Tacit Coordination via Asynchronous Play *

Yevgeny Tsodikovich†

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Abstract

We study infinitely repeated games in which players are limited to subsets of their action space in each stage – a generalization of asynchronous games (where these subsets are singletons). We show that such rigidity in the actions facilitates the creation and sustainment of coordination and identify the stage games that are prone to asynchronous behaviour. Consequently, publicly announcing prices, which is considered legal and even encouraged in many countries, can lead to tacit collusion and non-competitive results. Moreover, we indicate which of the players should be the asynchronous ones and identify a wide family of games in which tacit collusion via asynchronous play will arise. We use the worst case rational payoff, *the effective minimax*, to evaluate the collusive result and compare the outcome of different durations of inactivity.

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†School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel. E-mail: yevgenyt@post.tau.ac.il.

1 Introduction

In many economic situations players can cooperate. For example, in oligopolistic markets firms may hinder competition and increase their revenue close to the monopolistic level by coordinating production capacities or prices. The simplest method to achieve such agreement is by explicit communication (cheap talk), which is illegal under antitrust laws, such as The Sherman Act (USA) and Article 101 (EU). Instead, firms can tacitly collude and communicate through actions or public signals (e.g., commercials). Typically, the collusive outcome is the same and the two types of collusion are indistinguishable in real-life and in mathematical models. Nonetheless, while explicit collusion is illegal, the legality (as well as the definition) of tacit collusion varies: in some countries collusion is forbidden altogether while in others the formation of a cartel is prohibited but not the collusive behaviour [Harrington, 2012].

Commonly, collusion consists of two stages: initiation and implementation [Green et al., 2014]. We focus on the implementation stage as the initiation is done before the game and requires the definition of a meta-game that would include signalling, negotiation mechanism and so forth. One of the main methods to tacitly collude is to play asynchronously by replaying the same action for several stages. This allows other players to coordinate on the repeated action to achieve a cooperative result without communication, as in the price leadership mechanism [MacLeod, 1985] and other economic examples (see Maskin and Tirole [1988], Lagunoff and Matsui [1997], Libich and Stehlík [2010, 2011] to name a few). Although this method lacks a commitment power, it allows the players to signal collusive intent, as done in the "initial capital investment" mechanism [Green et al., 2014]. If the collusion is not achieved, the players can always revert to non-collusive actions and outcomes.

The purpose of this paper is to expand asynchronous play to situations where the actions are not fixed but still cannot be freely altered. We achieve this goal by introducing a new class of games, *Sub-Actions Repeated* (SAR) *Games*, which generalize standard repeated games. In our model, the players are limited to play a certain *subset* of the mixed actions set in each stage and not necessarily a singleton containing the action from the previous stage. We show how tacit collusion can be achieved in this relaxed version of asynchronous games and that the flexibility of the players relative to standard asynchronous games yields a better collusive payoff.

SAR games arise is many real-life scenarios. One example is price rigidity that is caused by publishing the price in a commercial or a catalogue. For some time after the price was published, the vendor cannot raise it because it is illegal (false advertisement) or costly in terms of reputation. Here, the set of available prices includes only prices that are lower than the price in the commercial. Another example is portfolio management. In some cases, it is cumbersome and costly prohibited to change the entire portfolio in one instance, while small deviations from a certain position are feasible. Thus, one can model portfolio management as a SAR game where each new diverse action must be close enough to the previous one, according to some metric.

The solution concept we use to evaluate different SAR games is the worst-case rational payoff, namely the *effective minimax value*, a version of which was first presented by Wen [2002], Takahashi and Wen [2003] and Yoon [2004]. For each player the effective minimax is defined as the lowest possible payoff in the feasible and individually rational set, i.e. the worst-case payoff as long as the other players receive more than their (standard) minimax value. The effective minimax value is a better lower bound on equilibrium payoffs than the standard minimax value, as it is tight.We show that playing asynchronously can have opposite effects on these values: the standard minimax value of the asynchronous players is lower while their effective minimax value might rise, which improves their situation. We conclude that whenever the effective minimax value in a SAR game is larger than in the simultaneous-move game for all the players, they will collude on the SAR game by somehow limiting their ability to revise their actions.

Our contribution is twofold. First we introduce the notion of colluding on an asynchronous game instead of on a particular strategy profile. This is equivalent to colluding on a *set* of outcomes instead of one particular outcome in the simultaneous-move game. This idea significantly simplifies the ability to create and maintain collusion as small deviations are not regarded as defecations and would not trigger penalizing reactions, which extends the "punishments proportional to defections" approach presented by Kalai and Stanford [1985].

Second, we show how the shape of the feasible set in the stage game can facilitate cooperation in the asynchronous game. This allows us to characterize the games in which collusion via asynchronous games is more plausible. In these games, the structure of the game itself limits the ability of the players to deviate from the collusive set, which results in simpler collusive strategies. We thereby expand the notion of players with equivalent utilities (see Yoon [2001, 2004]) to situation where the equivalence is partial.

Moreover, our model is applicable to the asymmetric case, in which collusion is harder to achieve and sustain [Ivaldi et al., 2003], and resolves the issue of choosing the leader in price leadership mechanisms. We show that players whose effective minimax value is strictly larger than their standard minimax value, i.e. punishing them lowers the payoff of other players below their worst-case payoff, should play asynchronously to form a collusive result.

It follows that publicly announcing prices, which is considered legal and even encouraged in many countries, can lead to tacit collusion and non-competitive results, and we predict which of the firms should announce its prices.

1.1 Literature Review

While the definition of explicit collusion is straightforward, the definition of tacit collusion varies between economists and legislators in different countries. For example, Green and Porter [1984] allow communication between the players before the game starts, in the initiation stage, and consider tacit collusion as coordination that is achieved without communication during the implementation stage of the agreed agreement. Alternatively, Green et al. [2014] defines tacit collusion as a long-run mutual interdependence among actors that generates an excessive outcome relative to the myopic interaction without direct communication. In this paper we follow the idea of the last definition and define collusion as a situation in which an asynchronous game yields a higher worst-case payoff than the simultaneous-move game, encouraging them to impose self-limitation on their ability to revise the actions.

Rees [1993] identifies four steps in the process of forming a stable collusion in oligopolistic market. First, the set of all possible agreements should be identified. Second, one of them should be chosen as the agreement to follow. Third, the agreement should be carried out by all the firms. Fourth, credible and effective punishments should be carried out against firms that deviate from the agreement, usually in the form of limited-time price wars.

Each of the above steps is challenging in the absence of direct communication and information exchange. For example, in the absence of communication and information about the profits of other firms, it is difficult to identify the set of all possible collusion opportunities and to agree upon a particular one. In the third stage, the lack of information has a direct impact on revenue, as firms might misinterpret random demand shocks as deviations and carry out unnecessary punishments. In fact, Garrod and Olczak [2017] show that under imperfect monitoring and stochastic price shocks, price wars are part of the equilibrium path. It follows that sophisticated collusion schemes that exists according to the Folk Theorem are too complicated to maintain, so simple strategies should be chosen to prevent the breakdown of the collusion. Our model facilitates collusion by limiting the strategy spaces of the players and by removing the requirement to play an equilibrium – the worst-case payoff of the players will be larger in the SAR game regardless of the exact strategies used. This can come on expense of maximizing the payoff but it is a well known result that players select a non-payoff maximizing equlibrium to avoid deviations and maintain collusion [Green and Porter, 1984].

One common method to collude is *price leadership* or *conscious parallelism* [MacLeod, 1985] in which one firm sets prices first and the rest follow. By following this practice, the firms can react to possible demand changes in the market and still maintain a collusive result. Nevertheless, some coordination is needed to determine which firm is the leader; when the price should be set; and what should be the new price. Our model resolves this issue and determines which of the firms should be asynchronous and by doing so, take the leadership and choose its actions "first".

Another problem, presented by Carlton and Gertner [1989], is that any price change bears a risk of starting a price war. To deal with this issue, different pooling schemes [Ivaldi et al., 2003, Hanazono and Yang, 2007, Garrod, 2012] suggest "rigid prices" for long periods of time to reduce the frequency of interaction. On the downside, prices are set non-optimally since they fail to respond to the demand level [Athey et al., 2004]. Our model lets the firms make small adaptations in their actions while maintaining the collusive result.

In this paper we study the effective minimax value which is the worst-case payoff a player receives as long as other players are rational. This extends equivalent utilities to situations where the utility functions are not equal upto an affine transformation. Typically, non-equivalent utilities (NEU condition) is assumed in the Folk Theorem with sub-game perfect equilibria in simultaneous-move games [Friedman, 1971, Abreu et al., 1994] and in asynchronous games [Yoon, 2001], since players with equivalent utilities will refrain from punishing each other – any punishment will lower both the payoff of the punishee and the punisher without the ability to compensate the punisher in the future. To generalize the Folk Theorem in the presence of equivalent utilities, Wen [2002] and Yoon [2004] defined the effective minimax value as the payoff of a player when he and all his equivalent maximize his utility while the rest of the players minimize it, and showed that this is the lowest possible equilibrium payoff. We generalize their work, and define the effective minimax value when the NEU condition holds but there exists some equivalence between the utility functions of the players which prevents them to minimize each other "too much". Thus, our effective minimax is a tighter lower bound on the equilibrium payoff of each player.

To exemplify our main result we consider a two-player game with complete information where one player plays in each stage and the other plays asynchronously. A deterministic version of this model was studied by Wen [2002] (see also Takahashi and Wen [2003]) and largely inspired this paper. He showed how the effective minimax value depends not only on the possible payoffs but also on the order of play and we extend his result to stochastic order of play. We show that if the asynchronous game is better in terms of effective minimax for both players for a particular intra-revision timing, it is better for *all* intra-revision timings, which makes collusion very easy to achieve and maintain.

We also consider a slightly different model, in which the players do not know the timings of the revisions of the asynchronous players. Similarly to Spiegler [2015], we show that when the revision probability is large enough, the value of the such zero-sum games remains the same, despite the lack of agility and the inherit disadvantage of the asynchronous player. In the second game we show that this is not true and that for some intra-revision timings the effective minimax in the asynchronous game remains the same as in the simultaneous-move game, so collusion is prevented.

The rest of the paper is organized as follows. Section 2 presents the model, our extension of repeated games and formally defines our figure of merit – the effective minimax. In section 3 we discuss the main result and present the conditions for collusion. The main result is applied to two versions of two-player games with different information structures in Section 4. Conclusions and extensions are presented in Section 5. To improve readability, all the proofs were relegated to the appendix.

2 The model

2.1 The stage game

Let $G = (I, (A_i)_{i=1}^n, (u_i)_{i=1}^n)$ denote a strategic-form game where $I = \{1, \ldots, n\}$ is the set of players, A_i is the finite set of pure actions for player i and $u_i : \times_{i=1}^n A_i \to \mathbb{R}$ is the stage game payoff function for player i. A mixed action α_i of player i is a distribution over A_i, i.e. an element of $\Delta(A_i)$, the set of all mixed actions. The expected stage payoff of player i, given the action profile $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Delta(A_1) \times \ldots \times \Delta(A_n)$ is

$$u_i(\alpha) = \sum_{a \in \times_{i=1}^n A_i} \alpha(a) u_i(a). \tag{1}$$

We write $-i$ to denote all players except player i, and define the (*standard*) *minimax value* of the stage game for player i by

$$v_i = \min_{\alpha_{-i} \in \Delta(A_{-i})} \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \alpha_{-i}). \tag{2}$$

The standard minimax value is the lower bound on the payoff that player i can guarantee when all players try to minimize his payoff in a single stage. Finally, let F be the convex hull of the set of feasible payoff vectors and F_{IR} the set of individually rational feasible payoffs, i.e. $F_{IR} = \{(x_1, \ldots, x_n) \in F | x_i \geqslant v_i\}$.

2.2 Sub-Actions Repeated (SAR) Games

To model asynchronous play we introduce a generalization of the standard repeated game model, namely the sub-actions repeated (SAR) game. In this game, the set of available actions in each stage for each player is a subset of the set of mixed actions. The SAR game unfolds as follows: The stage game is repeated infinitely many times. In stage $t = 0$, each player chooses his respective action from $\Delta(A_i)$. In period $t \geqslant 1$, a non-empty convex and closed subset $A_i(t) \subseteq \Delta(A_i)$ is randomly chosen for each player who is confined to choose a mixed action from the set $A_i(t)$.

To keep the definition applicable to many models, we do not impose here any restrictions on the stochastic processes $A_i(t)$ or their co-dependence. A model with specific restrictions will be presented in the examples bellow. We assume that each player knows at time t his own $A_i(t)$ and has a belief over $A_{j \neq i}(t)$, given the history of his own $A_i(t)$ upto stage t, and possibly the history of other signals. Players observe past actions, but might not know the realizations of past sub-action sets of other players. A SAR game with a common discount factor δ is denoted by $\Gamma = (G, \{A_i(t)\}_{i \in I}^{t \in \mathbb{N}}, \delta)$ and for simplicity will be referred to as *the repeated game*. This formulation is a generalization of many existing models:

1. Simultaneous-move game: For each player and every t, $A_i(t) = \Delta(A_i)$. This is the standard repeated game, denoted by Γ_0.

2. One-shot game: For each player, $A_i(t)$ is a singleton containing only the pure action played at stage $t = 0$. We will refer to this game as Γ_1.

3. Alternating-moves game [Maskin and Tirole, 1988, Lagunoff and Matsui, 1997]: A two-player game in which $A_1(2t-1) = \Delta(A_1)$, $A_2(2t) = \Delta(A_2)$ for $t \geqslant 1$ and in any other stage $A_i(t)$ is a singleton containing the pure action played in the previous stage.

4. Asynchronous-moves game with mixed actions [Yoon, 2001, 2004]: At each stage, a random set of players $I_t \subseteq I$ is chosen and only they can revise their action: $A_{i \in I_t}(t) = \Delta(A_i)$. The rest of the players play the *mixed* action they played in the previous stage.

2.3 The standard and effective minimax values

Denote by Σ_i the set of all possible (behavioural) strategies of player i in the repeated game. Each strategy specifies the action $\alpha_i^t \in A_i(t)$ of player i after every t-length history, which includes both the actions of the players and the realizations of $A_i(t)$ which he observed.

		Player 2	
		B	S
Player 1	B	$(2,1)$	$(0,0)$
	S	$(0,0)$	$(1,2)$
	P	$(0,-1)$	$(0,-1)$

Table 1: A modified "Battle of the Sexes" where the effective minimax of player 2 is strictly greater than his standard minimax.

Given the strategy profile $\sigma \in \bigtimes_{i=1}^{n} \Sigma_i$, the expected δ-discounted payoff of player i is

$$u_i(\sigma) = (1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}(u_i(\alpha_1^t, ..., \alpha_n^t)). \tag{3}$$

For each player the standard minimax value of the repeated game is defined according to

$$v_i(\Gamma) = \min_{\sigma_{-i}\in\Sigma_{-i}} \max_{\sigma_i\in\Sigma_i} u_i(\sigma_i, \sigma_{-i}). \tag{4}$$

The standard minimax is the worst case payoff that a player can guarantee, in case that other players act as adversaries. However, in the presence of possible partial equivalence of the utility functions such behaviour can harm them and lower their payoff as well, sometimes even below *their* minimax value. Thus, we define the *effective minimax value* as the minimum payoff that a player receives when the strategy profile produces an individually rational payoff for all the players:

$$v_i^e(\Gamma) = \min_{\sigma\in\Sigma}\{u_i(\sigma_i, \sigma_{-i}) | u_j(\sigma_i, \sigma_{-i}) \geqslant v_j(\Gamma) \text{ for all } j \in I\}. \tag{5}$$

When $v_i^e(\Gamma) > v_i(\Gamma)$, player i cannot guarantee the effective minimax by himself and must rely on the rationality of other players to obtain this value.

Example 1. *Effective minimax for a simultaneous-move game*

Consider a modified version of the "Battle of the Sexes" game which is shown in Table 1. In this game, the row player has an additional action, P, which is dominated by the other actions and does not affect his standard minimax value: $v_1(\Gamma_0) = \frac{2}{3}$. This action can serve as a minimizing strategy against the column player, setting $v_2(\Gamma_0) = -1$. Playing P too often is not individually rational for the row player as it will lower his payoff below $\frac{2}{3}$. Thus, whenever the row player receives at least $\frac{2}{3}$, the outcome of the game lies inside the grey shaded area in Figure 1, which results in a payoff of at least $v_2^e(\Gamma_0) = -\frac{1}{3}$ for the column

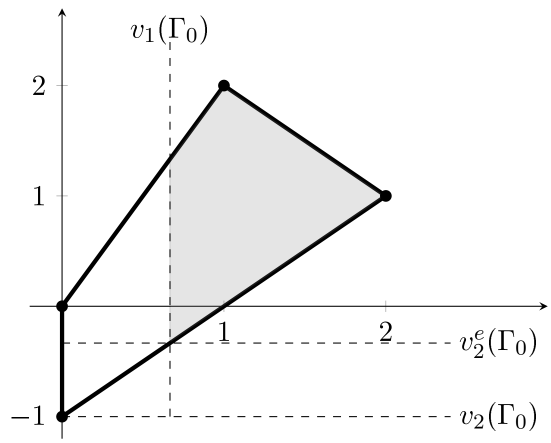

Figure 1: The set of feasible payoffs and the effective minimax of player 2 in the modified "Battle of the Sexes" (Table 1).

player. The column player cannot guarantee a payoff higher than -1 by himself, but thanks to the rationality of the row player and the partial equivalence of the payoff functions, he receives at least $-\frac{1}{3}$. $\triangle$

3 The Main Result

A tacit collusion on a SAR game takes place when all players prefer the SAR game over the simultaneous-move game in terms of the effective minimax. Tacitly colluding on a particular SAR game is challenging, as information needs to be shared between the agents regarding the sets $A_i(t)$. Instead, we focus on studying situations where a wide class of SAR games is better than a simultaneous-move game rather than a particular one. In this case, it is enough for agents to announce that they play asynchronously without providing too many details, to insure that the simultaneous-move game is no longer played and their effective minimax value is increased. Such information can be conveyed in practice using commercials or other public signals. This increase does not necessarily lead to increase in the equilibrium payoff, but since the effective minimax value is the lower bound on equilibrium payoffs, it leads to an increase of the worst-case equilibrium payoff.

In order to o demonstrate the main ideas, we start by considering a general two-player game. Clearly, a collusion will take place when the utility functions of both players are equivalent, i.e. one is an affine transformation of the other (Definition 1 in Yoon [2001]). Such strict equivalence is not required, and also partial equivalence of the utility functions

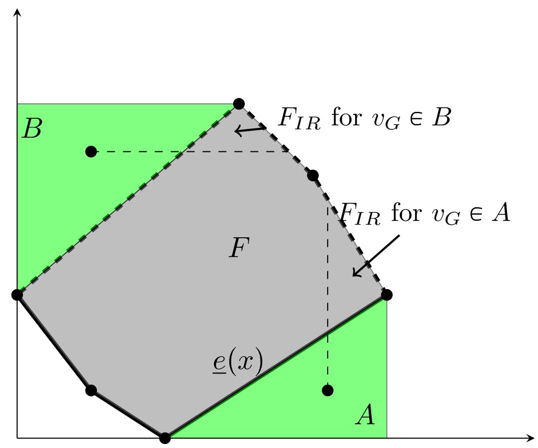

Figure 2: All possible locations of the standard minimax point $v_G = (v_1, v_2)$ in the payoff space. The location of v_G determines the feasible individually rational set and the effective minimax value of the players.

might create collusion. This partial equivalence can be studied by examining the feasible set F, and the location of the standard minimax point $v_G = (v_1, v_2)$ relative to it. Generally speaking, this point can lie in one of three areas (A,B and F), as shown in Figure 2.

When v_G is in the interior of F, according to the Folk Theorem, there are equilibria where the payoff for both players is arbitrary close to their respective standard minimax values. Different SAR games will exhibit different behaviours regarding the standard and effective minimax values. Our main result cannot be applied here and we leave this analysis to future studies.

When the minimax point is below the feasible set in region A,[1] any payoff that is individually rational for player 1 will result in a payoff for player 2 which is strictly larger than his standard minimax. Therefore, the effective minimax of player 1 is equal to his standard minimax ($v_1^e = v_1$) while for player 2 the former is larger, i.e., $v_2^e > v_2$. Since the lower envelope of the feasible set, $\underline{e}(x) = \min\{y|(x, y) \in F\}$, for $x \geqslant v_1$ is an increasing function, any SAR game in which the standard minimax value of player 1 is higher than v_1 will result in a higher than v_2^e worst-case rational payoff for player 2 – an improvement for both players. This is the type of partial equivalence between utility functions that we are studying in this paper – player 2 will prefer self-imposed limitations, such as asynchronous play, to prevent himself from lowering the payoff of player 1, increase the minimax of player 1 and in return, increase his own effective minimax value.

[1] Region B is equivalent to A with the roles of the players are reversed.

When generalizing this idea to the n-player game, we impose only one requirement on the SAR game – the standard minimax will rise in the SAR game for all the players whose effective minimax is equal to the standard minimax in the stage game. When there are players that satisfy $v_i^e(\Gamma_0) > v_i(\Gamma_0)$, the minimax point has to be in the region A and an opportunity to collude arises. These players should play asynchronously, increase the minimax value of the other players and consequently, increase their own effective minimax value.

Theorem 1. *Fix* $\Gamma = \left(G, \{A_i(t)\}_{i\in I}^{t\in\mathbb{N}}, \delta\right)$ *to be a SAR game and let* Γ_0 *be the corresponding simultaneous-move game. Denote the set of players whose effective minimax in* Γ_0 *is equal to the standard minimax value in* Γ_0 *by* I_1*.*

If $v_i(\Gamma) \geqslant v_i(\Gamma_0)$ *for every* $i \in I_1$ *then* $v_i^e(\Gamma) \geqslant v_i^e(\Gamma_0)$ *for all the players.*

Theorem 1 divides the players into two groups. The first group, I_1, is the group of players whose effective minimax value is equal to the standard minimax value in the simultaneous-move game. When these players play individually rational strategies, they help the players of the second group to receive at least their effective minimax payoff. In any SAR game where the first group receives even higher worst-case payoffs, the effective minimax value of the second group will surely rise as well. In a two-player case, the theorem can be applied to simultaneous-move game the effective minimax of one of the players is strictly greater than its standard minimax. If the latter occurs, any modification of the repeated game that strictly increases the standard minimax of the other player will also increase the effective minimax of the first player, making this modified game favourable for both players.

Verifying the conditions of Thoerem 1 is not trivial in the general case as the effective minimax value of all the players might be complicated to compute. Nonetheless, the theorem has two major applications. First, it transforms the problem from a question regarding SAR games to a simpler question regarding the stage game and the geometry of its feasible set. Second, when limiting the scope to only particular types of SAR games, applying the Theorem leads to case specific conditions that are more applicable. To demonstrate this, we study two models of two-player games where only one of the players is asynchronous. When applying Theorem 1, we get a condition on the inter-revision timings for which the asynchronous game is more favourable than the simultaneous-move game.

4 Two-Player Games with One-Sided Asynchronicity

We study a two-player game in which player 1 can revise her actions in every stage while player 2 receives revision opportunities according to some exogenous random variable. Whenever player 2 cannot revise his action, he is forced to repeat the same pure action from the previous stage. We consider two possibilities for the information the players have. In the first model, player 1 knows the schedule of revisions while in the second model she knows only the distribution of the timing of the next revision. This knowledge gap between the models affects the effective minimax value of both players and their ability to collude.

Formally, let G be a two-player stage game, $X \in \Delta(\mathbb{N})$ a random variable with finite support,[2] and $x_1, x_2, ...$ iid realizations of X. A repeated game with one-sided asynchronous play is a SAR game where player 1 can revise her action in any stage, $A_1(t) = \Delta(A_1)$, and player 2 can revise his action only in stages $t = x_1, x_1 + x_2, \ldots$. In the rest of the stages, $A_2(t) = \{a(t-1)\}$, where $a(t-1)$ is the pure action player by player 2 in stage $t-1$.

We assume that both players know the distribution of X and consider two different models with respect to the information about its realizations. In the complete information model, denoted by Γ_X, the realizations of X are known to both players in the outset of the game. In the unknown realizations model, denoted by $\tilde{\Gamma}_X$, player 1 does not know the realizations of X, so she does not know if a revision will be possible in each stage or not. Regardless, based on T, the number of stages since the previous revision, she can compute the conditional probability of a revision in the current stage, $q(T) = \Pr(X = T | X \geqslant T)$, and base her strategy on it.

Our models are an extension of an example shown by Wen [2002] in the exposition of his paper. Wen discussed the regular "Battle of the Sexes" game (Table 1 without the action "P") where player 1 can revise her actions in every stage while player 2 can revise his action only at the stages $0, T, 2T, \ldots$ for some constant known $1 < T$. In these settings, player 2 can be lowered to an undiscounted average payoff of $v_2 = \frac{2}{3T}$ per stage, while the minimax value of player 1 rises to $v_1 = 1$ regardless of T. Nevertheless, due to the structure of the feasibility set, the equilibrium payoff of player 2 cannot be lower than $v_2^e = \frac{1}{2}$.

4.1 One-Sided Asynchronous Games with Complete Information

We first develop a formula to compute the standard minimax value of each player in this model by analysing the corresponding zero-sum game. This formula is essential as one of the conditions of Theorem 1 is that the standard minimax value of some of the players is larger in the SAR game relative to the simultaneous-move game. Second, we apply Theorem 1 and find the class of stage games for which the effective minimax value of both players is larger in the asynchronous game relative to the simultaneously repeated game. Games that

[2] This is equivalent to the Finite Period of Inaction condition from Yoon [2001] and Wen [2002]. This assumption is not needed for our results but necessary for the Folk Theorem to hold [Dutta, 1995].

fulfil this condition are prone to tacit collusion via asynchronous play. The collusive result is easily maintained as player 2 announces the revision dates and deviation from these dates are easily detectable.

4.1.1 The Value of the Zero-Sum Repeated Game

Suppose that the underlying stage game is a zero-sum game. Naturally the value of the SAR game is strictly larger than the value of the simultaneous-move game (as player 2 has less strategies available). Whenever player 2 has no revision opportunity, player 1 will best respond to the anticipated pure action. Thus, in stages where there is a revision opportunity, player 2 needs to choose an action that will maximize his payoff in this stage while minimizing his losses in the upcoming stages of inactivity (except for trivial stage games where player 2 can achieve the value in pure actions). The result is the algorithms to compute the optimal strategies and the value of the game, as shown in Proposition 1.

Proposition 1. *The game $\Gamma_X = (G, X, \delta)$ where G is a zero-sum stage game has the value*

$$V_{\Gamma_X} = \frac{(1-\delta)\mathbb{E}(V_X)}{1-\mathbb{E}(\delta^X)}, \tag{6}$$

where V_n is the value of the one-shot zero-sum auxiliary game with the payoff function

$$u_n(a_1, a_2) = u(a_1, a_2) + \frac{\delta - \delta^n}{1-\delta} u\Big(b(a_2), a_2\Big), \tag{7}$$

$b(\cdot) : A_2 \to A_1$ *is the best response function and* $\mathbb{E}(V_X) = \sum_n V_n \cdot \Pr(X = n)$.

The auxiliary game represents the situation a player faces when a revision opportunity is given – player 2 chooses an action for his entire period of inactivity and player 1 needs only to choose an action for this stage; afterwards, the pure action is fixed and she can best respond to it. W.l.o.g. player 1 is the maximizer, her stage payoffs in the auxiliary game are larger than the payoffs in the original game, with strict inequality unless player 2 has a pure maximining action. Otherwise, the minimax value of the repeated game strictly increases for player 1 and strictly decreases for player 2 relative to the simultaneous-move games.

For large realizations of X and sufficiently patient players, the first term in Eq. (7) is insignificant relative to the second term, which means that the play is in-effect sequential: player 2 chooses first and player 1 responds. Therefore, when X is a "large" random variable (in the sense that $\mathbb{E}(\delta^X)$ is very small) the value of the game reaches its limit – the minimax value of the stage game in *pure* actions.

		Player 2	
		B	S
Player 1	B	2	0
	S	0	1
	P	0	0

	Player 2	
	B	S
B	$2(1+\delta+\delta^2)$	$\delta+\delta^2$
S	$2(\delta+\delta^2)$	$1+\delta+\delta^2$
P	$2(\delta+\delta^2)$	$\delta+\delta^2$

Table 2: The payoffs of the row player in the auxiliary one-shot zero-sum game derived from "Battle of the Sexes" presented in Table 1 for $n=1$ (left) and $n=3$ (right).

Example 1 (Continued)**.** *The minimax values in "Battle of the Sexes" when the realizations of X are common knowledge.*

Consider the non-zero-sum game presented in Example 1 and assume player 2 revises his actions according to the random variable

$$X = \begin{cases} 1 & \text{w.p. } \frac{1}{2}, \\ 3 & \text{w.p. } \frac{1}{2}. \end{cases} \tag{8}$$

Again, the row player can play "P" in every stage, setting $v_2(X) = -1$. To compute the standard minimax of the row player, we consider only her payoffs and, according to Proposition 1, compute the value of two one-shot auxiliary games presented in Table 2. For $n=1$ this is the regular one-shot game with $V_1 = \frac{2}{3}$. For $n=3$ and for large enough discount factor ($\delta > \frac{\sqrt{5}-1}{2} \approx 0.61$) the optimal strategies are $[S,S]$ and the value is $V_3 = 1+\delta+\delta^2$. Plugging these numbers into Eq. (6) leads to the standard minimax of the row player in the repeated zero-sum game:

$$v_1(X) = (1-\delta)\frac{\frac{5}{3}+\delta+\delta^2}{2-\delta-\delta^3}. \tag{9}$$

For patient enough players ($\delta \to 1$) the standard minimax value is $v_1(X) = \frac{11}{12}$ which is also the result in the undiscounted case: when $X=1$ the payoff is $\frac{2}{3}$ during one stage and when $X=3$ the payoff per stage is 1 and is given during 3 stages. The total payoff in these 4 stages is $3+\frac{2}{3}$ and the expected payoff per stage is $\frac{3+\frac{2}{3}}{4} = \frac{11}{12}$. △

4.1.2 Effective Minimax Value of the Non-Zero-Sum Game

As shown in the previous section, the standard minimax value of player 1 can be directly calculated and, expect for trivial games, is larger in the asynchronous game relative to the simultaneous-move game. Therefore, Theorem 1 can be applied only when the effective minimax value of player 2 in the stage game is larger than his standard minimax value.

This happens only if the minimax point in the stage game lies in the region "A", shown in Figure 2. The combination of these conditions defines the class of games where one-sided asynchronous play is better than the simultaneous-move repeated game for both players in terms of effective minimax.

Corollary 1. *If G is a two-player stage game such that*

1. *Player 2 has no pure minimaxing strategy against player 1; and*
2. $v_2 < \underline{e}(v_1) = \min\{y|(v_1, y) \in F\}$ *(i.e., $(v_1, v_2) \in A$),*

then for every random variable $X \in \Delta(\mathbb{N})$ and every discount factor $\delta \in (0, 1)$, the effective minimax value for both players in Γ_X is larger than the effective minimax value in the simultaneous-move game.

The proof is omitted as it is a direct result of Theorem 1, Proposition 1 and the discussion above. Instead, we show the implication of this corollary on the "Battle of the Sexes" example shown before.

Example 1 (Continued). *The effective minimax values in "Battle of the Sexes".*

Both conditions of Corollary 1 are satisfied for the game shown in Table 1. Thus, for any X, the effective minimax of both players in Γ_X is larger than the effective minimax value in Γ_0. For example, when considering X presented in Eq. (8) we showed that for patient enough players $v_1^e(\Gamma_X) = \frac{11}{12}$, which yields $v_2^e(\Gamma_X) = -\frac{1}{12} > v_2^e(\Gamma_0)$. △

Note that this result does not hold in the symmetric version of the "Battle of the Sexes", where the action P is unavailable since the minimax point lies within F. This statement is true for all symmetric games – if the game is symmetric, the standard minimax value for all the players is equal and v_G must lie inside F. Thus, the effective minimax is also the standard minimax and the conditions of Corollary 1 (and Theorem 1 in the general case) never hold. Our main contribution is studying tacit collusion in the asymmetric case and not the more discussed symmetric case.

4.2 One-Sided Asynchronous Games with Partial Information

We now consider the second model, where the realizations of X are unknown and only the distribution of X is common knowledge. We assume that revision opportunities are observable after the actions were chosen and start by studying the value of zero-sum games.In this case, unlike Section 4.1.1, there is no simple formula for the new value of the game.

Moreover, it is not clear that the value changes, even if the optimal strategy of player 2 in the zero-sum stage game is mixed.

Consider some stage $t \neq 0$ and suppose that the pure action a was played in the $T-1$ previous stages by player 2. The action a will be played in this stage either if player 2 has no revision (with probability $1-q(T)$) or has a revision and decides to replay it (with some probability). Thus, this stage is equivalent to a stage game where player 2 can choose any action in $\Delta(A_2)$ that satisfies $\Pr(a) \geqslant 1-q(T)$. If there exists an optimal strategy in which this condition is satisfied, player 2 can secure the value of the one-shot game for stage t. Otherwise, he must choose a mixed action from a set that does not contain any optimal action and will get a lower payoff for this stage and for the entire repeated game.

The existence of a maximin strategy that chooses a with probability larger than $1-q(T)$ for any T is not enough to secure the one-shot value. It is possible that the maximin strategy has a non-zero probability to choose an action that is never played in an optimal strategy with probability higher than $1-q(T)$ for some T. In this case, there is some positive probability that this action will be chosen and in one of the following stages player 2 will be forced to play non-optimally. To guarantee the value in each stage it is essential that every action that is chosen by the optimal strategy can be played by some optimal strategy with probability larger than $1-q(T)$ for all possible Ts. Therefore, the maximin strategies should choose this action with probability larger than

$$p_X = \sup_{T \leqslant \max(supp(X))} \big(1-q(T)\big). \tag{10}$$

The next definition formally defines the set of pure actions that can be used in this manner.

Definition 1. *A non-empty set of actions, $B_i \subseteq A_i$, is said to be a* p-min optimal set (for player i) *in the one-shot zero-sum game G if for every $a \in B_i$ there exists an optimal strategy $\alpha \in \Delta(B_i)$ that satisfies $\alpha(a) \geqslant p$.*

Following the discussion, the existence of a p_X-min optimal set for player 2 is sufficient to achieve the value of the one-shot game in the repeated game. The next proposition formalizes this discussion and proves that this is also a necessary condition. Otherwise, player 2 will receive less in the repeated game relative to the stage game.

Proposition 2. *Let G be a zero-sum two-player stage game, $X \in \Delta(\mathbb{N})$ and $\tilde{\Gamma}_X$ the corresponding repeated game where the realizations of X are unknown. The value of the repeated game, $V_{\tilde{\Gamma}_X}$ is equal to the value of the one-shot stage game, $v(\Gamma_1)$, iff player 2 has a p_X-min optimal set, where p_X is defined in Eq.* (10)*. Otherwise, $V_{\tilde{\Gamma}_X} > V_{\Gamma_1}$.*

Finally, consider a repeated game $\tilde{\Gamma}_X$ where G is a non-zero sum stage game. According to Proposition 2, the standard minimax value of player 1 can remain the same as in the one-shot game whenever player 2 has a p_X-min optimal set against player 1, despite the advantage she has. In this case Theorem 1 cannot be applied and the effective minimax value of player 2 might remain the same as well. This gap of knowledge prevents collusion from taking place and player 2 should do something to close it – either change his schedule to a different one that does not satisfy the conditions of Proposition 2 or publicly announce the dates of the next revisions (if possible). Only by revealing this information the tacit collusion will be possible and the situation will return to the one described in Section 4.1.

Example 2. *"Battle of the Sexes" with unknown realizations of X.*

We return to Example 1, this time assuming that the realizations of X are unknown. For the standard minimax value of the row player to rise, it is necessary and sufficient to show that in the zero-sum game shown in Table 2, the column player has no p_X-min action set. This is true, for example, for the X defined in Eq. (8) since $1 - q(2) = \Pr(X > 2 | X \geqslant 2) = 1$. In every stage that the column player did not receive a revision opportunity, the row player knows that $X = 3$ and the revision will take place only in the next stage. She will best respond to the anticipated pure action and the value of the game increases.

On the other hand, for $X \sim Geom(\frac{2}{3})$ and for every $T \in \mathbb{N}$, the probability to keep the previous action is at least $1 - q(T) = \frac{1}{3}$, thus $p_X = \frac{1}{3}$. The optimal strategy in the stage game is $\left[\frac{1}{3}(B), \frac{2}{3}(S)\right]$ thus $\{B, S\}$ is $\frac{1}{3}$-min action set and player 2 can obtain the value by using the following strategy: in every revision opportunity: always switch from S to B and switch from B to S with probability 0.5.Thus, neither the standard minimax value of the row player nor the effective minimax of both players change for this X. △

5 Concluding Remarks

In this paper we introduce a new solution concept, the effective minimax, and use it to compare different asynchronous games. This solution concept leads to a new definition of collusion in which the coordination is on a set of results rather than on a particular result. This allows firms to tacitly collude in a method that is easier to maintain as close inspection others' actions is no longer needed and punishing deviators rarely occurs. Moreover, this definition gives the players some flexibility if needed to change their actions, without loosing utility due to unnecessary rigidity in the prices or due to the provocation of a price war.

We showed that such collusion can take place in situations where some of the players commit to asynchronous play or to making only small adjustments in the mixed strategy. This phenomena of self imposed rigidity occurs naturally in the markets, for example, either when firms publish commercials or catalogues, or when there is not enough information available to justify a price change. Our model allows the firms to identify the ones that should act as price leaders in an asymmetric market, when executing an unspoken agreement which improves the worst-case payoff of all the firms.

Moreover, in the two-player scenario, we analysed two distinct models of information regarding the timing of price changes of the asynchronous player, and showed that it is better to publish not only that the action is fixed but also for how long. These results contribute to our understandings of the structure of the market that facilitate collusion and identify the conditions under which rigid prices emerge.

A key feature of our model is that it applies to asymmetric games, which are generally considered to be less prone to tacit collusion. We show that collusion can occur in these cases via asynchronous play, as long as there is coordination between the payoff functions of the players. This is achieved, in part, by abandoning symmetric solution concepts such as symmetric perfect public equilibrium (SPPE) in favour of the effective minimax value solution concept, which is more suitable in this case.

Our work applies to any n-player repeated game.As the framework of SAR games is general, so are the results. We leave the work of testing our model and its results in particular cases, such as Cornout competition with demand shocks to future papers, in which we shall extract more model-specific results and insights.

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A Proof of Theorems

Theorem 1. *Fix* $\Gamma = \big(G, \{A_i(t)\}_{i\in I}^{t\in\mathbb{N}}, \delta\big)$ *to be a SAR game and let* Γ_0 *be the corresponding simultaneous-move game. Denote the set of players whose effective minimax in* Γ_0 *is equal to the standard minimax value in* Γ_0 *by* I_1.

If $v_i(\Gamma) \geqslant v_i(\Gamma_0)$ *for every* $i \in I_1$ *then* $v_i^e(\Gamma) \geqslant v_i^e(\Gamma_0)$ *for all the players.*

Proof of Theorem 1. If $I_1 = I$, the result trivially holds, so assume that $I_1 \neq I$ and let j be a player for whom $v_j^e(\Gamma_0) > v_j^e(\Gamma)$. Therefore, there exists a feasible and individually rational payoff $\underline{x}$ in the game Γ so that $v_j^e(\Gamma_0) > x_j \geqslant v_j^e(\Gamma)$. In addition, by the closedness of the feasible and individually rational set of the game Γ_0, there exists a payoff $\underline{y} \in F_{IR}$ so that $y_j = v_j^e(\Gamma_0)$.

Define $\underline{z} = \epsilon\underline{y} + (1-\epsilon)\underline{x}$. Since F is convex, $\underline{z} \in F$ for any $\epsilon \in [0,1]$. For the players in I_1, the payoff $\underline{z}$ is individually rational in Γ_0 since $z_i = \epsilon y_i + (1-\epsilon)x_i \geqslant \epsilon v_i(\Gamma_0) + (1-\epsilon)v_i(\Gamma) \geqslant v_i(\Gamma_0)$. It is possible to choose ϵ very close to 1 so that the payoff will be rational also for the rest of the players, since $z_i = \epsilon y_i + (1-\epsilon)x_i \geqslant \epsilon v_i^e(\Gamma_0) + (1-\epsilon)x_i$ and $v_i^e(\Gamma_0) > v_i(\Gamma_0)$. However, $z_j = \epsilon v_j^e(\Gamma_0) + (1-\epsilon)x_j < v_j^e(\Gamma_0)$ which is a contradiction since $\underline{z}$ is an individually rational payoff in Γ_0 with a lower payoff for player j than its effective minimax value.

Therefore, in the game Γ the effective minimax of all the players in $I \backslash I_1$ is greater than in Γ_0. Trivially, $v_i^e(\Gamma) \geqslant v_i(\Gamma) \geqslant v_i(\Gamma_0) = v_i^e(\Gamma_0)$ for $i \in I_1$, so the effective minimax in Γ is greater than the effective minimax in Γ_0 for all. ■

Proposition 1. *The game* $\Gamma_X = (G, X, \delta)$ *where* G *is a zero-sum stage game has the value*

$$V_{\Gamma_X} = \frac{(1-\delta)\mathbb{E}(V_X)}{1-\mathbb{E}(\delta^X)}, \tag{6}$$

where V_n *is the value of the one-shot zero-sum auxiliary game with the payoff function*

$$u_n(a_1, a_2) = u(a_1, a_2) + \frac{\delta - \delta^n}{1-\delta} u\Big(b(a_2), a_2\Big), \tag{7}$$

$b(\cdot) : A_2 \to A_1$ *is the best response function and* $\mathbb{E}(V_X) = \sum_n V_n \cdot \Pr(X = n)$.

Proof of Proposition 1. First, note that the SAR game has a value according to standard arguments of contraction, so the rest of the proof deals with finding it. Second, note that if the realization of X at $t = 0$ is n, then starting from $t = n$ the situation is the same situation as the in $t = 0$, so the continuation payoff is $\delta^n V_{\Gamma_X}$, where V_{Γ_X} is the value of the game. In the first n stages the actions and payoffs can be found by writing explicitly the minimax of the payoff:

$$(1-\delta)\min_{\alpha_2\in\Delta(A_2)}\left(\max_{\alpha_1^0\in\Delta(A_1)} u(\alpha_1^0,\alpha_2)+\sum_{a_2\in A_2}\alpha_2(a_2)\big(\sum_{t=1}^{n-1}\max_{a_1^t\in A_1}\delta^t u(a_1^t,a_2)\big)\right), \tag{11}$$

where a_2 is the pure action that was chosen by the mixed action α_2 with probability $\alpha_2(a_2)$ at $t=0$ and a_1^t is the pure action of player 1 in stage t. For $1 \leqslant t \leqslant n-1$ the maximizing action would be the best response to a_2, $b(a_2) \in A_1$, since a_2 is known in advance in those stages. We can rewrite the payoff of the mixed actions as the expected value of the pure actions and use the fact that every $\alpha_1 \in \Delta(A_1)$ is a probability distribution over A_1 to turn the last formula into

$$(1-\delta)\min_{\alpha_2\in\Delta(A_2)}\max_{\alpha_1\in\Delta(A_1)}\sum_{a_2\in A_2}\sum_{a_1\in A_1}\alpha_2(a_2)\alpha_1(a_1)\left(u(a_1,a_2)+\frac{\delta-\delta^n}{1-\delta}u\Big(b(a_2),a_2\Big)\right). \tag{12}$$

For every $n \in \mathbb{N}$, consider the auxiliary zero-sum stage game with the same players and actions with the modified payoff function

$$u_n(a_1,a_2)=u(a_1,a_2)+\frac{\delta-\delta^n}{1-\delta}u\Big(b(a_2),a_2\Big). \tag{13}$$

This auxiliary game has a value in mixed actions, V_n, and it is exactly the minimax value from Eq. (12). Therefore, the payoff of the first n stages is $(1-\delta)V_n$. The expected δ-discounted payoff from $t=0$ until the next revision opportunity is therefore $\mathbb{E}(V_X)=\sum_n V_n\cdot\Pr(X=n)$ and the continuation payoff from this revision onward is $\mathbb{E}(\delta^X)V_{\Gamma_X}$.

To conclude, the value of the repeated zero-sum game must satisfy

$$V_{\Gamma_X}=(1-\delta)\mathbb{E}(V_X)+\mathbb{E}(\delta^X)V_{\Gamma_X}, \tag{14}$$

which is equivalent to Eq. (6) and the proof is complete. ■

Proposition 2. *Let G be a zero-sum two-player stage game, $X \in \Delta(\mathbb{N})$ and $\tilde{\Gamma}_X$ the corresponding repeated game where the realizations of X are unknown. The value of the repeated game, $V_{\tilde{\Gamma}_X}$ is equal to the value of the one-shot stage game, $v(\Gamma_1)$, iff player 2 has a p_X-min optimal set, where p_X is defined in Eq.* (10)*. Otherwise, $V_{\tilde{\Gamma}_X} > V_{\Gamma_1}$.*

Proof of Proposition 2. Suppose that player 2 has a p_X-min action set B and consider the following strategy for it:

1. At $t=0$ choose an action according to some maximin strategy with support over B.

2. Whenever a revision opportunity arrives after T consecutive stages playing the pure action a, play each pure action with the probability:

$$\alpha_2'(a') = \mathbf{1}_{a'\neq a}\frac{\alpha_2^*(a')}{q(T)} + \mathbf{1}_{a'=a}\frac{\alpha_2^*(a') - (1-q(T))}{q(T)}, \tag{15}$$

where α_2^* is some minimax strategy with support over B. [3]

Assume a was played T stages in a row, and player 2 follows this strategy. The probability that the pure action a' will be played next is

$$\begin{aligned}\Pr(a') &= \Pr(a'|\text{no revision})\Pr(\text{no revision}) + \Pr(a'|\text{revision})\Pr(\text{revision}) \\ &= \mathbb{1}_{a'=a}(1-q(T)) + \alpha_2'(a')q(T) = \alpha_2^*(a'). \end{aligned} \tag{16}$$

In this stage, player 1 plays against the mixed action α_2^* which yields the expected payoff of at least V_{Γ_1}. This is true for every stage, so the payoff for player 2 in the repeated game is at least V_{Γ_1} as well. Player 1, however, can make sure to pay at most V_{Γ_1} by playing the minimaxing action of the stage game in every stage, which sets the value of the game to be exactly $V_{\tilde{\Gamma}_X} = V_{\Gamma_1}$.

The existence of a p_X-min action set for player 2 is also a necessary condition. Otherwise, there is a positive probability to play in one of the stages an action that is played in maxmin strategies in lower than p_X probability. When such action is played, for large enough realization of X, there will be a stage in which $1 - q(T)$ is greater than the probability to play this action in equilibrium, forcing player 2 to choose this action with probability higher than the maxmin probability. In this stage, the best response of player 1 to the subset of mixed actions in which the former action is played with probability of at least $1 - q(T)$ would yield a higher payoff than V_{Γ_1}, resulting in $V_{\tilde{\Gamma}_1} > V_{\Gamma_1}$. ■

[3] Thus $\alpha_2^*(a) \geqslant (1-q(T))$ and α_2^* is indeed a probability function over B.