An axiomatic characterization of the Theil inequality ordering (preliminary)

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Abstract

We characterize the Theil ordering of income inequality by means of a few ordinal axioms.

1 Introduction

Economists have been interested in income inequality for quite some time now. Typical issues include the evolution of income inequality in some particular country or region over time, the differences in income inequality across different regions, and the effect of different policies on income inequality, and of income inequality on different economic variables. In order to address these or other similar questions one must first be able to measure income inequality.

The literature on the measurement of income inequality offers a plethora of inequality indices but the extent to which they are appropriate is not clear. In order to compare different measures one may apply them to various examples and check if these measures do not contradict one's intuitions about income inequality. For instance, we may apply a few inequality indices to two income distributions, one of which we believe is more unequal than the other, and discard all those indices that contradict our subjective judgement. Another way to evaluate inequality measures is not to observe their performance on particular examples but to evaluate their properties at a more abstract level. We may make a list of properties we think a reasonable inequality measure should satisfy and check which inequality measures actually satisfy them.

Some properties of inequality measures are uncontroversial. In fact, they are so uncontroversial that they are considered as defining properties of the bare concept of inequality measure. For example, the Pigou-Dalton principle of transfers, which postulates that the transfer of income from a rich individual to a poorer one increases inequality as long as the poor individual does not become richer than the rich one, is considered by Fields and Fey [2] as one of the basic axioms of inequality measurement. Other axioms are less uncontroversial, though. For example, some of them require from an inequality index that it be decomposable in some particular way into an inequality between regions, and an inequality within regions. This decomposability is definitely not an essential property of an inequality index.

This axiomatic approach has been successfully applied by Bourguignon [1], Foster [3], Shorrocks [4] and many others. In particular, Foster [3] has shown that the Theil index of income inequality is the only index that satisfies the three basic properties enumerated by Fields and Fey [2], as well as a simple decomposability property.

There is a distinction between properties that we believe to be important. Some axioms are ordinal in nature, and others are cardinal. Ordinal axioms impose restrictions on the way different income distributions are ranked. The Pigou-Dalton principle of transfers is an ordinal property in that it compares two particular distributions and tells us which one is more unequal. It does not tell us anything about the magnitude of the inequality. Cardinal axioms, on the other hand, impose restrictions on the functional form that is used to measure inequality. The decomposability property that Foster uses to characterize the Theil index is cardinal. Indeed, it requires that the total inequality of a distribution be a weighted sum of the inequalities of its regions and the inequality between these regions.

In this paper we characterize the Theil inequality measure by means of ordinal axioms only. In particular, we strip the decomposability property used by Foster (1983) from all its cardinal content, and maintain its ordinal content only.

2 Definitions

A *society* is a collection $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$ where y_k, $k = 1, \ldots, K$ are income levels, not all of them 0, and for each k, $n_k > 0$ is the mass of people with income level y_k. Elements (n_k, y_k) of a society are called *social classes*.

For a society $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$, we denote by $|S|$, the total level of income in S, and by $n(S)$ its total population. That is,

$$|S| = \sum_{k=1}^{K} n_k y_k \quad \text{and} \quad n(S) = \sum_{k=1}^{K} n_k.$$

Note that $|S| > 0$ and $n(S) > 0$.

For any two societies S_1 and S_2, $S_1 \cup S_2$ denotes the union of the two.

We denote by $\mathcal{S}_n$, the set of all societies with population mass n, and by $\mathcal{S} = \cup_{n>0} \mathcal{S}_n$ the set of all societies.

An *inequality ordering* is a binary relation $\succcurlyeq$ on $\mathcal{S}$. Some orderings can be represented by an *inequality index*. An inequality index is a function $I : \mathcal{S} \to \mathbb{R}$ that assigns to each society a real number, that represents the society's inequality level.

2.1 Examples of inequality indices

Example 1 *The* Theil index, $T : \mathcal{S} \to \mathbb{R}$, *is defined as follows:*
for all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$,

$$T(S) = \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \log_2 \left(n(S) \frac{y_k}{|S|} \right)$$

The Theil ordering *is the ordering represented by the Theil index.*

Note that the Theil index can be written as

$$T(S) = \log_2 n(S) - \sum_{k=1}^{K} \frac{n_k y_k}{|S|} \log_2 \left(\frac{|S|}{y_k} \right)$$

or as the difference between the maximum entropy in a population of mass n, and the entropy of the society.

Example 2 *The* Gini index*,* $G : \mathcal{S} \to [0,1]$*, is defined as follows: for all* $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$*,*

$$G(S) = 1 - \frac{\sum_{k=1}^{K} \frac{n_k}{n(S)} (\sum_{j=1}^{k-1} n_j y_j) + \sum_{j=1}^{k} n_j y_j)}{|S|}$$

The Gini ordering *is the ordering represented by the Gini index.*

Example 3 *The* Atkinson index*,* $A : \mathcal{S} \to [0,1]$*, is defined as follows: for all* $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$*,*

$$A(S) = 1 - \frac{n(S)}{|S|} \prod_{k=1}^{K} (y_k)^{\frac{n_k}{n(S)}}$$

The Atkinson ordering *is the ordering represented by the Atkinson index.*

Example 4 *The* Second Theil index*,* $T_0 : \mathcal{S}_+ \to [0, \infty)$*,is defined as follows: for all* $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}_+$*,*

$$T_0(S) = \log \left(\frac{|S|/n(S)}{\prod_{k=1}^{K} (y_k)^{\frac{n_k}{n(S)}}} \right)$$

The Second Theil ordering *is the ordering represented by the Second Theil index.*

Researchers are sometimes interested in decomposable inequality indices. Decomposable indices allow us to attribute total inequality to different factors. In particular, decomposable indices allow us to decompose total inequality into inequality *between* subsocieties and inequality *within* subsocieties. One general way to define decomposability is as follows.

Definition 5 *We say that inequality index* I *is* decomposable *if for any two societies* S_1 *and* S_2*,*

$$I(S_1 \cup S_2) = f(v_1, w_1) I(S_1) + f(v_2, w_2) I(S_2) + I(\overline{S_1} \cup \overline{S_2})$$

where $\overline{S_i} = \left\langle \left(n(S_i), \frac{|S_i|}{n(S_i)} \right) \right\rangle$*,* $v_i = \frac{n(S_i)}{n(S_1 \cup S_2)}$ *and* $w_i = \frac{|S_i|}{|S_1 \cup S_2|}$*, for* $i = 1, 2$*, and* f *is homogeneous of degree one in both its arguments.*

Note that decomposability is a cardinal axiom.

Foster (1983) used a more restrictive version of decomposability in his characterization of the Theil index.

Definition 6 *We say that inequality index I is* Theil-decomposable, *if for any two societies S_1 and S_2,*

$$I(S_1 \cup S_2) = \frac{|S_2|}{|S_1 \cup S_2|} I(S_1) + \frac{|S_2|}{|S_1 \cup S_2|} I(S_2) + I(\overline{S_1} \cup \overline{S_2})$$

Although Theil decomposability is a cardinal axiom, it has very strong ordinal implications. In this paper we identify some of these ordinal implications and, together with other ordinal axioms, use them to characterize the Theil income inequality ordering.

3 Axioms

We now present a set of axioms that an inequality ordering may satisfy. The first axioms embodies the idea that we are interested in *relative* measures of income inequality.

Definition 7 (HOM) *We say that $\succcurlyeq$ satisfies* Homogeneity *if for all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$, and for all $\alpha > 0$, we have $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \sim \langle (n_1, \alpha y_1), \ldots, (n_K, \alpha y_K) \rangle$.*

Homogeneity states that only the relative distribution of income determines inequality. In other words, one does not need to know the units in which income is measured (dollars, euros, etc.) to determine whether one society has a more or less equal distribution than another one.

The next axiom is similar to homogeneity in the sense that it is not the absolute number of people who has any given income level what matters, but their proportion in the population.

Definition 8 (RI) *We say that $\succcurlyeq$ satisfies* replication invariance *if for all $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$, and for all $\alpha > 0$, we have $\langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \sim \langle (\alpha n_1, y_1), \ldots, (\alpha n_K, y_K) \rangle$.*

Replication invariance, which is sometimes called Dalton's principle of population, states that if we replicate a society by multiplying each individual by a fixed constant then inequality remains unaffected.

The previous two axioms state that some particular changes in the society do not affect its income inequality. The next axiom, on the other hand, states that some other changes do affect income inequality in a certain way.

Definition 9 (TP) *We say that $\succcurlyeq$ satisfies* the transfer principle *if for all egalitarian societies $\langle (n, y) \rangle$, and for all $\langle (n_1, y_1), (n_2, y_2) \rangle$ such that $n_1 + n_2 = n$ and $ny = n_1 y_1 + n_2 y_2$ we have $\langle (n_1, y_1), (n_2, y_2) \rangle \succcurlyeq \langle (n, y) \rangle$, with equivalence ($\sim$) if and only if $y_1 = y_2$.*

According to the transfer principle, if one divides an egalitarian society into two social classes by transferring income from some individuals to others, then one obtains a new society with more unequal distribution of income. A stronger version of the transfer principle is stated in the following axiom. It requires that a division of any social class into two different social classes that results from a transfer of income from one group of individuals to another results in a society with more unequal distribution of income.

Definition 10 (SCDP) *We say that* $\succcurlyeq$ *satisfies* social class division property *if whenever* S' *is obtained from* S *by means of a subdivision of a social class* $(n_k, y_k) \in S$ *into two social classes* $(n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2})$ *such that* $n_k = n_{k_1} + n_{k_2}$ *and* $n_k y_k = n_{k_1} y_{k_1} + n_{k_2} y_{k_2}$*, then:* $S' \succcurlyeq S$*, with equivalence* $(\sim)$ *if and only if* $y_{k_1} = y_{k_2}$*.*

The next axiom is an ordinal implication of the decomposability axiom used in Foster's (1983) characterization of the Theil index.

Definition 11 (IND) *We say that* $\succcurlyeq$ *satisfies* Independence*, if for all* $S_1, S_2 \in \mathcal{S}$ *such that* $|S_1| = |S_2|$ *and* $n(S_1) = n(S_2)$*, and for all societies* $S \in \mathcal{S}$*,*

$$S_1 \succcurlyeq S_2 \Leftrightarrow S_1 \cup S \succcurlyeq S_2 \cup S.$$

Independence says that if a given society is composed of two regions or subsocieties, and one of its regions' income becomes more unequally distributed, then the income distribution of the whole society becomes more unequal as well.

Claim 12 *If the inequality order* $\succcurlyeq$ *satisfies Independence, and the transfer principle, then it also satisfies the social class division property.*

Proof. Let $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle$ be a society and let $S' = S \setminus \langle (n_k, y_k) \rangle \cup \langle (n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2}) \rangle$ be the society that is obtained from S by means of a subdivision of a social class $(n_k, y_k) \in S$ into two social classes $(n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2})$ such that $n_k = n_{k_1} + n_{k_2}$ and $n_k y_k = n_{k_1} y_{k_1} + n_{k_2} y_{k_2}$. By the transfer principle,

$$\langle (n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2}) \rangle \succcurlyeq \langle (n_k, y_k) \rangle .$$

By independence,

$$S \setminus \langle (n_k, y_k) \rangle \cup \langle (n_{k_1}, y_{k_1}), (n_{k_1}, y_{k_2}) \rangle \succcurlyeq S \setminus \langle (n_k, y_k) \rangle \cup \langle (n_k, y_k) \rangle$$

or, $S' \succcurlyeq S$. ■

The next axiom is another ordinal implication of Theil-decomposability.

Definition 13 (DEC) *We say that* $\succcurlyeq$ *satisfies* Decomposability, *if for all four societies* $S_1, S_2, S_3, S_4 \in \mathcal{S}$*, such that*

- $|S_1| = |S_2|$*, and*
- $|S_3| = |S_4|$*, and* $n(S_3) = n(S_4)$*,*

we have

$$(S_1 \cup S_3) \succcurlyeq (S_2 \cup S_3) \Leftrightarrow (S_1 \cup S_4) \succcurlyeq (S_2 \cup S_4).$$

Decomposability states the following. Suppose that a given society $(S_1 \cup S_3)$ is composed of two subsocieties and that the population of one of these subsocieties, S_1, changes (without changing its total income), resulting in the society $(S_2 \cup S_3)$. This population change obviously may affect the inequality of the whole society's income distribution in that the society's income might become either more or less equally distributed. Decomposability requires that this effect be independent of the distribution of income of the subsociety whose population remained unchanged.

The last axiom is a technical continuity requirement. It states that "similar" societies have "similar" levels of income inequality.

Definition 14 *The inequality ordering* $\succcurlyeq$ *satisfies* Continuity *if for all three societies* S*,* S' *and* S''*, the sets*

$$\left\{\alpha \in [0,1] : \alpha S \cup (1-\alpha) S' \succcurlyeq S''\right\} \quad \textit{and} \quad \left\{\alpha \in [0,1] : S'' \succcurlyeq \alpha S \cup (1-\alpha) S'\right\}$$

are closed.

3.1 Properties of The Theil Inequality Ordering

Claim 15 *The Theil ordering satisfies replication invariance and homogeneity.*

Proof. Left to the reader. ■

Claim 16 *The Theil ordering satisfies Independence.*

Proof. Let $S_1 = \langle (n_1, x_1), \ldots, (n_{K_1}, x_{K_1}) \rangle$ and $S_2 = \langle (m_1, y_1), \ldots, (m_{K_2}, y_{K_2}) \rangle$ be two societies such that $|S_1| = |S_2|$ and $n(S_1) = n(S_2) = n$, and let $S = \langle (p_1, z_1), \ldots, (p_{K_3}, z_{K_3}) \rangle \in \mathcal{S}$. Then, $|S_1 \cup S| = |S_2 \cup S|$. Since the Theil ordering satisfies homogeneity, we can assume without loss of generality that $|S_1 \cup S| = |S_2 \cup S| = 1$.

$$
\begin{aligned}
S_1 &\succcurlyeq S_2 \Leftrightarrow \log_2 n - \sum_{k=1}^{K_1} \frac{n_k x_k}{|S_1|} \log_2\left(\frac{|S_1|}{x_k}\right) \geq \log_2 n - \sum_{k=1}^{K_2} \frac{m_k y_k}{|S_2|} \log_2\left(\frac{|S_2|}{y_k}\right) \\
&\Leftrightarrow \sum_{k=1}^{K_1} \frac{n_k x_k}{|S_1|} \log_2\left(\frac{|S_1|}{x_k}\right) \leq \sum_{k=1}^{K_2} \frac{m_k y_k}{|S_2|} \log_2\left(\frac{|S_2|}{y_k}\right) \\
&\Leftrightarrow \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{|S_1|}{x_k}\right) \leq \sum_{k=1}^{K_2} m_k y_k \log_2\left(\frac{|S_2|}{y_k}\right) \\
&\Leftrightarrow \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) + \sum_{k=1}^{K_1} n_k x_k \log_2(|S_1|) \leq \sum_{k=1}^{K_2} m_k y_k \log_2\left(\frac{1}{y_k}\right) + \sum_{k=1}^{K_2} m_k y_k \log_2(|S_2|) \\
&\Leftrightarrow \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) + |S_1| \log_2(|S_1|) \leq \sum_{k=1}^{K_2} m_k y_k \log_2\left(\frac{1}{y_k}\right) + |S_2| \log_2(|S_2|) \\
&\Leftrightarrow \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) \leq \sum_{k=1}^{K_2} m_k y_k \log_2\left(\frac{1}{y_k}\right) \\
&\Leftrightarrow \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) + \sum_{k=1}^{K_3} p_k z_k \log_2\left(\frac{1}{z_k}\right) \leq \sum_{k=1}^{K_2} m_k y_k \log_2\left(\frac{1}{y_k}\right) + \sum_{k=1}^{K_3} p_k z_k \log_2\left(\frac{1}{z_k}\right) \\
&\Leftrightarrow S_1 \cup S \succcurlyeq S_2 \cup S
\end{aligned}
$$

■

Claim 17 *The Theil ordering satisfies Decomposability.*

Proof. Let $S_1 = \langle (n_1, x_1), \ldots, (n_{K_1}, x_{K_1}) \rangle, S_2 = \langle (t_1, w_1), \ldots, (t_{K_2}, w_{K_2}) \rangle \in \mathcal{S}$, and $S_3 = \langle (m_1, y_1), \ldots, (m_{K_3}, y_{K_3}) \rangle, S_4 = \langle (p_1, z_1), \ldots, (p_{K_4}, z_{K_4}) \rangle \in \mathcal{S}_m$ such that $|S_1| = |S_2|$, and $|S_3| = |S_4|$. Note then that $|(S_1 \cup S_3)| = |(S_2 \cup S_3)| = |S_2 \cup S_4| = |S_1 \cup S_4|$, which by homogeneity can be assumed to be equal 1.

$$\begin{aligned}
(S_1 \cup S_3) &\succcurlyeq (S_2 \cup S_3) \Leftrightarrow \log_2(n(S_1)+m) - \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) - \sum_{k=1}^{K_3} m_k y_k \log_2\left(\frac{1}{y_k}\right) \\
&\geq \log_2(n(S_2)+m) - \sum_{k=1}^{K_2} t_k w_k \log_2\left(\frac{1}{w_k}\right) - \sum_{k=1}^{K_3} m_k y_k \log_2\left(\frac{1}{y_k}\right) \\
&\Leftrightarrow \log_2(n(S_1)+m) - \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) \geq \log_2(n(S_2)+m) - \sum_{k=1}^{K_2} t_k w_k \log_2\left(\frac{1}{w_k}\right) \\
&\Leftrightarrow \log_2(n(S_1)+m) - \sum_{k=1}^{K_1} n_k x_k \log_2\left(\frac{1}{x_k}\right) - \sum_{k=1}^{K_4} p_k z_k \log_2\left(\frac{1}{z_k}\right) \\
&\geq \log_2(n(S_2)+m) - \sum_{k=1}^{K_2} t_k w_k \log_2\left(\frac{1}{w_k}\right) - \sum_{k=1}^{K_4} p_k z_k \log_2\left(\frac{1}{z_k}\right) \\
&\Leftrightarrow (S_1 \cup S_4) \succcurlyeq (S_2 \cup S_4)
\end{aligned}$$

■

We are now ready to state our main result.

Theorem 18 *There is a unique inequality ordering defined on $\mathcal{S}$ that satisfies, homogeneity, replication invariance, independence, decompoability, the transfer principle, and continuity. It is the Theil inequality ordering.*

4 Proof of the main theorem

Let $S_0 = \langle(1,1)\rangle$ be the society with population mass 1 and a uniformly distributed income of one.

For each $\alpha \in (0,1)$, let $S_\alpha = \langle(\alpha,0),(1-\alpha,1/(1-\alpha))\rangle$ be the society with population mass 1, in which a proportion α of the population has income 0, and a proportion $1-\alpha$ of the population has income $1/(1-\alpha)$. Note that for all α, S_α has population 1 and income 1.

Lemma 19 *For each society $S \in \mathcal{S}$, there is $\alpha \in (0,1)$ such that $S_\alpha \succcurlyeq S$.*

Proof. Let $S \in \mathcal{S}$. By RI we can assume that S has a population mass of 1. By homogeneity we can assume without loss of generality that the maximum level of income in S is 1: $\max\{y_k : k = 1, \ldots K\} = 1$. Note that $|S| = \sum n_k y_k \leq 1$. Denote S' the society that is obtained from S by subdividing each bracket (n_k, y_k) in S into two sub-brackets $\langle (n_k(1-y_k), 0), (n_k y_k, 1)\rangle$. By SCDP, $S' \succcurlyeq S$. By SCDP again,

$$S' \sim \left\langle (1 - \sum n_k y_k, 0), (\sum n_k y_k, 1) \right\rangle = \langle (1 - |S|, 0), (|S|, 1) \rangle \sim \langle (1 - |S|, 0), (|S|, 1/|S|) \rangle .$$

Therefore $\alpha = 1 - |S|$ is the number we are looking for. ■

Lemma 20 *All societies where total income is uniformly distributed have the same degree of income inequality. Further for all societies $S \in \mathcal{S}$, $S \succcurlyeq S_0$.*

Proof. Let S be a society with uniformly distributed income. By HOM, RI, and SCDP, $S \sim S_0$. Let now $S = \langle (n_1, y_1), \ldots, (n_K, y_K) \rangle \in \mathcal{S}$ be an arbitrary society. Let S^k be the society that results from combining social classes 1 to k, into one bracket $\left(\sum_{i=1}^k n_i, \sum_{i=1}^k n_i y_i / \sum_{i=1}^k n_i \right)$. By SCDP, $S^k \succcurlyeq S^{k+1}$. Therefore, $S = S^1 \succcurlyeq S^K$. But S^K has only one social class, and hence income is uniformly distributed there. ■

Lemma 21 *Let $0 \leq \alpha < \beta < 1$. Then, $S_\beta \succ S_\alpha$.*

Proof.

$$\begin{aligned} S_\beta &= \left\langle (\beta, 0), (1-\beta, \frac{1}{1-\beta}) \right\rangle \\ &\sim \left\langle (\alpha, 0), (\beta - \alpha, 0), (1-\beta, \frac{1}{(1-\beta)}) \right\rangle \text{ by SCDP} \\ &\succcurlyeq \left\langle (\alpha, 0), (1-\alpha, \frac{1}{(1-\alpha)}) \right\rangle = S_\alpha \text{ by SCDP.} \end{aligned}$$

■

Lemma 22 *Let $a \in (0,1)$ be such that $S_a \succ S_0$. Then, for $0 \leq \alpha < \beta < 1$,*

$$\beta S_a \cup (1-\beta) S_0 \succ \alpha S_a \cup (1-\alpha) S_0$$

Proof. By SCDP applied three times,

$$\begin{aligned} \beta S_a \cup (1-\beta) S_0 &\sim \alpha S_a \cup (\beta - \alpha) S_a \cup (1-\beta) S_0 \\ &\succ \alpha S_a \cup (\beta - \alpha) S_0 \cup (1-\beta) S_0 \\ &\sim \alpha S_a \cup (1-\alpha) S_0. \end{aligned}$$

■

Lemma 23 *Let $a \in (0,1)$ be such that $S_a \succ S_0$. For any society $S \in \mathcal{S}$ such that $S_a \succcurlyeq S \succcurlyeq S_0$, there is a unique $\alpha \in [0,1]$ such that*

$$S \sim \alpha S_a \cup (1-\alpha) S_0$$

Proof. By C, the sets $\{\alpha \in [0,1] : \alpha S_a \cup (1-\alpha)S_0 \succcurlyeq S\}$ and $\{\alpha \in [0,1] : S \succcurlyeq \alpha S_a \cup (1-\alpha)S_0\}$ are closed. Since $S_a \succcurlyeq S \succcurlyeq S_0$, they are not empty. Since $\succcurlyeq$ is complete, their union is $[0,1]$. Therefore, since the unit interval is connected, the intersection of the two sets is not empty. By Lemma 22, this intersection must contain a single element. This single element is the α we are looking for. ■

Let S be a society and let $a \in (1/2, 1)$ such that $S_a \succ S$ and $S_a \succ S_0$. By Lemmas 19 and 21 such a exists. Then we have $S_a \succ S \succcurlyeq S_0$. Let $\alpha(a)$ be the unique number identified in Lemma 23 that satisfies

$$S \sim \alpha(a) S_a \cup (1-\alpha(a)) S_0$$

Similarly, since $S_a \succ S_{1/2} \succ S_0$, let $\beta(a)$ the unique number that satisfies

$$S_{1/2} \sim \beta(a) S_a \cup (1-\beta(a)) S_0.$$

By Lemma 22 $\beta(a) > 0$. Therefore, we can define an index $r(S)$ to be the ratio

$$r(S) = \frac{\alpha(a)}{\beta(a)}.$$

Claim 24 *The ratio $r(S)$ is well-defined. Namely, it does not depend on the choice of a.*

Proof. Let $a, b \in (1/2, 1)$ such that $S_a \succ S$ and $S_b \succ S$. Let $\alpha(a)$ and $\beta(a)$ be defined by

$$S \sim \alpha(a) S_a \cup (1-\alpha(a)) S_0 \tag{1}$$

and

$$S_{1/2} \sim \beta(a) S_a \cup (1-\beta(a)) S_0. \tag{2}$$

By Lemma 23 such numbers exist. Similarly define $\alpha(b)$ and $\beta(b)$ to satisfy

$$S \sim \alpha(b) S_b \cup (1-\alpha(b)) S_0 \tag{3}$$

and

$$S_{1/2} \sim \beta(b) S_b \cup (1-\beta(b)) S_0. \tag{4}$$

Assume without loss of generality that $a > b$. Then $S_a \succcurlyeq S_b$. Let γ be defined by

$$S_b \sim \gamma S_a \cup (1-\gamma) S_0.$$

By Lemma 23 such γ exists. Therefore, it follows from (3) using replication invariance and independence, and then by SCDP,

$$\begin{aligned} S &\sim \alpha(b)\left[\gamma S_a \cup (1-\gamma)S_0\right] \cup (1-\alpha(b))S_0 \\ &\sim \alpha(b)\gamma S_a \cup (1-\alpha(b)\gamma)S_0. \end{aligned}$$

Comparing with (1), we obtain $\alpha(a) = \alpha(b)\gamma$. Similarly,

$$\begin{aligned} S_{1/2} &\sim \beta(b)\left[\gamma S_a \cup (1-\gamma)S_0\right] \cup (1-\beta(b))S_0 \\ &\sim \beta(b)\gamma S_a \cup (1-\beta(b)\gamma)S_0. \end{aligned}$$

Comparing with (2), we obtain $\beta(a) = \beta(b)\gamma$. Note that by Lemma 22 $\beta(a), \beta(b), \gamma > 0$. As a result we get

$$\frac{\alpha(a)}{\beta(a)} = \frac{\alpha(b)}{\beta(b)}.$$

■

Lemma 25 *The ratio r represents the inequality order $\succcurlyeq$.*

Proof. Let S and S' be two societies and assume $S' \succcurlyeq S$. Let $a \in (1/2, 1)$ such that $S_a \succcurlyeq S$. By Lemma 19 this can be done. Let α and β be defined by

$$S \sim \alpha S_a \cup (1-\alpha) S_0$$

and

$$S' \sim \beta S_a \cup (1-\beta) S_0$$

Then, by Lemma 22 $\alpha \leq \beta$. Letting $\beta(a)$ be defined by

$$S_{1/2} \sim \beta(a) S_a \cup (1-\beta(a)) S_0$$

we obtain

$$r(S) = \frac{\alpha}{\beta(a)} \leq \frac{\beta}{\beta(a)} = r(S').$$

■

Proposition 26 *Let S and S' be two societies. Then*

$$r(S \cup S') = \frac{|S|}{|S \cup S'|} r(S) + r\left(\left\langle \left(n(S), \frac{|S|}{n(S)}\right)\right\rangle \cup S'\right)$$

Proof. Let S and S' be two societies, with populations n and m, respectively. By RI and HOM, we can assume without loss of generality that $n+m=1$, and $|S \cup S'| = 1$. Let $a \in (1/2, 1)$ be such that $S_a \succcurlyeq S$, $S_a \succcurlyeq S'$, and $S_a \succcurlyeq S \cup S'$. By Lemma (20) this can be done. Let γ be defined by

$$\left\langle n, \frac{|S|}{n} \right\rangle \cup S' \sim \gamma S_a \cup (1-\gamma) S_0. \tag{5}$$

Similarly, let α be defined by

$$S \sim \alpha S_a \cup (1-\alpha) S_0. \tag{6}$$

It is enough to show that

$$S \cup S' \sim (|S|\,\alpha + \gamma) S_a \cup ((1-\gamma) - \alpha\,|S|) S_0. \tag{7}$$

We first show the following technical lemma.

Lemma 27 $\gamma \leq 1 - |S|\,\alpha$.

Proof. Denote $S_a^* = n\left\langle (a,0), (1-a, \frac{|S|}{(1-a)n}) \right\rangle$ and $S_0^* = \left\langle (n, \frac{|S|}{n}) \right\rangle$. Note that these two societies have a population n and an income $|S|$. That is, they have the same population and income as S. They are obtained, respectively, by multiplying the population and incomes of S_a and S_0 by n and $|S|/n$. Therefore, by homogeneity and replication invariance, it follows from (6) that

$$S \sim \alpha S_a^* \cup (1-\alpha) S_0^*.$$

By IND,

$$S \cup S' \sim \alpha S_a^* \cup (1-\alpha) S_0^* \cup S'. \tag{8}$$

Since $S_a \succcurlyeq S \cup S'$,

$$\begin{aligned} S_a^* &\succcurlyeq \alpha S_a^* \cup (1-\alpha) S_0^* \cup S' && \text{by HOM, RI and (8)} \\ \alpha S_a^* \cup \underbrace{(1-\alpha) S_a^* \cup \frac{|S'|}{|S|} S_a^*}_{Z_1} &\succcurlyeq \alpha S_a^* \cup \underbrace{(1-\alpha) S_0^* \cup S'}_{Z_2} && \text{by RI and SCDP.} \end{aligned}$$

Since Z_1 and Z_2 have the same income ($|Z_1| = |Z_2| = (1-\alpha)|S| + |S'|$), we can apply Decomposability and replace αS_a^* in both sides of the above expression by αS_0^* (both αS_a^* and αS_0^* have the same population and income), and obtain

$$\alpha S_0^* \cup \underbrace{(1-\alpha)S_a^* \cup \frac{|S'|}{|S|}S_a^*}_{Z_1} \succcurlyeq \alpha S_0^* \cup \underbrace{(1-\alpha)S_0^* \cup S'}_{Z_2}.$$

Now,

$$\begin{aligned} \alpha S_0^* \cup \left((1-\alpha) + \tfrac{|S'|}{|S|}\right) S_a^* &\succcurlyeq S_0^* \cup S' && \text{by SCDP} \\ \alpha|S|S_0 \cup (1-\alpha|S|)\, S_a &\succcurlyeq S_0^* \cup S' && \text{by RI} \\ \alpha|S|S_0 \cup (1-\alpha|S|)\, S_a &\succcurlyeq (1-\gamma)S_0 \cup \gamma S_a && \text{by (5).} \end{aligned}$$

Consequently, by Lemma 22 $1 - \alpha|S| \geq \gamma$. ■

Now we are ready to show that (7) holds. Since $\gamma \in (0,1)$, there exists $k \in \mathbb{N}$ such that $\gamma \leq 1 - \frac{|S|}{k}$. Denote $S_a^* = n\left\langle (a,0), (1-a, \frac{|S|}{(1-a)n})\right\rangle$ and $S_0^* = \left\langle (n, \frac{|S|}{n})\right\rangle$. By (5),

$$S_0^* \cup S' \sim \gamma S_a \cup (1-\gamma)S_0.$$

Therefore,

$$\begin{aligned} S_0^* \cup \overbrace{S' \cup (k-1)(\gamma S_a \cup (1-\gamma)S_0)}^{Z_1} &\sim k(\gamma S_a \cup (1-\gamma)S_0) && \text{by IND and SCDP} \\ &\sim \tfrac{k}{|S|}(\gamma S_a^* \cup (1-\gamma)S_0^*) && \text{by HOM and RI} \\ &\sim S_0^* \cup \overbrace{\frac{k}{|S|}(\gamma S_a^* \cup \underbrace{(1-\gamma - \frac{|S|}{k})}_{\geq 0} S_0^*)}^{Z_2} && \text{by SCDP.} \end{aligned}$$

By Lemma 27, $1 - \gamma - \frac{|S|}{k} > 0$. Since $|Z_1| = |Z_2| = |S'| + (k-1)$, by Decomposability,

$$S \cup S' \cup (k-1)(\gamma S_a \cup (1-\gamma)S_0) \sim S \cup \frac{k}{|S|}(\gamma S_a^* \cup (1-\gamma - \frac{|S|}{k})S_0^*)$$

and by IND (S and $\alpha S_a^* \cup (1-\alpha)S_0^*$ have the same population and income),

$$\begin{aligned} S \cup S' \cup (k-1)(\gamma S_a \cup (1-\gamma)S_0) &\sim (\alpha S_a^* \cup (1-\alpha)S_0^*) \cup \tfrac{k}{|S|}(\gamma S_a^* \cup (1-\gamma-\tfrac{|S|}{k})S_0^*) \\ &\sim (\alpha + \tfrac{k\gamma}{|S|})S_a^* \cup (\tfrac{k}{|S|}(1-\gamma) - \alpha)S_0^* && \text{by SCDP} \\ &\sim (\alpha|S| + k\gamma)S_a \cup (k(1-\gamma) - \alpha|S|)S_0 && \text{by HOM and RI} \end{aligned}$$

which, by SCDP, is equivalent to

$$(\alpha\,|S|+\gamma)S_a \cup \underbrace{((1-\gamma)-\alpha\,|S|)}_{\geq 0} S_0 \cup (k-1)(\gamma S_a \cup (1-\gamma)S_0).$$

Note that this society is well defined since, by Lemma 27, $(1-\gamma)-\alpha\,|S| \geq 0$. Therefore,

$$S \cup S' \cup (k-1)(\gamma S_a \cup (1-\gamma)S_0) \sim (\alpha\,|S|+\gamma)S_a \cup ((1-\gamma)-\alpha\,|S|)S_0 \cup (k-1)(\gamma S_a \cup (1-\gamma)S_0)$$

By IND ($S \cup S'$ and $(\alpha\,|S|+\gamma)S_a \cup ((1-\gamma)-\alpha\,|S|)S_0$ have the same population and income),

$$S \cup S' \sim (\alpha\,|S|+\gamma)S_a \cup ((1-\gamma)-\alpha\,|S|)S_0$$

■

Corollary 28 *Let $S_1,\ldots,S_K$ be K societies. And let $S=\bigcup_{k=1}^{K} S_k$. Then*

$$r\,(S)=\sum_{k=1}^{K}\frac{|S_k|}{|S|}r\,(S_k)+r\left(\bigcup_{k=1}^{K}\left\langle\left(n(S_k),\frac{|S_k|}{n(S_k)}\right)\right\rangle\right).$$

Proof. Left to the reader. ■

Proposition 29 *For all $\alpha \in (0,1]$, $r(S_{1-\alpha})=-\log_2 \alpha$.*

Proof. Let $h:[0,1)\to\mathbb{R}$ be defined by $h(\alpha)=r(S_{1-\alpha})$. By definition of r,

$$h(\alpha)\geq 0 \quad \text{for all } \alpha\in(0,1]. \tag{9}$$

Also,

$$h(1/2)=r(1/2)=1. \tag{10}$$

We will now show that

$$h(pq)=h(p)+h(q) \quad \text{for all } p,q\in(0,1]. \tag{11}$$

To see this, note that

$$\begin{aligned} S_{1-pq} &= \left\langle(1-pq,0)\,,\left(pq,\tfrac{1}{pq}\right)\right\rangle \\ &\sim \left\langle(1-q,0)\,,q\,(1-p,0)\,,\left(pq,\tfrac{1}{pq}\right)\right\rangle \qquad \text{by SCDP} \\ &= \left\langle q\,(1-p,0)\,,\left(pq,\tfrac{1}{pq}\right)\right\rangle \cup \langle(1-q,0)\rangle\,. \end{aligned}$$

Therefore, by Proposition 26,

$$\begin{aligned} r(S_{1-pq}) &= r\left(\left\langle q\,(1-p,0)\,,\left(pq,\tfrac{1}{pq}\right)\right\rangle\right)+r\left(\left\langle\left(q,\tfrac{1}{q}\right),(1-q,0)\right\rangle\right) \\ &= r\left(\left\langle(1-p,0)\,,\left(p,\tfrac{1}{p}\right)\right\rangle\right)+r\left(\left\langle\left(q,\tfrac{1}{q}\right),(1-q,0)\right\rangle\right) \qquad \text{by HOM and RI} \\ &= r\left(S_{1-p}\right)+r(S_{1-q}), \end{aligned}$$

which shows that (11) holds. It is known that the only function on $(0,1]$ that satisfies (9-11) is $-\log_2$.[1] ■

Proposition 30 *The index r is the Theil index.*

Proof. Let $S=\langle(n_1,y_1)\,,\ldots,(n_K,y_K)\rangle\in\mathcal{S}$ be a society. We need to show that $r(S)=T(S)$. If $K=1$, the result is obvious. So assume $K\geq 2$. By RI we can assume without loss of generality that $n\,(S)=1$. Similarly, by homogeneity we can assume without loss of generality that $\sum y_k=1$. Therefore $|S|^2<|S|=\sum n_k y_k<1$. Also, $y_k|S|<1$ for $k=1,\ldots K$. Let $S^0=S$, and define recursively,

$$S^K=\bigcup_{k=1}^{K}\left\langle(n_k(1-y_k|S|)),0)\,,\left(n_k y_k|S|,\frac{1}{|S|}\right)\right\rangle$$

That is, S^K is the result of replacing brackets $(n_k,y_k)\,,k=1,\ldots,K$, in S by $\left\langle(n_k(1-y_k|S|)),0)\,,\left(n_k y_k|S|,\frac{1}{|S|}\right)\right\rangle$. Therefore, by Corollary 28,

$$r(S^K)=r(S)+\sum_{k=1}^{K}\frac{n_k y_k}{|S|}r\left(\left\langle(n_k(1-y_k|S|)),0)\,,\left(n_k y_k|S|,\frac{1}{|S|}\right)\right\rangle\right)$$

and

$$r(S)=r(S^K)-\sum_{k=1}^{K}\frac{n_k y_k}{|S|}r\left(\left\langle(n_k(1-y_k|S|),0)\,,\left(n_k y_k|S|,\frac{1}{|S|}\right)\right\rangle\right).$$

Note that by RI, and homogeneity,

$$\begin{aligned} \left\langle(n_k(1-y_k|S|),0)\,,\left(n_k y_k|S|,\frac{1}{|S|}\right)\right\rangle &\sim \left\langle(1-y_k|S|,0)\,,\left(y_k|S|,\frac{1}{|S|}\right)\right\rangle \\ &\sim \left\langle(1-y_k|S|),0)\,,\left(y_k|S|,\frac{1}{y_k|S|}\right)\right\rangle=S_{1-y_k|S|}. \end{aligned}$$

[1] See Theorem 0.2.5 in Aczél and Daróczy (1975).

Also,

$$
\begin{aligned}
S^K &= \cup_{k=1}^K \left\langle (n_k(1 - y_k|S|)), 0) \, , \left(n_k y_k |S|, \frac{1}{|S|} \right) \right\rangle \\
&\sim \left\langle \left(\sum_{k=1}^K n_k(1 - y_k|S|), 0 \right) , \left(\sum_{k=1}^K n_k y_k |S|, \frac{1}{|S|} \right) \right\rangle \\
&\sim \left\langle \left(\sum_{k=1}^K n_k - \sum_{k=1}^K n_k y_k |S|), 0 \right) , \left(\sum_{k=1}^K n_k y_k |S|, \frac{1}{|S|} \right) \right\rangle \\
&\sim \left\langle \left(1 - |S|^2), 0\right) , \left(|S|^2, \frac{1}{|S|} \right) \right\rangle = S_{1-|S|^2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
r(S) &= r(S_{1-|S|^2}) - \sum_{k=1}^K \frac{n_k y_k}{|S|} r\left(S_{1-y_k|S|}\right) \\
&= \sum_{k=1}^K \frac{n_k y_k}{|S|} \left(r(S_{1-|S|^2}) - r\left(S_{1-y_k|S|}\right)\right) \\
&= \sum_{k=1}^K \frac{n_k y_k}{|S|} \left(\log_2 \frac{1}{|S|^2} - \log_2 y_k |S| \right) \\
&= \sum_{k=1}^K \frac{n_k y_k}{|S|} \left(\log_2 \frac{y_k}{|S|} \right) \\
&= T(S).
\end{aligned}
$$

■

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