from selling k items to a (single, additive) buyer whose valuation for the k items is jointly distributed according to $\mathcal{F}$:

- $\textsc{Rev}(\mathcal{F}) \equiv \textsc{Rev}_k(\mathcal{F})$ is the maximal revenue obtainable by any incentive compatible and individually rational mechanism.
- $\textsc{SRev}(\mathcal{F})$ is the maximal revenue obtainable by selling each item *separately.*
- $\textsc{BRev}(\mathcal{F})$ is the maximal revenue obtainable by *bundling* all items together.

Thus, $\textsc{Rev}(\mathcal{F}) = sup_b R(b; \mathcal{F})$ where b ranges over all convex functions with $0 \le \partial b(x)/\partial x_i \le 1$ for all i and $b(0) = 0$. It will be often convenient to use random variables rather than distributions, and thus we use $\textsc{Rev}(X)$ and $\textsc{Rev}(\mathcal{F})$ interchangeably when the buyer's valuation is a random variable $X = (X_1, \ldots, X_k)$ with values in $\mathbb{R}_+^k$ distributed according to $\mathcal{F}$. In this case we have $\textsc{SRev}(X) = \textsc{Rev}(X_1) + \cdots + \textsc{Rev}(X_k)$ and $\textsc{BRev}(X) = \textsc{Rev}(X_1 + \cdots + X_k)$.

This paper will only deal with *independently* distributed item values, that is, $\mathcal{F} = F_1 \times \cdots \times F_k$, where F_i is the distribution of item[16] i. We have[17] $\textsc{SRev}(\mathcal{F}) = \textsc{Rev}(F_1) +$

[14] We write this as $x = (x_1, \ldots, x_k) \sim \mathcal{F}$. We use $\mathcal{F}$ for multi-dimensional distributions and F for one-dimensional distributions.

[15] For each x with $s(x) < 0$ redefine $q(x)$ and $s(x)$ as $q(x')$ and $s(x')$ for x' that maximizes $\sum_i x_i q_i(x') - s(x')$ over those x' with $s(x') \ge 0$.

[16] As these are cumulative distribution functions, we have $\mathcal{F}(x_1, \ldots, x_k) = F_1(x_1) \cdot \ldots \cdot F_k(x_k)$.

[17] The formula for $\textsc{SRev}$ holds without independence, with F_i the i-th marginal distribution of $\mathcal{F}$.

$\cdots + \textsc{Rev}(F_k)$ and $\textsc{BRev}(\mathcal{F}) = \textsc{Rev}(F_1 * \cdots * F_k)$, where * denotes convolution. Our companion paper Hart and Nisan [2012] studies general distributions $\mathcal{F}$, i.e., interdependent values.

For $k = 1$ we have **Myerson's characterization** of the optimal revenue:

$$\textsc{Rev}_1(X) = \textsc{SRev}(X) = \textsc{BRev}(X) = \sup_{p \geq 0} p \cdot \mathbb{P}(X \geq p)$$

(which also equals $\sup_{p\geq 0} p \cdot \mathbb{P}(X > p) = \sup_{p \geq 0} p \cdot (1 - F(p)))$.

Note that for any k, both the separate revenue $\textsc{SRev}$ and the bundling revenue $\textsc{BRev}$ require solving only one-dimensional problems; by Myerson's characterization, the former is given by k item prices $p_1, \ldots, p_k$, and the latter by one price $\bar{p}$ for all items together.

3 Warm up: Selling Separately vs. Bundling

In this section we analyze the gaps between the two simple auctions: bundling and selling the items separately. Not only are these comparisons interesting in their own right, but they will be used as part of our general analysis, and will also provide the largest lower bounds we have on the approximation ratios of these two auctions relative to the optimal revenue.

We start with a particular distribution which will turn out to be key to our analysis. We then prove upper bounds on the bundling revenue in terms of the separate revenue, and finally we prove upper bounds on the separate revenue in terms of the bundling revenue.

3.1 The Equal-Revenue Distribution

We introduce the distribution which we will show is extremal in the sense of maximizing the ratio between the bundling auction revenue and the separate auction revenue.

Let us denote by ER – the *equal-revenue distribution* – the distribution with density function $f(x) = x^{-2}$ for $x \geq 1$; its cumulative distribution function is thus $F(x) = 1 - x^{-1}$ for $x \geq 1$ (and for $x < 1$ we have $f(x) = 0$ and $F(x) = 0$). (This is also called the *Pareto* distribution with parameter $\alpha = 1$.) It is easy to see that, on one hand, $\textsc{Rev}_1(ER) = 1$ and, moreover, this revenue is obtained by choosing any price $p \geq 1$. On the other hand its expected value is infinite: $\mathbb{E}(ER) = \int_1^\infty x \cdot x^{-2} dx = \infty$. We start with a computation of the distribution of the weighted sum of two ER distributions.

Lemma 6 *Let X_1, X_2 be*[18] *i.i.d.-ER and $\alpha, \beta > 0$. Then*[19]

$$\mathbb{P}(\alpha X_1 + \beta X_2 \geq z) = \frac{\alpha\beta}{z^2} \log\left(1 + \frac{z^2 - (\alpha+\beta)z}{\alpha\beta}\right) + \frac{\alpha+\beta}{z}$$

for $z \geq \alpha + \beta$, and $\mathbb{P}(\alpha X_1 + \beta X_2 \geq z) = 1$ for $z \leq \alpha + \beta$.

Proof. Let $Z = \alpha X_1 + \beta X_2$. For $z \leq \alpha + \beta$ we have $\mathbb{P}(Z \geq z) = 1$ since $X_i \geq 1$. For $z > \alpha + \beta$ we get

$$\begin{aligned}
\mathbb{P}(Z \geq z) &= \int f(x)\left(1 - F\left(\frac{z - \alpha x}{\beta}\right)\right) \mathrm{d}x \\
&= \int_1^{(z-\beta)/\alpha} \frac{1}{x^2} \frac{\beta}{z - \alpha x} \, \mathrm{d}x + \int_{(z-\beta)/\alpha}^{\infty} \frac{1}{x^2} 1 \, \mathrm{d}x \\
&= \frac{\beta}{z}\left[\frac{\alpha}{z}\log x - \frac{\alpha}{z}\log\left(\frac{z}{\alpha} - x\right) - \frac{1}{x}\right]_1^{(z-\beta)/\alpha} + \frac{\alpha}{z-\beta} \\
&= \frac{\alpha\beta}{z^2}\left(\log\left(\frac{z}{\beta} - 1\right) + \log\left(\frac{z}{\alpha} - 1\right)\right) - \frac{\alpha\beta}{z(z-\beta)} + \frac{\beta}{z} + \frac{\alpha}{z-\beta} \\
&= \frac{\alpha\beta}{z^2}\log\left(1 + \frac{z^2 - (\alpha+\beta)z}{\alpha\beta}\right) + \frac{\alpha+\beta}{z} \, .
\end{aligned}$$

■

We can now calculate the revenue obtainable from bundling several independent ER items.

Lemma 7 $\mathrm{BRev}(ER \times ER) = 2.5569...$, *where* $2.5569... = 2(w+1)$ *with w the solution of*[20] $we^w = 1/e$.

Remark. We will see below (Corollary 29) that bundling is optimal here, and so 2.5569... is in fact the *optimal revenue* for two i.i.d.-ER items.

Proof. Using Lemma 6 with $\alpha = \beta = 1$ yields $p \cdot \mathbb{P}(X_1 + X_2 \geq p) = p^{-1}\log(1 + p^2 - 2p) + 2 = 2p^{-1}\log(p-1) + 2$, which attains its maximum of $2w + 2$ at $p = 1 + 1/w$. ■

Lemma 8 *There exist constants $c_1, c_2 > 0$ such that for all $k \geq 2$,*

$$c_1 k \log k \leq \mathrm{BRev}(ER^{\times k}) \leq c_2 k \log k.$$

[18] For a one-dimensional distribution F, "i.i.d.-F" refers to a collection of independent random variables each distributed according to F.

[19] log denotes *natural* logarithm.

[20] Thus $w = W(1/e)$ where W is the so-called "Lambert-W" function.

In particular, this shows that selling separately may yield, as k increases, an arbitrarily small proportion of the optimal revenue: $\textsc{Rev}(ER^{\times k}) \geq \textsc{BRev}(ER^{\times k}) \geq c_1 k \log k = c_1 \log k \cdot \textsc{SRev}(ER^{\times k})$.

Proof. Let X be a random variable with distribution ER; for $M \geq 1$ let $X^M := \min\{X, M\}$ be X truncated at M. It is immediate to compute $\mathbb{E}(X^M) = \log M + 1$ and $\mathbb{V}\mathrm{ar}(X^M) \leq 2M$.

• *Lower bound*: Let $X_1, \ldots, X_k$ be i.i.d.-ER; for every $p, M > 0$ we have $\textsc{Rev}(\sum_i X_i) \geq p \cdot \mathbb{P}\left(\sum_i X_i \geq p\right) \geq p \cdot \mathbb{P}\left(\sum_i X_i^M \geq p\right)$.

When $M = k \log k$ and $p = (k \log k)/2$ we get $(k\mathbb{E}(X^M) - p)/\sqrt{k\mathbb{V}\mathrm{ar}(X^M)} \geq \sqrt{\log k/8}$, so p is at least $\sqrt{\log k/8}$ standard deviations below the mean of $\sum_{i=1}^k X_i^M$. Therefore, by Chebyshev's inequality, $\mathbb{P}(\sum_{i=1}^k X_i^M \geq p) \geq 1 - 8/\log k \geq 1/2$ for all k large enough, and then $\textsc{Rev}(\sum_{i=1}^k X_i) \geq p \cdot 1/2 = k \log k/4$.

• *Upper bound*: We need to bound $\sup_{p \geq 0} p \cdot \mathbb{P}(\sum_{i=1}^k X_i \geq p)$. If $p \leq 6k \log k$ then $p \cdot \mathbb{P}(\sum_{i=1}^k X_i \geq p) \leq p \leq 6k \log k$.

If $p \geq 6k \log k$ then (take $M = p$)

$$p \cdot \mathbb{P}\left(\sum_{i=1}^{k} X_i \geq p\right) \leq p \cdot \mathbb{P}\left(\sum_{i=1}^{k} X_i^p \geq p\right) + p \cdot \mathbb{P}\left(X_i > p \text{ for some } 1 \leq i \leq k\right). \quad (1)$$

The second term is at most $p \cdot k \cdot (1 - F(p)) = k$ (since $F(p) = 1 - 1/p$). To estimate the first term, we again use Chebyshev's inequality. When k is large enough we have $p/(k(\log p + 1)) \leq 2$ (recall that $p \geq 6k \log k$), and so p is at least $\sqrt{p/(8k)}$ standard deviations above the mean of $\sum_{i=1}^k X_i^p$. Thus $p \cdot \mathbb{P}(\sum_{i=1}^k X_i^p \geq p) \leq p \cdot (8k)/p = 8k$, and so $p \cdot \mathbb{P}(\sum_{i=1}^k X_i \geq p) \leq 9k$ (recall (1)).

Altogether, $\textsc{Rev}(\sum_{i=1}^k X_i) \leq \max\{6k \log k, 9k\} = 6k \log k$ for all k large enough. ∎

Remark. A more precise analysis, based on the "Generalized Central Limit Theorem,"[21] shows that $\textsc{BRev}(ER^{\times k})/(k \log k)$ converges to 1 as $k \to \infty$. Indeed, when X_i are i.i.d.-ER, the sequence $(\sum_{i=1}^k X_i - b_k)/a_k$ with $a_k = k\pi/2$ and $b_k = k \log k + \Theta(k)$ converges in distribution to the Cauchy distribution as $k \to \infty$. Since $\textsc{Rev}_1(\text{Cauchy})$ can be shown to be bounded (by $1/\pi$), it follows that $\textsc{Rev}(\sum_{i=1}^k X_i) = k \log k + \Theta(k)$.

3.2 Upper Bounds on the Bundling Revenue

It turns out that the equal revenue distribution exhibits the largest possible ratio between the bundling auction and selling separately. This is a simple corollary from the fact that the equal revenue distribution has the heaviest possible tail.

[21] See, e.g., Zaliapin et al. [2005].

Let X and Y be one-dimensional random variables. We say that X is *(first-order) stochastically dominated* by Y if for every real p we have $\mathbb{P}(X \geq p) \leq \mathbb{P}(Y \geq p)$. Thus, Y gets higher values than X.

Lemma 9 *If a one-dimensional X is stochastically dominated by a one-dimensional Y then* $\textsc{Rev}_1(X) \leq \textsc{Rev}_1(Y)$.

Proof. $\textsc{Rev}(X) = \sup_p p \cdot \mathbb{P}(X \geq p) \leq \sup_p p \cdot \mathbb{P}(Y \geq p) = \textsc{Rev}(Y)$ (by Myerson's characterization). ■

It should be noted that this monotonicity of the revenue with respect to stochastic dominance does *not* hold when there are two or more items Hart and Reny [2011].

Lemma 10 *For every one-dimensional X and every $r > 0$:* $\textsc{Rev}_1(X) \leq r$ *if and only if X is stochastically dominated by*[22] $r \cdot ER$.

Proof. By Myerson's characterization, $\textsc{Rev}(X) \leq r$ if and only if for every p we have $\mathbb{P}(X \geq p) \leq r/p$; but r/p is precisely the probability that $r \cdot ER$ is at least p. ■

We will thus need to consider sums of "scaled" versions of ER, i.e., linear combinations of independent ER random variables. What we will see next is that equalizing the scaling factors yields stochastic domination.

Lemma 11 *Let X_1, X_2 be i.i.d.-ER and let $\alpha, \beta, a', \beta' > 0$ satisfy $\alpha + \beta = \alpha' + \beta'$. If*[23] *$\alpha\beta \leq \alpha'\beta'$ then $\alpha X_1 + \beta X_2$ is stochastically dominated by $\alpha' X_1 + \beta' X_2$.*

Proof. Let $Z = \alpha X_1 + \beta X_2$ and $Z' = \alpha' X_1 + \beta' X_2$, and put $\gamma = \alpha + \beta = \alpha' + \beta'$. Using Lemma 6, for $z \leq \gamma$ we have $\mathbb{P}(Z \geq z) = \mathbb{P}(Z' \geq z) = 1$, and for $z > \gamma$ we get

$$\begin{aligned} \mathbb{P}(Z \geq z) &= \frac{\alpha\beta}{z^2} \log\left(1 + \frac{z^2 - \gamma z}{\alpha\beta}\right) + \frac{\gamma}{z} \\ &\leq \frac{\alpha'\beta'}{z^2} \log\left(1 + \frac{z^2 - \gamma z}{\alpha'\beta'}\right) + \frac{\gamma}{z} = \mathbb{P}(Z' \geq z), \end{aligned}$$

since $t \log(1 + 1/t)$ is increasing in t for $t > 0$, and $\alpha\beta/(z^2 - \gamma z) \leq \alpha'\beta'/(z^2 - \gamma z)$ by our assumption that $\alpha\beta \leq \alpha'\beta'$ together with $z > \gamma$. ■

We note the following useful fact: if for every i, X_i is stochastically dominated by Y_i, then $X_1 + \cdots + X_k$ is stochastically dominated by[24] $Y_1 + \cdots + Y_k$.

[22] We slightly abuse the notation and write $r \cdot ER$ for a random variable $r \cdot Y$ when Y is distributed according to ER.

[23] Equivalently, $|\alpha - \beta| \geq |\alpha' - \beta'|$.

[24] Think of all the random variables being defined on the same probability space and satisfying $X_i \leq Y_i$ pointwise (which can be obtained by the so-called "coupling" construction), and then $\sum X_i \leq \sum Y_i$ is immediate.

Corollary 12 *Let X_i be i.i.d.-ER and $\alpha_i > 0$. Then $\sum_{i=1}^{k} \alpha_i X_i$ is stochastically dominated by $\sum_{i=1}^{k} \bar{\alpha} X_i$, where $\bar{\alpha} = (\sum_{i=1}^{k} \alpha_i)/k$.*

Proof. If, say, $\alpha_1 < \bar{\alpha} < \alpha_2$, then the previous lemma implies that $\alpha_1 X_1 + \alpha_2 X_2$ is stochastically dominated by $\bar{\alpha} X_1 + \alpha_2' X_2$,where $\alpha_2' = \alpha_1 + \alpha_2 - \bar{\alpha}$, and so $\sum_{i=1}^{k} \alpha_i X_i$ is stochastically dominated by $\bar{\alpha} X_1 + \alpha_2' X_2 + \sum_{i=3}^{k} \alpha_i X_i$. Continue in the same way until all coefficients become $\bar{\alpha}$. ■

We can now provide our upper bounds on the bundling revenues.

Lemma 13 *(i) For every one-dimensional distributions F_1, F_2,*

$$\mathrm{BR{\scriptsize EV}}(F_1 \times F_2) \leq 1.278... \cdot (\mathrm{R{\scriptsize EV}}(F_1) + \mathrm{R{\scriptsize EV}}(F_2)) = 1.278... \cdot \mathrm{SR{\scriptsize EV}}(F_1 \times F_2),$$

where $1.278... = w + 1$ with w the solution of $we^w = 1/e$.

(ii) There exists a constant $c > 0$ such that for every $k \geq 2$ and every one-dimensional distributions $F_1, \ldots, F_k$,

$$\mathrm{BR{\scriptsize EV}}(F_1 \times \cdots \times F_k) \leq c \log k \cdot \sum_{i=1}^{k} \mathrm{R{\scriptsize EV}}(F_i) = c \log k \cdot \mathrm{SR{\scriptsize EV}}(F_1 \times \cdots \times F_k).$$

Proof. Let X_i be distributed according to F_i, and denote $r_i = \mathrm{R{\scriptsize EV}}(F_i)$, so X_i is stochastically dominated by $r_i Y_i$ where Y_i is distributed according to ER (see Lemma 10). Assume that the X_i are independent, and also that the Y_i are independent. Then $X_1 + \cdots + X_k$ is stochastically dominated by $r_1 Y_1 + \cdots + r_k Y_k$. By Corollary 12 the latter is stochastically dominated by $\bar{r} Y_1 + \cdots + \bar{r} Y_k$ where $\bar{r} = (\sum_i r_i)/k = (\sum_i \mathrm{R{\scriptsize EV}}(F_i))/k$. Therefore $\mathrm{BR{\scriptsize EV}}(F_1 \times \cdots \times F_k) \leq \bar{r} \mathrm{BR{\scriptsize EV}}(ER^{\times k})$, and the results (i) and (ii) follow from Lemmas 7 and 8 respectively. ■

3.3 Lower Bounds on the Bundling Revenue

In general, the bundling revenue obtainable from items that are independently distributed according to different distributions may be significantly smaller than the separate revenue.

Lemma 14 *For every integer $k \geq 1$ and every one-dimensional distributions $F_1, \ldots, F_k$,*

$$\mathrm{BR{\scriptsize EV}}(F_1 \times \cdots \times F_k) \geq \frac{1}{k} \cdot \sum_{i=1}^{k} \mathrm{R{\scriptsize EV}}(F_i) = \frac{1}{k} \cdot \mathrm{SR{\scriptsize EV}}(F_1 \times \cdots \times F_k).$$

Proof. For every i we have $\mathrm{R{\scriptsize EV}}(F_i) \leq \mathrm{BR{\scriptsize EV}}(F_1 \times \cdots \times F_k)$, and so $\sum_i \mathrm{R{\scriptsize EV}}(F_i) \leq k \cdot \mathrm{BR{\scriptsize EV}}(F_1 \times \cdots \times F_k)$. ■

This is tight:

Example 15 $\text{BRev}(F_1 \times \cdots \times F_k) = (1/k + \epsilon) \cdot \text{SRev}(F_1 \times \cdots \times F_k)$:
Take a large M and let F_i have support $\{0, M^i\}$ with $\mathbb{P}(M^i) = M^{-i}$. Then $\text{Rev}(F^i) = 1$ and so $\text{SRev}(F^1 \times \cdots \times F^k) = k$, while $\text{BRev}(F^1 \times \cdots \times F^k)$ is easily seen to be at most $\max_i M^i \cdot (M^{-i} + \cdots + M^{-k}) \leq 1 + 1/(M-1)$.

However, when the items are distributed according to *identical* distributions, the bundling revenue cannot be much smaller than the separate revenue, and this is the case that the rest of this section deals with.

Lemma 16 *For every one-dimensional distribution F,*

$$\text{BRev}(F \times F) \geq \frac{4}{3} \cdot \text{Rev}(F) = \frac{2}{3} \cdot \text{SRev}(F \times F).$$

Proof. Let X be distributed according to F; let p be the optimal Myerson price for X and $q = \mathbb{P}(X \geq p)$, so $\text{Rev}(F) = pq$. If $q \leq 2/3$ then the bundling auction can offer a price of p and the probability that the bundle will be sold is at least the probability that one of the items by itself has value p, which happens with probability $2q - q^2 = q(2-q) \geq 4q/3$, so the revenue will be at least $4q/3 \cdot p = (4/3)\text{Rev}(F)$. On the other hand, if $q \geq 2/3$ then the bundling auction can offer price $2p$, and the probability that it will be accepted is at least the probability that both items will get value of at least p, i.e. q^2. The revenue will be $2q^2 p \geq (4/3)qp = (4/3)\text{Rev}(F)$. ∎

This bound is tight:

Example 17 $\text{BRev}(F \times F) = (2/3) \cdot \text{SRev}(F \times F)$:
Let F have support $\{0, 1\}$ with $\mathbb{P}(1) = 2/3$, then $\text{Rev}(F) = 2/3$ while $\text{BRev}(F \times F) = 8/9$ (which is obtained both at price 1 and at price 2).[25]

We write F^{*k} for the k-times convolution of F; this is the distribution of the *sum* of k i.i.d. random variables each distributed according to F.

Lemma 18 *For every integer $k \geq 1$ and every one-dimensional distribution F,*

$$\text{BRev}(F^{\times k}) = \text{Rev}(F^{*k}) \geq \frac{1}{4} k \cdot \text{Rev}(F) = \frac{1}{4} \cdot \text{SRev}(F^{\times k}).$$

[25] It can be checked that the optimal revenue is attained here by the separate auction, i.e., $\text{Rev}(F \times F) = \text{SRev}(F \times F) = 4/3$.

Proof. Let X be distributed according to F; let p be the optimal Myerson price for X and $q = \mathbb{P}(X \geq p)$, so $\textsc{Rev}(F) = pq$. We separate between two cases. If $qk \leq 1$ then the bundling auction can offer price p and, using inclusion-exclusion, the probability that it will be taken is bounded from below by $kq - \binom{k}{2}q^2 \geq kq/2$ so the revenue will be at least $kqp/2 \geq k \cdot \textsc{Rev}(F)/2$. If $qk \geq 1$ then we can offer price $p\lfloor qk \rfloor$. Since the median in a Binomial(k, q) distribution is known to be at least $\lfloor qk \rfloor$, the probability that the buyer will buy is at least $1/2$. The revenue will be at least $p\lfloor qk \rfloor/2 \geq kqp/4 = k \cdot \textsc{Rev}(F)/4$. ■

We have not attempted optimizing this constant $1/4$, which can be easily improved. The largest gap that we know of is the following example where the bundling revenue is less than 57% than that of selling the items separately, and applies to all large enough k. We suspect that this is in fact the maximal possible gap.

Example 19 *For every k large enough, a one-dimensional distribution F such that* $\textsc{BRev}(F^{\times k})/\textsc{SRev}(F^{\times k}) \leq 0.57$:

Take a large k and consider the distribution F on $\{0, 1\}$ with $\mathbb{P}(1) = c/k$ where $c = 1.256...$ is the solution of $1 - e^{-c} = 2(1 - (c + 1)e^{-c})$, so the revenue from selling a single item is c/k. The bundling auction should clearly offer an integral price. If it offers price 1 then the probability of selling is $1 - (1 - c/k)^k \approx 1 - e^{-c} = 0.715...$, which is also the expected revenue. If it offers price 2 then the probability of selling is $1-(1-c/k)^k - k(c/k)(1-c/k)^{k-1} \approx 1-(c+1)e^{-c}$ and the revenue is twice that, again $0.715...$. If it offers price 3 then the probability of selling is $1 - (1 - c/k)^k - k(c/k)(1 - c/k)^{k-1} - \binom{k}{2}(c/k)^2(1 - c/k)^{k-2} \approx 1 - (1 + c + c^2/2)e^{-c} \approx 0.13...$, and the revenue is three times higher, which is less than 0.715. For higher integral prices t the probability of selling is bounded from above by $c^t/t!$, the revenue is t times that, and is even smaller. Thus $\textsc{BRev}(F^{\times k})/\textsc{SRev}(F^{\times k}) \approx 0.715/1.256 \leq 0.57$ for all k large enough.

4 Two Items

Our main result is an "approximate direct sum" theorem. We start with a short proof of Theorem 1 which deals with two independent items. The arguments used in this proof are then extended to a more general setup of two independent *sets* of items.

4.1 A Direct Proof of Theorem 1

In this section we provide a short and direct proof of Theorem 1 (see the Introduction), which says that $\textsc{Rev}(F_1 \times F_2) \leq 2(\textsc{Rev}(F_1) + \textsc{Rev}(F_2))$.

Proof of Theorem 1. Let X and Y be independent one-dimensional nonnegative random variables. Take any IC and IR mechanism (q, s). We will split its expected revenue into two parts, according to which one of X and Y is maximal: $\mathbb{E}(s(X,Y)) \leq \mathbb{E}(\mathbb{1}_{X \geq Y}\, s(X,Y)) + \mathbb{E}(\mathbb{1}_{Y \geq X}\, s(X,Y))$ (the inequality since $\mathbb{1}_{X=Y}\, s(X,Y)$ is counted twice; recall that $s \geq 0$ by NPT). We will show that

$$\mathbb{E}(\mathbb{1}_{X \geq Y}\, s(X,Y)) \leq 2\textsc{Rev}(X); \tag{2}$$

interchanging X and Y completes the proof.

To prove (2), for every fixed value y of Y define a mechanism $(\tilde{q}, \tilde{s})$ for X by $\tilde{q}(x) := q_1(x, y)$ and $\tilde{s}(x) := s(x,y) - yq_2(x,y)$ for every x (so the buyer's payoff remains the same: $\tilde{b}(x) = b(x,y)$). The mechanism $(\tilde{q}, \tilde{s})$ is IC and IR for X, since (q, s) was IC and IR for (X, Y) (only the IC constraints with y fixed, i.e., (x', y) vs. (x, y), matter). Therefore

$$\begin{aligned}
\textsc{Rev}(X) &\geq \textsc{Rev}(\mathbb{1}_{X \geq y}\, X) \geq \mathbb{E}(\mathbb{1}_{X \geq y}\, \tilde{s}(X)) \\
&\geq \mathbb{E}(\mathbb{1}_{X \geq y}\, (s(X,y) - y)) \geq \mathbb{E}(\mathbb{1}_{X \geq y}\, s(X,y)) - \textsc{Rev}(X),
\end{aligned}$$

where we have used NPT for the one-dimensional X for the first inequality; $\tilde{s}(x) = s(x,y) - yq_2(x,y) \geq s(x,y) - y$ (since $y \geq 0$ and $q_2 \leq 1$) for the third inequality; and $\mathbb{E}(\mathbb{1}_{X \geq y}\, y) = \mathbb{P}(X \geq y)\, y \leq \textsc{Rev}(X)$ (since posting a price of y is an IC and IR mechanism for X) for the last inequality. This holds for every value y of Y; taking expectation over y (recall that X is independent of Y) yields (2). ■

4.2 The Main Decomposition Result

We now generalize the decomposition of the previous section to two sets of items. In this section X is a k_1-dimensional nonnegative random variable and Y is a k_2-dimensional nonnegative random variable (with $k_1, k_2 \geq 1$). While we will assume that the vectors X and Y are *independent*, we allow for arbitrary interdependence among the coordinates of X, and the same for the coordinates of Y.

Theorem 20 (Generalization of Theorem 1) *Let X and Y be multi-dimensional random variables. If X and Y are independent then*

$$\textsc{Rev}(X, Y) \leq 2\,(\textsc{Rev}(X) + \textsc{Rev}(Y)).$$

The proof of this theorem is divided into a series of lemmas. The main insights are the "Marginal Mechanism" (Lemma 21) and the "Smaller Value" (Lemma 25).

The first attempt in bounding the revenue from two items, is to fix one of them and look at the induced marginal mechanism on the second. Let us use the notation $\text{VAL}(X) = \mathbb{E}(\sum_i X_i) = \sum_i \mathbb{E}(X_i)$, the expected total sum of values, for multi-dimensional X's (for one-dimensional X this is $\text{VAL}(X) = \mathbb{E}(X)$.)

Lemma 21 (Marginal Mechanism) *Let X and Y be multi-dimensional random variables (here X and Y may be dependent). Then*

$$\text{REV}(X, Y) \leq \text{VAL}(Y) + \mathbb{E}_Y[\text{REV}(X|Y)],$$

where $(X|Y)$ denotes the conditional distribution of X given Y.

Proof. Take a mechanism that obtains the optimal revenue from (X, Y), and fix some value of $y = (y_1, \ldots, y_{k_2})$. The induced mechanism on the X-items, which are distributed according to $(X|Y = y)$, is IC and IR, but also hands out quantities of the Y items. If we modify it so that instead of allocating y_j with probability $q_j = q_j(x, y)$, it pays back to the buyer an additional money amount of $q_j y_j$, we are left with an IC and IR mechanism for the X items. The revenue of this mechanism is that of the original mechanism conditioned on $Y = y$ minus the expected value of $\sum_j q_j y_j$, which is bounded from above by $\sum_j y_j$. Now take expectation over the values y of Y to get $\mathbb{E}_{y \sim Y}[\text{REV}(X|Y = y)] \geq \text{REV}(X, Y) - \text{VAL}(Y)$. ∎

Remark. When X and Y are *independent* then $(X|Y = y) = X$ for every y and thus $\text{REV}(X, Y) \leq \text{VAL}(Y) + \text{REV}(X)$.

Unfortunately this does not suffice to get good bounds since it is entirely possible for $\text{VAL}(Y)$ to be infinite even when $\text{REV}(Y)$ is finite (as happens, e.g., for the equal-revenue distribution ER.) To effectively use the marginal mechanism lemma we will have to carefully cut up the domain of (X, Y), bound the value of one of the items in each of these sub-domains, and then stitch the results together. We will use Z to denote an arbitrary multi-dimensional nonnegative random variable, but the reader may want to think of it as (X, Y).

Lemma 22 (Sub-Domain Restriction) *Let Z be a multi-dimensional random variable and let S be a set of values of*[26] *Z. Then*

$$\text{REV}(\mathbb{1}_{Z \in S} Z) \leq \text{REV}(Z).$$

[26]If Z is a k-dimensional random variable, then S is a (measurable) subset of $\mathbb{R}_+^k$. We use the notation $\mathbb{1}_{Z \in S}$ for the indicator random variable which takes the value 1 when $Z \in S$ and 0 otherwise.

Proof. The optimal mechanism for $\mathbb{1}_{Z\in S}Z$ will extract at least as much from Z. This follows directly from an optimal mechanism having No Positive Transfers (see the end of Section 2.1). ■

Lemma 23 (Sub-Domain Stitching) *Let Z be a multi-dimensional random variable and let S, T be two sets of values of Z such that $S \cup T$ contains the support of Z. Then*

$$\textsc{Rev}(\mathbb{1}_{Z\in S}Z) + \textsc{Rev}(\mathbb{1}_{z\in T}Z) \geq \textsc{Rev}(Z).$$

Proof. Take the optimal mechanism for Z. $\textsc{Rev}(Z)$ is the revenue extracted by this mechanism, which is at most the sum of what is extracted on S and on T. If you take the same mechanism and run it on the random variable $\mathbb{1}_{Z\in S}Z$, it will extract the same amount on S as it extracted from Z on S, and similarly for T which contains the complement of S. ■

Our trick will be to choose S so that we are able to bound $\textsc{Val}(\mathbb{1}_{(X,Y)\in S}Y)$. This will suffice since the marginal mechanism lemma actually implies:

Lemma 24 (Marginal Mechanism on Sub-Domain) *Let X and Y be multi-dimensional random variables, and let S be a set of values of (X, Y). If X and Y are independent then*

$$\textsc{Rev}(\mathbb{1}_{(X,Y)\in S} \cdot (X, Y)) \leq \textsc{Val}(\mathbb{1}_{(X,Y)\in S}Y) + \textsc{Rev}(X).$$

Proof. For every y let $S_y = \{x|(x, y) \in S\}$. Note that $\textsc{Rev}(\mathbb{1}_{(X,Y)\in S} \cdot (X, Y)) = \textsc{Rev}(\mathbb{1}_{(X,Y)\in S}X, \mathbb{1}_{(X,Y)\in S}Y)$, and $\textsc{Rev}(\mathbb{1}_{(X,Y)\in S}X|Y = y) = \textsc{Rev}(\mathbb{1}_{X\in S_y}X)$. The Marginal Mechanism Lemma 21 yields $\textsc{Rev}(\mathbb{1}_{(X,Y)\in S}\cdot(X, Y)) \leq \textsc{Val}(\mathbb{1}_{(X,Y)\in S}Y)+\mathbb{E}_{y\sim Y}[\textsc{Rev}(\mathbb{1}_{X\in S_y}X)]$, and by the Sub-Domain Restriction Lemma 22 we have $\textsc{Rev}(\mathbb{1}_{X\in S_y}X) \leq \textsc{Rev}(X)$ for every y. ■

In the case of two items, i.e. one-dimensional X and Y, the set of values S for which we bound $\textsc{Val}(\mathbb{1}_{(X,Y)\in S}Y)$ will be the set $\{Y \leq X\}$.

Lemma 25 (Smaller Value) *Let X and Y be one-dimensional random variables. If X and Y are independent then*

$$\mathbb{E}(\mathbb{1}_{Y\leq X}Y) \leq \textsc{Rev}(X).$$

Proof. A possible mechanism for X that yields revenue of $\textsc{Val}(\mathbb{1}_{Y\leq X}Y)$ is the following: choose a random y according to Y and offer this as the price. The expected revenue of this mechanism is $\mathbb{E}_{y\sim Y}(y \cdot \mathbb{P}(X \geq y)) = \mathbb{E}_{y\sim Y}(\mathbb{E}(Y\mathbb{1}_{Y\leq X}|Y = y)) = \mathbb{E}(Y\mathbb{1}_{Y\leq X})$, so this is a lower bound on $\textsc{Rev}(X)$. ■

The proof of Theorem 1 can now be restated as follows:

Proof of Theorem 20 – one-dimensional case. Using the Sub-Domain Stitching Lemma 23, we will cut the space as follows: $\textsc{Rev}(X,Y) \le \textsc{Rev}(\mathbb{1}_{Y\le X}(X,Y)) + \textsc{Rev}(\mathbb{1}_{X\le Y}(X,Y))$. By the Marginal Mechanism on Sub-Domain Lemma 24, the first term is bounded by $\mathbb{E}(\mathbb{1}_{Y\le X}Y) + \textsc{Rev}(X) \le 2\textsc{Rev}(X)$, where the inequality uses the Smaller Value Lemma 25. The second term is bounded similarly. ■

The multi-dimensional case is almost identical. The Smaller Value Lemma 25 becomes:

Lemma 26 *Let X and Y be multi-dimensional random variables. If X and Y are independent then*

$$\textsc{Val}(\mathbb{1}_{\sum_j Y_j \le \sum_i X_i} Y) \le \textsc{BRev}(X).$$

Proof. Apply Lemma 25 to the one-dimensional random variables $\sum_i X_i$ and $\sum_j Y_j$, and recall that $\textsc{Rev}(\sum_i X_i) = \textsc{BRev}(X)$. ■

From this we get a slightly stronger version of Theorem 20 for multi-dimensional variables (which will be used in Section 5 to get bounds for any fixed number of items).

Theorem 27 *Let X and Y be multi-dimensional random variables. If X and Y are independent then*

$$\textsc{Rev}(X,Y) \le \textsc{Rev}(X) + \textsc{Rev}(Y) + \textsc{BRev}(X) + \textsc{BRev}(Y).$$

Proof. The proof is almost identical to that of the main theorem. We will cut the space by $\textsc{Rev}(X,Y) \le \textsc{Rev}(\mathbb{1}_{\sum_j Y_j \le \sum_i X_i} \cdot (X,Y)) + \textsc{Rev}(\mathbb{1}_{\sum_j Y_j \ge \sum_i X_i} \cdot (X,Y))$, and bound the first term by $\textsc{Val}(\mathbb{1}_{\sum_j Y_j + \textsc{Rev}(X) \le \sum_i X_i} Y) \le \textsc{BRev}(X) + \textsc{Rev}(X)$ using Lemmas 24 and 26. The second term is bounded similarly. ■

Proof of Theorem 20 – multi-dimensional case. Use the previous theorem and $\textsc{BRev} \le \textsc{Rev}$. ■

Remark. The decomposition of this section holds in more general setups than the totally additive valuation of this paper (where the value to the buyer of the outcome $q \in [0,1]^k$ is $\sum_i q_i x_i$). Indeed, consider an abstract mechanism design problem with a set of alternatives A, valuated by the buyer according to a function $v : A \to \mathbb{R}_+$ (known to him, whereas the seller only knows that the function v is drawn from a certain distribution). If the set of alternatives A is in fact a product $A = A_1 \times A_2$ with the valuation additive between the two sets, i.e., $v(a_1, a_2) = v_1(a_1) + v_2(a_2)$, with v_1 distributed according to X and v_2 acording to Y, then Theorem 20 holds as stated. The proof now uses $\textsc{Val}(Y) = \mathbb{E}(\sup_{a_2 \in A_2} v_2(a_2))$ (which, in our case, where $A_2 = [0,1]^{k_2}$ and $v_2(q) = \sum_j q_j y_j$, is indeed $\textsc{Val}(Y) = \mathbb{E}(\sum_j Y_j)$ since $\sup_q v_2(q) = \sum_j y_j$).

4.3 A Tighter Result for Two I.I.D. Items

For the special case of two independent and *identically distributed* items we have a tighter result, namely Theorem 2 stated in the Introduction. The proof is more technical and is relegated to Appendix A.

4.4 A Class of Distributions Where Bundling Is Optimal

For some special cases we are able to fully characterize the optimal auction for two items. We will show that bundling is optimal for distributions whose density function decreases fast enough; this includes the equal-revenue distribution.

Theorem 28 *Let F be a one-dimensional cumulative distribution function with density function f. Assume that there is $a > 0$ such that for $x < a$ we have $f(x) = 0$ and for $x > a$ the function $f(x)$ is differentiable and satisfies*

$$xf'(x) + \frac{3}{2}f(x) \leq 0. \tag{3}$$

Then bundling is optimal for two items: $\textsc{Rev}(F \times F) = \textsc{BRev}(F \times F)$.

Theorem 28 is proved in Appendix B. Condition (3) is equivalent to $\left(x^{3/2}f(x)\right)' \leq 0$, i.e., $x^{3/2}f(x)$ is nonincreasing in x (the support of F is thus either a finite interval $[a, A]$, or the half-line $[a, \infty)$). When $f(x) = cx^{-\gamma}$, (3) holds whenever $\gamma \geq 3/2$. In particular, ER satisfies (3); thus, by Lemma 7, we have:

Corollary 29 $\textsc{Rev}(ER \times ER) = \textsc{BRev}(ER \times ER) = 2.5569...$.

Thus $\textsc{SRev}(ER \times ER)/\textsc{Rev}(ER \times ER) = 2/2.559... = 0.78...$, which the largest gap we have obtained between the separate auction and the optimal one.

4.5 Multiple Buyers

Up to now we dealt a single buyer, but it turns out that the main decomposition result generalizes to the case of multiple buyers. We consider selling the two items (with a single unit of each) to n buyers, where buyer j's valuation for the first item is X^j, and for the second item is Y^j (with $X^j + Y^j$ being the value for getting both). Let the auction allocate the first item to buyer j with probability q_1^j, and the second item with probability q_2^j; of course, here $\sum_{j=1}^n q_1^j \leq 1$ and $\sum_{j=1}^n q_2^j \leq 1$.

Unlike the simple decision-theoretic problem facing the single buyer, we now have a multi-person game among the buyers. Thus, we consider two main notions of incentive compatibility: *dominant-strategy* IC and *Bayes-Nash* IC. Our result below applies equally well to both notions, and with an identical proof.

For either one of these notions, we denote by $\text{Rev}^{[n]}(X, Y)$ the revenue that is obtainable by the optimal auction. Similarly, selling the two items separately yields a maximal revenue of $\text{SRev}^{[n]}(X, Y) = \text{Rev}^{[n]}(X) + \text{Rev}^{[n]}(Y)$.

We allow the values of the different buyers for each single item to be arbitrarily correlated; however, we assume that independence between the two items.

Theorem 30 *Let $X = (X^1, \ldots, X^n) \in \mathbb{R}_+^n$ be the values of the first item to the n buyers, and let $Y = (Y^1, \ldots, Y^n) \in \mathbb{R}_+^n$ be the values of the second item to the n buyers. If X and Y are independent then*

$$\text{Rev}^{[n]}(X) + \text{Rev}^{[n]}(Y) \geq \frac{1}{2} \cdot \text{Rev}^{[n]}(X, Y),$$

where $\text{Rev}^{[n]}$ is taken throughout either with respect to dominant-strategy implementation, or with respect to Bayes-Nash implementation.

Thus selling the two items separately yields at least half the maximal revenue, i.e., $\text{SRev}^{[n]}(X, Y) \geq (1/2) \cdot \text{Rev}^{[n]}(X, Y)$.

The proof of Theorem 30 is almost identical to the proof of the Theorem 20 and is spelled out in Appendix C (we also point out there why we could not extend it to multiple buyers *and* more than 2 items). We emphasize that the proof does not use the characterization of the optimal revenue for a single item and n buyers (just like the proof of Theorem 20 did not use Myerson's characterization for one buyer).

5 More Than Two Items

The multi-dimensional decomposition results of Section 4.2 can be used recursively, by viewing k items as two sets of $k/2$ items each. Using Theorem 20 we can prove by induction that $\text{Rev}(F_1 \times \cdots \times F_k) \leq k \sum_{i=1}^k \text{Rev}(F_i)$, as follows: $\text{Rev}(F_1 \times \cdots \times F_k) \leq 2(\text{Rev}(F_1 \times \cdots \times F_{k/2}) + \text{Rev}(F_{k/2+1} \times \cdots \times F_k)) \leq 2(k/2 \sum_{i=1}^{k/2} \text{Rev}(F_i) + k/2 \sum_{i=k/2+1}^{k} \text{Rev}(F_i)) = k \sum_{i=1}^k \text{Rev}(F_i)$, where the first inequality is by Theorem 20, and the second by the induction hypothesis.

However, using the stronger statement of Theorem 27, as well as the relations we have shown between the bundling revenue and the separate revenue, will give us the better bound of $c \log^2 k$ (instead of k) of Theorem 3, stated in the Introduction.

Proof of Theorem 3. Assume first that $k \geq 2$ is a power of 2, and we will prove by induction that $\textsc{Rev}(F_1 \times \cdots \times F_k) \leq c \log^2(2k) \sum_{i=1}^k \textsc{Rev}(F_i)$, where c is the constant of Lemma 13. By applying Theorem 27 to $(F_1 \times \cdots \times F_{k/2}) \times (F_{k/2+1} \times \cdots \times F_k)$ we get

$$\begin{aligned}\textsc{Rev}(F_1 \times \cdots \times F_k) \quad &\leq \quad \textsc{BRev}(F_1 \times \cdots \times F_{k/2}) + \textsc{BRev}(F_{k/2+1} \times \cdots \times F_k) \\ &\quad + \textsc{Rev}(F_1 \times \cdots \times F_{k/2}) + \textsc{Rev}(F_{k/2+1} \times \cdots \times F_k). \qquad (4)\end{aligned}$$

Using Lemma 13 on each of the BRev terms, their sum is bounded by $c \log k \sum_{i=1}^k \textsc{Rev}(F_i)$. Using the induction hypothesis on each of the Rev terms, their sum is bounded by $c \log^2 k \sum_{i=1}^k \textsc{Rev}(F_i)$. Now $\log k + \log^2 k \leq \log^2(2k)$, and so the coefficient of each $\textsc{Rev}(F_i)$ is at most $c \log^2(2k)$ as required.

When $2^{m-1} < k < 2^m$ we can pad to 2^m with items that have value identically zero, and so do not contribute anything to the revenue. This at most doubles k. ■

As we have seen in Example 15, the bundling auction may, in contrast, extract only $1/k$ fraction of the optimal revenue. This we can show is tight.

Lemma 31 *There exists a constant $c > 0$ such that for every $k \geq 2$ and every one-dimensional distributions $F_1, ..., F_k$,*

$$\textsc{BRev}(F_1 \times \cdots \times F_k) \geq \frac{c}{k} \cdot \textsc{Rev}(F_1 \times \cdots \times F_k).$$

Proof. For k a power of two, we use as in the previous proof the decomposition of (4) to obtain by induction $\textsc{Rev}(F_1 \times \cdots \times F_k) \leq (3k-2)\textsc{BRev}(F_1 \times \cdots \times F_k)$, where the induction step uses the fact that the bundled revenue from a subset of the items is at most the bundled revenue from all of them. Again, when k is not a power of 2 we can pad to the next power of 2 with items that have value identically zero, which at most doubles k. ■

However, for identically distributed items the bundling auction does much better, and in fact we can prove a tighter result, with $\log k$ instead of k : Theorem 4, stated in the Introduction.

Proof of Theorem 4. For $k \geq 2$ a power of two we apply Theorem 27 inductively to obtain: $\textsc{Rev}(F^{\times k}) \leq 2\,\textsc{BRev}(F^{\times (k/2)}) + 4\,\textsc{BRev}(F^{\times (k/4)}) + \ldots + (k/2)\,\textsc{BRev}(F^{\times 2}) + k\,\textsc{BRev}(F) + k\,\textsc{Rev}(F)$. Each of the $\log_2 k + 1$ terms in this sum is of the form

$(k/m)\,\mathrm{BR}\textsc{ev}(F^{\times m}) = (k/m)\,\textsc{Rev}(F^{*m})$ and is thus bounded from above, using Lemma 18 applied to the distribution F^{*m}, by $4\,\textsc{Rev}(F^{*k}) = 4\,\mathrm{BR}\textsc{ev}(F^{\times k})$. Altogether we have $\textsc{Rev}(F^{\times k}) \leq 4(\log_2 k + 1)\,\mathrm{BR}\textsc{ev}(F^{\times k})$.

When $2^{m-1} < k < 2^m$ we have $\textsc{Rev}(F^{\times k}) \leq \textsc{Rev}(F^{\times 2^m}) \leq 4(\log_2 2^m{+}1)\,\mathrm{BR}\textsc{ev}(F^{\times 2^m}) \leq 4(\log_2 k + 2) \cdot 2 \cdot 1.3 \cdot \mathrm{BR}\textsc{ev}(F^{\times 2^{m-1}}) \leq 4(\log_2 k + 2) \cdot 2 \cdot 1.3 \cdot \mathrm{BR}\textsc{ev}(F^{\times k})$ (we have used Lemma 13 with $F_1 = F_2 = F^{*2^{m-1}}$ and $1 + w \leq 1.3$). ■

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Appendices

A A Tighter Bound for Two Items

In this appendix we prove Theorem 2 which is stated in the Introduction (see also Section 4.2), which says that selling two i.i.d. items separately yields at least $e/(e+1) = 0.73...$ of the optimal revenue.

Proof of Theorem 2. Let X and Y be i.i.d.-F. Without loss of generality we will restrict ourselves to symmetric mechanisms, i.e., b such that $b(x,y) = b(y,x)$ (indeed: if $b(x,y)$ is optimal, then so are $\hat{b}(x,y) := b(y,x)$ and their average $\bar{b}(x,y) := (b(x,y) + \hat{b}(x,y))/2$, which is symmetric). Put $R := \textsc{Rev}(X) = \textsc{Rev}(Y) = \sup_{t\geq 0} t \cdot \bar{F}(t)$, where $\bar{F}(t) := \mathbb{P}(X \geq t) = \lim_{u\to t^+}(1 - F(u))$.

Define $\varphi(x) := q_1(x,x) = q_2(x,x)(= b_x(x,x))$ and $\Phi(x) := b(x,x)/2$, then $\Phi(x) = \int_0^x \varphi(t)\,\mathrm{d}t$ (recall that $b(0,0) = 0$ by IR and NPT).

We will consider the two regions $X \geq Y$ and $Y \geq X$ separately; by symmetry, the expected revenue in the two regions is the same, and so it suffices to show that

$$\mathbb{E}(s(X,Y)\mathbb{1}_{X\geq Y}) \leq \left(1 + \frac{1}{e}\right) R.$$

As in Lemma 21 and Appendix 4.1, fix y and define a mechanism $(\tilde{q}^y, \tilde{s}^y)$ for X by $\tilde{q}^y(x) := q_1(x,y)$ and $\tilde{s}^y(x) := s(x,y) - yq_2(x,y)$ for every x (note that the buyer's payoff remains the same: $\tilde{b}^y(x) = b(x,y)$). The mechanism $(\tilde{q}^y, \tilde{s}^y)$ is IC and IR for X, since (q,s) was IC and IR for (X,Y). Now apply the mechanism $(\tilde{q}^y, \tilde{s}^y)$ to the random variable X conditional on $[X \geq y]$, which we write X_y for short. Since $X_y \geq y$ we have $\tilde{b}^y(X_y) = b(X_y,y) \geq b(y,y) = 2\Phi(y)$ and $\tilde{q}^y(X_y) = q_1(X_y,y) \geq q_1(y,y) = \varphi(y)$, and so applying Lemma 32 below to X_y yields

$$\mathbb{E}(\tilde{s}^y(X)|X \geq y) = \mathbb{E}(\tilde{s}^y(X_y)) \leq (1 - \varphi(y))\textsc{Rev}(X_y) + y\varphi(y) - 2\Phi(y). \tag{5}$$

Since $\mathbb{P}(X_y \geq t) = \mathbb{P}(X \geq t)/\mathbb{P}(X \geq y) = \bar{F}(t)/\bar{F}(y)$ for all $t \geq y$, we get

$$\textsc{Rev}(X_y) = \sup_{z\geq 0} z \cdot \mathbb{P}(X_y \geq z) = \sup_{z\geq y} z \cdot \frac{\bar{F}(z)}{\bar{F}(y)} \leq \sup_{z\geq 0} z \cdot \frac{\bar{F}(z)}{\bar{F}(y)} = \frac{R}{\bar{F}(y)}.$$

Multiply (5) by $\mathbb{P}(X \geq y) = \bar{F}(y)$ to get

$$\mathbb{E}(\tilde{s}^y(X)\mathbb{1}_{X\geq y}) \leq (1 - \varphi(y))R + (y\varphi(y) - 2\Phi(y))\bar{F}(y),$$

and then take expectation over $Y = y$:

$$\mathbb{E}(\tilde{s}^Y(X)\mathbb{1}_{X\geq Y}) \leq R\mathbb{E}(1-\varphi(Y)) + \mathbb{E}((Y\varphi(Y) - 2\Phi(Y)\mathbb{1}_{X\geq Y}).$$

Since $s(x,y) = \tilde{s}^y(x) + yq_2(x,y) \leq \tilde{s}^y(x) + yq_2(x,x) = \tilde{s}^y(x) + y\varphi(x)$ (use $y \geq 0$ and the monotonicity of $q_2(x,y)$ in y), we finally get

$$\begin{aligned}
\mathbb{E}(s(X,Y)\mathbb{1}_{X\geq Y}) &= \mathbb{E}(\tilde{s}^Y(X)\mathbb{1}_{X\geq Y}) + \mathbb{E}(Y\phi(X)\mathbb{1}_{X\geq Y}) \\
&\leq R - R\mathbb{E}(\varphi(Y)) + \mathbb{E}(W\mathbb{1}_{X\geq Y}),
\end{aligned} \tag{6}$$

where

$$W := Y\varphi(X) + Y\varphi(Y) - 2\Phi(Y).$$

The expression (6) is affine in φ (recall that $\Phi(x) = \int_0^x \varphi(s)\,\mathrm{d}s$), and φ is a nondecreasing function with values in $[0,1]$. The set of such functions φ is the closed convex hull of the functions $\varphi(x) = \mathbb{1}_{[t,\infty)}(x)$ for $t \geq 0$. Therefore, in order to bound (6), it suffices to consider these extreme functions.

When $\varphi(x) = \mathbb{1}_{[t,\infty)}(x)$ we get $\Phi(x) = \max\{x-t, 0\}$ and

$$W = \begin{cases} 2Y - 2(Y-t) = 2t, & \text{if } X \geq Y \geq t, \\ Y - 0 = Y, & \text{if } X \geq t > Y, \\ 0, & \text{if } t > X \geq Y. \end{cases}$$

Thus

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{X\geq Y}) &= 2t\mathbb{P}(X \geq Y \geq t) + \mathbb{E}(Y\mathbb{1}_{X\geq t>Y}) \\
&= t\mathbb{P}(X \geq t)\mathbb{P}(Y \geq t) + \mathbb{P}(X \geq t)\mathbb{E}(Y\mathbb{1}_{t>Y}) \\
&= \mathbb{P}(X \geq t)\mathbb{E}(\min\{Y,t\}) = \bar{F}(t)\mathbb{E}(\min\{Y,t\})
\end{aligned}$$

(we have used the fact that X, Y are i.i.d., and $\min\{Y,t\} = t\mathbb{1}_{Y\geq t} + Y\mathbb{1}_{t>Y}$). Together with $\mathbb{E}(\varphi(Y)) = \mathbb{P}(Y \geq t) = \bar{F}(t)$, (6) becomes

$$R - R\bar{F}(t) + \bar{F}(t)\mathbb{E}(\min\{Y,t\}) = R + \bar{F}(t)\left(\mathbb{E}(\min\{Y,t\}) - R\right). \tag{7}$$

Let $r(t)$ denote the expression in (7). When $t \leq R$ we have $\mathbb{E}(\min\{Y,t\}) \leq R$, and so $r(t) \leq R$. When $t > R$ we have

$$\mathbb{E}(\min\{Y,t\}) = \int_0^\infty \mathbb{P}(\min\{Y,t\} \geq u)\,\mathrm{d}u = \int_0^t \mathbb{P}(Y \geq u)\,\mathrm{d}u$$

$$= \int_0^t \bar{F}(u)\,\mathrm{d}u \le \int_0^R 1\,\mathrm{d}u + \int_R^t \frac{R}{u}\,\mathrm{d}u = R + R\log\left(\frac{t}{R}\right),$$

where we have used $\bar{F}(u) \le \min\{R/u, 1\}$ (which follows from $R = \sup_{u\ge 0} u\bar{F}(u)$). Therefore in this case

$$r(t) \le R + \frac{R}{t}\left(R + R\log\left(\frac{t}{R}\right) - R\right) = R\left(1 + \frac{\log\tau}{\tau}\right),$$

where $\tau := t/R > 1$. Since $\max_{\tau\ge 1} \tau^{-1}\log\tau = 1/e$ (attained at $\tau = e$), it follows that $r(t) \le (1+1/e)R$ for all $t > R$, and thus also for all $t \ge 0$. Recalling (6) and (7) therefore yields $\mathbb{E}(s(X,Y)\mathbb{1}_{X\ge Y}) \le (1+1/e)R$, and so $\mathbb{E}(s(X,Y)) \le 2(1+1/e)R = (1+1/e)\cdot \mathrm{SRev}(F\times F)$. ■

Lemma 32 *Let X be a one-dimensional random variable whose support is included in $[x_0, \infty)$ for some $x_0 \ge 0$, and let $b_0 \ge 0$ and $0 \le q_0 \le 1$ be given. Then the maximal revenue the seller can obtain from X subject to guaranteeing to the buyer a payoff of at least b_0 and a probability of getting the item of at least q_0 (i.e., $b(x) \ge b_0$ and $q(x) \ge q_0$ for all $x \ge x_0$) is*

$$(1-q_0)\mathrm{Rev}(X) + q_0x_0 - b_0.$$

Proof. A mechanism satisfying these constraints is plainly seen to correspond to a one-dimensional convex function b with $q_0 \le b'(x) \le 1$ and $b(x_0) = b_0$. When $q_0 < 1$ (if $q_0 = 1$ the result is immediate) put $\tilde{b}(x) := (b(x) - q_0(x-x_0) - b_0)/(1-q_0)$, then $\tilde{b}$ is a convex function with $0 \le \tilde{b}(x) \le 1$ and $\tilde{b}(x_0) = 0$, and so $\mathrm{Rev}(F) \ge R(\tilde{b}; F) = (R(b;F) - q_0x_0 + b_0)/(1-q_0)$. ■

B When Bundling Is Optimal

In this appendix we prove Theorem 28 which is stated in Section 4.3: for two i.i.d. items, if the one-item value distribution satisfies condition (3), then bundling is optimal.

Proof of Theorem 28. Let b correspond to a two-dimensional IC and IR mechanism; assume without loss of generality that b is symmetric, i.e., $b(x,y) = b(y,x)$ (cf. the proof of Theorem 2 in Appendix A above). Thus $\mathbb{E}(s) = \mathbb{E}(xb_x + yb_y - b) = \mathbb{E}(2xb_x - b)$, and so

$$R(b, F\times F) = \int_a^\infty \int_a^\infty (2xb_x(x,y) - b(x,y))\, f(x)\,\mathrm{d}x\, f(y)\,\mathrm{d}y = \sup_{M>a} r_M(b),$$

where

$$r_M(b) := \int_a^M \int_a^M (2xb_x(x,y) - b(x,y))\, f(x)\,\mathrm{d}x\, f(y)\,\mathrm{d}y. \tag{8}$$

For each y we integrate by parts the $2xb_x(x,y)f(x)$ term:

$$\begin{aligned}\int_a^M 2b_x(x,y)xf(x)\,\mathrm{d}x &= [2b(x,y)xf(x)]_a^M - \int_a^M 2b(x,y)\left(f(x)+xf'(x)\right)\,\mathrm{d}x\\ &= 2b(M,y)Mf(M) - 2b(a,y)af(a)\\ &\quad - \int_a^M 2b(x,y)\left(f(x)+xf'(x)\right)\,\mathrm{d}x.\end{aligned}$$

Substituting this in (8) yields

$$\begin{aligned}r_M(b) &= 2Mf(M)\int_a^M b(M,y)f(y)\,\mathrm{d}y - 2af(a)\int_a^M b(a,y)f(y)\,\mathrm{d}y\\ &\quad +2\int_a^M\int_a^M b(x,y)\left(-\frac{3}{2}f(x)-xf'(x)\right)f(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Define $\tilde{b}(x,y) := b(x+y-a,a) = b(a,x+y-a)$ for every (x,y) with $x,y\geq a$, then $\tilde{b}$ is a convex function on $[a,\infty)\times[a,\infty)$ with $0\leq \tilde{b}_x,\tilde{b}_y\leq 1$, and so it corresponds to a two-dimensional IC & IR mechanism. Moreover, since b is convex we have for every $x,y\geq a$

$$b(x,y)\leq \lambda\, b(x+y-a,a)+(1-\lambda)\, b(a,x+y-a)=\tilde{b}(x,y),$$

where $\lambda=(x-a)/(x+y-2a)$. Therefore replacing b with $\tilde{b}$ can only increase r_M, i.e., $r_M(b)\leq r_M(\tilde{b})$; indeed, in the first and third terms the coefficients of $b(x,y)$ are nonnegative (recall our assumption (3)); and in the second term, $b(a,y)=\tilde{b}(a,y)$. Hence $R(b,F\times F)=\sup_M r_M(b)\leq \sup_M r_M(\tilde{b}) = R(\tilde{b},F\times F)$.

It only remains to observe that $\tilde{b}(x,y)$ is a function of $x+y$, and so it corresponds to a bundled mechanism. Formally, put $\beta(t):=\tilde{b}(t-a,a)$, then $\beta:[2a,\infty)\to\mathbb{R}_+$ is a one-dimensional convex function with $0\leq\beta'(t)\leq 1$. For all $x,y\geq a$ with $x+y=t$ we have $\tilde{b}(x,y)=\beta(t)$ and $x\tilde{b}_x(x,y)+y\tilde{b}_y(x,y)-\tilde{b}(x,y)=t\beta'(t)-\beta(t)$, and so $R(\tilde{b},F\times F)=R(\beta,F*F)\leq \textsc{Rev}(F*F)=\textsc{BRev}(F\times F)$. ■

C Multiple Buyers

We prove here Theorem 30 (see Section 4.5): selling separately two independent items to n buyers yields at least one half of the optimal revenue.

Let $X^{\max}=\max_{1\leq j\leq n}X^j$ and $Y^{\max}=\max_{1\leq j\leq n}Y^j$ be the highest values for the two items. Define $\textsc{Val}^{[n]}(X)=\mathbb{E}(X^{\max})$ and $\textsc{Val}^{[n]}(Y)=\mathbb{E}(Y^{\max})$ (these are the values obtained by always allocating each item to the highest-value buyer).

We proceed along the same lines as the proof of Theorem 20 in Section 4.2. In the lemmas below, X and Y are independent n-dimensional random variables, Z is a $2n$-

dimensional random variable (for instance, (X,Y)), and S and T are sets of values of Z.

Lemma 33 $\textsc{Rev}^{[n]}(X,Y) \le \textsc{Val}^{[n]}(Y) + \textsc{Rev}^{[n]}(X)$.

Proof. The proof is similar to the case of a single buyer (Lemma 21), except that the amount of money we need to return to compensate for the y's is exactly $\textsc{Val}^{[n]}(Y)$ since if each buyer j gets q_2^j units (or probability) of the y item then we have $\sum_j q_2^j \le 1$ and thus $\sum_j q_2^j y^j \le y^{\max}$. We emphasize that if the original mechanism for (X,Y) was a dominant-strategy mechanism, so will be the conditional-on-y mechanism for X; and the same for Bayes-Nash mechanisms. ■

Lemma 34 $\textsc{Rev}^{[n]}(\mathbb{1}_{Z\in S} \cdot Z) \le \textsc{Rev}^{[n]}(Z)$.

The proof is identical to the case $n = 1$ (Lemma 22).

Lemma 35 *If* $S \cup T$ *contains the support of* Z *then* $\textsc{Rev}^{[n]}(\mathbb{1}_{Z\in S} \cdot Z) + \textsc{Rev}^{[n]}(\mathbb{1}_{Z\in T} \cdot Z) \ge \textsc{Rev}^{[n]}(Z)$.

The proof is identical to the case $n = 1$ (Lemma 23).

Lemma 36 $\textsc{Rev}^{[n]}(\mathbb{1}_{(X,Y)\in S} \cdot (X,Y)) \le \textsc{Val}^{[n]}(\mathbb{1}_{(X,Y)\in S} \cdot Y) + \textsc{Rev}^{[n]}(X)$.

The proof is identical to the case $n = 1$ (Lemma 24).

The set according to which we will cut our space will be the following one:

Lemma 37 $\textsc{Val}^{[n]}(\mathbb{1}_{Y^{\max} \le X^{\max}} \cdot Y) \le \textsc{Rev}^{[n]}(X)$.

Proof. Here is a possible mechanism for X: choose a random $y = (y^1, \ldots, y^n)$ according to Y and offer $y^{\max}$ as the take-it-or-leave-it price to the buyers sequentially (the first one in lexicographic order to accept gets it). The expected revenue of this mechanism is exactly $\textsc{Val}^{[n]}(\mathbb{1}_{Y^{\max} \le X^{\max}} \cdot Y)$ so this is a lower bound on $\textsc{Rev}^{[n]}(X)$. ■

We can now complete our proof.

Proof of Theorem 30. Using lemma 35 we cut the space into two parts, $\textsc{Rev}^{[n]}(X,Y) \le \textsc{Rev}^{[n]}(\mathbb{1}_{Y^{\max} \le X^{\max}} \cdot (X,Y)) + \textsc{Rev}^{[n]}(\mathbb{1}_{X^{\max} \le Y^{\max}} \cdot (X,Y))$, and bound the revenue in each one. By Lemma 36, the revenue on the first part is bounded by $\textsc{Val}^{[n]}(1_{Y^{\max} \le X^{max}} \cdot Y) + \textsc{Rev}^{[n]}(X)$ which using Lemma 37 is bounded from above by $2\,\textsc{Rev}^{[n]}(X)$. The revenue in the second part is bounded similarly by $2\,\textsc{Rev}^{[n]}(Y)$. ■

Remark. The problem when trying to extend this method to more than 2 items is that when Y is a *set* of items we do not have a "Smaller Value" counterpart to Lemma 37 (recall also Lemma 25).

D Many I.I.D. Items

It turns out that when the items are independent and identically distributed, and their number k tends to infinity, then the bundling revenue approaches the optimal revenue. Even more, essentially all the buyer's surplus can be extracted by the bundling auction. The logic is quite simple: the law of large numbers tells us that there is almost no uncertainty about the sum of many i.i.d. random variables, and so the seller essentially knows this sum and may ask for it as the bundle price. For completeness we state this result and provide a short proof, which also covers the case where the expectation $\mathbb{E}(F)$ is infinite.

Theorem 38 (Armstrong [1999], Bakos and Brynjolfsson [1999]) *For every one-dimensional distribution F,*

$$\lim_{k\to\infty} \frac{\text{BRev}(F^{\times k})}{k} = \lim_{k\to\infty} \frac{\text{Rev}(F^{\times k})}{k} = \mathbb{E}(F).$$

Proof. We always have $\text{BRev}(F^{\times k}) \leq \text{Rev}(F^{\times k}) \leq k\,\mathbb{E}(F)$ (the second inequality follows from NPT). Let us first assume that our distribution F has finite expectation and finite variance. In this case if we charge price $(1-\epsilon)k\,\mathbb{E}(F)$ for the bundle then by Chebyshev's inequality the probability that the bundle will not be bought is at most $\mathbb{V}\text{ar}(F)/(\epsilon^2\mathbb{E}(F)\sqrt{k})$, where $\mathbb{V}\text{ar}(F)$ is the variance of F, and this goes to zero as k increases.

If the expectation or variance are infinite, then just consider the truncated distribution where values above a certain M are replaced by M, which certainly has finite expectation and variance. We can choose the finite M so as to get the expectation of the truncated distribution as close as we desire to the original one (including as high as we desire, if the original distribution has infinite expectation). ■

Despite the apparent strength of this theorem, it does not provide any approximation guarantees for any fixed value of k. In particular, for $k = 2$ we have already seen an example where the bundling auction gets only 2/3 of the revenue of selling the items separately (Example 17), and for every large enough k we have seen an example where the bundling auction's revenue is less than 57% than that of selling the items separately (Example 19); of course, as a fraction of the optimal revenue this can only be smaller. The results of Section 4 provide approximation bounds for each fixed k.

E Summary of Approximation Results

The table below summarizes the approximation results of this paper. The four main results are in bold, and the arrows [$\rightarrow$] and [$\leftarrow$] indicate that the result in that box is a special case of the one in the next box to the right or left, respectively.

$\mathcal{F} =$	$F_1 \times F_2$	$F \times F$	$F_1 \times \cdots \times F_k$	$F^{\times k}$
$\forall \mathcal{F}$ $\frac{\text{SRev}(\mathcal{F})}{\text{Rev}(\mathcal{F})} \geq$	$\mathbf{\frac{1}{2}}$ [Th 1]	$\mathbf{\frac{e}{e+1} \approx 0.73}$ [Th 2]	$\mathbf{\Omega\left(\frac{1}{\log^2 k}\right)}$ [Th 3]	$\Omega\left(\frac{1}{\log^2 k}\right)$ [$\leftarrow$]
$\exists \mathcal{F}$ $\frac{\text{SRev}(\mathcal{F})}{\text{Rev}(\mathcal{F})} \leq$	$\frac{1}{1+w} \approx 0.78$ [$\rightarrow$]	$\frac{1}{1+w} \approx 0.78$ [Co 29]	$O\left(\frac{1}{\log k}\right)$ [$\rightarrow$]	$O\left(\frac{1}{\log k}\right)$ [Le 8]
$\forall \mathcal{F}$ $\frac{\text{BRev}(\mathcal{F})}{\text{Rev}(\mathcal{F})} \geq$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ [Th 1 + Le 14]	$\frac{2}{3} \cdot \frac{e}{e+1}$ [Th 2 + Le 16]	$\Omega\left(\frac{1}{k}\right)$ [Le 31]	$\mathbf{\Omega\left(\frac{1}{\log k}\right)}$ [Th 4]
$\exists \mathcal{F}$ $\frac{\text{BRev}(\mathcal{F})}{\text{Rev}(\mathcal{F})} \leq$	$\frac{1}{2} + \varepsilon$ [Ex 15]	$\frac{2}{3}$ [Ex 17]	$\frac{1}{k} + \varepsilon$ [Ex 15]	$\approx 0.57 + o(1)$ [Ex 19]

The Menu-Size Complexity of Auctions (Working Paper)

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Abstract

We consider the *menu size* of auctions as a measure of auction complexity and study how it affects revenue. Our setting has a single revenue-maximizing seller selling $k \geq 2$ heterogenous items to a single buyer whose private values for the items are drawn from a (possibly correlated) known distribution, and whose valuation for sets of items ("bundles") is additive over the items. We show that the revenue may increase arbitrarily with menu size and that a bounded menu size can *not* ensure any positive fraction of the optimal revenue. (For bounded valuations, we show that a finite menu size can ensure an arbitrarily good *additive* approximation of revenue.)

The menu size turns out to "nail down" the revenue properties of *deterministic* auctions: their menu size may be at most exponential in the number of items and indeed their revenue may be larger than that achievable by the simplest types of auctions by a factor that is exponential in the number of items but no larger. In particular our results imply an infinite separation between the revenues achievable by deterministic and general randomized auctions even when selling two items, answering a question left open in Briest et al. [2010].

1 Introduction

Are complex auctions better than simple ones? Myerson's classic result (Myerson [1981]) shows that if you are aiming to maximize revenue when selling a single item, then the

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answer is "no." The optimal auction is very simple, allocating to the highest bidder (using either 1st or 2nd price) as long as he bids above a single deterministically chosen reserve price.

However, when selling multiple items the situation turns out to be more complex. There has been significant work both in the economics literature and in the computer science literature showing that, when selling multiple items, simple auctions are no longer optimal. Specifically, it is known that randomized auctions may yield more revenue than deterministic ones[1] and that bundling the items may yield higher (or lower) revenue than selling each of them separately. This is true even in the very simple setting where there is a single bidder (buyer) and where his valuation is additive over the different items.

In this paper we consider such a simple setting: a single seller, who aims to maximize his expected revenue, is selling k heterogenous items to a single buyer whose private values for the items are drawn from an arbitrary (possibly correlated) but known prior distribution, and whose value for bundles is additive over the items in the bundle. Since we are considering only a single buyer, this work may alternatively be interpreted as dealing with monopolist pricing for multiple items.[2]

In a previous paper (Hart and Nisan [2012]) we considered the case where the buyer's values for the different items are *independent*, in which case we showed that simple auctions are *approximately* optimal: Selling each item separately (deterministically) for its Myerson price extracts a constant fraction of the optimal revenue. Specifically, for the case of two items, it gives at least a half of the optimal revenue, and when selling k items, at least a $\Omega(1/\log^2 k)$ fraction of the optimal revenue. In this paper we start by showing that the picture changes completely when the item valuations are correlated.

For a prior distribution $\mathcal{F}$ on $\Re^k_+$, let us denote by $\textsc{Rev}(\mathcal{F})$ the optimal revenue achievable by an auction when selling k items to a single buyer whose values for the items are jointly distributed according to the distribution $\mathcal{F}$. We would like to compare this revenue to the one achievable by "simple" auctions. For start let us take "deterministic" as an "upper bound" of modeling "simple"; denote by $\textsc{DRev}(\mathcal{F})$ the optimal revenue achievable by any *deterministic* auction. We have:

Theorem A *For every $k \geq 2$ there exists a k-item distribution $\mathcal{F}$ on $\Re^k_+$ such that*

$$\textsc{Rev}(\mathcal{F}) = \infty \quad \text{and} \quad \textsc{DRev}(\mathcal{F}) = 1.$$

Notice the contrast with the independent case for which, as mentioned above, for

[1] See Hart and Reny [2012] for a most simple and transparent such example, together with a discussion of why this phenomenon can occur only when there are two or more items.

[2] Our main separation results obtained in the single-buyer setting apply directly also to multiple-buyer auctions. In Appendix 4 we discuss the multiple-buyer case further.

every $k \geq 2$ there exists a fixed c_k such that $\text{Rev}(\mathcal{F}) \leq c_k \cdot \text{DRev}(\mathcal{F})$ (for two items we have $c_2 \leq 2$).

In Briest et al. [2010] a similar question was considered in a combinatorially simpler model where the buyer has a unit demand. They gave an infinite separation for the case $k \geq 3$ and left the case $k = 2$ as an open problem, providing some partial results indicating that for this case the gap may be bounded. While the models are different, we show in Appendix 1 that the gap between our models is at most exponential in the number of items and thus our infinite separation carries over to their model as well, answering the open problem.

Of course, to get infinite revenue the support of $\mathcal{F}$ must be unbounded. However, by truncating and then rescaling the distribution one immediately gets a similar result for distributions with bounded support: deterministic auctions may get only an arbitrarily small fraction of the optimal revenue.

Corollary 1.1 *For every $k \geq 2$ and every $\varepsilon > 0$ there exists a k-item distribution $\mathcal{F}$ on $[0, 1]^k$ such that*

$$\text{DRev}(\mathcal{F}) < \varepsilon \cdot \text{Rev}(\mathcal{F}).$$

Looking at the proof of this separation as well as the earlier one in Briest et al. [2010], one observes that having a large menu size seems to be the crucial attribute of the high-revenue auctions: it enables the sophisticated screening between different buyer types required for high revenue extraction. So, in the rest of this paper we focus on the ***menu size*** as a ***complexity measure of auctions*** and study the revenue extraction capabilities of auctions that are limited in their menu size. The menu size of an auction is simply the number of possible outcomes of the auction, where an outcome is a vector $(q_1, \ldots, q_k)$ in which q_i specifies the probability of allocating item[3] i. It is well known ("the taxation principle") that in our setting any auction can be put into the normal form of offering a fixed menu with a fixed price for every menu item and letting the buyer choose among these options. Notice that while deterministic auctions can have a menu size of at most $2^k - 1$ (since each q_i must be 0 or 1, and we are not counting $(0, \ldots, 0)$), randomized auctions may have either finite or infinite menu size.

For a prior distribution $\mathcal{F}$ on $\Re_+^k$, we will use $[m]$-$\text{Rev}(\mathcal{F})$ to denote the optimal revenue achievable by an auction whose menu size is at most m (when selling k items to a single buyer whose values for the items are distributed according to the distribution $\mathcal{F}$.) For a single item, $k = 1$, Myerson's result implies that $[1]$-$\text{Rev}(\mathcal{F}) = \text{Rev}(\mathcal{F})$, but this is not true for more than a single item: the revenue may increase as we allow the menu size to increase.

[3]Our menu-size measure does not count the "empty" allocation $(0, \ldots, 0)$ that we assume without loss of generality is always available, by individual rationality.

We start by looking at the simplest auctions according to this complexity measure, those with a single menu item. It is not difficult to verify that the optimal single-menu-item auction is always a *bundling* auction that sells the whole bundle (i.e., $q_i = 1$ for all i) for the optimal (Myerson) bundle-price: $[1]\text{-}\textsc{Rev}(\mathcal{F}) = \textsc{BRev}(\mathcal{F})$, where BRev denotes the optimal bundling revenue. Next, one easily observes that the revenue may increase at most linearly in the menu size; thus

$$[m]\text{-}\textsc{Rev}(\mathcal{F}) \leq m \cdot [1]\text{-}\textsc{Rev}(\mathcal{F}) = m \cdot \textsc{BRev}(\mathcal{F}). \tag{1}$$

Our proof of Theorem A implies that the revenue may indeed strictly increase with the menu size. A more precise estimate is the following:[4]

Theorem B *For every $k \geq 2$ there exists a k-item distribution $\mathcal{F}$ on $\Re_+^k$ with $0 < [1]\text{-}\textsc{Rev}(\mathcal{F}) < \infty$ such that for all $m \geq 1$,*

$$[m]\text{-}\textsc{Rev}(\mathcal{F}) \geq \Omega(m^{1/7}) \cdot [1]\text{-}\textsc{Rev}(\mathcal{F}).$$

The proof constructs the distribution together with the menu of allocations: for every menu entry it puts a mass point on a valuation that chooses that menu entry. The trick is to have a *large* number of menu entries, and to extract a significant amount of revenue from each one—while not letting the simple bundling auction, which uses a *single* price, do the same.

As mentioned above, deterministic auctions on k items can have menu size of at most $2^k - 1$ and thus we have $\textsc{DRev}(\mathcal{F}) \leq (2^k - 1) \cdot [1]\text{-}\textsc{Rev}(\mathcal{F})$ while $\textsc{Rev}(\mathcal{F}) \geq \lim_{m\to\infty}[m]\text{-}\textsc{Rev}(\mathcal{F}) = \infty$ so this theorem actually implies Theorem A. We continue by showing that the limited menu size—rather than not using lotteries—is the main bottleneck of deterministic auctions. Specifically we show that the exponential-in-k upper bound on deterministic revenue implied by menu size (see (1)),

$$\textsc{DRev}(\mathcal{F}) \leq [2^k - 1]\text{-}\textsc{Rev}(\mathcal{F}) \leq (2^k - 1) \cdot [1]\text{-}\textsc{Rev}(\mathcal{F}),$$

is essentially tight:

Theorem C *For every $k \geq 2$ there exists a k-item distribution $\mathcal{F}$ on $[0,1]^k$ with $0 < [1]\text{-}\textsc{Rev}(\mathcal{F}) < \infty$ such that*

$$\textsc{DRev}(\mathcal{F}) \geq \Omega\left(\frac{2^k}{k}\right) \cdot [1]\text{-}\textsc{Rev}(\mathcal{F}).$$

[4] The increase is at a polynomial rate in m. For menu sizes that are at most exponential in the number of items, the proof of Theorem C below actually shows that the growth can be almost linear.

This again is in contrast to the independent case studied in Hart and Nisan [2012] for which[5] $\text{DR}{\scriptstyle\text{EV}}(\mathcal{F}) \le \text{O}(k) \cdot [1]\text{-R}{\scriptstyle\text{EV}}(\mathcal{F})$.

Extending the results of Hart and Nisan [2012] to the case of correlated valuations, we compare the revenue of such single-menu-item auctions to that of the other type of "simple" auctions, those that sell the items *separately* in completely separate auctions, whose revenue is denoted SRev. As opposed to the previous theorems, the bounds that we get here are the same as in the independent case and as Hart and Nisan [2012] shows are tight. Specifically, single-menu-item auctions (i.e. bundling auctions) may raise at most $\text{O}(\log k)$ times more revenue than selling the items separately and at least a $1/k$ fraction of that: $\Omega(1/\log k)\cdot[1]\text{-Rev}(\mathcal{F}) \le \text{SRev}(\mathcal{F}) \le k\cdot[1]\text{-Rev}(\mathcal{F})$. The linear upper bound on the gap holds despite the fact that, technically, the separate auction may have exponential menu size since any subset of the items may be acquired. This suggests a more refined complexity measure, presented in section 8, which we term "additive-menu-size," a measure to which all previous results extend as well.

While we have seen that auctions with bounded menu size cannot guarantee any fixed fraction of revenue, it turns out that they can provide a good *additive approximation* of revenue. For an additive approximation to make sense we need of course to normalize the range of values, so we consider without loss of generality distributions $\mathcal{F}$ on $[0,1]^k$. As in Corollary 1.1, this bounded domain does not matter for multiplicative approximation as one can easily modify the distribution of Theorem B (by truncating and then rescaling) to get, for every $m \ge 1$ and $\varepsilon > 0$, a distribution $\mathcal{F}_{m,\varepsilon}$ on $[0,1]^2$ such that

$$[m]\text{-Rev}(\mathcal{F}_{m,\varepsilon}) < \varepsilon \cdot \text{Rev}(\mathcal{F}_{m,\varepsilon}).$$

Yet, for additive approximation we show:

Theorem D *For every $k \ge 2$ and $\delta > 0$ there is $m_0 = (k/\delta)^{\text{O}(k)}$ such that for every k-item distribution $\mathcal{F}$ on $[0,1]^k$ and every $m \ge m_0$,*

$$[m]\text{-Rev}(\mathcal{F}) \ge \text{Rev}(\mathcal{F}) - \delta.$$

One may get a multiplicative version of this theorem if $\mathcal{F}$'s support is bounded both from below and from above, that is, if the ratio between the highest and the lowest values is bounded. In particular if $\mathcal{F}$ is a distribution on, say, $[1,H]^k$ then this theorem implies that $[m]\text{-Rev}(\mathcal{F}) \ge (1-\delta)\text{Rev}(\mathcal{F})$ whenever[6] $m > m_0$ for $m_0 = (kH/\delta)^{\text{O}(k)}$.

[5] If one looks at the gap between $\text{DRev}(\mathcal{F})$ and the revenue obtained by selling the items separately $\text{SRev}(\mathcal{F})$, then the same exponential gap holds, a rare *doubly-exponential* contrast with the independent case considered in Hart and Nisan [2012] where $\text{DRev}(\mathcal{F}) \le \text{O}(\log^2 k)\cdot\text{SRev}(\mathcal{F})$.

[6] In Appendix 2, we tighten the dependence on H to be poly-logarithmic, for the case of two items.

Organization of the Paper

Next, in Section 2, we briefly go over related previous work; and then in Section 3 we define our model and notations. We start in Section 4 by listing the basic properties of the menu-size complexity measure. Next, in Section 5 we describe our basic reduction from questions of auction revenue to combinatorial constructions and as a demonstration we also provide a construction that implies Theorem C. The heart of the paper, which is the combinatorial construction implying Theorems A and B, appears in Section 6. Section 7 provides positive results on approximation by simple auctions and proves Theorem D. Finally, in section, 8, we prove the relations between the menu-size notion of simple auctions and the separate auction, and introduce the more refined "additive menu size" complexity measure. Several additional results are postponed to appendices. In Appendix 1, we prove the close relationship between our model and that of Briest et al. [2010] showing why our results answer their question as well. Appendix 2 gives the tighter multiplicative approximation when the items' values are bounded both from below and from above. Appendix 3 is devoted to precise comparisons between the bundling auction and other classes of auctions, and Appendix 4 deals with multiple buyers.

2 Existing Work

Within the economics literature the issue of simple vs. complex auctions is mostly implicit in the study of randomized vs. deterministic auctions. This was first studied in McAfee and McMillan [1988] who identified sufficient conditions for the optimal mechanism to be deterministic. However, Thanassoulis [2004] and Manelli and Vincent [2006] found a technical error in the paper and exhibited counter-examples. Other examples where randomization helps, including in the case where the values of the two items are independent and identically distributed, are provided in Pavlov [2011] and Hart and Reny [2012] (the latter is essentially the simplest possible such example). In general, it is still not clear what optimal mechanisms for selling two items look like and, in particular, it is not clear when they are deterministic. Good surveys of the work within economic theory on such questions appear in Thanassoulis [2004] and Manelli and Vincent [2006], with more recent work in Fang and Norman [2006], Jehiel et al. [2007], Hart and Reny [2010], Lev [2011], Hart and Reny [2012].

This question was studied more explicitly in a line of work in computer science (Chawla et al. [2007], Chawla et al. [2010a], Chawla et al. [2010b], Daskalakis and Weinberg [2011]) that considered approximating the optimal revenue by simple mechanisms and quantified the amount of the loss incurred by simple mechanisms relative to the optimal. This was done for various settings, especially unit-demand settings and some generalizations. In particular, in unit-demand settings, Briest et al. [2010] show that the gap between the revenue of randomized and deterministic mechanisms may be unbounded for three or more items, and leave the case of two items open, while showing that for a restricted class of randomized auctions, the gap is only constant.

3 Notation and Preliminaries

3.1 Mechanisms

A mechanism for selling k items specifies a (possibly randomized) protocol for interaction between a seller (who has no private information and commits to the mechanism) and a buyer who has a private valuation for the items. The outcome of the mechanism is an allocation specifying the probability of getting each of the k items and an (expected)[7] payment that the buyer gives to the seller. We will use the following notations:

- **Buyer valuation:** $x = (x_1, \ldots, x_k)$ where $x_i \geq 0$ denotes the value of the buyer for getting item i.
- **Allocation:** $q = (q_1, \ldots, q_k) \in [0, 1]^k$, where $q_i = q_i(x)$ denotes the probability that item i is allocated to the buyer when his valuation is x (alternatively, one may interpret q_i as the fractional quantity of item i that the buyer gets).
- **Seller revenue:** $s = s(x)$ denotes the expected payment[8] that the seller receives from the buyer when the buyer's valuation is x.
- **Buyer utility:** $b = b(x)$ denotes the utility of the buyer when his valuation is x, i.e., $b(x) = \sum_i x_i q_i(x) - s(x) = x \cdot q(x) - s(x)$.

We will be discussing mechanisms that are:

- **IR – (Ex-post) Individually Rational**: $b(x) \geq 0$ for all x.
- **IC – Incentive Compatible**: For all x, x': $\sum_i x_i q_i(x) - s(x) \geq \sum_i x_i q_i(x') - s(x')$.

The IC requirement simply captures the notion that the buyer acts strategically in the mechanism. Since we are discussing a single buyer, this is in a simple decision-theoretic sense and in particular there is no distinction between the dominant strategy and the Bayes-Nash implementation notions.

[7] We only consider risk-neutral agents.

[8] In the literature this is also called transfer, cost, price, revenue, and denoted by p, t, c, etc. We hope that using the mnemonic s for the ***S***eller's final payoff and b for the ***B***uyer's final payoff will avoid confusion.

The following lemma gives well known and easily proven equivalent conditions for incentive compatibility. A short proof using our notations can be found in Hart and Nisan [2012] (see also Hart and Reny [2012] for a slightly tighter characterization).

Lemma 3.1 *The following three definitions are equivalent for a mechanism with* $b(x) = x \cdot q(x) - s(x) = \sum_i x_i q_i(x) - s(x)$*:*

1. *The mechanism is IC.*

2. *The allocation* q *is weakly monotone, in the sense that for all* x, x' *we have* $(x - x') \cdot (q(x) - q(x')) \geq 0$*, and the payment to the seller satisfies* $x' \cdot (q(x) - q(x')) \leq s(x) - s(x') \leq x \cdot (q(x) - q(x'))$ *for all* x, x'*.*

3. *The buyer's utility* b *is a convex function of* x *and for all* x *the allocation* $q(x)$ *is a subgradient of* b *at* x*, i.e., for all* x' *we have* $b(x') - b(x) \geq q(x) \cdot (x' - x)$*. In particular* b *is differentiable almost everywhere and there* $q_i(x) = \partial b(x)/\partial x_i$*.*

Note that this in particular implies that any convex function b with $0 \leq \partial b(x)/\partial x_i \leq 1$ for all i defines an incentive compatible mechanism by setting $q_i(x) = \partial b(x)/\partial x_i$ (at non-differentiability points take q to be an arbitrary subgradient of b) and $s(x) = x \cdot q(x) - b(x)$.

When $x_1, \ldots, x_k$ are distributed according to the joint cumulative distribution function $\mathcal{F}$ on[9] $\Re_+^k$, the expected revenue of the mechanism given by b is

$$R(b; \mathcal{F}) = \mathbb{E}_{x \sim \mathcal{F}}(s(x)) = \int \cdots \int \left(\sum_{i=1}^{k} x_i \frac{\partial b(x)}{\partial x_i} - b(x) \right) \mathrm{d}\mathcal{F}(x_1, \ldots, x_k).$$

3.2 Revenue

For a cumulative distribution $\mathcal{F}$ on $\Re_+^k$ (for $k \geq 1$), we consider the optimal revenue obtainable from selling k items to a (single, additive) buyer whose valuation for the k items is jointly distributed according to $\mathcal{F}$:

- $\textsc{Rev}(\mathcal{F})$ is the maximal revenue obtainable by any incentive compatible and individually rational mechanism. Formally, $\textsc{Rev}(\mathcal{F}) = sup_b R(b; \mathcal{F})$ where b ranges over all convex functions with $0 \leq \partial b(x)/\partial x_i \leq 1$ for all i and $b(0) = 0$.

- $\textsc{DRev}(\mathcal{F})$ is the maximum revenue obtainable by a *deterministic* auction, i.e., in our notation when $q_i(x) \in \{0, 1\}$ for all i and x.

[9] We write this as $x = (x_1, \ldots, x_k) \sim \mathcal{F}$.

- $\text{SRev}(\mathcal{F})$ is the maximal revenue obtainable by selling each item *separately.* Formally, by Myerson's characterization, this means considering those deterministic auctions where there are single-item prices $p_1, \ldots, p_k$ such that $s(x) = \sum_i q_i(x)p_i$ for all x.

- $\text{BRev}(\mathcal{F})$ is the maximal revenue obtainable by *bundling* all items together. By Myerson's characterization, this means that for each x, $q(x)$ is either $(1, \ldots, 1)$ or $(0, \ldots, 0)$.

- $[m]\text{-Rev}(\mathcal{F})$ is the maximal revenue obtainable by an auction whose menu size is at most m; that is, by auctions that have at most m possible outcomes, i.e., such that[10] $|\{q(x) : x \in \Re^k_+\}\backslash\{(0, \ldots, 0)\}| \le m$.

4 Basic Results for Menu Size

In this section we list the basic relations related to menu-size complexity. All the statements here are immediate.

We start by showing that the optimal auction with menu of size 1 is a bundling one. Recall the notation $[m]\text{-Rev}(\mathcal{F})$ for the revenue achievable by an auction whose menu size is at most m.

Proposition 4.1 $[1]\text{-Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F})$.

Proof. Consider the optimal single-item-menu auction selling some fractions of items for some price p, and now consider selling the whole bundle for the same price p. The buyer will buy whenever he did in the original auction, so the revenue can only increase. ∎

For higher menu complexities, the revenue can increase at most linearly.

Proposition 4.2 $[m]\text{-Rev}(\mathcal{F}) \le m \cdot \text{BRev}(\mathcal{F})$.

Proof. Just consider the menu entry that brings in the largest fraction of revenue. ∎

This is essentially tight e.g. for the case of a distribution that puts value 2^i on item i with probability 2^{-i} (and zero otherwise), where these events are disjoint. The bundling auction can get revenue of at most 2, while for each $1 \le m \le k$ selling each of the items $i = 1, \ldots, m$ for a price of 2^i gets revenue m. We will later see that the increase with m need not be bounded by k, and in fact even for 2 items, increasing m can increase revenue by a factor of at least $m^{1/7}$.

[10] In our counting m does not include the pay-nothing-get-nothing outcome that we always assume without loss of generality is available, by individual rationality.

This directly implies the basic relation between deterministic auctions and simple ones.

Proposition 4.3 $\text{DR}\textsc{ev}(\mathcal{F}) \leq (2^k - 1) \cdot \text{BR}\textsc{ev}(\mathcal{F})$.

Proof. A deterministic auction has at most $2^k - 1$ menu items, one for each non-empty set of items. ∎

For the special case of $k = 2$ we will show in Appendix 3 the tight bound $\text{DR}\textsc{ev}(\mathcal{F}) \leq 5/2 \cdot \text{BR}\textsc{ev}(\mathcal{F})$.

5 Combinatorial Constructions for the Support of Distributions

Our main results are gaps between the revenue achievable by different types of auctions. To find distributions for which this gap materializes we will use various combinatorial constructions for the support of the distribution.

Let $q^1, q^2, \ldots$ be a finite or countably infinite sequence of points in $[0,1]^k$. Define

$$\text{gap}^n := q^n \cdot q^n - \max_{j<n} q^j \cdot q^n,$$

($q' \cdot q''$ is the scalar product $\sum_{i=1}^k q'_i q''_i$).

Proposition 5.1 *For every (finite or countably infinite) sequence $q^1, q^2, \ldots$ of points in $[0,1]^k$ there exists a distribution $\mathcal{F}$ on $\Re_+^k$ such that $\text{SR}\textsc{ev}(\mathcal{F}) \leq 2k$, $\text{BR}\textsc{ev}(\mathcal{F}) \leq 2k$, and $\text{R}\textsc{ev}(\mathcal{F}) \geq \sum_n \text{gap}^n$. Moreover, in the auction yielding revenue of $\sum_n \text{gap}^n$, for each n there is a menu item that gives allocation q^n and extracts expected revenue of gap^n from buyer types that choose it.*

In Appendix 3 we will state and prove a somewhat tighter relation between $\text{BR}\textsc{ev}(\mathcal{F})$ and $\text{R}\textsc{ev}(\mathcal{F})$, again in terms of $\sum_n \text{gap}^n$.

Proof. Without loss of generality assume that $\text{gap}^n > 0$ for all n (remove all points with non-positive gap; this can only increase the gaps for the remaining points). For each n, let $M^n = (2k)^n / (\Pi_{j=1}^n \text{gap}^j)$ (so $M^n \geq 2M^{n-1}$ since $\text{gap}^n \leq k$). For every n, let $\mathcal{F}$ be the distribution that for each n puts probability $1/M^n$ on the point $x^n := M^n q^n \in \Re_+^k$ (these probabilities sum up to at most 1 since $M^n \geq 2^n$), and puts the remaining probability on the point 0.

First notice that the revenue obtainable from any single item x_i is at most 2. This is a direct consequence of the fact that for every n the probability that x_i is at least M^n is at most $\sum_{j\geq n} 1/M^j$ (since $x_i^j = M^j\, q_i^j \leq M^j < M^n$ for $j < n$), and this sum is at most $2/M^n$ because the M^j grow by a factor of at least 2. A similar reasoning shows that $\text{BRev}(\mathcal{F}) \leq 2k$, as the probability that $\sum_i x_i$ is at least kM^n is at most $2/M^n$.

Now consider the auction whose menu contains, for all n in the sequence, the allocation q^n with payment $s^n := M^n\,\text{gap}^n$. The utility that a buyer with valuation x^n gets from the j-th menu entry (q^j, s^j) is $M^n(q^j \cdot q^n) - M^j\,\text{gap}^j$. First, for $j < n$ this utility is bounded from above by $M^n(q^j \cdot q^n)$, which by the definition of gap^n is bounded from above by $M^n(q^n \cdot q^n) - M^n\,\text{gap}^n$, which is precisely the utility that the buyer gets from the n-th menu item (q^n, s^n). Second, for $j > n$ we have $M^n(q^j \cdot q^n) - M^j\,\text{gap}^j \leq M^n\,k - M^{j-1}\,2k < 0$. Thus the buyer with valuation x^n will choose the n-th menu item and pay for it $M^n\,\text{gap}^n$, and since this happens with probability $1/M^n$, the expected revenue from this type of buyer is exactly gap^n. ∎

To demonstrate the use of this result let us show that the separation between the simple auctions of bundling or selling separately and an arbitrary deterministic auction may be exponential in the number of items; a statement that includes that of Theorem C.

Lemma 5.2 *There exists a k-item distribution $\mathcal{F}$ such that* $\text{SRev}(\mathcal{F}) \leq 2k$, $\text{BRev}(\mathcal{F}) \leq 2k$, *and* $\text{DRev}(\mathcal{F}) \geq 2^k - 1$.

Proof. Let us enumerate the $2^k - 1$ non-empty subsets L^n of $\{1, \ldots, k\}$ such that the size of L^n is weakly increasing in n, and let q^n be the indicator vector of L^n (i.e., $q_i^n = 1$ for $i \in L^n$ and $q_i^n = 0$ for $i \notin L^n$). For $j < n$ we have $q^j \cdot q^n = |L^j \cap L^n| < |L^n| = q^n \cdot q^n$, and thus $\text{gap}^n \geq 1$. Use the "moreover" statement of Proposition 5.1. ∎

6 Complex Auctions May Be Infinitely Better

We will construct the sequence of points q^n for Proposition 5.1, in order to prove an infinite separation between the revenue of an arbitrary auction for $k = 2$ items and that of selling the items separately (for larger k one may add items with zero values).

Proposition 6.1 *There exists an infinite sequence $q^1, q^2, \ldots$ of points in $[0,1]^2$ with $||q^n|| \leq 1$ such that for all n,* $\text{gap}^n = \Omega(n^{-6/7})$.

Proof. The sequence of points that we will build is composed of a sequence of "shells", each containing multiple points. The shells will be getting closer and closer to each other, approaching, as the shell, N, goes to infinity, the unit sphere: All the points q^j in the N-th shell will be of length $||q^j|| = \sum_{\ell=1}^{N} \ell^{-3/2}/\alpha$, where $\alpha = \sum_{\ell=1}^{\infty} \ell^{-3/2}$ (which indeed converges). Each shell N will contain $N^{3/4}$ different points in it so the angle between any two of them is at least $\Omega(N^{-3/4})$.

Now we estimate $q^j \cdot q^{j'} = ||q^j|| \cdot ||q^{j'}|| \cdot \cos(\theta)$ where θ is the angle between q^j and $q^{j'}$. Let N denote j's shell. For $j' < j$ there are two possibilities, either $q^{j'}$ is in the same shell as q^j or it is in a smaller shell. In the first case we have $\theta \geq \Omega(N^{-3/4})$ and thus $\cos(\theta) \leq 1 - \Omega(N^{-3/2})$ (since $\cos(x) = 1 - x^2/2 + x^4/24 - \ldots$) and since $||q^j|| = \Theta(1)$ we have $q^j \cdot q^j - q^{j'} \cdot q^j \geq \Omega(N^{-3/2})$. In the second case, $||q^j|| - ||q^{j'}|| \geq N^{-3/2}$, so again since $||q^j|| = \Theta(1)$ we have $(q^j \cdot q^j - q^{j'} \cdot q^j \geq \Omega(N^{-3/2})$. Thus for any point q^j in the N-th shell we have $\text{gap}^j = \Omega(N^{-3/2})$. Since the first N shells contain together $\sum_{\ell=1}^{N} \ell^{3/4} = \Theta(N^{7/4})$ points, we have $j = \Theta(N^{7/4})$ and thus $\text{gap}^j = \Omega(N^{-3/2}) = \Omega(j^{-6/7})$. ∎

This directly implies Theorems A and B.

Proof of Theorems A and B and Corollary 1.1. Take the countably infinite sequence of points q^n constructed in Proposition 6.1, and apply Proposition 5.1 to get the distribution $\mathcal{F}$. An auction that has only the first m menu items $q^1, \ldots, q^m$ will extract $\sum_{j=1}^{m} \text{gap}^j = \Omega(\sum_{j=1}^{m} j^{-6/7}) = \Omega(m^{1/7})$ from $\mathcal{F}$ (by the "moreover" statement of Proposition 5.1), and thus $[m]$-$\text{REV}(\mathcal{F}) \geq \Omega(m^{1/7})$. However for this $\mathcal{F}$, $\text{SREV}(\mathcal{F}) \leq 4$ and $\text{BREV}(\mathcal{F}) \leq 4$, so this proves Theorem B. As $\text{DREV}(\mathcal{F}) \leq 3 \cdot \text{BREV}(\mathcal{F}) \leq 12$ by Proposition 8.5, scaling $\mathcal{F}$ by the constant factor 12 yields Theorem A. Finally, for Corollary 1.1 we take the distribution $\mathcal{F}_n$ generated by Proposition 5.1 for the finite sequence $q^1, \ldots, q^n$ with large enough n. ∎

7 Additive Approximation

Theorem D is a direct corollary of the following lemma.

Lemma 7.1 *Let n be a positive integer and $m = (n+1)^k - 1$. Then for every distribution $\mathcal{F}$ on $[0,1]^k$ we have $[m]$-$\text{REV}(\mathcal{F}) \geq Rev(\mathcal{F}) - (2k)/\sqrt{n}$.*

Proof. Let $\mathcal{F}$ have support in $[0,1]^k$, and let b be a mechanism with corresponding (q, s).

Define a new mechanism $\tilde{b}$ as follows: for each $x \in [0,1]^n$, let $\tilde{q}(x)$ be the rounding up of $q(x)$ to the $1/n$-grid on[11] $[0,1]^k$, and let $\tilde{s}(x) := (1 - 1/\sqrt{n})s(x)$. Since $\tilde{q}$ can take at most $(n+1)^k$ different values, the menu size of $\tilde{b}$ is at most $(n+1)^k - 1 = m$.

If $\tilde{q}(x) \cdot x - \tilde{s}(x) \leq \tilde{q}(y) \cdot x - \tilde{s}(y)$ then (recall that $q(x) \cdot x - s(x) \geq q(y) \cdot x - s(y)$) we must have $(1/n) \sum_{i=1}^{k} x_i \geq (1/\sqrt{n})(s(x) - s(y))$, hence $s(y) \geq s(x) - k/\sqrt{n}$ (since $\sum_i x_i \leq k$),

[11] I.e., $\tilde{q}_i(x) = \lceil nq_i(x) \rceil / n$ for each i and x.

which implies that the seller's revenue at x from $\tilde{b}$ must be $\geq (1-1/\sqrt{n})(s(x)-k/\sqrt{n})$. Therefore $R(\tilde{b};\mathcal{F}) \geq (1-1/\sqrt{n})R(b;\mathcal{F}) - k/\sqrt{n} \geq R(b;\mathcal{F}) - 2k/\sqrt{n}$ (since $R(b;\mathcal{F}) \leq \sum_i x_i \leq k$). ∎

One may get a relative approximation result from this one by bounding the values both from above and from below. Specifically by letting $\mathcal{F}$ be a distribution on $[1,H]^k$. In this case, the following variant of Theorem D may be stated:

Theorem 7.2 *For every $k \geq 2$, $\delta > 0$, and $H > 0$ there is $m_0 \geq (kH/\delta)^{O(k)}$ such that for every k-item distribution $\mathcal{F}$ on $[1,H]^k$ and every $m \geq m_0$,*

$$[m]\text{-}\mathrm{Rev}(\mathcal{F}) \geq (1-\delta)\cdot \mathrm{Rev}(\mathcal{F}).$$

Proof. We first scale $[1,H]$ to $[1/H,1]$, which for multiplicative approximation is the same. We then design an auction which gives an additive approximation to within ε/H, which using Theorem D requires a menu size m as stated. Now, since $\mathcal{F}$ is bounded from below by $1/H$, its revenue is similarly bounded $\mathrm{Rev}(\mathcal{F}) \geq 1/H$, and thus an ε/H additive approximation is also a $(1-\varepsilon)$ multiplicative approximation as required. ∎

We do not know whether the dependence on k needs to be exponential, however we do show, in appendix 2, that, at least for $k=2$ items, the dependence on H may in fact be poly-logarithmic rather than polynomial.

8 The Separate Auction and Additive Menu Size

In this section we study the separate auction and its relationship to the bundling auction.

Proposition 8.1 $\mathrm{BRev}(\mathcal{F}) \leq k \cdot \mathrm{SRev}(\mathcal{F})$.

Proof. Let BRev be achieved with price p. If the separate auction offers price p/k for each item then whenever $\sum_i X_i \geq p$ we have $X_i \geq p/k$ for some i, and so one of the k items will be acquired in the separate auction. ∎

This is tight for $k=2$ and correlated values. Consider the distribution of $(x, 1-x) \in \Re^2_+$ where x is chosen uniformly in $[0,1]$. The bundling auction can get all the value by offering price 1 and selling always. Each item is distributed uniformly in $[0,1]$ so the best obtainable revenue for each is $1/4$, obtained at price $1/2$.

For larger values of k, we can get a stronger result.

Proposition 8.2 $\mathrm{BRev}(\mathcal{F}) \leq O(\log k) \cdot \mathrm{SRev}(\mathcal{F})$.

Proof. Let BRev($\mathcal{F}$) be achieved with price p. We can first assume without loss of generality that for each x in the support of $\mathcal{F}$, either $\sum_i x_i = p$ or $\sum_i x_i = 0$. (This is without loss of generality since for this new "truncated distribution" $\mathcal{F}'$, BRev($\mathcal{F}'$) = BRev($\mathcal{F}$) while SRev($\mathcal{F}'$) $\leq$ SRev($\mathcal{F}$) using Myerson's characterization.) We can now make another assumption without loss of generality, that in fact we always have that $\sum_i x_i = p$ in the support of $\mathcal{F}$ (so in fact BRev($\mathcal{F}$) $= p$). (This is without loss of generality as if we look at the conditional distribution on those x's with $\sum_i x_i = p$, both BRev and SRev increase by a factor of exactly $Pr[\sum_i x_i = p]$.)

At this point there are two different ways to proceed; we present both, as they may lead to different extensions.

Proof 1: Denote $e_i = \mathbb{E}_{X\sim\mathcal{F}}(X_i)$ be the expected value of the i'th item, so now (using our assumptions) $\sum_i e_i = p$. The claim is that the i'th item can be sold in a separate auction yielding a revenue of at least $(e_i - p/(2k))/(2(1 + log_2 k))$. The lemma is implied by summing over all i. Consider the distribution of X_i and split the range of values of X_i into $(2 + \log_2 k)$ sub-ranges: A "low" range for which $X_i \leq p/(2k)$ and for each $j = 0 \ldots \log_2 k$ a subrange where: $p/(2^{j+1}) < X_i \leq p/(2^j)$ (notice that since $X_i \leq p$ we have covered the whole support of X_i). The low subrange contributes at most $p/(2k)$ to the expectation of X_i, and thus one of the other $1 + \log_2 k$ sub-ranges contributes at least $((e_i - p/(2k))/(1 + \log_2 k)$ to this expectation. The lower bound of this sub-range, $p/(2^{j+1})$, is smaller by a factor of at most 2 than any value in the subrange, so asking it as the item price will yield revenue which is at least half of the contribution of this sub-range to the expectation, i.e. at least $((e_i - p/(2k))/(2(1 + \log_2 k))$.

Proof 2: Let $r_i := \text{Rev}(X_i) = \sup_t t(1 - F_i(t))$ (where F_i is the i-th marginal of $\mathcal{F}$), then $1 - F_i(t) \leq \min\{r_i/t, 1\}$ and so

$$\mathbb{E}[X_i] = \int_0^\infty (1 - F_i(t))dt \leq \int_0^{r_i} 1dt + \int_{r_i}^p \frac{r_i}{t}dt = r_i(1 + \log p - \log r_i).$$

The function $r(1 + \log p - \log r)$ is concave in r, hence

$$\frac{p}{k} = \frac{1}{k}\sum_{i=1}^k \mathbb{E}[X_i] \leq \bar{r}\left(1 + \log p - \log \bar{r}\right),$$

where $\bar{r} := (1/k)\sum_i r_i$. Thus $p/(k\bar{r}) \leq 1 + \log k + \log(p/(k\bar{r}))$, from which it follows that[12] BRev($\mathcal{F}$)/SRev($\mathcal{F}$) $= p/(k\bar{r}) < 4\log k$. ∎

The relation in the opposite direction is simple and given by:

Proposition 8.3 SRev($\mathcal{F}$) $\leq k \cdot$ BRev($\mathcal{F}$).

[12] The function $f(x) = x - \log x - 1 - \log k$ is increasing, and $f(4\log k) > 0$ for every $k \geq 2$.

Proof. Let F_i denote the marginal distribution on the i'th item, so by definition $\text{SRev}(\mathcal{F}) = \sum_i \text{Rev}(F_i)$. But the optimal auction for item i separately has, by Myerson, a single menu entry so is at most $[1]\text{-Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F})$. ∎

This is tight for every k, even for independent items; see Hart and Nisan [2012].

Combining with Proposition 4.3 we get the following corollary, which by Theorem C is essentially tight.

Corollary 8.4 $\text{DRev}(\mathcal{F}) \leq \text{O}(2^k \log k) \cdot \text{SRev}(\mathcal{F})$.

For the special case of $k = 2$ items, we have the somewhat tighter

Proposition 8.5 *For $k = 2$, $\text{DRev}(\mathcal{F}) \leq 3 \cdot \text{SRev}(\mathcal{F})$.*

Proof. Take the optimal deterministic auction which has three menu items: either selling just one of the items, or selling the bundle. Its revenue is the sum of that obtained for a single item and that obtained for the bundle, thus $\text{DRev}(\mathcal{F}) \leq \text{SRev}(\mathcal{F}) + \text{BRev}(\mathcal{F})$. The proof is completed using Proposition 8.1. ∎

Notice that the separate auction has menu size $2^k - 1$ (the buyer can buy any set $L \subset \{1, \ldots, k\}$ for $\sum_{i \in L} p_i$), and yet its revenue is bounded by only k times that of the bundling revenue rather than $2^k - 1$ (as is the case for general deterministic auctions; recall Proposition 4.3). Intuitively, this seems connected to the fact that the separate auction only has k "degrees of freedom" or "parameters" (the k prices). This may be formalized by defining a more refined "additive-menu-size" complexity:

- An auction has *additive menu size* m if it offers at most m "basic" menu entries to the buyer, each with its own price, but here the buyer may buy any *combination* of basic menu entries, provided that no item is allocated more than once in total, for the sum of the entry prices.[13]
- $[m]^*\text{-Rev}(\mathcal{F})$ is the maximal revenue obtainable by an auction whose additive menu size is at most m.

Clearly $[m]^*\text{-Rev}(\mathcal{F}) \geq [m]\text{-Rev}(\mathcal{F})$, and $[1]^*\text{-Rev}(\mathcal{F}) = [1]\text{-Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F})$ (cf. Proposition 4.1). Interestingly, the linear bound of Proposition 4.2 holds here too:

Proposition 8.6 $[m]^*\text{-Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F})$.

[13]Formally, (q, s) is a *combination* of $(q^1, s^1), \ldots, (q^n, s^n)$ if $q = q^1 + \cdots + q^n$, $s = s^1 + \cdots + s^n$, and $q \in [0, 1]^k$ (i.e., (q, s) is a feasible choice).

Proof. For each $n = 1, \ldots, m$, let s^n be the seller's payment from the n-th basic menu item, and let α^n be the *total* probability that the buyer takes this menu item (thus $\sum_n \alpha^n$ may be as high as k). The total revenue is $\sum_n s^n \alpha^n$, and, as in the proofs of Propositions 4.1 and 4.2, each term is bounded from above by $\text{BRev}(\mathcal{F})$. ∎

This essentially implies that the results in this paper apply also to this more refined complexity measure as well. This complexity measure really captures the power of selling the items separately as the additive menu complexity of the separate auction is at most k (cf. Propositions 8.3 and 8.6).

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Appendix 1: The Unit Demand Model

In this section we shortly compare our model to the unit-demand model considered in several papers, in particular in Briest et al. [2010]. There are k items for sale and a single buyer. There are two basic differences between our model and the unit-demand one. First, in the unit-demand model, the buyers are modelled as having unit-demand valuations. Additionally, the unit-demand model was defined to only allow the auction to offer single items rather than bundles of items as in our model. This second restriction does not turn out to matter.

More formally, in the unit demand model there is a single buyer with a unit demand valuation, i.e., for a set $L \subset \{1, \ldots, k\}$ of items, its value is $\max_{i \in S} x_i$ (rather than $\sum_{i \in L} x_i$). A deterministic auction is this setting would offer a price p_i for each item i. For unit-demand buyers this is equivalent to a completely general deterministic auction as there is no need to offer prices for bundles since the buyer is not interested in them. Thus for example an auction offering price p_1 for item 1, p_2 for item 2 and p_{12} for both items, would be the same as asking price $\min(p_1, p_{12})$ for item 1 and price $\min(p_1, p_{12})$ for item 2.

A randomized auction in this model is allowed to offer a set of lotteries, each with its own price, where a lottery is a vector of probabilities $\alpha_1, \ldots, \alpha_k$ of getting the items, with $\sum_i \alpha_i \leq 1$ (in contrast to our additive buyer, where $q_i \leq 1$ for each i). Again, for unit-demand buyers this is equivalent to general randomized auctions that are also allowed to offer lotteries for bundles of items. For example a menu item offering the lottery "item 1 with probability 2/9; item 2 with probability 3/9; and both items with probability 4/9" for a certain price can be replaced by the two menu items "item 1 with probability 6/9; item 2 with probability 3/9" and "item 1 with probability 2/9; item two with probability 7/9", each for the same price as the original menu item.

Let us use the notation $\textsc{Rev}^{\text{UD}}(\mathcal{F})$ to denote the revenue obtainable from a unit demand buyer whose valuation for the k single items is distributed according to $\mathcal{F}$. Similarly $\textsc{DRev}^{\text{UD}}(\mathcal{F})$ will denote the revenue achievable by deterministic auctions. We can compare these revenues to those achievable in our model from an additive buyer whose valuation for the k single items is distributed according to the same $\mathcal{F}$.

Proposition 9.1 *For every $k \geq 2$ and every k-item distribution $\mathcal{F}$ on $\Re_{+}^{k}$,*

- $\textsc{Rev}^{\text{UD}}(\mathcal{F}) \leq \textsc{Rev}(\mathcal{F}) \leq k \cdot \textsc{Rev}^{\text{UD}}(\mathcal{F})$.
- $\textsc{DRev}^{\text{UD}}(\mathcal{F}) \leq \textsc{DRev}(\mathcal{F}) \leq k2^k \cdot \textsc{DRev}^{\text{UD}}(\mathcal{F})$.

Proof. The lower bounds in both cases follow since any auction in the unit-demand model offers only unit-demand menu entries, and on these the unit-demand buyer and the additive buyer have the same preferences; thus offering the same menu in our setting would give exactly the same revenue as it does in the unit-demand setting.

For the upper bound for randomized algorithms notice that if we replace each menu entry of an auction in our model that asks price s for allocation $(q_1, \ldots, q_k)$ (where $0 \leq q_i \leq 1$ for each i) by a menu entry asking price s/k for the allocation $(q_1/k, \ldots, q_k/k)$, then we do not change the preferences of the buyer between the different menu items, and thus the revenue drops by a factor of exactly k. However, the new auction only gives unit-demand allocations, and moreover the unit-demand buyer and the additive buyer behave the same.

For the deterministic upper bound we start with a deterministic auction in our model. Since it has at most $2^k - 1$ menu entries, a fraction of at least 2^{-k} of the revenue must come when allocating one of them to, say, the set L of items. An auction that only offers to sell the set L of items for the same price s as the original one did, will thus make at least a 2^{-k} fraction of the revenue of the original one. Now consider the unit-demand auction that offers the price $s/|L|$ for each of the items in L: whenever the additive buyer in the additive auction bought L we are guaranteed that his value for at least one of the items in L was at least $s/|L|$, in which case the unit-demand buyer will also acquire an item. ∎

The interesting gap in the above lemma is the exponential one for deterministic auctions, and indeed we can show that this is tight.

Lemma 9.2 *There exists a k-item distribution $\mathcal{F}$ such that* $\mathrm{DRev}(\mathcal{F}) \geq \Omega(2^k/k) \cdot \mathrm{DRev}^{\mathrm{UD}}(\mathcal{F})$.

Proof. Take the distribution $\mathcal{F}$ constructed in the proof of Lemma 5.2 that satisfies $\mathrm{SRev}(\mathcal{F}) \leq 2k$ and $\mathrm{DRev}(\mathcal{F}) \geq 2^k - 1$. Now $\mathrm{DRev}^{\mathrm{UD}}(\mathcal{F}) \leq \mathrm{SRev}(\mathcal{F})$ as the item prices used in any deterministic auction in the unit-demand model can only yield more revenue in our additive model where the buyer may buy more than a single item. ∎

Despite the exponential separation, for finite k it is still constant, and so a super-constant separation between randomized and deterministic auctions in our setting is equivalent to the same separation in the unit-demand setting.

Appendix 2: Multiplicative Approximation with Bounded Domains

In this appendix we give a stronger version of Theorem 7.2, with a poly-logarithmic dependence on H, for the case of two items.

Theorem 10.1 *For every $\delta > 0$ and $H > 0$ there is $m_0 \geq \Omega(\delta^{-5} \log^2 H)$ such that for every two-item distribution $\mathcal{F}$ on $[1, H]^2$ and every $m \geq m_0$,*

$$[m]\text{-}\mathrm{Rev}(\mathcal{F}) \geq (1 - \delta) \cdot \mathrm{Rev}(\mathcal{F}).$$

This is a direct corollary of the following lemma.

Lemma 10.2 *For every two-item mechanism (q, s), every $H > 1$ and every $\delta > 0$, there exists a mechanism $(\tilde{q}, \tilde{s})$ with menu size at most $\mathrm{O}(\delta^{-5} \log^2 H)$, such that for every x with $1 \leq s(x) \leq H$ we have $\tilde{s}(x) \geq (1 - \delta)s(x)$.*

Proof. We will discretize the menu of the the original mechanism. Our first step will be to discretize the payments s, and the second to discretize the allocations $q = (q_1, q_2)$.

We start by splitting the range $[1, H]$ into K sub-ranges, each of them with ratio at most $H^{1/K}$ between its end-points, where K is chosen so that $H^{1/K} \leq \delta^2$, i.e. $K = \mathrm{O}(\delta^{-2} \log H)$. We define a real function $\phi(s)$ by rounding s up to the top of its range and then multiplying by $1 - \delta$. So we have $(1 - \delta)s < \phi(s) < (1 - \delta)(1 + \delta^2)s$. Then for any $s' \leq s(1 - \delta)$ we have $\phi(s) - \phi(s') \leq (1 - \delta)(s - s' + s\delta^2) \leq (s - s') - s\delta^2 + s\delta^2 = s - s'$.

Now we take every menu entry (q, s) of the original mechanism and replace s with $\phi(s)$. The previous property of ϕ ensures that any buyer who previously preferred (q, s) to some other menu item (q', s') with $s' < (1 - \delta)s$ will still prefer $(q, \phi(s))$ in the new menu; thus in the new menu he will pay $\phi(s')$ for some $s' \geq (1-\delta)s$, and $\phi(s') > (1-\delta)s' \geq (1-\delta)^2 s$; his payment in the new menu is therefore at least $(1-\delta)^2$ times his payment in the original menu.

We now have a menu with only K distinct price levels $s^1 < \cdots < s^K$. Before we continue, we scale it down by a factor of $(1-\delta)$, i.e. multiply both the q's and the s's by $(1-\delta)$. This does not change the menu choice of any buyer; reduces the payments by a factor of exactly $1-\delta$; and ensures that $q_1, q_2 \leq 1-\delta$. We now round down each q_1 and each q_2 to an integer multiple of δ/K, and then add $\delta i/K$ to each menu entry whose price is s^i. Notice that rounding down reduced each q by at most δ/K, and since higher paying menu entries got a boost which is larger by at least δ/K than any lower paying menu entry, any buyer that previously chose an entry paying s, can now only choose an entry paying some $s' \geq s$.

All in all we have reached a new mechanism whose payment is at least $(1-\delta)^3 \geq 1-3\delta$ times that of the original one (so redefine the δ in the proof to be 1/3 of the δ in the statement). There are $K = \mathrm{O}(\delta^{-2}\log H)$ price levels and $\delta^{-1}K = \mathrm{O}(\delta^{-3}\log H)$ different allocation levels for both q_1 and for q_2. However notice that for a fixed price level s and a fixed q_1 there can only be a single value of q_2 that is actually used in the menu (as lower ones will be dominated), and so the total number of possible allocations is $\mathrm{O}(\delta^{-5}\log^2 H)$.
∎

Notice that the poly-logarithmic dependence of m on H is "about right" since the distribution $\mathcal{F}$ induced by the first m_0 points in the construction used for proving theorem A has $H = m_0^{\mathrm{O}(m_0)}$, and the $m_0^{1/7}$ gap between $\textsc{Rev}(\mathcal{F})$ and $[1]$-$\textsc{Rev}(\mathcal{F})$ implies that for $m = \mathrm{O}((\log H)^{1/8})$ we have $[m]$-$\textsc{Rev}(\mathcal{F}) = o(\textsc{Rev}(\mathcal{F}))$.

Appendix 3: Exact Comparisons to Bundling

In this section we will provide bounds on the bundling revenue: on the one hand, it can be arbitrarily low relative to the optimal revenue, and on the other hand it guarantees a certain fixed fraction (such as 2/3, or 1/2, or 5/2) of the optimal revenue over certain interesting subclasses of mechanisms (such as deterministic, or separate prices, mechanisms).

The following proposition provides a precise tool for all these comparisons.

Proposition 11.1 *Let $b(x) = \max_{0 \leq n \leq m}\{q^n \cdot x - s^n\}$ be a finite k-dimensional mechanism, where $q^n \in [0,1]^k$, $s^n \geq 0$, $q^0 = (0,\ldots,0)$, $s^0 = 0$, and the indices are such that*

$s^{n-1} \le s^n$ for all $n \ge 1$. For each $n \ge 1$ let $t^n := \inf\{\sum_{i=1}^k x_i : s(x) \ge s^n\}$, and define

$$T(b) := \sum_{n=1}^{m} \frac{s^n - s^{n-1}}{t^n}$$

(where we take $(0-0)/0$ as 0, and the inf *of an empty set as*[14] *∞). Then:*

(i) *For any k-item distribution $\mathcal{F}$ on $\Re^k_+$,*

$$R(b;\mathcal{F}) \le T(b) \cdot \mathrm{BR{\small EV}}(\mathcal{F}). \tag{2}$$

(ii) *There exists a k-item distribution $\mathcal{F}$ on $[0,1]^k$ such that*

$$R(b;\mathcal{F}) = T(b) \cdot \mathrm{BR{\small EV}}(\mathcal{F}). \tag{3}$$

Moreover, if b is symmetric (i.e., $b(x) = b(\pi x)$ for every $x \in \Re^k_+$ and every permutation of the indices[15] *π, then $\mathcal{F}$ can be taken to be a symmetric distribution*[16] *(i.e., $\mathcal{F}(x) = \mathcal{F}(\pi x)$ for every $x \in \Re^k_+$ and every permutation of the indices π).*

Before proving the proposition it is instructive to see the computation of $T(b)$ in a few examples with $k = 2$ items.

- For $b(x) = \max\{0, x_1-1, x_2-2, x_1+x_2-4\}$ we have $(s^0, s^1, s^2, s^3) = (0, 1, 2, 4)$ and $(t^1, t^2, t^3) = (1, 2, 5)$ (attained at the points $(1,0), (0,2)$, and $(2,3)$, respectively), and so $T(b) = (1-0)/1 + (2-1)/2 + (4-2)/5 = 12/5$.
- For $b(x) = \max\{0, x_1-2, x_2-2, x_1+x_2-4\}$ we have $(s^0, s^1, s^2, s^3) = (0, 2, 2, 4)$ and $(t^1, t^2, t^3) = (2, 2, 4)$ ($t^1 = t^2 = 2$ is attained at both $(2,0)$ and $(0,2)$, and $t^3 = 4$ at $(2,2)$), and so $T(b) = (2-0)/2 + (2-2)/2 + (4-2)/4 = 3/2$.
- For $b(x) = \max\{0, x_1-5, x_2-2, x_1+x_2-4\}$ we have $(s^0, s^1, s^2, s^3) = (0, 2, 4, 5)$ and $(t^1, t^2, t^3) = (2, 4, \infty)$, and ($t^1 = 2$ is attained at $(0,2)$ and $t^2 = 4$ at $(x_1, 4-x_1)$ with $2 \le x_1 \le 4$), and so $T(b) = (2-0)/2 + (4-2)/4 + (5-4)/\infty = 3/2$.

In general, the t^n are computed as follows: First, for each $n \ge 1$ let τ^n be the minimum of $x_1 + \cdots + x_k$ over the set $\{x \ge 0 : q^n \cdot x - s^n \ge q^j \cdot x - s^j \text{ for all } j\}$; this is a linear programming problem. Then, for each $n \ge 1$ put $t^n = \min\{\tau^j : s^j \ge s^n\}$.

[14] $t^n = 0$ if and only if $s^n = 0$ (and then $s^{n-1} = 0$). The sum in $T(b)$ can thus be started with $n = n_0$, the first index with $s^{n_0} > 0$, and ending at $n = m_0$, the first index with $t^{m_0+1} = \infty$, where we put $t^{m+1} := \infty$.

[15] I.e., π is a one-to-one mapping from $\{1, \ldots, k\}$ onto itself, and $\pi(x_1, \ldots, x_k) = (x_{\pi(1)}, \ldots, x_{\pi(k)})$.

[16] Also known as an "exchangeable" distribution.

Proof of Proposition 11.1. If $s^n = 0$ for all n then $R(b; F) = 0$ for all F and $T(b) = 0$ and there is nothing to prove. Otherwise $T(b) > 0$, and, as in footnote 14, we only consider those n between n_0 and m_0 for which $0 < t^n < \infty$; for simplicity assume that these are all n, i.e., $1 \leq n \leq m$. Let X be a k-dimensional random variable with distribution $\mathcal{F}$. By definition of t^n we have $[\sum_i X_i \geq t^n] \supset [s(X) \geq s^n]$, and so

$$\text{BRev}(\mathcal{F}) \geq t^n \cdot \mathbb{P}\left[\sum_{i=1}^k X_i \geq t^n\right] \geq t^n \cdot \mathbb{P}\left[s(X) \geq s^n\right] = t^n \cdot \sum_{j=n}^m \sigma^j,$$

where $\sigma^j := \mathbb{P}\left[s^j \leq s(X) < s^{j+1}\right]$ (thus $\sigma^j = 0$ when $s^j = s^{j+1}$, and $\sigma^j = \mathbb{P}\left[s(X) = s^j\right]$ when $s^j < s^{j+1}$). Therefore

$$\begin{aligned} \sum_{i=1}^m \frac{s^n - s^{n-1}}{t^n} \text{BRev}(\mathcal{F}) &\geq \sum_{i=1}^m (s^n - s^{n-1}) \sum_{j=i}^m \sigma_j = \sum_{j=1}^m \sigma_j \sum_{i=1}^j (s^n - s^{n-1}) \\ &= \sum_{j=1}^m \sigma_j s_j = \mathbb{E}\left[s(X)\right] = R(b; \mathcal{F}). \end{aligned}$$

(recall that $s_0 = 0$). The left-hand side is precisely $T(b) \cdot \text{BRev}(\mathcal{F})$, and we have proved (2).

To construct a distribution for which we have equality, for each n let $x^n \in \Re^k_+$ be a point where the minimum t^n is attained; thus $\sum_i x_i^n = t^n$ and $s(x^n) \geq s^n$. Let $\mathcal{F}_b$ put probability $\alpha^n := t^1/t^n - t^1/t^{n+1} \geq 0$ on the point x^n; note that $\sum_{j \geq n} \alpha^j = t^1/t^n$, (recall that $t^{m+1} = \infty$) and in particular $\sum_{j=1}^\infty \alpha^j = 1$. The sequence t^n is nondecreasing (since the sequence s^n is nondecreasing), and so the revenue of the bundled mechanism $b^n(x) = \max\{0, \sum_i x_i - t^n\}$ is $R(b^n; \mathcal{F}_b) = t^n \sum_{j \geq n} \alpha^j = t^n(t^1/t^n) = t^1$. Therefore $\text{BRev}(\mathcal{F}_b) = t^1$ (prices different from some t^n can only yield lower revenues). For the original mechanism b, its revenue at x^n is $s(x^n) \geq s^n$, and so

$$\begin{aligned} R(b; \mathcal{F}_b) &= \sum_{n=1}^m s(x^n)\ \alpha^n \geq \sum_{n=1}^m s^n \left(\frac{t^1}{t^n} - \frac{t^1}{t^{n+1}}\right) = t^1 \sum_{n=1}^m \left(\frac{s^n}{t^n} - \frac{s^{n-1}}{t^n}\right) \\ &= T(b)\ t^1 = T(b)\ \text{BRev}(\mathcal{F}_b) \geq R(b; \mathcal{F}_b) \end{aligned}$$

(recall that $s^0/t^1 = s^m/t^{m+1} = 0$), where the last inequality is (2). We thus have equality throughout.

Now rescaling all s^n by $\lambda > 0$ rescales all the t^n, and thus the support of F_b, by λ, while it does not affect T. Specifically, let $\hat{b}$ be the mechanism $\{(\hat{q}^n, \hat{s}^n)\}_{n=0}^m$ where $\hat{q}^n = q^n$ and $\hat{s}^n = \lambda s^n$; then $t^n = \lambda t^n$, and so $T(\hat{b}) = T(b)$. Thus the distribution $F_{\hat{b}}$ constructed above for $\hat{b}$ also satisfies (3) (with the same $T(b)$), and its support is included in $\{\sum_i x_i \leq \hat{t}^m = \lambda t^m\}$; take $\lambda = 1/t^m$.

Finally, the revenue of any symmetric mechanism b remains the same when we "symmetrize" $\mathcal{F}$; i.e., $R(b;\tilde{\mathcal{F}}) = R(b;\mathcal{F})$, where $\tilde{\mathcal{F}}(x) := (k!)^{-1}\sum_\pi \mathcal{F}(\pi x)$ for all x. Bundled mechanisms are symmetric, thus $\text{BR}\textsc{ev}(\tilde{\mathcal{F}}) = \text{BR}\textsc{ev}(\mathcal{F})$, and so one can use the symmetric distribution $\tilde{\mathcal{F}}_b$ instead of $\mathcal{F}_b$ in (3). ∎

As we will see below (Proposition 11.3), the bound $T(b)$ can be arbitrarily large. However, there are various interesting classes of mechanisms for which $T(b)$ is uniformly bounded, and then Proposition 11.1 yields uniform bounds.

For a class C of k-dimensional mechanisms (for instance: all mechanisms, or all deterministic mechanisms), we denote by $\text{C-R}\textsc{ev}(\mathcal{F})$ the maximal revenue obtainable over all mechanisms in C; i.e., $\text{C-R}\textsc{ev}(\mathcal{F}) := \sup_{b\in\text{C}} R(b;\mathcal{F})$.

Corollary 11.2 *Let* C *be a class of finite k-dimensional mechanisms, and define*

$$T(\text{C}) := \sup_{b\in\text{C}} T(b).$$

Then for every k-item distribution $\mathcal{F}$ on $\Re^k_+$

$$\text{C-R}\textsc{ev}(\mathcal{F}) \leq T(\text{C}) \cdot \text{BR}\textsc{ev}(\mathcal{F}); \tag{4}$$

moreover, the bound $T(\text{C})$ is tight.

Proof. By (2) and the definition of $T(\text{C})$, we have $R(b;\mathcal{F})/\text{BR}\textsc{ev}(\mathcal{F}) \leq T(b) \leq T(\text{C})$ for every mechanism $b \in \text{C}$; taking sup over $b \in \text{C}$ gives (4).

To see that the bound $T(\text{C})$ is tight, take any $T' < T(\text{C})$, then there exists $b \in \text{C}$ with $T(b) > T'$. Let $\mathcal{F}_b$ be a distribution satisfying (3), then $\text{C-R}\textsc{ev}(\mathcal{F}_b)/\text{BR}\textsc{ev}(\mathcal{F}_b) \geq R(b;\mathcal{F}_b)/\text{BR}\textsc{ev}(\mathcal{F}_b) = T(b) > T'$; thus no $T' < T(\text{C})$ can serve as an upper bound in (4). ∎

Proposition 11.3 *Let* $[m]\text{-F}\textsc{in}$ *be the class of all k-dimensional mechanisms with menu of size at most m, then*

$$T([m]\text{-F}\textsc{in}) = \Omega(m^{1/7}).$$

Proof. Let $q^1, q^2, \ldots, q^m \in [0,1]^k$ be given by Proposition 6.1 so $\text{gap}^n = \Omega(n^{-6/7})$. Let b be given by the menu of size m consisting of (q^n, s^n) for $1 \leq n \leq m$, where $s^n := 2^n$.

For each $n \geq 1$ let $\mu^n := s^n/\text{gap}^n$ and $x^n := \mu^n q^n$. We have

$$q^n \cdot x^n - s^n = \mu^n(q^n \cdot q^n - \text{gap}^n) = \mu^n \max_{j<n} q^j \cdot q^n = \max_{j<n} q^j \cdot x^n \geq \max_{j<n}(q^j \cdot x^n - s^j),$$

and so a buyer with valuation x^n will choose only among those menu items (q^j, s^j) with[17] $j \geq n$, and all these s^j are $\geq s^n$; thus $s(x^n) \geq s^n$.

Therefore $t^n = \inf\{x \cdot e : s(x) \geq s^n\} \leq x^n \cdot e = \mu^n\ q^n \cdot e \leq \mu^n k = k2^n/\text{gap}^n$, and we get

$$T(b) = \sum_{n=1}^{m} \frac{s^n - s^{n-1}}{t^n} \geq \sum_{n=1}^{m} \frac{2^n - 2^{n-1}}{k2^n/\text{gap}^n} = \frac{1}{2k} \sum_{n=1}^{m} \text{gap}^n = \Omega(m^{1/7}).$$

∎

Remark. One can easily modify the construction of the mechanism above as to obtain a *symmetric* mechanism b with $T(b) = \Omega(m^{1/7})$. Indeed, the menu will include each (q^n, s^n) together with all its permutations $(\pi q^n, s^n)$, which increases its size by a factor of at most 2^k and does not affect $T(b)$ (since all permutations of q^n come with the same s^n).

Corollary 11.4

$$\sup_{\mathcal{F}} \frac{[m]\text{-}\textsc{Rev}(\mathcal{F})}{\textsc{BRev}(\mathcal{F})} \geq \Omega\left(m^{1/7}\right) \qquad \text{and} \qquad \sup_{\mathcal{F}} \frac{\textsc{Rev}(\mathcal{F})}{\textsc{BRev}(\mathcal{F})} = \infty,$$

where the supremum is over all k-item distributions $\mathcal{F}$ on $\Re^k_+$, or over all symmetric such distributions.

We will now provide precise bounds on how good the bundling auction is relative to deterministic auctions and to separate-items auctions when there are two items.

Proposition 11.5 (i) *For every two-item distribution $\mathcal{F}$ on $\Re^2_+$,*

$$\textsc{BRev}(\mathcal{F}) \geq \frac{2}{5}\ \textsc{DRev}(\mathcal{F}) \qquad \text{and} \qquad \textsc{BRev}(\mathcal{F}) \geq \frac{1}{2}\ \textsc{SRev}(\mathcal{F}).$$

(ii) *For every two-item symmetric distribution $\mathcal{F}$ on $\Re^2_+$,*

$$\textsc{BRev}(\mathcal{F}) \geq \frac{2}{3}\ \textsc{DRev}(\mathcal{F}) \qquad \text{and} \qquad \textsc{BRev}(\mathcal{F}) \geq \frac{2}{3}\ \textsc{SRev}(\mathcal{F}).$$

Moreover, all four bounds are tight.

Proof. We will compute $T(\mathrm{C})$ for the four appropriate classes of mechanisms: (2) and (4) give (i), and (1) and (3) give (ii) (when $\mathcal{F}$ is symmetric we can limit ourselves to symmetric mechanisms).

[17] When indifferent, the buyer chooses the menu item with highest payment to the seller s; this can be assumed without loss of generality, see Hart and Reny [2012].

(1) A *symmetric deterministic* mechanism is of the form $b(x_1, x_2) = \max\{0, x_1 - p_1, x_2 - p_1, x_1 + x_2 - p_2\}$ with $p_i \geq 0$.

If $p_1 \leq p_2$ then $s_1 = p_1$ and $s_2 = p_2$, and we get $t_1 = p_1$ and $t_2 \leq 2(p_2 - p_1)$ (since $x_1 + x_2 - p_2 \geq x_1 - p_1$ implies $x_2 \geq p_2 - p_1$, and $x_1 + x_2 - p_2 \geq x_2 - p_1$ implies $x_1 \geq p_2 - p_1$, and so $x_1 + x_2 \geq 2(p_2 - p_1)$); therefore $T(b) \leq (p_1 - 0)/p_1 + (p_2 - p_1)/(2(p_2 - p_1)) \leq 3/2$ (when $p_1 = 0$ the first term is 0 and then $T(b) \leq 1/2$).

If $p_1 > p_2$ then $b(x_1, x_2) = \max\{0, x_1 + x_2 - p_2\}$ (which is bundled) and then $T(b) = (p_2 - 0)/p_2 \leq 1$.

Thus $T(b) \leq 3/2$. A mechanism b with $T(b) = 3/2$ is, for example, $b(x_1, x_2) = \max\{0, x_1 - 1, x_2 - 1, x_1 + x_2 - 3\}$.

(2) A *deterministic* mechanism is of the form $b(x_1, x_2) = \max\{0, x_1 - p_1, x_2 - p_2, x_1 + x_2 - p_3\}$; without loss of generality assume that $p_1 \leq p_2$.

If $0 \leq p_1 \leq p_2 \leq p_3$ then $s_i = p_i$ and we have: $t_1 \geq p_1$ (since $x_1 - p_1 \geq 0$ implies $x_1 + x_2 \geq p_1$); next, $t_2 \geq p_2 \geq p_2 - p_1$ (since $x_2 - p_2 \geq 0$ implies $x_1 + x_2 \geq p_2$); and finally $t_3 \geq 2(p_3 - p_2)$ (since $x_1 + x_2 - p_3 \geq x_1 - p_1$ implies $x_2 \geq p_3 - p_1 \geq p_3 - p_2$, and $x_1 + x_2 - p_3 \geq x_2 - p_2$ implies $x_1 \geq p_3 - p_2$, and thus $x_1 + x_2 \geq 2(p_3 - p_2)$). Therefore $T(b) \leq 1 + 1 + 1/2 = 5/2$.

If $0 \leq p_1 \leq p_3 \leq p_2$ then $b(x_1, x_2) = \max\{x_1 - p_1, x_1 + x_2 - p_3\}$ and then $s_1 = p_1, s_2 = p_3, t_1 = p_1$, and $t_2 \geq p_3 - p_1$ (since $x_1 + x_2 - p_3 \geq x_1 - p_1$ implies $x_2 \geq p_3 - p_1$), thus $T(b) \leq 1 + 1 = 2$.

If $0 \leq p_3 \leq p_1 \leq p_2$ then $b(x_1, x_2) = \max\{0, x_1 + x_2 - p_3\}$ and $T(b) = 1$.

Thus $T(b) \leq 5/2$. The computations above show that the bound 5/2 cannot be attained; to approach it, for large M take $b(x_1, x_2) = \max\{0, x_1 - 1, x_2 - M, x_1 + x_2 - M^2\}$, then $T(b) > 5/2 - 2/M$.

(3) A *symmetric separate-price* mechanism is of the form $b(x_1, x_2) = \max\{0, x_1 - p, x_2 - p, x_1 + x_2 - 2p\}$.We have $s_1 = p_1 = t_1$ (attained at $(p, 0)$ and $(0, p)$) and $s_2 = 2p = t_2$ (attained at (p, p); therefore $T(b) = p/p + (2p - p)/2p = 3/2$.

(4) A *separate-price* mechanism is of the form $b(x_1, x_2) = \max\{0, x_1 - p_1, x_2 - p_2, x_1 + x_2 - p_1 - p_2\}$; without loss of generality assume that $p_1 \leq p_2$. As above, we get: $s_1 = p_1 = t_1$ (attained at $(p_1, 0)$); $s_2 = p_2 = t_2$ (attained at $(0, p_2)$); and $s_3 = p_1 + p_2 = t_3$ (attained at (p_1, p_2)). Therefore $T(b) = p_1/p_1 + (p_2 - p_1)/p_2 + p_1/(p_1 + p_2) = 2 - p_1^2/((p_2(p_1 + p_2)) < 2$, and the sup is 2 (take $p_2/p_1 \to \infty$). ∎

Examples of distributions for which these bounds are tight are easily obtained from the above proof together with the construction in Proposition 11.1 (ii) (we have slightly simplified them):

- For DRev, take the distribution $\mathcal{F}_M$ that puts probabilities $c, c/M$, and $c/(2M^2)$, on the points $(1,0)$, $(0,M)$, and (M^2,M^2), respectively, where $c = (1+1/M+1/(2M^2))^{-1}$ (note that $c \to 1$ as $M \to \infty$). Then $\mathrm{BR}\textsc{ev}(\mathcal{F}_M) = \max\{1 \cdot 1,\ M \cdot (c/M + c/(2M^2)),\ 2M^2 \cdot c/2M^2\} \to 1$, and the deterministic mechanism $b(x_1,x_2) = \max\{0, x_1-1, x_2-M, x_1+x_2-M^2\}$ yields $R(b;\mathcal{F}_M) = 1 \cdot c + M \cdot c/M + M^2 \cdot c/(2M^2) = 5/2c \to 5/2$.

- For $\mathrm{SR}\textsc{ev}$, take X_1 and X_2 to be independent, $\mathbb{P}[X_1 = 0] = \mathbb{P}[X_1 = 2] = 1/2$, $\mathbb{P}[X_2 = 0] = 1 - 1/M$, $\mathbb{P}[X_2 = M] = 1/M$ (where $M > 2$), then $\mathrm{R}\textsc{ev}(X_1) = 2 \cdot 1/2 = 1$, $\mathrm{R}\textsc{ev}(X_2) = M \cdot 1/M = 1$, and so $\mathrm{SR}\textsc{ev}(X) = 2$; and $\mathrm{BR}\textsc{ev}(X) = \max\{0 \cdot 1, 2 \cdot (1/2 + 1/(2M)), M \cdot 1/M, (M+1) \cdot 1/(2M)\} = 1 + 1/M \to 1$.

- For the symmetric case, take the distribution $\mathcal{F}$ that puts probability 1/4 on each of $(1,0)$ and $(0,1)$ and probability 1/2 on $(1,1)$. We have $\mathrm{BR}\textsc{ev}(\mathcal{F}) = \max\{1 \cdot 1, 2 \cdot 1/2\} = 1$, and the mechanism $b(x_1,x_2) = \max\{0, x_1-1, x_2-1, x_1+x_2-2\}$ yields $\mathrm{DR}\textsc{ev}(\mathcal{F}) = \mathrm{SR}\textsc{ev}(\mathcal{F}) = R(b;\mathcal{F}) = 3/2$.

Appendix 4: Multiple Buyers

This paper has concentrated on a single-buyer scenario that may also be interpreted to be a monopolist pricing setting. One may naturally ask the same questions for more general auction settings involving multiple buyers. An immediate observation is that since our main results (Theorems A, B, and C) are separations, they apply directly also to multiple-buyer settings, simply by considering a single "significant" buyer together with multiple "negligible" (in the extreme, 0-value for all items) buyers. The issue of extending the results to the multiple-buyer settings is thus relevant for the upper bounds in the paper, both the significant ones (Theorem D and Proposition 8.2) and the simple ones (Propositions 4.1, 4.2, and 4.3). In this appendix we discuss why these can all be extended to the multi-buyer scenario, at least if we are willing to incur a *loss that is linear in the number of buyers.* It is not completely clear where and how this loss may be avoided.

In the case of multiple buyers, we must first choose our notion of implementation: Bayes-Nash or Dominant-strategy. Also, we need to specify whether we are assuming independence between buyers' valuations or allow them to be correlated. The discussion here will be coarse enough as to apply to all these variants at the same time, with differences noted explicitly.

The next issue is how should we define the menu size for the case of multiple buyers. In the single-buyer case we defined it to be simply the size of the outcome space: $|\{q(x)|x \in \Re^k_+\}\backslash\{(0,\ldots,0)\}|$ and interpreted it as the number of options from which the buyer may choose. In the case of multiple buyers, these are two separate notions. For example consider deterministic auctions of k items among n buyers. There are a total of $(n+1)^k$ different outcomes (each item may go to any buyer or to no one), but each buyer only considers 2^k possibilities (whether *he* gets each item or not). Moreover, the total set of outcomes cannot be interpreted as a menu from which the buyers may choose, since each buyer can only choose among the possibilities that he sees in front of him (and these choices need not be overall feasible). It takes the combined actions of all the buyers together in order for the auction outcome to be determined. For this reason we prefer to define the menu size of a multi-buyer auction by considering its menu size from the point of view of the different buyers. Since the menu that a buyer sees is a function of the bids of the others, we will take the maximum.

- An n-buyer auction has *menu size* at most m if for every buyer $j = 1, \ldots, n$ and every (direct) bids of the other buyers[18] $x^{-j} \in (\Re^k_+)^{n-1}$, the number of non-zero choices that buyer j faces is at most m, i.e., $|\{q^j(x^j, x^{-j}) : x^j \in \Re^k_+\} \backslash \{(0, \ldots, 0)\}| \leq m$.

Note: If the original mechanism was incentive compatible in dominant strategies then the mechanism induced by x^{-j} on player j is also incentive compatible. If the original mechanism was incentive compatible in the Bayesian sense then this need not be the case, but we will still have individual rationality[19] of the induced mechanism which suffices for what comes next.

Let us first analyze the simplest auctions, those with a single non-trivial menu item for each buyer. Clearly bundling auctions satisfy this property, however not every auction that has a single non-trivial menu-item for each buyer can be converted to a bundling auction. Let us also be careful with the meaning of bundling auction. Clearly in the case of correlated buyer valuations, the optimal auction for selling even a single item (the whole bundle in our case) is not necessarily to sell it to the highest bidder, but rather use the bids of the others as to set the reserve price to each bidder. (Consider for example the case of two buyers with a common value, where the bid of one of them should be used as the asking price from the other.) Thus in the rest of the discussion below we used BRev to denote the best auction that always sells the bundle as a whole – not necessarily to the highest bidder or with a uniform reserve price. For the case of independent buyer values, the simpler version that sells it to the highest bidder with a fixed reserve price will do too.

[18] Superscripts are used here for the buyers.

[19] This assumes that the original mechanism was ex-post individually rational, which one may verify does not really loose generality relative to ex-ante individual rationality.

What can be easily observed is that by focusing solely on the buyer that yields the largest fraction of revenue we can reduce back to the single-buyer case and extract at least $1/n$ fraction of revenue by selling the bundle to that single buyer. A full bundling auction can only make more, so this gives us the analog to Proposition 4.1 for the case of n buyers:[20] $\text{BRev}^{[n]}(\mathcal{F}) \leq [1]\text{-Rev}^{[n]}(\mathcal{F}) \leq n \cdot \text{BRev}^{[n]}(\mathcal{F})$. The loss of the factor of n can be seen to be justified when considering independent buyer values and the restricted definition of bundling auctions even when there is one good (i.e., $k = 1$) by considering the distribution where each buyer j gets value M^j with probability M^{-j}, and zero otherwise (all independently), for a large enough but fixed M.

A similar argument that focuses on the single buyer that provides the largest fraction of revenue provides the generalization of Proposition 4.2: $[m]\text{-Rev}^{[n]}(\mathcal{F}) \leq nm \cdot \text{BRev}^{[n]}(\mathcal{F})$, and of Proposition 4.3: $\text{DRev}^{[n]}(\mathcal{F}) \leq n \cdot (2^k - 1) \cdot \text{BRev}^{[n]}(\mathcal{F})$. It turns out that the linear loss in n is required here too, again for independent buyer values and the restricted interpretation of bundling auctions: take the construction of Theorem C on each of the n different buyers and combine it with the argument above. That is, whenever the construction has a valuation x with probability p, let buyer j have valuation $M^j x$ with probability $M^{-j}p$ (independently over the buyers).

Versions of Theorem D and Proposition 8.2 that incur a linear loss in n are also easily implied, but do not seem to be interesting. It would seem that in both cases one might get sharper results that avoid the additional loss due to the number of buyers.

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[20]The superscript $[n]$ on the various revenues denotes the number of buyers.

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Maximal Revenue with Multiple Goods: Nonmonotonicity and Other Observations*

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Abstract

Consider the problem of maximizing the revenue from selling a number of goods to a single buyer. We show that, unlike the case of one good, when the buyer's values for the goods increase the seller's maximal revenue may well *decrease*. We also provide a characterization of revenue-maximizing mechanisms (more generally, of "seller-favorable" mechanisms) that circumvents nondifferentiability issues. Finally, through simple and transparent examples, we clarify the need for and the use of randomization when maximizing revenue in the multiple-goods versus the one-good case.

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1 Introduction

Consider the problem of a seller who wishes to maximize the revenue from selling multiple goods to a single buyer with private information about his value for the goods.

In Section 3, we exhibit the surprising phenomenon that the seller's maximal revenue may well *decrease* when the buyer's values for the goods *increase.* This revenue nonmonotonicity can occur only when there is *more than one good*: revenue is easily seen to be nondecreasing in the buyer's value when there is only a single good.

In Section 2, we restrict attention to "seller-favorable" mechanisms and characterize their revenue using directional derivatives, which exist everywhere (and therefore circumvent nondifferentiability issues arising from incentive compatibility).

In Section 4, we present a simple example where randomization is necessary for revenue maximization, and clarify why randomization is needed *only* when there are multiple goods.

Since the maximal revenue problem appears significantly less well behaved when the values of the goods are correlated (cf.[1] Hart and Nisan 2012a, 2012b), it is important to obtain examples with independent, and even independent and identically distributed, values. We do so both for revenue-nonmonotonicity and for randomization.

1.1 Preliminaries

The seller possesses $k \geq 1$ goods (or "items"), which are worth nothing to him (and there are no costs). The valuation of the goods to the buyer is given by a vector[2] $x = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$, where x_i is his value for good i. The valuation is assumed to be additive over the goods: the value of a set $L \subset \{1, 2, ..., k\}$ of goods is $\sum_{i \in L} x_i$. The buyer knows the valuation vector

[1] For instance, deterministic mechanisms always ensure at least one half of the maximal revenue in the independent case, versus an arbitrarily small fraction in the general (correlated) case.

[2] $\mathbb{R}$ denotes the real line, $\mathbb{R}^k$ the k-dimensional Euclidean space, and $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x \geq 0\}$ its nonnegative orthant. Negative valuations are not ruled out.

x, whereas the seller knows only that x is drawn from a given probability distribution $\mathcal{F}$ on $\mathbb{R}^k$ with support D. We make no further assumptions on $\mathcal{F}$. In particular, $\mathcal{F}$ may possess atoms and its support D may be finite or infinite and hence need not be convex or even connected. The seller and the buyer are each risk-neutral and have quasilinear utilities.

A *(direct) mechanism* for selling the k goods is given by a pair of functions (q, s), where $q : D \to [0, 1]^k$ and $s : D \to \mathbb{R}$. If the buyer reports that his valuation is x, then $q_i(x) \in [0, 1]$ is the probability that he receives good i (for $i = 1, ..., k$), and $s(x)$ is the payment that the seller receives from the buyer. When the buyer reports his valuation x truthfully, his payoff is $b(x) = \sum_{i=1}^k q_i(x)x_i - s(x) = q(x) \cdot x - s(x)$, where $q(x) \equiv (q_1(x), ..., q_k(x))$, and the seller's payoff is[3] $s(x)$. A mechanism (q, s) is *individually rational* (*IR*) if $b(x) \geq 0$ for all $x \in D$ and it is *incentive compatible* (*IC*) if $b(x) \geq q(y) \cdot x - s(y)$ for all $x, y \in D$. By the Revelation Principle, the *maximal revenue* from the distribution $\mathcal{F}$ is $\textsc{Rev}(\mathcal{F}) := \sup \mathbb{E}_{\mathcal{F}} [s(x)]$, where x is distributed according to $\mathcal{F}$, and the supremum is over all IC and IR mechanisms (q, s).

If (q, s) is IC, then it is useful to extend the buyer's payoff function b from D to all $\mathbb{R}^k$ by $b(x) := \sup_{(p,t)\in R}(p \cdot x - t)$, where $R := \{(q(x), s(x)) : x \in D\}$ is the range of (q, s). So defined, b is a convex function, being the pointwise supremum of affine functions. The IC property of (q, s) ensures that the values of b remain unchanged on D, and also ensures that b is finite for every $x \in \mathbb{R}^k$. Henceforth, "the buyer's payoff function b" will mean the above extension of b to all of[4] $\mathbb{R}^k$.

Let f be a real convex function defined on $\mathbb{R}^k$. The *directional derivative at $x \in \mathbb{R}^k$ in the direction $y \in \mathbb{R}^k$* is $f'(x; y) := \lim_{\delta\to0^+}(f(x + \delta y) - f(x))/\delta$. Since f is convex, $f'(x; y)$ always exists. If $0 \leq f(x + z) - f(x) \leq \sum_{i=1}^k z_i$ holds for every $x, z \in \mathbb{R}^k$ with $z \geq 0$ then the function f is *nondecreasing* and *nonexpansive.*[5]

[3]In the literature this is called transfer, cost, price, or revenue, and denoted by t, c, p, and so on. We hope that using the mnemonic s for the *s*eller's final payoff and b for the *b*uyer's final payoff will avoid confusion.

[4]The domain D is irrelevant, as any IC mechanism can be extended to the whole space $\mathbb{R}^k$ (see footnote 10 below).

[5]For convex f, this is equivalent to $0 \leq \partial f(x)/\partial x_i \leq 1$ for all i and all x where the derivative exists (i.e., a.e.).

Let $\mathcal{B}^k$ be the collection of all real functions on $\mathbb{R}^k$ that are nondecreasing, nonexpansive, and convex.

2 Seller-Favorable Mechanisms

When maximizing revenue one may without loss of generality consider only mechanisms that are "seller-favorable," which means that whenever the buyer is indifferent he chooses an outcome that maximizes the seller's revenue (i.e., ties are broken by the buyer in favor of the seller). Formally, an incentive-compatible mechanism (q, s) with buyer's payoff function b is *seller-favorable* if there is no other incentive-compatible mechanism $(\bar{q}, \bar{s})$ having the same payoff function b for the buyer (i.e., $\bar{q}(x) \cdot x - \bar{s}(x) = b(x)$ for all x in D) and such that $\bar{s}(x) \geq s(x)$ for every $x \in D$, with strict inequality for some x. In this section we will see that the restriction to seller-favorable mechanisms simplifies the analysis (it circumvents nondifferentiability issues); moreover, seller-favorable mechanisms arise not only from revenue-maximization considerations, but also from strict implementation.[6]

The characterization of IC mechanisms (q, s) as being those whose assignment function, q, is a subgradient of the buyer's convex payoff function is well known (starting with Rochet 1985). It is an inconvenient and often technically annoying fact that the buyer's convex payoff function, while differentiable almost everywhere, need not be differentiable everywhere. Proofs that are otherwise simple and elegant often require detours through subgradient selection arguments.[7]

Such detours can be avoided when one restricts attention to seller-favorable mechanisms. The reason is that the buyer's payoff function is not differentiable only when he is indifferent between a number of reports. But if the mechanism (q, s) is seller-favorable, the buyer's truthful report must maximize the seller's payoff among all of the buyer's optimal reports. As we show, this implies that $q(x) \cdot x = b'(x; x)$ for every buyer valuation[8] $x \in D$.

[6]See the last paragraph of Remark *(a)* at the end of this section.

[7]E.g., Lemma A.4 in Manelli and Vincent (2007).

[8]This formula holds even when D is a finite set since b is a convex function defined on

Consequently, in a seller-favorable mechanism the buyer's payoff function, b, completely determines the seller's payoff function s at *every* $x \in D$, whether a point of differentiability of b or not, and $s(x) = b'(x; x) - b(x)$ for all $x \in D$.

Lemma 1 *If (q, s) is IC then the buyer's payoff function b belongs to $\mathcal{B}^k$ and $s(x) \leq b'(x; x) - b(x)$ for every $x \in D$.*

Proof. Recall (Section 1.1) that b is a convex function and $b(x) = \sup_{(p,t) \in R}(p \cdot x - t)$, where $R = \{(q(x), s(x)) : x \in D\}$ is the range of (q, s). IC also implies that the range of $s(\cdot)$ is bounded. Thus, $\bar{R}$, the closure of R, is a compact subset of $[0, 1]^k \times \mathbb{R}$, and so for every $x \in \mathbb{R}^k$ there is $(p^*(x), t^*(x)) \in \bar{R}$ such that $b(x) = p^*(x) \cdot x - t^*(x)$ (for $x \in D$ take $(p^*(x), t^*(x)) = (q(x), s(x))$. Therefore, for every $x, y \in \mathbb{R}^k$ we get

$$b(y) - b(x) \geq (p^*(x) \cdot y - t^*(x)) - (p^*(x) \cdot x - t^*(x)) = p^*(x) \cdot (y - x), \quad (1)$$

which says that $p^*(x)$ is a subgradient of[9] b at x. Thus, $p^*(x) \cdot x \leq \sup\{p \cdot x : p \in \partial b(x)\} = b'(x; x)$ and so $s(x) = q(x) \cdot x - b(x) \leq b'(x; x) - b(x)$ for every[10] $x \in D$.

Taking $y = x + z$ with $z \geq 0$ in (1) implies that $0 \leq p^*(x) \cdot z \leq b(x + z) - b(x) \leq p^*(x + z) \cdot z \leq \sum_{i=1}^k z_i$, and so b is nondecreasing and nonexpansive. ■

Lemma 2 *Let $b \in \mathcal{B}^k$. Then there is an IC mechanism $(\bar{q}, \bar{s})$ such that the buyer's payoff function is b and the seller's payoff is $\bar{s}(x) = b'(x; x) - b(x)$ for all x.*

Proof. Being nondecreasing, nonexpansive, and convex on $\mathbb{R}^k$, the function b satisfies $0 \leq b(x) - b(x - z) \leq p \cdot z \leq b(x + z) - b(x) \leq \sum_{i=1}^k z_i$ for

all of $\mathbb{R}^k$. Hence $b'(x; x)$ is well defined for every $x \in \mathbb{R}^k$, and in particular for $x \in D$.

[9] For a convex function f on $\mathbb{R}^k$, a vector $p \in \mathbb{R}^k$ is a *subgradient* of f at $x \in \mathbb{R}^k$ if $f(y) - f(x) \geq p \cdot (y - x)$ for all $y \in \mathbb{R}^k$. Letting $\partial f(x)$ denote the set of subgradients of f at x (which is always a nonempty closed set), we have $f'(x; y) = \sup\{p \cdot y : p \in \partial f(x)\}$ (see Rockafellar 1970).

[10] Note that (p^*, t^*) is an IC mechanism on all of $\mathbb{R}^k$ that extends the given IC mechanism (q, s) on D. This shows that it is without loss of generality to require the incentive constraints to hold on all of $\mathbb{R}^k$, and not merely on D.

every $x \in \mathbb{R}^k$, every $p \in \partial b(x)$, and every $z \in \mathbb{R}^k_+$. In particular, $\partial b(x) \subset [0,1]^k$ and so $b'(x;x) = \sup_{p \in \partial b(x)} p \cdot x$ is attained at some $\bar{q}(x) \in [0,1]^k$, i.e., $b'(x;x) = \bar{q}(x) \cdot x$. Define $\bar{s}(x) := b'(x;x) - b(x) = \bar{q}(x) \cdot x - b(x)$. Then $\bar{q}(y) \cdot y - \bar{s}(y) = b(y) \geq \bar{q}(x) \cdot (y - x) + b(x) = \bar{q}(x) \cdot y - \bar{s}(x)$ (using the definitions of $\bar{s}(y)$ and $\bar{s}(x)$, and $\bar{q}(x) \in \partial b(x)$), and so $(\bar{q}, \bar{s})$ is IC. ■

Together, Lemmas 1 and 2 imply the following.

Corollary 3 *Let (q, s) be an IC mechanism with buyer's payoff function b. Then (q, s) is seller-favorable if and only if $q(x) \cdot x = b'(x;x)$ and $s(x) = b'(x;x) - b(x)$ for every x.*

Consider now the problem of maximizing the seller's expected revenue subject to individual rationality (IR) for the buyer (i.e., $b \geq 0$). Since it is without loss of generality to restrict attention to seller-favorable mechanisms, a consequence of Corollary 3 is the following.

Corollary 4 *The seller's maximal expected revenue is*

$$\textsc{Rev}(\mathcal{F}) = \sup_{b \in \mathcal{B}^k, b \geq 0} \mathbb{E}_{\mathcal{F}} \left[b'(x;x) - b(x) \right]. \tag{2}$$

Remarks. *(a) Strict implementation.* Given any IC mechanism (q, s), there are numerous ways to eliminate, at arbitrarily small cost, the problem of the buyer having only weak incentives to report truthfully. For example, one can introduce an arbitrarily small positive probability that, after the buyer reports his valuation, the given (IC) mechanism is replaced by a random reserve price on each good.

Another alternative is to choose any arbitrarily small $\varepsilon > 0$, and use instead the mechanism $(q, (1 - \varepsilon)s)$ (it need not be IC), which amounts to giving a constant discount (fraction ε) on all prices. This mechanism *guarantees* to the seller, for *any* optimal choices of the buyer, a payoff of at least $(1 - \varepsilon)s(x)$ for every valuation x of the buyer.[11] Thus, the seller is

[11] If y is an optimal report of a buyer with valuation x (as in the proof of Lemma 1, one

guaranteed at least $1-\varepsilon$ times his payoff in the original mechanism, regardless of which optimal report the buyer makes.[12]

In fact, the $(q,(1-\varepsilon)s)$ mechanism guarantees to the seller not merely $(1-\varepsilon)s(x)$ for every x, but $(1-\varepsilon)\bar{s}(x) = (1-\varepsilon)(b'(x;x)-b(x))$, the maximal seller-favorable payoffs, for every[13] x (indeed, in the argument of footnote 11 replace $(q(x),s(x))$ with a $(q(z),s(z))$ that satisfies $b'(x;x) = q(z)\cdot x$ and $s(z) = q(z)\cdot x - b(x)$).

(b) Boundary points. Let $C \subset \mathbb{R}^k$ be a convex set that includes D, the support of $\mathcal{F}$, let $\bar{x}$ be a boundary point of C, and let $\lambda \neq 0$ belong to the normal cone to C at $\bar{x}$, i.e., $\lambda\cdot\bar{x} \geq \lambda\cdot x$ for every $x \in C$. If $\lambda\cdot\bar{x} \geq 0$ and (q,s) is seller-favorable, then we can assume w.l.o.g. that $\bar{q} := q(\bar{x})$ is maximal in the direction λ, i.e., $\tilde{q} := \bar{q}+\varepsilon\lambda \notin [0,1]^k$ for every $\varepsilon > 0$. Indeed, $\tilde{q} \in \partial b(\bar{x})$ (since $\lambda\cdot(x-\bar{x}) \leq 0$ and $\bar{q} \in \partial b(\bar{x})$), and so $b'(\bar{x};\bar{x}) \geq \tilde{q}\cdot\bar{x}$; but $\tilde{q}\cdot\bar{x} \geq \bar{q}\cdot\bar{x} = b'(\bar{x};\bar{x})$ (by Corollary 3) and so $b'(\bar{x};\bar{x}) = \tilde{q}\cdot\bar{x}$ and we can replace $\bar{q}$ by $\tilde{q}$. Moreover, if $\lambda\cdot\bar{x} > 0$ then $\tilde{q}\cdot\bar{x} > \bar{q}\cdot\bar{x} = b'(\bar{x};\bar{x})$, a contradiction, and so $\bar{q}$ *must* be maximal in the direction λ.

In particular, we have:

- w.l.o.g. $q_i(\bar{x}) = 0$ when $\bar{x}_i = 0$ (take $\lambda = -e^{(i)}$, where $e^{(i)} \in \mathbb{R}^k_+$ is the i-th unit vector);
- $q_i(\bar{x}) = 1$ when $\bar{x}_i = \max\{x_i : x \in C\} > 0$ (for instance, if $C = [0,1]^k$, then $q_i(\bar{x}) = 1$ when $\bar{x}_i = 1$; take $\lambda = e^{(i)}$);

may need to consider the closure of the range of $(q,(1-\varepsilon)s)$), then

$$\begin{aligned} q(y)\cdot x-(1-\varepsilon)s(y) &\geq q(x)\cdot x-(1-\varepsilon)s(x) \\ &= [q(x)\cdot x - s(x)] + \varepsilon s(x) \\ &\geq [q(y)\cdot x - s(y)] + \varepsilon s(x) \end{aligned}$$

(by IC of (q,s)). Hence $s(y) \geq s(x)$ (subtract and divide by ε), and so the seller's payoff, $(1-\varepsilon)s(y)$, is at least $(1-\varepsilon)s(x)$.

[12]Thus the possibility of multiple optimal reports for the buyer, which is sometimes described as problematic (see for instance footnote 3 in Manelli and Vincent 2007), in fact isn't.

[13]Thus the tie-breaking rule in favor of the seller is obtained as the limit of *any* optimal behavior of the buyer in the perturbed mechanisms.

- $\max_i q_i(\bar{x}) = 1$ when $\sum_i \bar{x}_i = \max\{\sum_i x_i : x \in C\} > 0$ (for instance, if C is the unit simplex in $\mathbb{R}^k_+$; take $\lambda = (1, 1, ..., 1)$).

(c) $b'(x; x) = \lim_{\delta \to 0^+}(b((1+\delta)x) - b(x))/\delta$ is the right-derivative of the function $t \to b(tx)$ at[14] $t = 1$, and $s(x) = b'(x; x) - b(x)$ is the right-derivative of the function $t \to b(tx) - tb(x)$ at $t = 1$ (these functions relate to the local returns to scale of b). If $b(0) = 0$ (which, when maximizing revenue, can always be assumed when[15] $C \subset \mathbb{R}^k_+$), then $b'(x; x) \geq b(x)$ and[16] $s(x) \geq 0$ (i.e., there are no positive transfers from seller to buyer).

3 Nonmonotonicity: Increasing Values May Decrease Revenue

When the buyer's values for the goods increase, what happens to the seller's maximal revenue? It stands to reason that the revenue should also increase, as there is now more value for the seller to "extract."[17] While this can easily be seen to be true when there is one good,[18] it is perhaps a surprise that it no longer holds when there are multiple goods. As a consequence, if the distribution $\mathcal{F}$ of the buyer's valuation is not precisely known, but a certain lower bound $\mathcal{F}_0$ to $\mathcal{F}$ is given (in the first-order stochastic dominance sense), then computing the optimal revenue for $\mathcal{F}_0$ does *not* necessarily yield a lower bound on the optimal revenue for $\mathcal{F}$.

3.1 Monotonicity for one good

When there is only one good, i.e., $k = 1$, incentive compatibility (IC) implies that a buyer with a higher valuation pays no less than a buyer with a lower

[14] In the one-dimensional case ($k = 1$) we have $b'(x; x) = xb'_+(x)$. A useful property is $\int_{t_1}^{t_2} b'(tx; tx)dt = b(t_2 x) - b(t_1 x)$ (cf. Rockafellar 1970, Corollary 24.2.1).

[15] If $b(0) > 0$ then the revenue from $\tilde{b}(x) = b(x) - b(0)$ is higher by the amount $b(0)$ than the revenue from b.

[16] Since $0 = b(0) \geq b(x) + q(x) \cdot (0 - x) = -s(x)$.

[17] What we compare is the maximal revenue from two given distributions, one having higher values than the other (formally, this means first-order stochastic dominance).

[18] Another case where the revenue is easily seen to increase is when all valuations increase *uniformly* by the same amount (i.e., each x is replaced by $x + z$ for a fixed vector $z \gneqq 0$).

valuation. Thus increasing the valuation of the buyer can only increase the revenue.

Proposition 5 *When there is one good, i.e., $k = 1$, if F_2 fist-order stochastically dominates F_1 then* $\text{Rev}(F_2) \geq \text{Rev}(F_1)$.

Proof. First, we claim that every IC mechanism is monotonic in the sense that the seller's payoff increases weakly with the buyer's value: if $x > y$ then $s(x) \geq s(y)$. Indeed, for all x, y, the IC inequalities at x and at y imply $(q(x) - q(y))x \geq s(x) - s(y) \geq (q(x) - q(y))y$; when $x > y$ it follows that $q(x) - q(y) \geq 0$ and thus $s(x) - s(y) \geq 0$.

Second, the first-order stochastic dominance implies that $\mathbb{E}_{F_1}[s(x)] \leq \mathbb{E}_{F_2}[s(x)]$ for every IC mechanism, since s is a nondecreasing function. ■

Remark. Note that Proposition 5 also follows easily from Myerson's (1981) characterization of the optimal revenue when there is one good as $\text{Rev}(F) = \sup_{p \geq 0} p \cdot (1 - F(p))$; however, the proof above shows that the monotonicity of the revenue holds not only for optimal mechanisms, but also for *any* incentive-compatible mechanism.

3.2 Nonmonotonicity for multiple goods

Now, does the above hold when there are more goods? That is, does increasing the buyer's valuations yield higher revenue to the seller? The surprising answer is that this is no longer true when there is more than one good.

When there are multiple goods one can construct examples of IR and IC mechanisms that are not monotonic.[19] Take for instance the mechanism where the buyer is offered a choice among the following four outcomes: get nothing and pay nothing (with payoff = 0); or get good 1 for price 1 (with payoff = $x_1 - 1$); or get good 2 for price 2 (with payoff = $x_2 - 2$); or get both goods for price 4 (with payoff = $x_1 + x_2 - 4$); thus,

$$b(x_1, x_2) = \max\{0,\ x_1 - 1,\ x_2 - 2,\ x_1 + x_2 - 4\}. \tag{3}$$

[19] The first such example was constructed with Noam Nisan.

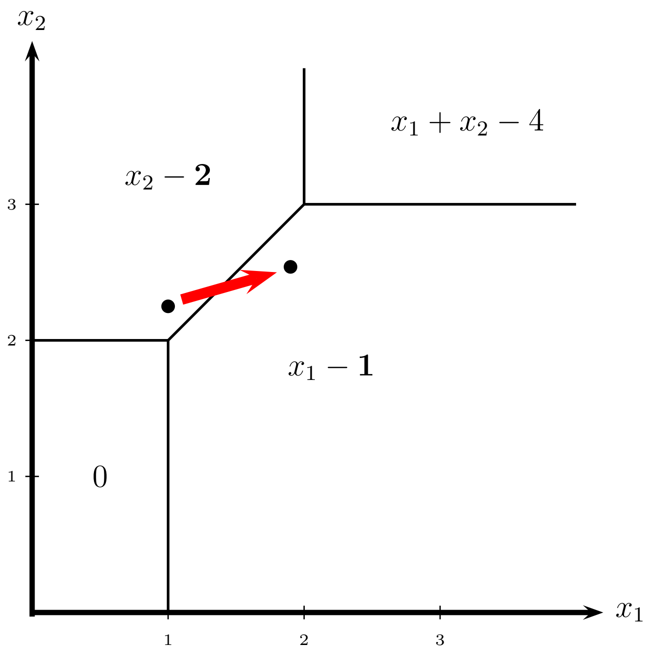

Figure 1: The nonmonotonic mechanism (3)

See Figure 1 for the regions in the buyer's valuation space where each outcome is chosen. If the valuation of the buyer is, say, $(1.3, 2.4)$, then his optimal choice is to pay 2 for good 2, whereas if his values increase to, say, $(1.7, 2.6)$, then his optimal choice is to pay 1 for good 1. Thus the seller receives a lower payment (1 instead of 2) when the buyer's values increase.

The more difficult question is whether this nonmonotonicity can also occur for the *maximal* revenue. The two examples below, a simpler one where the unique optimal mechanism is precisely the above deterministic mechanism[20] (3) but the valuations of the two goods are correlated, and a more complicated one where the valuations of the two goods are independent and identically distributed, show that the maximal revenue can indeed be nonmonotonic.

Example E1. For every $0 \leq \alpha \leq 1/4$, let $\mathcal{F}_\alpha$ be the following distribution

[20]This explains the reason for including the outcome $x_1 + x_2 - 4$ in the mechanism.

on $\mathbb{R}^2$:

$$\mathcal{F}_\alpha = \begin{cases} (1,1), & \text{with probability } 1/4, \\ (1,2), & \text{with probability } 1/4-\alpha, \\ (2,2), & \text{with probability } \alpha, \\ (2,3), & \text{with probability } 1/2. \end{cases}$$

As α increases, probability mass is moved from $(1,2)$ to $(2,2)$, and so $\mathcal{F}_\alpha$ *first-order stochastically dominates* $\mathcal{F}_{\alpha'}$ when $\alpha > \alpha'$. Nevertheless, the maximal revenue $\textsc{Rev}(\mathcal{F}_\alpha)$ *decreases* with α (in the region $0 \le \alpha \le 1/12$).

Proposition 6 *In Example E1: for every* $0 \le \alpha \le 1/12$,

$$\textsc{Rev}(\mathcal{F}_\alpha) = 11/4 - \alpha.$$

Proof. First, the revenue of $11/4-\alpha$ is achieved by the mechanism with b given by (3): $(1/4)\cdot 1 + (1/4-\alpha)\cdot 2 + \alpha \cdot 1 + (1/2)\cdot 4 = 11/4-\alpha$.

Second, we show that a higher revenue cannot be obtained. Consider the following inequalities:

$$\begin{array}{rcl|l} q_1^{11}+q_2^{11}-s^{11} & \ge & 0 & 1 \\ q_1^{12}+2q_2^{12}-s^{12} & \ge & q_1^{11}+2q_2^{11}-s^{11} & 1/2 \\ 2q_1^{22}+2q_2^{22}-s^{22} & \ge & 2q_1^{11}+2q_2^{11}-s^{11} & 3\alpha \\ 2q_1^{23}+3q_2^{23}-s^{23} & \ge & 2q_1^{11}+3q_2^{11}-s^{11} & 1/4-3\alpha \\ 2q_1^{23}+3q_2^{23}-s^{23} & \ge & 2q_1^{12}+3q_2^{12}-s^{12} & 1/4+\alpha \\ 2q_1^{23}+3q_2^{23}-s^{23} & \ge & 2q_1^{22}+3q_2^{22}-s^{22} & 2\alpha \end{array} \tag{4}$$

(the first inequality is IR at $(1,1)$, and the others are various IC constraints). Multiplying these inequalities by the multipliers on the right (which are all nonnegative when $0 \le \alpha \le 1/12$) and then adding them up yields:

$$\begin{aligned} &-(3/4-3\alpha)\,q_2^{11} - 2\alpha q_1^{12} + (1/4-3\alpha)\,q_2^{12} + 2\alpha q_1^{22} + q_1^{23} + (3/2)q_2^{23} \\ &\qquad \ge (1/4)s^{11} + (1/4-\alpha)\,s^{12} + \alpha s^{22} + (1/2)s^{23}. \end{aligned}$$

The right-hand side is precisely the expected revenue at $\mathcal{F}_\alpha$, and the left-hand side is at most $0+0+(1/4-3\alpha)+2\alpha+1+3/2 = 11/4-\alpha$ (since $q_2^{11}, q_1^{12} \ge 0$

and $q_2^{12}, q_1^{22}, q_1^{23}, q_2^{23} \le 1$). Therefore the revenue cannot exceed $11/4 - \alpha$, and so the revenue of $11/4 - \alpha$ achieved by b of (3) is indeed maximal. ■

To get some intuition: the mechanism of (3) is:

Valuation	**Outcome**	
$x = (x_1, x_2)$	$q(x) = (q_1(x), q_2(x))$	$s(x)$
$(1,1)$	$(0,0)$	0
$(2,2)$	$(1,0)$	1
$(1,2)$	$(0,1)$	2
$(2,3)$	$(1,1)$	4

(5)

When the value of good 1 goes up (e.g., from $x = (1,2)$ to $x' = (2,2)$), the probability of getting good 1 also goes up (i.e., $q_1(x') = 1 > 0 = q_1(x)$); this is always so, as it is a consequence of the convexity of the buyer's payoff function b). However, at the same time the probability of getting good 2 may go down (e.g., $q_2(x') = 0 < 1 = q_2(x)$); moreover, it can do so in such a way that the allocation is worth less to the buyer, and so his payment to the seller goes down (i.e., $s(x') = 1 < 2 = s(x)$).

Remarks. *(a)* The mechanism (5) is the *unique* optimal mechanism at each $\mathcal{F}_\alpha$ with $0 \le \alpha < 1/12$; indeed, in order to get the revenue of $11/4 - \alpha$ one needs all relevant inequalities to become equalities (thus $q_2^{11} = q_1^{12} = 0$ and $q_2^{12} = q_1^{23} = q_1^{23} = q_2^{23} = 1$, which together with (4) as equalities can be easily shown to yield $q_1^{11} = 1, q_2^{22} = 0, s^{11} = 1, s^{12} = 2, s^{22} = 1, s^{23} = 4$—which is precisely (5)).

(b) Any small enough perturbation of the example—such as having full support on a square like $[0,3]^2$, or increasing all valuations as α increases—will not affect the nonmonotonicity, since the inequality $\textsc{Rev}(\mathcal{F}_0) > \textsc{Rev}(\mathcal{F}_{1/12})$ is strict.

3.3 Nonmonotonicity for independent and identically distributed goods

We now provide an example of nonmonotonicity where the goods are independent and identically distributed.

Example E2. Let F_1 and F_2 be the following one-dimensional distributions:

$$F_1 = \begin{cases} 10, & \text{with probability } \frac{4}{15}, \\ 46, & \text{with probability } \frac{1}{90}, \\ 47, & \text{with probability } \frac{1}{3}, \\ 80, & \text{with probability } \frac{7}{30}, \\ 100, & \text{with probability } \frac{7}{45}. \end{cases} \qquad F_2 = \begin{cases} 10, & \text{with probability } \frac{2399}{9000}, \\ 13, & \text{with probability } \frac{1}{9000}, \\ 46, & \text{with probability } \frac{1}{90}, \\ 47, & \text{with probability } \frac{1}{3}, \\ 80, & \text{with probability } \frac{7}{30}, \\ 100, & \text{with probability } \frac{7}{45}. \end{cases}$$

Clearly F_2 first-order stochastically dominates F_1 (since F_2 is obtained from F_1 by moving a probability mass of 1/9000 from 10 to 13), which of course implies that $F_2 \times F_2$ first-order stochastically dominates $F_1 \times F_1$. However, the optimal revenue from $F_1 \times F_1$ turns out to be *higher* than the optimal revenue from $F_2 \times F_2$.

Proposition 7 *In Example E2:*

$$\textsc{Rev}(F_1 \times F_1) \approx 69.47145 > \textsc{Rev}(F_2 \times F_2) \approx 69.47126.$$

Proof. Maximizing revenue for a distribution with finite support is a linear programming problem (the unknowns are the $q_i(x)$ and $s(x)$ for all x in the support, the constraints are the IR and IC inequalities, and the objective function is the expected revenue). Using MAPLE yields the following.

The unique[21] optimal mechanism for $F_2 \times F_2$ consists of 11 outcomes

[21] Uniqueness is proved using the dual linear progamming problem, as in the previous sections.

(ordered in the table below according to increasing payment to the seller s):

Valuations	**Outcome**	
x	$q(x)$	$s(x)$
$(10,10),(10,13),(13,10),(13,13),$ $(10,46),(46,10),(13,46),(46,13),(46,46),$ $(13,47),(47,13),(10,47),(47,10)$	$(0,0)$	0
$(46,47)$	$(\frac{32}{1187},\frac{384}{13057})$	$\frac{34240}{13057}\approx 2.6$
$(47,46)$	$(\frac{384}{13057},\frac{32}{1187})$	$\frac{34240}{13057}\approx 2.6$
$(47,47)$	$(\frac{35}{1187},\frac{35}{1187})$	$\frac{3258}{1187}\approx 2.7$
$(13,80)$	$(\frac{32}{1187},\frac{5647}{5935})$	$\frac{90672}{1187}\approx 76.4$
$(80,13)$	$(\frac{5647}{5935},\frac{32}{1187})$	$\frac{90672}{1187}\approx 76.4$
$(46,80)$	$(\frac{35}{1187},\frac{5647}{5935})$	$\frac{90810}{1187}\approx 76.5$
$(80,46)$	$(\frac{5647}{5935},\frac{35}{1187})$	$\frac{90810}{1187}\approx 76.5$
$(10,80),(10,100),(13,100)$	$(0,1)$	80
$(80,10),(100,10),(100,13)$	$(1,0)$	80
$(46,100),(100,46),$ $(47,80),(80,47),(47,100),(100,47),$ $(80,80),(80,100),(100,80),(100,100)$	$(1,1)$	126

For $F_1 \times F_1$ the same mechanism is optimal; however, the 5th and 6th outcomes are not used (the value 13 has probability 0) and may be dropped. This yields:

$$\begin{aligned}\textsc{Rev}(F_1 \times F_1) &= \frac{408189937}{5875650} = 69.47145\ldots\ , \\ \textsc{Rev}(F_2 \times F_2) &= \frac{30614162731}{440673750} = 69.47126\ldots\ .\end{aligned}$$

The nonmonotonicity of the payments is seen at $s(10,80) > s(13,80), s(46,80)$ and $s(80,10) > s(80,13), s(80,46)$. ■

4 Lotteries and Revenue

In the case of a single good (i.e., $k = 1$), in order to maximize revenue it suffices to consider deterministic mechanisms (specifically, "posted-price" mechanisms; see Myerson 1981). That is *not* so in the multi-good case. Examples where the optimal mechanism requires randomization (i.e., in some of the outcomes the probability of getting a good is strictly between 0 and 1) have been provided by Thanassoulis (2004) (in the slightly different context where the buyer's demand is limited to one good), Pycia (2006), Manelli and Vincent (2006, 2007),[22] and Pavlov (2011, Example 3(ii)). However, most of these examples are relatively complicated and require non-trivial computations, and it is not clear why and how randomization helps only when there are multiple goods.

We will provide two examples that are simple and transparent enough that the need for randomization becomes clear. In the first, the values of the two goods are correlated; in the second, the values are independent and identically distributed.

4.1 Lotteries for multiple goods

Consider the following example with two goods and three possible valuations[23] (the values of the two goods are correlated).

Example E3. Let $\mathcal{F}$ be the following two-dimensional probability distribu-

[22] Manelli and Vincent (2007) provide an example (Example 1) of an "undominated mechanism" that requires lotteries. While it is clear that an undominated mechanism is optimal for some distribution $\mathcal{F}$, it is claimed there (Theorem 9) that any undominated mechanism is optimal for some distribution with *independent* goods (i.e., a product distribution). However, there is an error in the proof of Theorem 9, as the set of product distributions (specifically, the set G in their proof) is not convex.

[23] Pycia (2006) solves the seller's problem when there are exactly two valuations and shows that randomizations may be needed. For instance, when the valuations are $(2, 3)$ and $(6, 1)$ with equal probabilities, the *unique* optimal mechanism gives buyer $(2, 3)$, for the total price of 4, good 2 and a 1/2 chance of getting good 1; and gives buyer $(6, 1)$ both goods for the total price of 7. However, we have found that Example E1, with three possible valuations, provides slightly more transparent insights (as there is a clearer separation between the IC and IR constraints).

tion:

$$\mathcal{F} = \begin{cases} (1,0), & \text{with probability } 1/3, \\ (0,2), & \text{with probability } 1/3, \\ (3,3), & \text{with probability } 1/3. \end{cases}$$

Proposition 8 *The mechanism (q,s) defined by*

Valuation	**Outcome**	
x	$q(x)$	$s(x)$
$(1,0)$	$(\frac{1}{2},0)$	$\frac{1}{2}$
$(0,2)$	$(0,1)$	2
$(3,3)$	$(1,1)$	5

(6)

with

$$b(x_1,x_2) = \max\left\{0, \frac{1}{2}x_1 - \frac{1}{2},\ x_2 - 2,\ x_1 + x_2 - 5\right\} \tag{7}$$

is the unique *revenue-maximizing IC and IR mechanism for $\mathcal{F}$ of Example E3.*

Thus, the buyer can get both goods for price 5, or get good 2 for price 2, or *get good* 1 *with probability* 1/2 for price 1/2; the optimal revenue is $5/2 = 2.5$. It can be shown[24] that if the seller were restricted to deterministic mechanisms (where each q_i is either 0 or 1), then the optimal revenue would decrease to $7/3 = 2.33...$ (which is attained for instance by selling separately, at the optimal-single-good prices of 3 for good 1 and 2 for good 2). A detailed explanation of the role of randomization, and why it is needed only when there are multiple goods, follows the proof below.

Proof. Let $\langle(\alpha_1,\beta_1);\sigma_1\rangle, \langle(\alpha_2,\beta_2);\sigma_2\rangle$, and $\langle(\alpha_3,\beta_3);\sigma_3\rangle$ be the outcome $\langle(q_1(x),q_2(x));s(x)\rangle$ at $x=(1,0),(0,2)$, and $(3,3)$, respectively (thus $\alpha_i,\beta_i \in [0,1]$). The objective function is $S := \sigma_1+\sigma_2+\sigma_3$ (this is 3 times the revenue). Consider the relaxed problem of maximizing S subject only to the

[24]See footnote 25.

individual rationality constraints at $(1,0)$ and $(0,2)$, and to the two incentive compatibility constraints at $(3,3)$, i.e.,

$$\begin{aligned} \alpha_1 - \sigma_1 &\geq 0, \\ 2\beta_2 - \sigma_2 &\geq 0, \\ 3\alpha_3 + 3\beta_3 - \sigma_3 &\geq 3\alpha_1 + 3\beta_1 - \sigma_1, \\ 3\alpha_3 + 3\beta_3 - \sigma_3 &\geq 3\alpha_2 + 3\beta_2 - \sigma_2. \end{aligned}$$

These inequalities can be rewritten as:

$$\begin{aligned} \sigma_3 + 3\alpha_1 + 3\beta_1 - 3\alpha_3 - 3\beta_3 \;\leq\; \sigma_1 \;\leq\; \alpha_1, \\ \sigma_3 + 3\alpha_2 + 3\beta_2 - 3\alpha_3 - 3\beta_3 \;\leq\; \sigma_2 \;\leq\; 2\beta_2. \end{aligned}$$

Therefore, in order to maximize $S = \sigma_1 + \sigma_2 + \sigma_3$ we must take $\sigma_1 = \alpha_1$ and $\sigma_2 = 2\beta_2$, which gives:

$$\begin{aligned} \sigma_3 &\leq 3\alpha_3 + 3\beta_3 - 2\alpha_1 - 3\beta_1, \\ \sigma_3 &\leq 3\alpha_3 + 3\beta_3 - 3\alpha_2 - \beta_2. \end{aligned}$$

Thus we must take $\alpha_3 = \beta_3 = 1$, $\beta_1 = \alpha_2 = 0$, and then $\sigma_3 = \min\{6 - 2\alpha_1, 6 - \beta_2\}$, and so $S = \alpha_1 + 2\beta_2 + \min\{6 - 2\alpha_1, 6 - \beta_2\} = \min\{2\beta_2 - \alpha_1, \beta_2 + \alpha_1\} + 6$. Since S is increasing in β_2 we must take $\beta_2 = 1$, and then $S = \min\{2 - \alpha_1, 1 + \alpha_1\} + 6$ is maximized at[25] $\alpha_1 = 1/2$. This is pecisely the mechanism (6), which is easily seen to satisfy also all the other IR and IC constraints. ■

To understand the use of randomization, consider the outcome $\langle(1/2, 0); 1/2\rangle$ at $x = (1, 0)$ in (6): it is a lottery ticket that costs $1/2$ and gives a $1/2$ probability of getting good 1; alternatively,[26] it is a $1/2 - 1/2$ lottery between getting good 1 for the price 1 (i.e., $\langle(1, 0); 1\rangle$), and getting nothing

[25] For deterministic mechanisms (i.e., $\alpha_i, \beta_i \in \{0, 1\}$), everything is the same up to this point, but now S is maximized at both $\alpha_1 = 0$ and $\alpha_1 = 1$; the optimal revenue for deterministic mechanisms is thus $S/3 = 7/3$.

[26] Because of risk-neutrality.

and paying nothing (i.e., $\langle(0,0);0\rangle$). It is thus the average of these two deterministic outcomes, and we now consider what happens when we replace the lottery by either one of them (see Table 1 below). It turns out that in *both* cases the *revenue strictly decreases*. In the first case, replacing $\langle(1/2,0);1/2\rangle$ by $\langle(1,0);1\rangle$ forces the price of the bundle to decrease to 4 (otherwise the $(3,3)$-buyer would switch from paying 5 for the bundle to paying 1 for good 1); therefore the net change in the revenue is $1/3\cdot(1-1/2)+1/3\cdot(4-5)$, which is negative.[27] In the second case, replacing $\langle(1/2,0);1/2\rangle$ by $\langle(0,0);0\rangle$ results in the loss of the revenue from the $(1,0)$-buyer, without, however, increasing the revenue from the $(3,3)$-buyer: indeed, if we were to increase the bundle price, then $(3,3)$ would switch to $\langle(0,1);2\rangle$, i.e., would get good 2 for price 2 (and, if we were to drop this outcome $\langle(0,1);2\rangle$ altogether in order to increase the bundle price to 6, the total revenue would again decrease).[28]

x	$q(x)$	$s(x)$	$q^{(1)}(x)$	$s^{(1)}(x)$	$q^{(2)}(x)$	$s^{(2)}(x)$
$(1,0)$	$(\frac{1}{2},0)$	$\frac{1}{2}$	$(1,0)$	1	$(0,0)$	0
$(0,2)$	$(0,1)$	2	$(0,1)$	2	$(0,1)$	2
$(3,3)$	$(1,1)$	$\underline{\mathbf{5}}$	$(1,1)$	4	$(1,1)$	$\underline{\mathbf{5}}$

Table 1: Replacing a lottery outcome when there are *two* goods

It is instructive to compare this with a similar example but with a single good. Assume the values are $x=1,0,3$, with equal probabilities of 1/3 each (just like good 1 in Example E3). Take the mechanism with outcomes $\langle 1/2;1/2\rangle$, $\langle 0;0\rangle$, $\langle 1;2\rangle$ (see Table 2 below); it is easy to see that it is IC and IR, and its revenue is 5/6. The lottery item $\langle 1/2;1/2\rangle$—get the good with probability 1/2 for price 1/2—is the average of $\langle 0;0\rangle$ and $\langle 1;1\rangle$. Replacing the lottery $\langle 1/2;1/2\rangle$ by $\langle 1;1\rangle$ lowers the revenue to 2/3: the 3-buyer switches to $\langle 1;1\rangle$. Replacing the lottery $\langle 1/2;1/2\rangle$ by $\langle 0;0\rangle$ *increases* the revenue to 1: the 3-buyer is now offered, and chooses, $\langle 1;3\rangle$. The revenue of 5/6 of

[27]The buyer's payoff function in this mechanism is $b^{(1)}(x)=\max\{x_1-1,x_2-2,x_1+x_2-4\}$.

[28]The buyer's payoff function in this mechanism is $b^{(2)}(x)=\max\{0,x_2-2,x_1+x_2-5\}$.

the original mechanism with the lottery item is precisely the average of the revenues from these two resulting mechanisms, 2/3 and 1 (this averaging property holds at each valuation x).

x	q	s	$q^{(1)}$	$s^{(1)}$	$q^{(2)}$	$s^{(2)}$
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	0
0	0	0	0	0	0	0
3	1	$\underline{\mathbf{2}}$	1	1	1	$\underline{\mathbf{3}}$

Table 2: Replacing a lottery outcome when there is *one* good

This is a general phenomenon when there is only one good: the revenue from a mechanism that includes an outcome that is a probabilistic mixture of two outcomes (a "lottery outcome") is the *average* of the revenues obtained by replacing the lottery with each one of these two outcomes and then adapting the remaining outcomes.[29] Formally, this is the counterpart of expressing the corresponding $b \in \mathcal{B}^1$ as an average of two functions in $\mathcal{B}^1$; in the example above, $b(x) = \max\{0, x/2 - 1/2, x - 2\}$ is the $1/2 - 1/2$ average of $b^{(1)}(x) = \max\{0, x - 1\}$ and $b^{(2)}(x) = \max\{0, x - 3\}$. Thus lotteries are indeed not needed when there is only one good.

Example E3 illustrates why this is *not* the case for multiple items: replacing the lottery outcome with $\langle(0,0);0\rangle$ yields the mechanism $(q^{(2)}, s^{(2)})$, whose revenue is *lower* than that of (q, s) (whereas replacing $\langle 1/2; 1/2\rangle$ with $\langle 0; 0\rangle$ yields a *higher* revenue). In fact, the function b of (7) is an extreme point in $\mathcal{B}^2$ (in particular, it is *not* the average of $b^{(1)}$ and $b^{(2)}$).

This is exactly where having more than one good matters. In the case of *one* good there is only *one* binding constraint per value x, namely, the outcome chosen by the next lower value. Consequently, dropping an outcome (such as a lottery outcome) chosen by x enables the seller to increase the revenue obtained from all higher-valuation buyers, as they can no longer switch to the outcome that has been removed and they strictly prefer their own

[29]This statement, which is easily proved in general, provides another proof of Myerson's result that in the one-good case it suffices to consider deterministic mechanisms.

outcome to any of the outcomes chosen by values below x. In contrast, when there are *multiple goods*, such an increase in revenue may not be possible because there may be *multiple* binding constraints per each valuation x (in our example, buyer $(3,3)$ is indifferent between reporting truthfully and reporting either $(1,0)$ or $(0,2)$). These buyer types may switch to other outcomes that involve other goods, and so the total revenue may well decrease.

Next, how does a lottery outcome increase revenue? The seller would like to earn positive revenue from selling good 1 to the $(1,0)$ buyer, but without jeopardizing the higher revenue obtained from selling the bundle of both goods to the $(3,3)$ buyer (and, as we have seen, he cannot increase the price of the bundle because of the "good 2 for price 2," alternative, i.e., $\langle(0,1);2\rangle$). If the price of good 1 is above 1 then $(1,0)$ will not buy it; if it is below 1, then $(3,3)$ will switch from buying the bundle to buying good 1 (since his payoff will increase from 1 to 2 or more).[30] Thus selling good 1 does not help. What does help is selling only a *fractional part* of good 1, which has the effect of making this option less attractive to the high-valuation buyer $(3,3)$ (since his possible gain is smaller: it is only *that fraction* of the difference in values). Thus, the two conflicting desiderata—getting some revenue from a low-valuation buyer, and not jeopardizing the higher revenue from a higher-valuation buyer—are reconciled by offering for sale fractions of the goods, i.e., lotteries. In the present example, that optimal fraction turns out to be $1/2$; it comes from balancing the incentives between the two goods (specifically, it is the ratio of two value differences, $3-2$ for good 2 and $3-1$ for good 1; see the Proof of Proposition 8 above).[31]

Finally, we note that mechanism design is a sequential game, with the seller moving first. In such games, the use of randomization may in general be strictly advantageous to the first mover (take for instance the sequential "matching pennies" game). Thus, the surprising fact here is not that randomizations can increase revenue (when there are multiple goods), but that

[30] As we saw above, lowering the price of the bundle to 4 (while keeping the price of good 1 at 1) will not help either, because the total revenue decreases.

[31] Thus one can easily get other probabilities by changing the values. Moreover, the example is highly robust: it has a large neighborhood where the optimal mechanisms always require lotteries.

they *cannot* do so when there is only one good.[32,33]

4.2 Lotteries for independent and identically distributed goods

We now provide a simple example where lotteries are necessary to achieve the maximal revenue for two goods that are independent and identically distributed.

Example E4. Let F be the following one-dimensional probability distribution:

$$F = \begin{cases} 1, & \text{with probability } 1/6, \\ 2, & \text{with probability } 1/2, \\ 4, & \text{with probability } 1/3, \end{cases}$$

and take two independent F-distributed goods, i.e., $\mathcal{F} = F \times F$.

Proposition 9 *The mechanism (q, s) defined by*

Valuations	**Outcome**	
x	$q(x)$	$s(x)$
$(1, 1)$	$(0, 0)$	0
$(2, 1)$	$(\frac{1}{2}, 0)$	1
$(1, 2)$	$(0, \frac{1}{2})$	1
$(1, 4), (4, 1), (2, 2), (2, 4), (4, 2), (4, 4)$	$(1, 1)$	4

(8)

with

$$b(x_1, x_2) = \max\left\{0,\ \frac{1}{2}x_1 - 1,\ \frac{1}{2}x_2 - 1,\ x_1 + x_2 - 4\right\} \tag{9}$$

is the unique *optimal mechanism for $\mathcal{F} = F \times F$ of Example E4.*

Proof. First, the revenue from the mechanism (8) is easily computed: it equals 61/18.

[32] We thank Bob Aumann for this comment.

[33] Pycia (2006) shows how in the multiple-goods case non-deterministic mechanisms are generically needed to maximize revenue.

Second, take the following inequalities, which are various individual rationality and incentive compatibility constraints[34]:

$$
\begin{array}{rcl|l}
q_1^{11} + q_2^{11} - s^{11} & \geq & 0 & 3 \\
q_1^{12} + 2q_2^{12} - s^{12} & \geq & 0 & 8 \\
2q_1^{21} + q_2^{21} - s^{21} & \geq & 0 & 8 \\
2q_1^{22} + 2q_2^{22} - s^{22} & \geq & 0 & 17 \\
q_1^{12} + 2q_2^{12} - s^{12} & \geq & q_1^{11} + 2q_2^{11} - s^{11} & 1 \\
2q_1^{21} + q_2^{21} - s^{21} & \geq & 2q_1^{11} + q_2^{11} - s^{11} & 1 \\
2q_1^{22} + 2q_2^{22} - s^{22} & \geq & 2q_1^{12} + 2q_2^{12} - s^{12} & 3 \\
2q_1^{22} + 2q_2^{22} - s^{22} & \geq & 2q_1^{21} + 2q_2^{21} - s^{21} & 3 \\
q_1^{14} + 4q_2^{14} - s^{14} & \geq & q_1^{12} + 4q_2^{12} - s^{12} & 3 \\
4q_1^{41} + q_2^{41} - s^{41} & \geq & 4q_1^{21} + q_2^{21} - s^{21} & 3 \\
2q_1^{22} + 2q_2^{22} - s^{22} & \geq & 2q_1^{14} + 2q_2^{14} - s^{14} & 1 \\
2q_1^{22} + 2q_2^{22} - s^{22} & \geq & 2q_1^{41} + 2q_2^{41} - s^{41} & 1 \\
2q_1^{24} + 4q_2^{24} - s^{24} & \geq & 2q_1^{22} + 4q_2^{22} - s^{22} & 8 \\
4q_1^{42} + 2q_2^{42} - s^{42} & \geq & 4q_1^{22} + 2q_2^{22} - s^{22} & 8 \\
4q_1^{44} + 4q_2^{44} - s^{44} & \geq & 4q_1^{24} + 4q_2^{24} - s^{24} & 2 \\
4q_1^{44} + 4q_2^{44} - s^{44} & \geq & 4q_1^{42} + 4q_2^{42} - s^{42} & 2
\end{array}
\tag{10}
$$

Multiplying each inequality by the weight on the right and adding up yields:

$$
\begin{aligned}
& s^{11} + 3s^{12} + 3s^{21} + 9s^{22} + 2s^{14} + 2s^{41} + 6s^{24} + 6s^{42} + 4s^{44} \\
& \leq\ 2q_1^{22} + q_1^{14} + 10q_1^{41} + 8q_1^{24} + 24q_1^{42} + 16q_1^{44} \\
& \quad + 2q_2^{22} + 10q_2^{14} + q_2^{41} + 24q_2^{24} + 8q_2^{42} + 16q_2^{44}.
\end{aligned}
\tag{11}
$$

The left-hand side turns out to be precisely 36 times the expected revenue of the seller for the distribution $\mathcal{F} = F \times F$, i.e., $36\mathbb{E}_{\mathcal{F}}[s(x)]$, and the right-hand side is bounded from above by 122 (replace all q_1 and q_2 there by their upper bound of 1). Therefore $\mathbb{E}_{\mathcal{F}}[s(x)] \leq 122/36 = 61/18$. Recalling that $61/18$ is precisely the revenue of the mechanism (8) shows that (8) is optimal.

Finally, to see that (8) is the only optimal mechanism: by the proof above,

[34]These specific inequalities and their corresponding multipliers below were obtained by solving the dual of the linear programming problem of maximizing the revenue.

for the maximal revenue of 61/18 to be achieved, all the inequalities must become equalities. First, all the q_1 and q_2 appearing on the right-hand side of (11) must equal 1:

$$\begin{aligned} 1 &= q_1^{22} = q_1^{14} = q_1^{41} = q_1^{24} = q_1^{42} = q_1^{44} \\ &= q_2^{22} = q_2^{14} = q_2^{41} = q_2^{24} = q_2^{42} = q_2^{44}. \end{aligned} \tag{12}$$

Second, the inequalities in (10), which are now equalities, yield after substituting (12):

$$\begin{aligned} &s^{44} = s^{24} = s^{42} = s^{22} = s^{14} = s^{41} = 4, \quad s^{12} = s^{21} = 1, \quad s^{11} = 0, \\ &q_1^{11} = q_2^{11} = q_1^{12} = q_2^{21} = 0, \;\; q_1^{21} = q_2^{12} = \frac{1}{2}. \end{aligned}$$

Together with (12) this yields precisely the mechanism (8). ■

It can be checked that the maximal revenue achievable by a deterministic mechanism is 10/3 (obtained by the mechanism with price 2 for each good).

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