Quitting Games and Linear Complementarity Problems*

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Abstract

We prove that every multiplayer quitting game admits a sunspot ε-equilibrium for every $\varepsilon > 0$, that is, an ε-equilibrium in an extended game in which the players observe a public signal at every stage. We also prove that if a certain matrix that is derived from the payoffs in the game is not a Q-matrix in the sense of linear complementarity problems, then the game admits a uniform ε-equilibrium for every $\varepsilon > 0$.

Keywords: Stochastic games, quitting games, stopping games, sunspot equilibrium, linear complementarity problems, Q-matrices.

1 Introduction

Shapley (1953) introduced the model of stochastic games as a model of dynamic interactions in which players' actions affect both the stage payoffs and the evolution of the state variable. Shapley studied the two-player zero-sum model, and proved that the discounted value always exists and that both

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players have stationary optimal strategies. This result was extended to the existence of discounted equilibria in multiplayer stochastic games by Fink (1964) and Takahashi (1964).

The equilibrium strategies are not robust, as they depend on the discount factor. Mertens and Neyman (1981) proposed a solution concept that is robust to variation in the discount factor:[1] given $\varepsilon > 0$, a strategy profile is a *uniform ε-equilibrium* if it is a discounted ε-equilibrium for every discount factor sufficiently close to 0. Mertens and Neyman (1981) proved that every two-player zero-sum stochastic game admits a uniform ε-equilibrium, for every $\varepsilon > 0$, and Vieille (2000a, 2000b) extended this result to two-player nonzero-sum games. Solan (1999) proved that a uniform ε-equilibrium exists in three-player absorbing games, which are stochastic games with a single nonabsorbing state. It is still not known whether every multiplayer stochastic game admits a uniform ε-equilibrium, for every $\varepsilon > 0$.

In their study of uniform equilibrium in multiplayer stochasic games, Solan and Vieille (2001) introduced a new class of absorbing games, called *quitting games*, which is inspired by the game studied by Flesch, Thuijsman, and Vrieze (1997). In a quitting game, each one of N players decides at every stage whether to continue or to quit. As long as all players continue, the game continues. Once at least one player quits, the game terminates, and the terminal payoff depends on the set of players who decide to quit at the terminal stage. Solan and Vieille (2001) proved that if each player prefers to quit alone rather than to quit with other players, then a uniform ε-equilibrium exists, for every $\varepsilon > 0$. This result was extended to a more general class of quitting games by Simon (2012).

Aumann (1974, 1987) introduced the concept of correlated equilibrium in strategic-form games. A *correlated equilibrium* in a strategic-form game is an equilibrium in an extended game that includes a correlation device, which sends to each player a private signal before the play starts. In dynamic games several variations of correlated equilibrium come to mind (see Forges (1986) and Fudenberg and Tirole (1991)). The most general concept is *communication equilibrium*, that corresponds to an equilibrium in an extended game in which at every stage the device receives a private message from each player and sends a private signal to each player, which may depend on past mes-

[1]The concept of uniform equilibrium as defined by Mertens and Neyman (1981) is stronger than the one we provide here: it also requires the strategy profile to be also ε-equilibrium in the finite-stage game for every sufficiently long horizon, as well as to the infinite-horizon game with the limit of the averages payoff.

sages and signals of all players. A more restricted concept is *extensive-form correlated equilibrium*, which corresponds to the situation in which the device sends a private message to each player at the beginning of every stage, and does not receive any messages from the players. Yet a more restricted concept is *normal-form correlated equilibrium*, which corresponds to the situation in which the device sends one private message to each player at the beginning of the game. Cass and Shell (1984) proposed the concept of *sunspot equilibrium*, which is an equilibrium in a game extended by a correlation device that publicly sends to the players at every stage a uniformly distributed random variable in $[0, 1]$ that is chosen independently of past signals and past play.

Solan and Vieille (2002) proved that every multiplayer stochastic game admits an extensive-form uniform correlated ε-equilibrium, for every $\varepsilon > 0$, and Solan and Vohra (2002) proved that every multiplayer absorbing game admits a normal-form uniform correlated ε-equilibrium, for every $\varepsilon > 0$.

In this paper we prove that every multiplayer quitting game admits an undiscounted sunspot ε-equilibrium, for every $\varepsilon > 0$. Solan and Vieille (2001) proved that in quitting games, every undiscounted ε-equilibrium is also uniform, and their argument carries over to correlated equilibria. It follows that every multiplayer quitting game admits a sunspot uniform ε-equilibrium, for every $\varepsilon > 0$.

Our proof uses heavily the notion of Q-matrices from linear complementarity problems (see, e.g., Murty, 1988). Given an $n \times n$ matrix R and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem* $\mathrm{LCP}(R, q)$ is the problem of finding two vectors $w, z \in \mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n \colon x_i \geq 0, \quad \forall i \in [n]\}$ such that (a) $w = Rz + q$ and (b) $w_i = 0$ or $z_i = 0$ for every $i \in [n]$, where $[n] = \{1, 2, \ldots, n\}$. A matrix R is called a *Q-matrix* if a solution to the problem $\mathrm{LCP}(R, q)$ exists for every $q \in \mathbb{R}^n$.

Denote by $r^i \in \mathbb{R}^n$ the terminal payoff in the quitting game if player i quits alone. By adding a constant to the payoffs, we can assume without loss of generality that $r^i_i = 0$; that is, each player who quits alone receives 0. Call a player *normal* if his min-max value is nonpositive; that is, the player's min-max value is not higher than the payoff if that player quits alone. Let I_* be the set of all normal players and rename the players so that the normal players are players number $1, 2, \cdots, |I_*|$. Let $\widehat{r}^i \in \mathbb{R}^{I_*}$ be the vector r^i restricted to the coordinates that correspond to normal players and let $\widehat{R}$ be the $|I_*| \times |I_*|$ matrix whose i'th column coincides with $\widehat{r}^i$.

We show that if the matrix $\widehat{R}$ is not a Q-matrix, then for every $\varepsilon > 0$

the quitting game admits an ε-equilibrium which is almost stationary, in the sense that the ε-equilibrium profile is composed of a stationary strategy profile supplemented with threats of punishment. We also show that if the matrix $\widehat{R}$ is a Q-matrix, then for every $\varepsilon > 0$ the quitting game admits a sunspot uniform ε-equilibrium, in which at every stage at most one player quits with positive probability.

Our contribution is threefold. First, we prove that every quitting game admits a sunspot uniform ε-equilibrium. Second, we identify a general condition that ensures that a uniform ε-equilibrium exists in quitting games. Third, we relate the question of existence of uniform equilibrium in stochastic games to linear complementarity problems. In particular, our work limits the class of quitting games for which the existence of a uniform ε-equilibrium is not known. In the followup paper Solan and Solan (2018), our main result is used to show that every quitting game in which players may have more than one continue action (but they still have a single quitting action) admit a sunspot uniform ε-equilibrium. This result is then used in Solan, Solan, and Solan (2018) to show that every quitting game in which at least two players have at least two continue actions, and all players have one quit action, has a uniform ε-equilibrium. We hope that our result will pave the road to proving the existence of a sunspot uniform ε-equilibrium and a uniform ε-equilibrium in additional classes of stochastic games.

Unfortunately we are not aware of a characterization of Q-matrices, hence it is not clear whether the family of games that admit a uniform ε-equilibrium by our result is small or large. Yet, given an $n \times n$ matrix R whose diagonal entries are 0, once we find that R is not a Q-matrix, our result implies that a uniform ε-equilibrium exists in any n-player quitting game in which the payoff vector if each player i quits alone is the i'th row of R.

The paper is organized as follows. The model and the main results are described in Section 2. The proof of the main result appears in Section 3. In Section 4 we discuss the characterization of sunspot equilibrium payoffs when the matrix $\widehat{R}$ is an M-matrix, namely, a Q-matrix for which in each row and each column there is exactly one positive entry. Discussion and open problems, including a discussion on the extension of our result to stopping games, appear in Section 5.

2 The Model and the Main Results

2.1 The Model

A *quitting game* $\Gamma((r^S)_{S\subseteq I})$ is given by

- A finite set of players $I = [N] := \{1, 2, \cdots, N\}$.
- For every subset S of I, a vector $r^S \in [-1, 1]^N$.

Note that we assume w.l.o.g. that payoffs are bounded by 1.

The game evolves as follows. At every stage $t \in \mathbb{N}$ each player decides whether to continue or quit. Denote by t^* the first stage in which at least one player quits, and by S^* the set of players who quit at stage t^*. If no player ever quits, then $t^* = \infty$ and $S^* = \emptyset$. The payoff to the players is r^{S^*}.

For convenience, whenever $S = \{i\}$ contains one element we write r^i instead of $r^{\{i\}}$. We will maintain the following assumption, which states that if a player quits alone, his payoff is 0. This assumption is made without loss of generality, since adding a constant to the payoffs of a player does not change his strategic considerations.

Assumption 2.1 *For every $i \in I$ we have $r^i_i = 0$.*

Remark 2.2 *Assumption 2.1 is not usual in the literature of quitting games, where it is customary to normalize the payoff vector $r^\emptyset$ to $\vec{0} := (0, 0, \cdots, 0)$. As we will see, the vector $r^\emptyset$ will hardly play any role in the analysis, while to relate our approach to linear complementarity problems it will be more convenient to use the normalization expressed by Assumption 2.1.*

A (behavior) *strategy* of player i is a sequence $x_i = (x^t_i)_{t\in\mathbb{N}}$ of numbers in $[0, 1]$, with the interpretation that x^t_i is the conditional probability that player i quits at stage t, provided no player quit before that stage. Denote by X_i the set of strategies of player i, by $X_{-i} := \times_{j\in I\setminus\{i\}} X_i$ the set of strategy profiles of the other players, and by $X := \times_{i\in I} X_i$ the set of *strategy profiles.*

Every strategy profile $x = (x_i)_{i\in I} \in X$ induces a probability distribution $\mathbf{P}_x$ over the set of plays. We denote by $\mathbf{E}_x[\cdot]$ the corresponding expectation operator. Denote by $\gamma(x) := \mathbf{E}_x[r^{S^*}]$ the *payoff* under strategy profile x. A strategy profile x is an *ε-equilibrium* if for every $i \in I$ and every strategy $x'_i \in X_i$ of player i we have

$$\gamma_i(x) \geq \gamma_i(x'_i, x_{-i}) - \varepsilon.$$

Using the insights of Flesch, Thuijsman, and Vrieze (1997), Solan (1999) proved that every three-player absorbing game, hence in particular every three-player quitting game, admits an ε-equilibrium for every $\varepsilon > 0$. Solan and Vieille (2001) and Simon (2012) extended this result to multi-player quitting games that satisfy various conditions.

A strategy $x_i = (x_i^t)_{t\in\mathbb{N}} \in X_i$ is *stationary* if $x_i^t = x_i^{t'}$ for every $t, t' \in \mathbb{N}$. In this case we denote by x_i the probability by which player i quits at every stage, and we view a stationary strategy profile $x = (x_i)_{i\in I}$ as a vector in $[0,1]^N$. We denote by C_i (resp. Q_i) the pure stationary strategy of player i in which he always continues (resp. always quits).

Solan (1999) proved that in every three-player quitting game there is an ε-equilibrium of one of two simple forms: there is always a stationary ε-equilibrium or an ε-equilibrium in which at every stage at most one player quits with positive probability. Solan and Vieille (2002) provided a four-player quitting game in which there is an ε-equilibrium, yet for $\varepsilon > 0$ sufficiently small there is neither a stationary ε-equilibrium nor an ε-equilibrium in which at every stage at most one player quits with positive probability. To date it is not known whether four-player quitting games admit ε-equilibria for every $\varepsilon > 0$.

2.2 Sunspot Equilibrium

We enrich the game by introducing a public correlation device. That is, at the beginning of every stage $t \in \mathbb{N}$ the players observe a public signal $s^t \in [0,1]$ that is drawn by the uniform distribution, independently of past signals.

A *strategy* of player i in the game with public correlation device is a sequence of measurable functions $\xi_i = (\xi_i^t)_{t\in\mathbb{N}}$, where $\xi_i^t : [0,1]^t \to [0,1]$. The interpretation of ξ_i^t is that if no player quits before stage t, then at stage t player i quits with probability $\xi_i^t(s^1, s^2, \cdots, s^t)$.

Every strategy profile $\xi = (\xi_i)_{i\in I}$ induces a probability distribution $\mathbf{P}_\xi$ over the set of plays in the game with public correlation device, with a corresponding expectation operator that is denoted by $\mathbf{E}_\xi[\cdot]$. Denote by $\gamma(\xi) := \mathbf{E}_\xi[r^{S_*}]$ the *payoff* under strategy profile ξ.

Definition 2.3 *A strategy profile ξ is a* sunspot ε-equilibrium *if it is an ε-equilibrium in the game with public correlation device; that is, if for every*

$i \in I$ and every strategy ξ_i' of player i we have

$$\gamma_i(\xi) \geq \gamma_i(\xi_i', \xi_{-i}) - \varepsilon.$$

The main result of this paper is the following.

Theorem 2.4 *Every quitting game admits a sunspot ε-equilibrium, for every $\varepsilon > 0$.*

2.3 Normal and Abnormal Players

Simon (2012) defined the concepts of normal and abnormal players in quitting games. Denote by v_i player i's min-max value.

Definition 2.5 *Player i is* normal *if $v_i \leq 0$, and he is* abnormal *if $v_i > 0$.*

An abnormal player will never quit alone in an equilibrium, because by quitting alone he gets strictly less than his min-max value.

We denote by I_* the set of all normal players, and by $n = |I_*|$ the number of normal players. An important property of abnormal players is that they prefer other players to stop alone than stopping alone themselves.

Lemma 2.6 *Let $i \notin I_*$ be an abnormal player, and let $j \neq i$ be any player. Then $r_i^j > 0$.*

Proof. Since player i is abnormal, we can choose $\varepsilon \in (0, v_i)$. Consider the stationary strategy profile x_{-i} defined as follows:

$$x_k = \begin{cases} C_k & k \neq \{i,j\}, \\ (1-\varepsilon)C_j + \varepsilon Q_j & k = j. \end{cases}$$

We have

$$\gamma_i(Q_i, x_{-i}) = (1-\varepsilon) r_i^i + \varepsilon r_i^{\{i,j\}} \leq \varepsilon < v_i.$$

Since player i has a response to x_{-i} that yields to him at least v_i, we necessarily have

$$r_i^j = \gamma_i(C_i, x_{-i}) \geq v_i > 0,$$

as desired. ■

Below we will construct sunspot ε-equilibria in which at every stage at most one player quits. Every player who quits alone receives 0, while by

Lemma 2.6 every abnormal player receives a positive payoff when any other player quits. It is therefore not a wonder that in those equilibria we can ignore abnormal players: if we can construct an equilibrium in which at every stage only one player quits, this player is a normal player, and the game terminates with probability 1, then the payoff of an abnormal player is positive, hence he will not quit alone.

Lemma 2.6 provides a sufficient condition for the normality of a player: player i is normal as soon as there is $j \neq i$ such that $r_i^j \leq 0$. As we now show, one consequence of Lemma 2.6 is that if all players are abnormal, a stationary ε-equilibrium exists. We will therefore assume in the rest of the paper that there is at least one normal player.

Lemma 2.7 *If $I_* = \emptyset$ then the game admits a stationary ε-equilibrium for every $\varepsilon > 0$.*

Proof. Suppose that $I_* = \emptyset$. If $r_i^{\emptyset} \geq 0$ for every $i \in I$, then the stationary strategy profile in which all players always continue is a 0-equilibrium. Suppose then that there exists a player $i \in I$ such that $r_i^i < 0$. The reader can verify that for every $\varepsilon > 0$ the following strategy profile x is a stationary 2ε-equilibrium:

$$x_k = \begin{cases} C_k & k \neq \{i\}, \\ (1-\varepsilon)C_i + \varepsilon Q_i & k = i. \end{cases}$$

■

A second condition that guarantees the existence of a stationary ε-equilibrium is the following.

Lemma 2.8 *Suppose that there exists a probability distribution z over the set of normal players I_* that satisfies the following conditions: $\sum_{j \in I_*} z_j r_i^j \geq 0$ for every player $i \in I_*$, with an equality whenever $z_i > 0$. Then the game admits a stationary ε-equilibrium for every $\varepsilon > 0$.*

Proof. Fix $\varepsilon > 0$ and consider the following stationary strategy profile x:

$$x_i = \begin{cases} C_i & i \notin I_*, \\ (1-\varepsilon z_i)C_i + \varepsilon z_i Q_i & i \in I_*. \end{cases}$$

It is standard to verify that the strategy profile x is 2ε-equilibrium. Note that by Lemma 2.6 abnormal players cannot profit more than 2ε by deviating at the strategy profile x. ■

2.4 Linear Complementarity Problems and the Main Result

Recall that $[n] = \{1, 2, \ldots, n\}$. Let R ne an $n \times n$ matrix, denote its columns by $r^1, r^2, \cdots, r^n$, and let $q \in \mathbb{R}^n$. The *linear complementarity problem* $\mathrm{LCP}(R, q)$ is the following problem that consists of linear equalities and inequalities:

$$\begin{aligned} \text{Find} \quad & w \in \mathbb{R}^n_{\geq 0}, \text{ and } z = (z_0, z_1, \cdots, z_n) \in \Delta(\{0, 1, \cdots, n\}), \\ \text{such that} \quad & w = z_0 q + \sum_{i=1}^{n} z_i r^i, \\ & z_i = 0 \text{ or } w_i = 0, \quad \forall i \in [n]. \end{aligned} \tag{1}$$

The last condition in the problem (1) is the *complementarity condition.*

We note that for $q \in \mathbb{R}^n_{\geq 0}$ there is always at least one solution to the problem (1), namely, $z = (1, 0, \cdots, 0)$ and $w = q$. This solution is called the *trivial solution.*

Remark 2.9 *Let R be an $n \times n$ matrix, and let $q \in \mathbb{R}^n$. In the literature, the linear complementarity problem* $\mathrm{LCP}(R, q)$ *is the following problem that consists of linear equalities and inequalities:*

$$\begin{aligned} \textit{Find} \quad & z, w \in \mathbb{R}^n_{\geq 0}, \\ \textit{such that} \quad & w = q + Rz, \\ & z_i = 0 \textit{ or } w_i = 0, \quad \forall i \in [n]. \end{aligned} \tag{2}$$

We call the problem (1) a linear complementarity problem since it is obtained from problem (2) by division by the positive real number z_0, provided $z_0 > 0$. That is, if (w, z) is a solution of the problem (1) that satisfies $z_0 \neq 0$, then the following vector $(\widehat{w}, \widehat{z})$ is a solution of the problem (2):

$$\begin{aligned} \widehat{z}_i &:= \frac{z_i}{z_0}, \quad \forall i \in [n], \\ \widehat{w}_i &:= \frac{w_i}{z_0}, \quad \forall i \in [n]. \end{aligned}$$

Conversely, if $(\widehat{w}, \widehat{z})$ is a solution of the problem (2), then the following vector

(w,z) is a solution of the problem (1):

$$z_0 := \frac{1}{1+\sum_{j=1}^n z_j}, \quad z_i := \frac{z_i}{1+\sum_{j=1}^n z_j}, \quad \forall i \in [n],$$
$$w_i := \frac{w_i}{1+\sum_{j=1}^n z_j}, \quad \forall i \in [n].$$

Lemma 2.8 will imply that in our application, when $z_0 = 0$ a stationary ε-equilibrium exists, hence in the cases we will study below the two problems (1) and (2) will be equivalent.

Definition 2.10 *An $n \times n$ matrix R is called a Q-*matrix *if for every $q \in \mathbb{R}^n$ the linear complementarity problem* $\mathrm{LCP}(R,q)$ *has at least one solution.*

The authors are not aware of any characterization of Q-matrices. The following example illustrates the concept of Q-matrices.

Example 2.11 *Let R be a 3×3 matrix that has the following sign form:*

$$R = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{pmatrix}.$$

Theorem 6.2.7 in Berman and Plemmons (1994) implies that the matrix R is a Q-matrix if and only its determinant is positive.

Recall that the number of normal players is denoted by n. Order the players so that the set of normal players is $I_* = [n]$. For every normal player $i \in I_*$ we denote by $\widehat{r}^i \in \mathbb{R}^n$ the restriction of the vector r^i to the coordinates in I_*. Denote by $\widehat{R}$ the $n \times n$ matrix whose i'th column is $\widehat{r}^i$. By Assumption 2.1, all entries on the diagonal of $\widehat{R}$ are 0.

Any nontrivial solution (w,z) of the linear complementarity problem $\mathrm{LCP}(\widehat{R},q)$ that satisfies $z_0 = 0$ induces a nontrivial solution to the problem $\mathrm{LCP}(\widehat{R},\vec{0})$. Any nontrivial solution of the linear complementarity problem $\mathrm{LCP}(\widehat{R},\vec{0})$ induces a probability distribution z over the set of normal players I_*. By Lemma 2.8 this solution induces a stationary ε-equilibrium, for every $\varepsilon > 0$. We therefore obtain the following result.

Lemma 2.12 *Suppose that at least one of the following conditions hold:*

- *The linear complementarity problem* $\mathrm{LCP}(\widehat{R},\vec{0})$ *has a nontrivial solution.*
- *For some* $q \in \mathbb{R}^n$ *the linear complementarity problem* $\mathrm{LCP}(\widehat{R},q)$ *has a solution* (w,z) *with* $z_0 = 0$.

Then the game admits a stationary ε*-equilibrium for every* $\varepsilon > 0$.

Lemmas 2.7 and 2.12 show that for every $\varepsilon > 0$ a stationary ε-equilibrium exists as soon as $I_* = \emptyset$, or $I_* \neq \emptyset$ and the linear complementarity problem $\mathrm{LCP}(\widehat{R},\vec{0})$ has a nontrivial solution. The following theorem, which is proven in the next section, completes the proof of Theorem 2.4. In addition to proving that in every quitting game a sunspot ε-equilibrium exists, it links the property of $\widehat{R}$ being a Q-matrix to the structure of the sunspot ε-equilibria in the quitting game.

Theorem 2.13 *Suppose that* $I_* \neq \emptyset$ *and the linear complementarity problem* $\mathrm{LCP}(\widehat{R},\vec{0})$ *does not have a nontrivial solution.*

1. *If the matrix* $\widehat{R}$ *is not a* Q*-matrix, then the quitting game* $\Gamma((r^S)_{S\subseteq I})$ *has an* ε*-equilibrium, for every* $\varepsilon > 0$.
2. *If the matrix* $\widehat{R}$ *is a* Q*-matrix, then for every* $\varepsilon > 0$ *the quitting game* $\Gamma((r^S)_{S\subseteq I})$ *has a sunspot* ε*-equilibrium in which at every stage at most one player quits with positive probability.*

As mentioned earlier, we do not know whether the class of quitting games for which the matrix $\widehat{R}$ is not a Q-matrix is rich. Nevertheless, for specific quitting games one can check whether this matrix is a Q-matrix.

Example 2.14 *Suppose that there are four players, where the vectors* $(r^i)_{i=1}^4$ *are given by*[2]

$$r^1 = (0,2,-1,1),\ \ r^2 = (-1,0,2,1),\ \ r^3 = (2,-1,0,-2),\ \ r^4 = (-1,2,-1,0).$$

Note that the restriction of (r^1,r^2,r^3) *to the first three coordinates are the payoff vectors used in the game studied by Flesch, Thuijsman, and Vrieze (1997). The* 3×3 *matrix that is composed by the first three coordinates of the vectors* r^1, r^2, *and* r^3 *is a* Q*-matrix; see Example 2.11. For every player* i

[2] In the examples we deviate from the assumption that payoffs are bounded by 1.

there is $j \neq i$ with $r_i^j \leq 0$, hence all players are normal. Using the vector $q = (-1,-1,-1,-1)$ one can show that the matrix $\widehat{R}$ whose i'th column is equal to r^i, for $i = 1,2,3,4$, is not a Q-matrix. Consequently, every quitting game in which the payoffs of unilateral quittings is given by the vectors $(r^i)_{i=1}^4$ has a uniform ε-equilibrium, for every $\varepsilon > 0$.

2.5 An Example

To illustrate the solution concept and our approach, in this section we provide the construction of a sunspot ε-equilibrium that uses only unilateral quittings in a specific game. We will provide two constructions; the first will be used in Section 4 to characterize the set of sunspot equilibrium payoffs in a certain class of quitting games. Unfortunately we could not generalize it to all quitting games. The second construction will serve us in the proof of the general case.

Consider a quitting game with four players, where the vectors $(r^i)_{i=1}^4$ are given by

$$r^1 = (0,4,-1,-1),\ r^2 = (4,0,-1,-1),\ r^3 = (-1,-1,0,4),\ r^4 = (-1,-1,4,0).$$

The rest of the payoff function, namely, the vectors $r^\emptyset$ and $(r^S)_{|S|\geq 2}$, will not affect the analysis, hence is omitted. We note that all players are normal, and that these payoffs are essentially the same payoffs that were used by Solan and Vieille (2002) to construct a quitting game in which there is neither a stationary ε-equilibrium nor an ε-equilibrium in which at every stage at most one player quits with positive probability.

2.5.1 First Construction

Observe that

$$\begin{aligned}(1,1,0,0) &= \tfrac{1}{2}(2,0,0,0) + \tfrac{1}{2}(0,2,0,0) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}(4,0,-1,-1) + \tfrac{1}{2}(0,0,1,1)\right) + \tfrac{1}{2}\left(\tfrac{1}{2}(0,4,-1,-1) + \tfrac{1}{2}(0,0,1,1)\right),\end{aligned} \tag{3}$$

and similarly

$$\begin{aligned}(0,0,1,1) &= \tfrac{1}{2}(0,0,2,0) + \tfrac{1}{2}(0,0,0,2) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}(-1,-1,4,0) + \tfrac{1}{2}(1,1,0,0)\right) + \tfrac{1}{2}\left(\tfrac{1}{2}(-1,-1,0,4) + \tfrac{1}{2}(1,1,0,0)\right).\end{aligned} \tag{4}$$

Fix $\varepsilon > 0$ such that $\frac{1}{\varepsilon}$ is an integer. The following recursive construction, in which the players implement the payoff vector $(1,1,0,0)$ as a sunspot equilibrium payoff, suggests itself:

- Nature chooses whether the players implement the vector $(2,0,0,0)$ (if the current signal is smaller than $\frac{1}{2}$) or the vector $(0,2,0,0)$ (if the current signal is at least $\frac{1}{2}$).
- If Nature chose to implement the vector $(2,0,0,0)$, in each one of the next $\frac{1}{\varepsilon}$ stages Player 2 quits with probability δ, where $(1-\delta)^{1/\varepsilon} = \frac{1}{2}$. That is, in each of these stages Player 2 quits with a small probability, and during these stages the total probability that he quits is $\frac{1}{2}$.
- If Nature chose to implement the vector $(0,2,0,0)$, in each one of the next $\frac{1}{\varepsilon}$ stages Player 1 quits with probability δ, where $(1-\delta)^{1/\varepsilon} = \frac{1}{2}$.
- At the end of the $\frac{1}{\varepsilon}$ stages, if no player quits, the players turn to implement the vector $(0,0,1,1)$ in an analogous way, and the construction continues recursively.

Denote by ξ^* the strategy profile that was just defined. Under ξ^* the game terminates with probability 1. Moreover, even if one of the players deviates, the game terminates with probability 1. Eqs. (3) and (4) imply that $\gamma(\xi^*) = (1,1,0,0)$, and, more generally, that when the players attempt to implement a certain vector, say $(2,0,0,0)$, their expected payoff is that vector.

We now argue that no player can profit much by deviating from ξ^*. To this end we note that the expected continuation payoff of all players after every history is nonnegative. In each stage in which a player is supposed to quit with positive probability, his continuation payoff is 0, hence at such stages the player is indifferent between continuing and quitting.

A player who is supposed to quit with a positive probability at a given stage, does so with probability δ, which is small. Consequently, since payoffs are bounded by 1 and by Assumption 2.1, if a player deviates and quits at a stage in which he is supposed to continue, his payoff is at most δ. Since the continuation payoff of all players after every history is nonnegative, this implies that no player who is supposed to continue at a given stage can profit more than δ by quitting at that stage.

What is the driving force behind the construction? We identified a certain finite set

$$Y = \{(0,0,1,1),(1,1,0,0),(2,0,0,0),(0,2,0,0),(0,0,2,0),(0,0,0,2)\},$$

which satisfies the following property $\mathcal{P}$: Each of the vectors $y \in Y$ is either (a) a convex combination of the other vectors in the set, or (b) a convex combination of some other vector in the set and one of the vectors $\{r^1, r^2, r^3, r^4\}$, where, if this convex combination is with the vector r^i, then necessarily $y_i = 0$. We can then construct for every vector $y \in Y$ a sunspot ε-equilibrium with payoff y: If $y = \sum_{j=1}^J \beta_j y^{[j]}$ is a convex combination of some other vectors in the set, then to implement y as a sunspot ε-equilibrium payoff we let Nature choose one of the vectors $(y^{[j]})_{j=1}^J$ according to the probability distribution $(\beta_j)_{j=1}^J$, and then recursively implement the chosen vector as a sunspot ε-equilibrium payoff. If $y = \beta y' + (1-\beta) r^i$ for some $y' \in Y \setminus \{y\}$, then we let player i quit with small probability for some stages, until the total probability by which he quits is $1-\beta$, and if he did not quit, we continue by implementing y' as a sunspot ε-equilibrium payoff. Since the set Y is finite, this recursive procedure ensures that the play eventually terminates, and we obtain a sunspot ε-equilibrium.

Since the sunspot ε-equilibria that we construct involve only unilateral quittings, the payoff vectors that can be implemented this way lie in the convex hull of $\{r^1, r^2, r^3, r^4\}$. Since at each stage the player who quits does it with small probability, and since a player who quits alone receives 0, only payoff vectors in the nonnegative orthant can be constructed by this procedure. In our example, the intersection of the convex hull of $\{r^1, r^2, r^3, r^4\}$ and the nonnegative orthant is

$$\mathrm{conv}\{(2,0,0,0),(0,2,0,0),(0,0,2,0),(0,0,0,2)\}.$$

Since the extreme points of this set belong to the set Y, it follows that we can implement all payoff vectors in this convex hull as sunspot uniform equilibrium payoffs. As we will see in Section 4, this construction can be generalized to the case in which the matrix $\widehat{R}$ contains exactly one positive entry in each row and each column.

In the general case we could not identify a set Y that satisfies Property $\mathcal{P}$ and contains the extreme points of the intersection of the nonnegative orthant and the convex hull of $\{\widehat{r}^i, i \in I_*\}$. Moreover, we could not show the existence of a finite set Y that satisfies Property $\mathcal{P}$. In the next subsection we provide another finite set Y that satisfies Property $\mathcal{P}$, which resembles the construction that we use in the general case.

2.5.2 Second Construction

Fix $\varepsilon \in (0,1)$. We will use the following identities:[3]

$$\begin{aligned}
(0,0,2,0) &= \tfrac{\varepsilon}{6+\varepsilon}(0,0,0,2) + \tfrac{1}{6+\varepsilon}(0,4\varepsilon,2-3\varepsilon,-\varepsilon) \\
&\quad + \tfrac{1}{6+\varepsilon}(4\varepsilon,0,2-3\varepsilon,-\varepsilon) + \tfrac{4}{6+\varepsilon}(-\varepsilon,-\varepsilon,2+2\varepsilon,0) \\
&= \tfrac{\varepsilon}{6+\varepsilon}(0,0,0,2) \\
&\quad + \tfrac{1}{6+\varepsilon}\big((1-\varepsilon)(0,0,2,0) + \varepsilon(0,4,-1,-1)\big) \\
&\quad + \tfrac{1}{6+\varepsilon}\big((1-\varepsilon)(0,0,2,0) + \varepsilon(4,0,-1,-1)\big) \\
&\quad + \tfrac{4}{6+\varepsilon}\big((1-\varepsilon)(0,0,2,0) + \varepsilon(-1,-1,4,0)\big).
\end{aligned} \tag{5}$$

This equality suggest the following recursive construction of a sunspot 2ε-equilibrium ξ^* with $\gamma(\xi^*) = (0,0,2,0)$.

- Nature chooses whether to implement the vector $(0,0,0,2)$ (with probability $\frac{\varepsilon}{6+\varepsilon}$), the vector $(0,4\varepsilon,2-3\varepsilon,-\varepsilon)$ (with probability $\frac{1}{6+\varepsilon}$), the vector $(4\varepsilon,0,2-3\varepsilon,-\varepsilon)$ (with probability $\frac{1}{6+\varepsilon}$), or the vector $(-\varepsilon,-\varepsilon,2+2\varepsilon,0)$ (with probability $\frac{4}{6+\varepsilon}$).
- If Nature chose to implement the vector $(0,0,0,2)$, then the players recursively repeat the analogous construction with the appropriate amendments.
- If Nature chose to implement the vector $(0,4\varepsilon,2-3\varepsilon,-\varepsilon)$, then Player 1 quits with probability ε and continues with probability $1-\varepsilon$.
- If Nature chose to implement the vector $(4\varepsilon,0,2-3\varepsilon,-\varepsilon)$, then Player 2 quits with probability ε and continues with probability $1-\varepsilon$.
- If Nature chose to implement the vector $(-\varepsilon,-\varepsilon,2+2\varepsilon,0)$, then Player 4 quits with probability ε and continues with probability $1-\varepsilon$.
- If no player quit, then the players recursively implement the payoff $(0,0,2,0)$ as indicated above.

As in the first construction, under the strategy profile ξ^* the game terminates with probability 1, hence by Eq. (5) the expected payoff under ξ^* after every

[3] The verification of the calculations in this construction can be simplified by observing that the total sum of the coordinates of all vectors in the construction is 2.

finite history is the payoff vector that the players implement beginning at that stage.

We argue that no player can profit more than ε by deviating from ξ^*. Note that the play terminates with probability 1 even if a single player deviates. Consequently, when a player is supposed to quit with positive probability, he is indifferent between continuing and quitting. Moreover, if some player, say player i, is supposed to continue after some finite history, then his continuation payoff is at least $-\varepsilon$, while, since the player who quits with positive probability does so with probability ε, by quitting player i will obtain at most ε. Thus, no player can profit more than 2ε by deviating from ξ^*.

We now point at the main difference between the two constructions. Denote by y^t the expected payoff under the sunspot ε-equilibrium from stage t and on. In the first construction, the sequence $(y^t)_{t\in\mathbb{N}}$ is alternating: it is $(0,0,1,1)$ for odd t and $(1,1,0,0)$ for even t. In contrast, in the second construction $(y^t)_{t\in\mathbb{N}}$ is a stochastic process. In particular, to implement the sunspot ε-equilibrium in the second construction, the players need to recall all past signals they obtained in the past.

3 Proof of Theorem 2.13

In this section we prove Theorem 2.13. We start by describing the discounted game, which will be used in the proof of the first claim of the theorem. This claim will be proven in Section 3.2 and the second claim will be proven in Section 3.3.

3.1 The Discounted Game

In this section we consider the discounted game, in which the vector $r^\emptyset$ represents the stage payoff until the game terminates. Formally, given a discount factor $\lambda \in (0,1]$, the λ-*discounted game* $\Gamma_\lambda((r^S)_{S\subseteq I})$ is the strategic-form game $(I,(X_i)_{i\in I},\gamma^\lambda)$, where the set of players coincides with the set of players in the original quitting game $\Gamma((r^S)_{S\subseteq I})$, the set of strategies of each player $i\in I$ is X_i, his set of behavior strategies in the original quitting game, and the payoff function is given by

$$\begin{aligned}\gamma^\lambda(x) &:= \mathbf{E}_x\left[\lambda\sum_{t=1}^{\infty}(1-\lambda)^{t-1}\left(\mathbf{1}_{\{t<t_*\}}r^{\emptyset}+\mathbf{1}_{\{t\geq t_*\}}r^{S_*}\right)\right]\\ &= \mathbf{E}_x\left[\left(1-(1-\lambda)^{t_*-1}\right)r^{\emptyset}+(1-\lambda)^{t_*-1}r^{S_*}\right], \quad \forall x\in X.\end{aligned}$$

When x is a stationary strategy profile, we have

$$\gamma^\lambda(x) = \frac{\lambda\left(\prod_{i\in I}(1-x_i)\right)r^{\emptyset}+\sum_{\emptyset\neq S\subseteq I}\left(\prod_{i\in S}x_i\right)\left(\prod_{i\notin S}(1-x_i)\right)r^S}{\lambda\left(\prod_{i\in I}(1-x_i)\right)+\sum_{\emptyset\neq S\subseteq I}\left(\prod_{i\in S}x_i\right)\left(\prod_{i\notin S}(1-x_i)\right)}. \quad (6)$$

A strategy profile $x\in X$ is a λ-*discounted equilibrium* if for every player $i\in I$ and every strategy $x_i'\in X_i$ of player i we have $\gamma_i^\lambda(x)\geq\gamma_i^\lambda(x_i',x_{-i})$.

By Fink (1964) or Takahashi (1964) the λ-discounted game admits a λ-discounted equilibrium in stationary strategies. By Bewley and Kohlberg (1976) one can choose a semialgebraic function $\lambda\mapsto x^\lambda$ that assigns a stationary discounted equilibrium to each discount factor. In particular, we can assume w.l.o.g. that the limit $x^0:=\lim_{\lambda\to 0}x^\lambda$ exists. Moreover, we can assume that either $x_i^\lambda=0$ for every λ sufficiently close to 0 or $x_i^\lambda>0$ for every λ sufficiently close to 0.

3.2 ε-Equilibria

In this section we prove the first statement of Theorem 2.13. Let x be a stationary strategy profile. If $\sum_{i\in I}x_i=0$, then $x=\vec{0}$, and under x the game continue forever. If $\sum_{i\in I}x_i>0$ then under x at least one player quits at every period with positive probability, and the game terminates a.s.

Suppose that the matrix $\widehat{R}$ is not a Q-matrix. Then there is a vector $\widehat{q}\in\mathbb{R}^n$ such that the linear complementarity problem LCP$(\widehat{R},\widehat{q})$ does not have any solution. In particular, $\widehat{q}\notin\mathbb{R}^n_{\geq 0}$. Extend $\widehat{q}$ to a vector in $\mathbb{R}^N$ by setting all coordinates that are not[4] in $[n]$ to 1, and denote by q the resulting vector.

Consider the auxiliary quitting game $\Gamma((r^S)_{\emptyset\neq S\subseteq I},q)$, where q is the payoff if no player ever quits, and the λ-discounted version of this game. To distinguish the payoff (resp. the λ-discounted payoff) in the original game

[4]The new coordinates can be set to any positive number, and not necessarily to 1.

from the payoff (resp. the λ-discounted payoff) in the auxiliary game, we denote the former by $\gamma(x)$ (resp. $\gamma^\lambda(x)$), and the latter by $\widehat{\gamma}(x)$ (resp. $\widehat{\gamma}^\lambda(x)$).

Let $\lambda \mapsto x^\lambda$ be a function that assigns a stationary strategy profile to every discount factor $\lambda \in (0,1]$. It is well known (see, e.g., Vrieze and Thuijsman (1989) or Solan (1999)) that if the limit $x^0 := \lim_{\lambda\to 0} x^\lambda$ exists, and if $\sum_{i\in I} x_i^0 > 0$, then

$$\lim_{\lambda\to 0} \gamma^\lambda(x^\lambda) = \gamma(x^0) = \widehat{\gamma}(x^0) = \lim_{\lambda\to 0} \widehat{\gamma}^\lambda(x^\lambda). \tag{7}$$

Fix now a semialgebraic function $\lambda \mapsto x^\lambda$ that assigns a stationary equilibrium of the auxiliary quitting game $\Gamma((r^S)_{\emptyset\neq S\subseteq I}, q)$ to every discount factor $\lambda \in (0,1]$, and denote $x^0 := \lim_{\lambda\to 0} x^\lambda$. We will show that since $\widehat{R}$ is not a Q-matrix, x^0 must be *absorbing*; that is, under x^0 at least one player quits with positive probability. We will then show that if under x^0 at least two players quit with positive probability, then x^0 is a stationary 0-equilibrium. We will finally show that if under x^0 a single player i quits, then this player is a normal player, and hence x^0 can be transformed into an ε-equilibrium, by threatening player i that if he does not quit he will be punished at his min-max level.

Step 1: The stationary strategy profile x^0 is absorbing.

Assume to the contrary that x^0 is nonabsorbing. We argue that the linear complementarity problem LCP$(\widehat{R}, \widehat{q})$ has a solution, which contradicts the choice of $\widehat{q}$. Denote

$$z_i^\lambda := \frac{x_i^\lambda}{\lambda + \sum_{j\in I_*} x_j^\lambda}, \quad i \in I,$$

and

$$z_0^\lambda := \frac{\lambda}{\lambda + \sum_{j\in I_*} x_j^\lambda}.$$

Set

$$w := \lim_{\lambda\to 0} \widehat{\gamma}^\lambda(x^\lambda) = \lim_{\lambda\to 0} \left(z_0^\lambda q + \sum_{i\in I_*} z_i^\lambda r^i \right), \tag{8}$$

where the equality holds since x^0 is nonabsorbing. Define $z^0 := \lim_{\lambda\to 0} z^\lambda$. Let $\widehat{w}$ (resp. $\widehat{z}$) be the restriction of w (resp. z^0) to the first n coordinates (resp. first $n+1$ coordinates). We verify that $(\widehat{w}, \widehat{z})$ is a solution of the linear complementarity problem LCP$(\widehat{R}, \widehat{q})$, contradicting the assumption that this problem has no solution.

We first argue that $\widehat{z}$ is a probability distribution over $\{0, 1, \cdots, n\}$. Since z^λ is a probability distribution over $\{0, 1, \cdots, N\}$, it is sufficient to show that $z_i^\lambda = 0$ for every abnormal player i and every λ sufficiently close to 0. Fix then an abnormal player i. By Lemma 2.6, for every player $j \neq i$ we have $r_i^j > 0$, and by definition $q_i > 0$. Consequently,

$$\lim_{\lambda \to 0} \gamma_i^\lambda(C_i, x_{-i}^\lambda) > 0 = r_i^i = \gamma_i(Q_i, C_{-i}) = \lim_{\lambda \to 0} \gamma_i^\lambda(Q_i, x_{-i}^\lambda). \tag{9}$$

It follows that $x_i^\lambda = 0$ for every λ sufficiently small, and consequently $z_i^\lambda = 0$ for every λ sufficiently small.

Together with the definition of w and Eq. (6) this implies that

$$\widehat{w} = \widehat{z}_0 q + \sum_{i=1}^{n} \widehat{z}_i r^i.$$

We next show that $\widehat{w}$ lies in the nonnegative orthant and that the complementarity condition holds. Since x^λ is a λ-discounted equilibrium of the auxiliary game $\Gamma_\lambda((r^S)_{\emptyset \neq S \subseteq I}, q)$ and by Eq. (7), we have

$$\widehat{w}_i = \lim_{\lambda \to 0} \widehat{\gamma}_i^\lambda(x^\lambda) \geq \lim_{\lambda \to 0} \widehat{\gamma}_i^\lambda(Q_i, x_{-i}^\lambda) = \widehat{\gamma}_i(Q_i, C_{-i}) = 0, \quad \forall i \in I_*, \tag{10}$$

and therefore the vector $\widehat{w}$ lines in the nonnegative orthant. If $z_i^0 > 0$ for some player $i \in I_*$, then $z_i^\lambda > 0$ for every λ sufficiently close to 0, hence $x_i^\lambda > 0$ for every λ sufficiently close to 0, which implies that we have equality in Eq. (10). Thus, $(\widehat{w}, \widehat{z})$ is a solution to the linear complementarity problem LCP$(\widehat{R}, \widehat{q})$, as desired.

Step 2: If under x^0 at least two players quits with positive probability then x^0 is a stationary 0-equilibrium.

Since under x^0 at least two players quit with positive probability, the play eventually terminates even if one player deviates. By Eq. (7), since x^λ is a λ-discounted equilibrium of the auxiliary game $\Gamma_\lambda((r^S)_{\emptyset \neq S \subseteq I}, q)$, and by Eq. (7) once again, we deduce that for every player $i \in I$ we have

$$\gamma_i(x^0) = \widehat{\gamma}_i(x^0) = \lim_{\lambda \to 0} \widehat{\gamma}_i^\lambda(x^\lambda) \geq \lim_{\lambda \to 0} \widehat{\gamma}_i^\lambda(Q_i, x_{-i}^\lambda) = \widehat{\gamma}_i(Q_i, x_{-i}^0) = \gamma_i(Q_i, x_{-i}^0) \tag{11}$$

and

$$\gamma_i(x^0) = \widehat{\gamma}_i(x^0) = \lim_{\lambda\to 0} \widehat{\gamma}_i^\lambda(x^\lambda) \geq \lim_{\lambda\to 0} \widehat{\gamma}_i^\lambda(C_i, x_{-i}^\lambda) = \widehat{\gamma}_i(C_i, x_{-i}^0) = \gamma_i(C_i, x_{-i}^0). \tag{12}$$

It follows that no player can profit by deviating in the original quitting game.

Step 3: If under x^0 there is a single player who quits with positive probability then x^0 can be transformed into an ε-equilibrium, for every $\varepsilon > 0$.

Fix $\varepsilon > 0$. Denote by i the unique player who quits with positive probability under x^0. As in Step 1, player i is normal. A possible deviation of player $j \neq i$ is to quit at some stage. As in Eq. (11), such a deviation is not profitable for player j.

A possible deviation of player i is to continue forever. In this case his payoff will be q_i rather than 0, so this deviation may be profitable. Since i is a normal player, his min-max value is nonpositive. Consequently, the following strategy profile is an ε-equilibrium, where $T \in \mathbb{N}$ is sufficiently large such that $(1 - x_i^0)^T \leq \frac{\varepsilon}{2}$.[5] The players play the stationary strategy profile x^0 for T stages; if the game has not been terminated by stage T, players $I \setminus \{i\}$ switch to a strategy profile that reduces player i's payoff to $v_i + \frac{\varepsilon}{2}$.

3.3 Sunspot Equilibria in which at Most One Player Quits at Every Stage

In this section we prove the second statement of Theorem 2.13. We start by observing that the vectors $\{\widehat{r}^i, i \in I_*\}$ do not lie in the nonnegative orthant.

Lemma 3.1 *If the linear complementarity problem* LCP$(\widehat{R}, \vec{0})$ *has no nontrivial solution, then for every normal player $i \in I_*$ we have $\widehat{r}^i \notin \mathbb{R}^n_{\geq 0}$.*

Proof. Assume to the contrary that $\widehat{r}^i \in \mathbb{R}^n_{\geq 0}$. Then the vector $w := \widehat{r}^i$ and the probability distribution z over $\{0, 1, \ldots, n\}$ defined by $z_i := 1$ and $z_j := 0$ for every $j \neq i$ form a solution of the linear complementarity problem LCP$(\widehat{R}, \vec{0})$, a contradiction. ■

[5] Recall that $x_i^0(C)$ is the probability by which player i continues at every stage.

3.3.1 A Candidate Set of Sunspot Equilibrium Payoffs

Our goal is to construct a sunspot ε-equilibrium in which only normal players quit, at every stage at most one player quits, that player quit with a low probability, and the play eventually terminates. This has two consequences. First, the equilibrium payoff will be in $\mathrm{conv}(r^1, \cdots, r^n)$. Second, since $r^i_i = 0$ for every player $i \in I$, a player who deviates and quits when he should not, receives an amount close to 0. Hence the equilibrium payoff will be close to the nonnegative orthant. By Lemma 2.6 all abnormal players receive a positive payoff when normal players quit. Since $r^i_i = 0$ for every player $i \in [N]$, the abnormal players are content having only normal players quit.

Since such a sunspot ε-equilibrium depends only on the restriction of the vectors $(r^i)_{i\in I}$ to normal players, we will consider in this section the $n \times n$ matrix $\widehat{R}$ whose i'th column coincides with the vector $\widehat{r}^i$. Restricted to those coordinates, the set of sunspot equilibrium payoffs in which only normal players quit, at each stage at most one player quits, and he does so with small probability, is, therefore a subset of

$$D := \mathrm{conv}(\widehat{r}^1, \cdots, \widehat{r}^n) \cap \mathbb{R}^n_{\geq 0}.$$

Denote by D_0 the points in D that have at least one coordinate that is equal to 0:

$$D_0 := \{w \in D \colon w_i = 0 \text{ for some player } i \in I_*\}.$$

Note that D_0 is a subset of the relative boundary of the set D. We will show that when the matrix $\widehat{R}$ is a Q-matrix, there is a sunspot equilibrium payoff in the set D_0.

The following observation states that whenever y is in the convex hull of $\{\widehat{r}^1, \cdots, \widehat{r}^n\}$ and is not in the nonnegative orthant, then any vector w that is part of a solution of the linear complementarity problem $\mathrm{LCP}(\widehat{R}, q)$ lies in D_0.

Lemma 3.2 *If $\widehat{R}$ is a Q-matrix and $y \in \mathrm{conv}(\widehat{r}^1, \cdots, \widehat{r}^n) \setminus \mathbb{R}^n_{\geq 0}$, then every solution (w, z) of $\mathrm{LCP}(\widehat{R}, y)$ satisfies $w \in D_0$.*

Proof. Fix a solution (w, z) of $\mathrm{LCP}(\widehat{R}, y)$. Since $y \notin \mathbb{R}^n_{\geq 0}$, the solution cannot be the trivial solution, and therefore $z_0 < 1$. Since $y \in \mathrm{conv}(\widehat{r}^1, \cdots, \widehat{r}^n)$ we have $w \in \mathrm{conv}(\widehat{r}^1, \cdots, \widehat{r}^n)$. Since $w \in \mathbb{R}^n_{\geq 0}$ it follows that $w \in D$. Since $z_0 < 1$ there is a player $i \in I$ such that $z_i > 0$, hence by the complementarity condition $w_i = 0$, and therefore $w \in D_0$. ■

3.3.2 The Basic Building Block

As mentioned before, when $y \in \mathbb{R}^n_{\geq 0}$ one solution (w, z) of the problem $\text{LCP}(\widehat{R}, y)$ is the trivial solution. The following theorem asserts that for every $y \in D_0$ there exists a nontrivial solution to a certain system that is related to problem (1). This theorem is the basic building block of our construction of a sunspot ε-equilibrium.

Theorem 3.3 *Suppose that $I_* \neq \emptyset$, that the linear complementarity problem $\text{LCP}(\widehat{R}, \vec{0})$ does not have a nontrivial solution, and that the matrix $\widehat{R}$ is a Q-matrix. Then for every $y \in D_0$ and every $\varepsilon > 0$ sufficiently small there are $w \in D_0$, $w^1, \cdots, w^n \in \mathbb{R}^n$, and $z \in \Delta(\{0, 1, 2, \ldots, n\})$ that satisfy the following conditions:*

(F.1) *For every $i \in [n]$, either $w^i \in \text{conv}(w, \widehat{r}^i) \setminus \{w\}$ or $w^i \in \text{conv}(y, \widehat{r}^i) \setminus \{y\}$.*

(F.2) *$w^i_j \geq -\varepsilon$ for every $i, j \in [n]$.*

(F.3) *$w = z_0 y + \sum_{i=1}^n z_i w^i$.*

(F.4) *If $i \in [n]$ and $z_i > 0$, then $w^i_i = 0$.*

(F.5) *$z_0 < 1$.*

Conditions (F.1) and (F.2) state that each w^i is a convex combination of $\widehat{r}^i$ and either w or y that gives positive weight to $\widehat{r}^i$, and each of its coordinates is at least $-\varepsilon$. Conditions (F.3) and (F.4) state that (w, z) is a solution of problem that has some similarity to a linear complementarity problem. Condition (F.5) states that the solution to this problem is not trivial.

We note that Lemma 3.1 implies that, provided ε is sufficiently small, the equation $w^i = \delta_i \widehat{r}^i + (1 - \delta_i) w$ has a unique solution $\delta_i \in (0, 1]$, and this solution satisfies $\delta_i < 1$. Indeed, if $\delta_i = 1$ then $w^i = \widehat{r}^i$. If ε is sufficiently small this implies that $\widehat{r}^i \in \mathbb{R}^n_{\geq 0}$, contradicting Lemma 3.1.

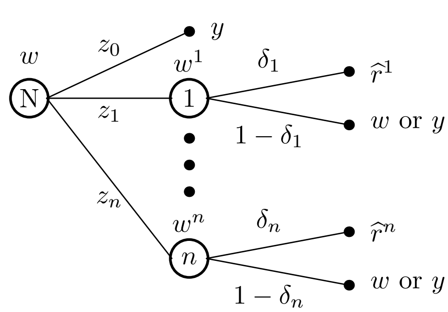

Figure 1: A graphical depiction of Theorem 3.3.

We now explain intuitively how Theorem 3.3 will be used to prove Theorem 2.13(2). Figure 1 provides a graphical interpretation to Theorem 3.3. Nature chooses an element $i \in \{0, 1, \cdots, n\}$ according the distribution z. If Nature chooses 0, the outcome is y. If Nature chooses $i \in [n]$, then player i quits with probability δ_i. If player i quits, the outcome is $\widehat{r}^i$, and otherwise it is w or y, depending on whether in Condition (F.1) $w^i \in \operatorname{conv}(w, \widehat{r}^i) \setminus \{w\}$ or $w^i \in \operatorname{conv}(y, \widehat{r}^i) \setminus \{y\}$.

Condition (F.4) asserts that every player who may be chosen is indifferent between quitting and continuing. Condition (F.1) asserts that the expected outcome if player i is chosen is w^i. This condition moreover implies that $\delta_i > 0$ for every $i \in [n]$, so that every player who is chosen, quits with positive probability. Condition (F.3) asserts that the expected outcome at the root is w. By Condition (F.5) we have $\sum_{i=1}^n z_i > 0$, hence some player i quits with positive probability.

Figure 1 can describe the behavior of the players in a single stage of the quitting game: Nature's signal chooses an element of $\{0, 1, \cdots, n\}$ according to the distribution z. If the choice is 0, no player quits; if the choice is i, player i quits with probability δ_i, while all other players continue. We will use a proper concatenation of this behavior to construct a sunspot ε-equilibrium in the quitting game $\Gamma((r^S)_{S \subseteq I})$.

In the above interpretation, if player i is chosen by nature, he quits with probability δ_i. In quitting games players can quit simultaneously, and thus, if player j quits when player i is chosen, the expected outcome will be $\delta_i \widehat{r}^{\{i,j\}} + (1-\delta_i)\widehat{r}^j$. Since player j's payoff in this case, $\delta_i \widehat{r}^{\{i,j\}}_j + (1-\delta_i)\widehat{r}^j_j = \delta_i \widehat{r}^{\{i,j\}}_j$, may be higher than w^i_j, which is his expected outcome given that player i is chosen, player j may find it beneficial to quit when player i is chosen. As in the examples in Section 2.5, to ensure that this type of deviation is not profitable, when player i is chosen, he will not quit in a single stage of the quitting game, but rather along a block of K stages, where K is sufficiently large; that is, in each stage of the block, he will quit with probability $1 - (1 - \delta_i)^{1/K}$. The expected continuation payoff along the block will thus be in the convex hull of w^i and w. Since $w \in D_0 \subset \mathbb{R}^n_{\geq 0}$ and $w^i_j \geq -\varepsilon$ for every $j \in [n]$, this will imply that the expected continuation payoff for all players along the block is at least $-\varepsilon$, so that a player who is supposed to continue throughout the block cannot profit much by deviating and quitting.

Since w is both a possible outcome at some of the leaves of the tree that appears in Figure 1 and the expected outcome of this interaction, we can create a repeated version of this game, in which, if one of the players is chosen, this player does not quit, and the outcome at the corresponding leaf is w, then another copy of the game is played, see Figure 2. Since $z_0 + \sum_{i=1}^n \delta_i z_i > 0$, the length of a play in the tree that appears in Figure 2 is distributed according to a geometric distribution. We call this auxiliary game $G(y)$. Note that the possible outcomes of $G(y)$ are $\widehat{r}^1, \widehat{r}^2, \cdots, \widehat{r}^n, y$, and the payoff under the behavior described above is w.

Denote by $w(y)$ the vector w that corresponds to $y \in D_0$ in Theorem 3.3. The natural approach to construct a sunspot ε-equilibrium in the original quitting game would be to find a sequence $(y^k)_{k\in\mathbb{N}}$ such that $y^{k+1} = w(y^k)$, and to concatenate the games that appear in Figure 2 one after the other. This is the approach that we take, though it requires some significant amendments.

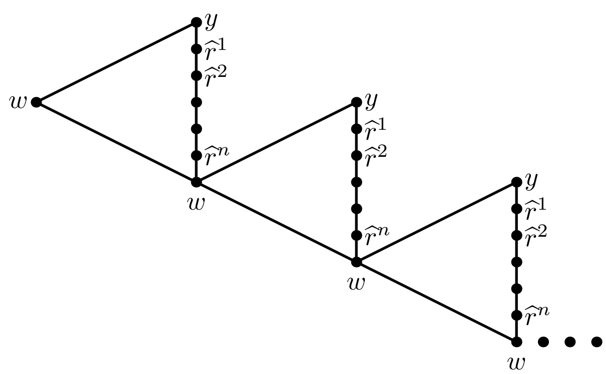

Figure 2: The auxiliary game $G(y)$ with geometric length.

By Theorem 3.3 we can choose for every $y \in D_0$ a point $w(y) \in D_0$, points $(w^i(y))_{i\in J(y)} \subset \mathbb{R}^n$, and a probability distribution $z(y) \in \Delta(\{0, 1, \cdots, n\})$ that satisfy Conditions (F.1)–(F.5). Denote by $\delta_i(y) \in (0, 1)$ the unique real number that satisfies $w^i(y) = \delta_i(y)\widehat{r}^i + (1 - \delta_i(y))w(y)$ or $w^i(y) = \delta_i(y)\widehat{r}^i + (1 - \delta_i(y))y$, depending on the exact form that Condition (F.1) takes. As described above, these quantities reflect an ε-equilibrium behavior in an auxiliary quitting game with geometric length: $w(y)$ is a sunspot ε-equilibrium payoff in the game with continuation payoff y. To construct a sunspot ε-equilibrium in the quitting game, we would like to concatenate such ε-equilibria.

If y were a sunspot ε-equilibrium in the game with continuation payoff $w(y)$, this could be done as follows: we would choose an arbitrary $y^0 \in D_0$

and define inductively $y^{k+1} := w(y^k)$. We would then implement a sunspot ε-equilibrium in the original game by playing first the ε-equilibrium that corresponds to the payoff y^0 in the auxiliary game with geometric length $G(y^1)$ with continuation payoff y^1, then the ε-equilibrium that corresponds to the payoff y^1 in the auxiliary game with geometric length $G(y^2)$ with continuation payoff y^2, and so on.

As soon as the total probability of termination under this construction is 1, the resulting strategy profile would be a sunspot ε-equilibrium in the original quitting game. There are two problems in implementing this approach.

First, $w(y)$ is a sunspot ε-equilibrium payoff in the auxiliary game with continuation payoff y, and not vice versa. Hence, if we choose $y^0 \in D_0$ arbitrarily and define inductively $y^{k+1} := w(y^k)$, then time goes backwards: we should choose some large $K \in \mathbb{N}$, play first a sunspot ε-equilibrium with payoff y^K in the auxiliary game with geometric length $G(y^{K-1})$ with continuation payoff y^{K-1}, then a sunspot ε-equilibrium with payoff y^{K-1} in the auxiliary game with geometric length $G(y^{K-2})$ with continuation payoff y^{K-2}, and so on, until we play a sunspot ε-equilibrium with payoff y^1 in the auxiliary game with geometric length $G(y^0)$ with continuation payoff y^0. After that we let the player play in an arbitrary way. If the probability that the game is not terminated by a player before we end playing the sequence of sunspot ε-equilibria is small, then the way players play after implementing the sunspot ε-equilibrium in $G(y^0)$ does not affect much the payoff, and we would still obtain a sunspot approximate equilibrium.

The second issue concerns the probability of termination. The probability that the play in the auxiliary game with geometric length $G(y^k)$ with continuation payoff y^k terminates by a player under the sunspot ε-equilibrium with payoff y^{k+1} is $\sum_{i \in I_*} z_i(y^k)$. By Condition (F.5) this quantity is positive, but we do not have a uniform lower bound on it. Hence, we cannot ensure that the probability of termination by a player under the finite concatenation of sunspot ε-equilibria in auxiliary games with geometric length can be made arbitrarily high. To overcome this difficulty we will construct a sequence $(y^k)_{k=1}^K$ that satisfies the required properties approximately. This approach is close to Theorem 3 in Simon (2007).

One way to construct the approximating sequence is by transfinite induction. Instead of defining a countable sequence $(y^k)_{k \in \mathbb{N}}$ such that $y^{k+1} := w(y^k)$, we define a transfinite sequence $(y^\alpha)_\alpha$ that assigns to every successor ordinal $\alpha+1$ the element $y^{\alpha+1} := w(y^\alpha)$, and to every limit ordinal α the element $y^\alpha := \lim_{\beta<\alpha} y^\beta$ (assuming the limit exists). Since $z(y^\alpha) > 0$ for every ordinal α, there is a minimal ordinal α^* that satisfies $\sum_{\alpha<\alpha_*} z(y^\alpha) = \infty$. We show that y^α is well defined for every $\alpha < \alpha_*$. Using the definition of a general sum of numbers, we argue that there is a finite sequence $(\alpha^k)_{k=1}^K$, which satisfies that $y^{\alpha^{k+1}}$ is close to $w(y^{\alpha^k})$ for every k, and that the noise added by this approximation is negligible relative to the probabilities of termination $(z(y^{\alpha^k}))_{k=1}^K$.

3.3.3 Proof of Theorem 3.3

To prove Theorem 3.3 we need two notations. For every nonempty set $J \subseteq I_*$ of normal players define

$$S(J) := \mathrm{conv}\{\widehat{r}^i, i \in J\}.$$

For every $y \in \mathbb{R}^n$ denote

$$J_y := \{i \in [n] \colon y_i = 0\}.$$

The proof of Theorem 3.3 is divided into three cases. Let $y \in D_0$.

- If $y \in S(J_y)$, then the claim holds trivially, because we can take $w^i = (1-\varepsilon)y + \varepsilon\widehat{r}^i$ for each $i \in [n]$, $w = y$ and choose $(z_0, z_1, \cdots, z_n)$ so that $y = \sum_{i \in J_y} z_i \widehat{r}^i$.
- The case $y \notin S(J_y)$ and $\mathrm{conv}(S(J_y), y) \cap D = \{y\}$ is handled in Lemma 3.4. In this case we use the assumption that the matrix $\widehat{R}$ is a Q-matrix.
- The case $y \notin S(J_y)$ and $\mathrm{conv}(S(J_y), y) \cap D \supsetneq \{y\}$ is handled in Lemma 3.5.

Lemma 3.4 *Let $y \in D_0$ be such that $y \notin S(J_y)$. If $\mathrm{conv}(S(J_y), y) \cap D = \{y\}$ then the conclusion of Theorem 3.3 holds.*

The proof of the lemma goes as follows. Using the fact that the matrix $\widehat{R}$ is a Q-matrix we will show that there is a vector $w \in D_0 \setminus \{y\}$ that lies in the convex hull of y and $S(J_y)$ (see Figure 3(B)) such that $z_i = 0$ or $w_i = 0$ for every $i \in I_*$. We will then define w^i to be a point in the convex hull of w and $\widehat{r}^i$ close to w.

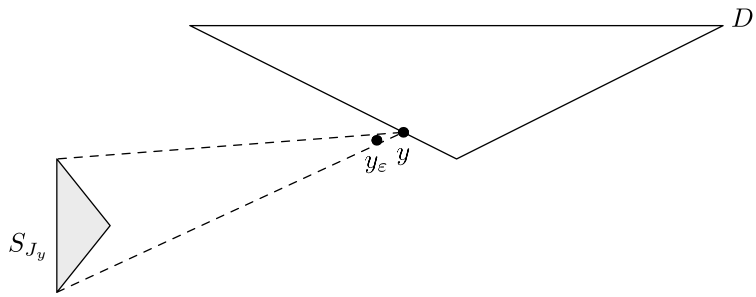

Figure 3, Part A: The construction in Lemma 3.4.

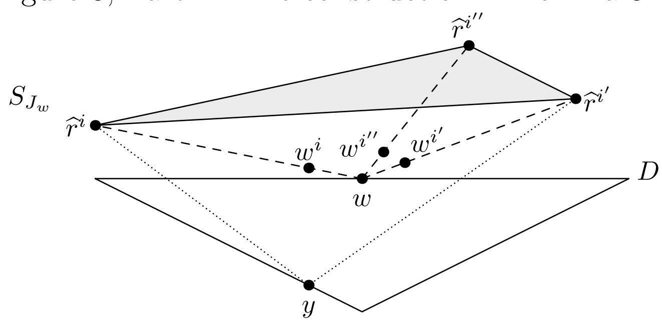

Figure 3, Part B: The construction in Lemma 3.4.

Proof. For every $\varepsilon > 0$ let $y_\varepsilon \in \text{conv}(S(J_y), y) \setminus \{y\}$ satisfy $d(y_\varepsilon, y) \le \varepsilon$ (see Figure 3(A)). Since $y \in D \subseteq \text{conv}(\widehat{r}^1, \cdots, \widehat{r}^n)$, it follows that $y_\varepsilon \in \text{conv}(\widehat{r}^1, \cdots, \widehat{r}^n)$. Since $\text{conv}(S(J_y), y) \cap D = \{y\}$ we have $y_\varepsilon \notin D = \text{conv}(\widehat{r}^1, \cdots, \widehat{r}^n) \cap \mathbb{R}^n_{\ge 0}$, and therefore $y_\varepsilon \notin \mathbb{R}^n_{\ge 0}$.

Let $(w_\varepsilon, z_\varepsilon) \in \mathbb{R}^n_{\ge 0} \times \Delta(\{0, 1, \cdots, n\})$ be a solution of the linear complementarity problem $\text{LCP}(R, y_\varepsilon)$, so that $w_{\varepsilon,i} = 0$ or $z_{\varepsilon,i} = 0$ for every $i \in [n]$, and

$$w_\varepsilon = z_{\varepsilon,0} y_\varepsilon + \sum_{i \in I_*} z_{\varepsilon,i} \widehat{r}^i.$$

By Lemma 3.2, $w_\varepsilon \in D_0$. In particular, $w_\varepsilon \in \text{conv}(S(J_{w_\varepsilon}), y_\varepsilon) \cap D$. By taking a subsequence we can assume that the three limits

$$w := \lim_{\varepsilon \to 0} w_\varepsilon, \quad z := \lim_{\varepsilon \to 0} z_\varepsilon, \quad y := \lim_{\varepsilon \to 0} y_\varepsilon$$

exist. We can moreover assume that the sets $(J(w_\varepsilon))_{\varepsilon > 0}$ are independent of ε. Since $w_\varepsilon \in D_0$ for every $\varepsilon > 0$ it follows that $w \in D_0$. Note that $w = z_0 y + \sum_{i \in I_*} z_i \widehat{r}^i$. Furthermore, for every $i \in [n]$ we have $w_i = 0$ or $z_i = 0$. We argue that $w \neq y$.

Indeed, assume by contradiction that $y = w = \lim_{\varepsilon \to 0} w_\varepsilon$. It follows that $J_{w_\varepsilon} \subseteq J_y$ for every $\varepsilon > 0$ sufficiently small. In particular, $S(J_{w_\varepsilon}) \subseteq S(J_y)$

for every $\varepsilon > 0$ sufficiently small. Since $\operatorname{conv}(S(J_y), y) \cap D = \{y\}$ and since $y_\varepsilon \in \operatorname{conv}(S(J_y), y) \setminus \{y\}$, we conclude that $\operatorname{conv}(S(J_{w_\varepsilon}), y_\varepsilon) \cap D = \emptyset$. But $w_\varepsilon \in \operatorname{conv}(S(J_{w_\varepsilon}), y_\varepsilon) \cap D$, a contradiction.

Fix now $\varepsilon > 0$. Define

$$\widehat{z}_0 := \frac{\varepsilon z_0}{\varepsilon z_0 + \sum_{i=1}^n z_i}, \qquad \widehat{z}_i := \frac{z_i}{\varepsilon z_0 + \sum_{i=1}^n z_i}, \quad \forall i \in [n].$$

Note that $z_i > 0$ if and only if $\widehat{z}_i > 0$. Therefore $w_i = 0$ or $\widehat{z}_i = 0$ for every $i \in [n]$. Define for every $i \in J_w$,

$$w^i := (1 - \varepsilon) w + \varepsilon \widehat{r}^i, \tag{13}$$

see Figure 3(B).

We argue that the conclusion of Theorem 3.3 holds for w, $(w^i)_{i \in [n]}$, and $(\widehat{z}^i)_{i=0}^n$. By construction Conditions (F.1) and (F.2) hold. If $\widehat{z}_i > 0$ then $z_i > 0$, hence $w_i = 0$, which implies that $w^i_i = 0$, so that Condition (F.4) holds as well. Since $w = z_0 y + \sum_{i \in I_*} z_i \widehat{r}^i$, and since $w \neq y$, it follows that $\sum_{i \in [n]} z_i > 0$, and therefore $\sum_{i \in [n]} \widehat{z}_i > 0$, implying that Condition (F.5) holds. We now verify that Condition (F.3) holds as well. By Condition (F.2) and Eq. (13),

$$\varepsilon w = \varepsilon z_0 y + \sum_{i \in I_*} \varepsilon z_i \widehat{r}^i = \varepsilon z_0 y + \sum_{i \in I_*} z_i (w^i - (1 - \varepsilon) w).$$

This implies that

$$w = \frac{\varepsilon z_0 y + \sum_{i \in I_*} z_i w^i}{\varepsilon z_0 + \sum_{i \in I_*} z_i} = \widehat{z}_0 y + \sum_{i \in I_*} \widehat{z}_i w^i,$$

and Condition (F.3) holds as well. ∎

Lemma 3.5 *Let $y \in D_0$ such that $y \notin S(J_y)$. If $\operatorname{conv}(S(J_y), y) \cap D \supsetneq \{y\}$ then the conclusion of Theorem 3.3 holds.*

The proof of Lemma 3.5 will be divided into two cases. If there is $i_0 \in J_y$ such that $\operatorname{conv}(\widehat{r}^{i_0}, y) \cap D \supsetneq \{y\}$ (see Figure 4(A)), then w will be a point in D_0 on the line segment between y and $\widehat{r}^{i_0}$, z will be the Dirac measure on i_0, and w^{i_0} will be a convex combination of w and $\widehat{r}^{i_0}$ close to w^{i_0}. If there is no such $i_0 \in J_y$ (see Figure 4(B)), then for $\varepsilon > 0$ sufficiently small the set $(1-\varepsilon)y + \varepsilon S(J_y)$ is close to y, intersects D, and its extreme points are outside D. We then take w to be a point in this set that lies in D_0, and (w^i) to be the extreme points of this set, which are close to D and satisfy that each w^i lies on the line segment between y and $\widehat{r}^i$.

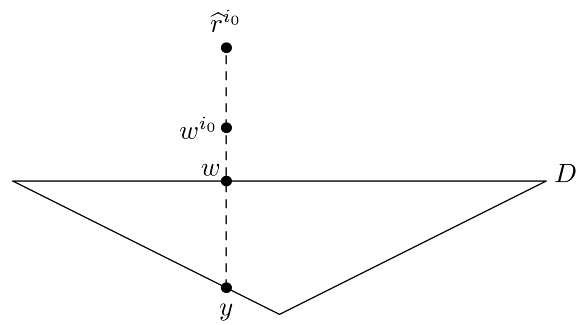

Figure 4, Part A: The construction in Lemma 3.5.

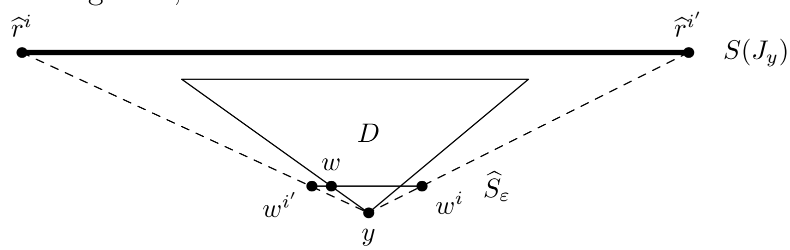

Figure 4, Part B: The construction in Lemma 3.5.

Proof. Fix $\varepsilon > 0$ sufficiently small. Assume first that there is $i_0 \in J_y$ such that $\mathrm{conv}(\widehat{r}^{i_0}, y) \cap D \supsetneq \{y\}$, see Figure 4(A). Since $\widehat{r}^{i_0} \notin D$ while $y \in D_0$ and $\mathrm{conv}(\widehat{r}^{i_0}, y) \cap D \supsetneq \{y\}$, it follows that there is $w \in \mathrm{conv}(\widehat{r}^{i_0}, y)$ that lies on the relative boundary of D. Hence, there is $\delta \in (0,1)$ such that $w = \delta\widehat{r}^{i_0} + (1-\delta)y$. Since $i_0 \in J(y)$ it follows that $w_{i_0} = 0$, and therefore $w \in D_0$. Thus, Theorem 3.3 holds with $w^{i_0} = (1-\varepsilon)w + \varepsilon\widehat{r}^{i_0}$, and z that is defined by

$$z_0 := \frac{\varepsilon(1-\delta)}{\varepsilon(1-\delta)+\delta}, \qquad z_{i_0} := \frac{\delta}{\varepsilon(1-\delta)+\delta}.$$

Indeed, since $w = \delta r^i + (1-\delta)y$ we have

$$\varepsilon w = \varepsilon\delta\widehat{r}^i + \varepsilon(1-\delta)y = \delta(w^i = (1-\varepsilon)w) + \varepsilon(1-\delta)y,$$

so that

$$w = \frac{\varepsilon(1-\delta)}{\varepsilon(1-\delta)+\delta}y + \frac{\delta}{\varepsilon(1-\delta)+\delta}\widehat{r}^i.$$

Assume now that $\text{conv}(\widehat{r}^i, y) \cap D = \{y\}$ for every $i \in J_y$ and consider the set (see Figure 4(B))

$$\widehat{S}_\varepsilon := (1-\varepsilon)y + \varepsilon S(J_y).$$

Since $\text{conv}(\widehat{r}^i, y) \cap D = \{y\}$, the set $\widehat{S}_\varepsilon$ is not a subset of D. Moreover, provided ε is sufficiently small, this set intersects D. By the definition of $S(J_y)$, this implies that there is a point $w \in D_0 \cap \widehat{S}_\varepsilon$. Since $\delta > 0$, we have $w \neq y$.

For every player $i \in J(y)$ define

$$w^i := (1-\varepsilon)y + \varepsilon r^i.$$

The points $(w^i)_{i \in J(y)}$ are the extreme points of the set $\widehat{S}_\varepsilon$, and therefore there is a probability distribution $z \in \Delta(J_y)$ such that $w = \sum_{i \in I_*} z_i w^i$. The reader can verify that the conclusion of Theorem 3.3 holds with w, $(w^i)_{i \in J_y}$, and z.
■

3.3.4 An Approximation Result

We start by a technical observation that will serve as an approximation tool.

Theorem 3.6 *Let (X, d) be a complete metric space and let $f : X \to X$ be a function that does not have any fixed point. For every $c, C > 0$ there are $K \in \mathbb{N}$ and a sequence $(x^k)_{k=1}^K$ of points in X such that the following properties hold:*

(A.1) $\sum_{k=1}^{K} d(x^k, f(x^k)) > C$.

(A.2) $\sum_{k=1}^{K-1} d(x^{k+1}, f(x^k)) < c$.

Figure 5 provides a graphical depiction of Theorem 3.6; each solid line represents the distance between some x^k and $f(x^k)$, and each dashed line represents the distance between some $f(x^k)$ and x^{k+1}. The theorem claims that the total length of the solid lines is above C, while the total length of the dashed lines is less than c.

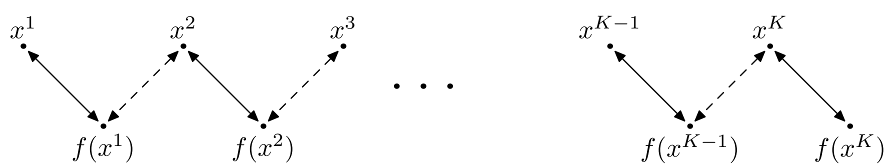

Figure 5: The construction in Theorem 3.6.

Proof. The proof uses a transfinite construction. We define an ordinal α_* and a sequence $(x^\alpha)_{\alpha<\alpha_*}$ as follows:

(TI.1) $x^0 \in X$ is arbitrary.

(TI.2) If α is a successor ordinal, set $x^\alpha := f(x^{\alpha-1})$.

(TI.3) If α is a limit ordinal and $\sum_{\beta<\alpha} d(x^\beta, f(x^\beta)) = \infty$, set $\alpha_* := \alpha$ and terminate the definition of the sequence.

(TI.4) If α is a limit ordinal and $\sum_{\beta<\alpha} d(x^\beta, f(x^\beta)) < \infty$, set $x^\alpha := \lim_{\beta<\alpha} x^\beta$.

We note that if α is a limit ordinal and $\sum_{\beta<\alpha} d(x^\beta, f(x^\beta)) < \infty$, then for every $\varepsilon > 0$ there is an ordinal $\alpha^\varepsilon < \alpha$ for which $\sum_{\alpha^\varepsilon\leq\beta<\alpha} d(x^\beta, f(x^\beta)) < \varepsilon$, which implies the existence of the limit $\lim_{\beta<\alpha} x^\beta$. Since the space X is complete, the limit $\lim_{\beta<\alpha} x^\beta$ in Case (TI.4) is in X, hence the definition of the sequence $(x^\alpha)_{\alpha<\alpha_*}$ is valid.

Since f has no fixed point, $d(x, f(x)) > 0$ for every $x \in X$, and therefore the construction ends at some ordinal α_*. In fact, since the set of rational numbers is dense in the set of real numbers, the ordinal α_* is a countable ordinal. Since $\sum_{\beta<\alpha_*} d(x^\beta, f(x^\beta)) = \infty$, there is an ordinal α_1 such that $\sum_{\beta<\alpha_1} d(x^\beta, f(x^\beta)) > C + 1$. By definition,

$$\sum_{\alpha<\alpha_*} d(x^\alpha, f(x^\alpha)) = \sup_A \sum_{\alpha\in A} d(x^\alpha, f(x^\alpha)),$$

where A ranges over all finite sets of ordinals smaller than α_*, hence there is a finite set A of ordinals smaller than α_1 such that the following two conditions hold:

(A.1') $\sum_{\alpha\in A} d(x^\alpha, f(x^\alpha)) > C$.

(A.2') $\sum_{\alpha\notin A,\alpha<\alpha_1} d(x^\alpha, f(x^\alpha)) < c$.

Denote $K := |A|$ and $A = \{\alpha^1, \alpha^2, \cdots, \alpha^K\}$, and assume that $\alpha^1 < \alpha^2 < \cdots < \alpha^K$. By Condition (A.1'),

$$\sum_{k=1}^{K} d(x^{\alpha^k}, f(x^{\alpha^k})) = \sum_{\alpha\in A} d(x^\alpha, f(x^\alpha)) > C.$$

By the triangle inequality and Condition (A.2'),

$$\begin{aligned}\sum_{k=1}^{K} d(x^{\alpha^k}, f(x^{\alpha^{k+1}})) &\leq \sum_{\alpha^k \leq \alpha < \alpha^{k+1}} d(x^\alpha, f(x^\alpha)) \\ &\leq \sum_{\alpha \notin A, \alpha < \alpha_1} d(x^\alpha, f(x^\alpha)) < c.\end{aligned}$$

The result follows. ■

Remark 3.7 *When X is compact, as is the case in our application, Theorem 3.6 can be proven by standard arguments as follows. Denote by T the diameter of X, that is, $d(x,y) \leq T$ for every $x, y \in X$. For every $x \in X$ let $D(x)$ be the open ball with radius $\frac{c}{C+T} d(x, f(x))$ around x. Since X is compact, there are $x_1, x_2, \cdots, x_L$ such that $\cup_{l=1}^{L} D(x_l) = X$. For every $y \in X$ denote by $l(y) \in [L]$ an index that satisfies $y \in D(x_{l(y)})$. Let $x^0 \in X$ be arbitrary, and define recursively $x^{k+1} := x_{l(f(x^k))}$, for every $k \geq 0$. By definition, the distance $d(x^{k+1}, f(x^k))$ is smaller than the radius of $D(x^{k+1})$, which is $\frac{c}{C+T} d(x^{k+1}, f(x^{k+1}))$.*

Since f has no fixed points, $\min_{l \in [L]} d(x_l, f(x_l)) > 0$. Denote by K the minimal integer such that $\sum_{k=1}^{K} d(x^k, f(x^k)) > C$. We have

$$\begin{aligned}\sum_{k=1}^{K-1} d(x^{k+1}, f(x^k)) &\leq \tfrac{c}{C+T} \sum_{k=2}^{K} d(x^k, f(x^k)) \\ &\leq \tfrac{c}{C+T} \times (C+T) = c,\end{aligned}$$

where the last inequality follows from the minimality of K. It follows that the sequence $(x^k)_{k=1}^{K}$ satisfies Conditions (A.1) and (A.2).

3.3.5 Constructing a Strategy Profile ξ^*

Fix $\varepsilon > 0$ sufficiently small so that each of the vectors $\widehat{r}^i$ contains an entry that is smaller than $-\varepsilon$. We now define a strategy profile ξ^* in the quitting game, which will turn out to be a sunspot 7ε-equilibrium.

We note that under the assumptions of Theorem 2.13, the function $w : D_0 \to D_0$ does not have a fixed point. Indeed, the existence of such a fixed point implies that the linear complementarity problem LCP$(\widehat{R}, \vec{0})$ has a nontrivial solution.

By Theorem 3.6 applied to $C = \frac{\binom{n}{2}\cdot 2(1+\varepsilon)}{\varepsilon^2}$, $c = \varepsilon$, $X = D_0$ endowed with the supremum norm, and $f(y) = w(y)$ for every $y \in D_0$, there are $K \in \mathbb{N}$ and a sequence $(y^k)_{k=1}^K$ that satisfy

(A.1") $\sum_{k=1}^K \|y^k - w(y^k)\|_\infty > \frac{\binom{n}{2}\cdot 2(1+\varepsilon)}{\varepsilon^2}$.

(A.2") $\sum_{k=1}^{K-1} \|y^{k+1} - w(y^k)\|_\infty < \varepsilon$.

For every $k \in [K]$ we have $\delta_i(y^k) \in (0,1)$, hence there is $C_k \in \mathbb{N}$ sufficiently large such that

$$1 - (1 - \delta_i(y^k))^{1/C_k} < \varepsilon, \quad \forall i \in I_*.$$

The strategy profile ξ^* that we will construct will yield a payoff close to $w(y^K)$. We will partition the set of stages $\mathbb{N}$ into $K+1$ kiloblocks of random (possibly infinite) size. For each $k \in [K]$, kiloblock k will mimic the sunspot ε-equilibrium in the auxiliary game with geometric length $G(y^{K-k+1})$ with continuation payoff y^{K-k+1} that yields equilibrium payoff $w(y^{K-k+1})$. The last kiloblock will represent the rest of the game. Condition (A.1") will imply that under ξ^* with high probability the play terminates in one of the first K kiloblocks. Condition (A.2") will imply that the fact that the equilibrium payoff of the play in the k'th kiloblock, namely, $w(y^{K-k+1})$, differs from the continuation payoff in the auxiliary game $G(y^{K-k})$, does not affect much the payoffs of the players.

The formal partition of the set of stages $\mathbb{N}$ into $K+1$ kiloblocks of random (possibly infinite) size is as follows. For $k \in [K]$, the kiloblock is divided into blocks of size C_k; each block has a *type* from the set $\{0, 1, \cdots, n\}$. At the beginning of the kiloblock, as well as at the end of each block, the type of the coming block is chosen by nature.

- With probability $z_i(y^{K-k})$ the type of the next block is i.
- With probability $z_0(y^{K-k})$ the type of the next block is 0, this block is the last block of the kiloblock, and the next kiloblock starts once this block ends.

The last kiloblock, which is the $(K+1)$'th kiloblock, is not divided into blocks and contains all remaining stages.

Define a strategy profile ξ^* as follows. At stage t,

- If t lies in a block of type $i \in [n]$ in kiloblock $k \in [K]$, then at stage t player i quits with probability $1 - (1 - \delta_i(y^{K-k+1}))^{1/C_{K-k+1}}$, and all other players continue.

- If t lies in the last block of a kiloblock (and then its type is necessarily 0), or in the $(K+1)$'th kiloblock, then all players continue at stage t.

3.4 The Strategy Profile ξ^* is a Sunspot 7ε-equilibrium.

In this section we prove that the strategy profile ξ^* is a sunspot 7ε-equilibrium. We will use the following inequality, which holds since, by Condition (F.3), $w(y) - y = \sum_{i=1}^n z_i(y) \cdot (w^i - y)$:

$$\|w(y) - y\|_\infty \leq 2 \sum_{i=1}^n z_i(y), \quad \forall y \in D_0. \tag{14}$$

We first prove that the expected payoff of the normal players under the strategy profile ξ^* is close to $w(y^K)$.

Lemma 3.8 *For every normal player $i \in I_*$ we have $|\gamma_i(\xi^*) - w_i(y^K)| < 2\varepsilon$.*

Proof. Fix a normal player $i \in I_*$. Define a stochastic process $\eta_i = (\eta_i^k)_{k=1}^{K+1}$ as follows:

- If the play was terminated at some stage t_* before kiloblock k, set

$$\eta_i^k := \widehat{r}_i^{S_*} + \sum_{l<k} \|y_i^{K-l+1} - w_i(y^{K-l})\|_\infty, \tag{15}$$

  where S_* is the set of players who quit at stage t_*.

- If the play was not terminated before kiloblock k, set

$$\eta_i^k := w_i(y^{K-k+1}) + \sum_{l<k} \|y_i^{K-l+1} - w_i(y^{K-l})\|_\infty. \tag{16}$$

By Eq. (14) and Condition (A.1”) we have $\sum_{k=1}^K z_i(y^k) \geq \frac{1}{\varepsilon}$, hence under the strategy profile ξ^* the play terminates during the first K kiloblocks with probability at least $1 - \varepsilon$. By Condition (F.3), the process η_i is a submartingale under the strategy profile ξ^*, hence

$$w_i(y^K) = \eta_i^1 \leq \mathbf{E}_{\xi^*}[\eta_i^{K+1}] \leq \gamma_i(\xi^*) + 2\varepsilon,$$

where the last inequality holds by the choice of c, Condition (A.2"), and since with high probability the play terminates during the first K kiloblocks.

Similarly, if one replaces the plus sign in Eqs. (15) and (16) with a minus sign, the process η_i becomes a supermartingale under the strategy profile ξ^*, hence

$$w_i(y^K) = \eta_i^1 \geq \mathbf{E}_{\xi^*}[\eta_i^{K+1}] \geq \gamma_i(\xi^*) - 2\varepsilon,$$

and the claim follows. ■

The next lemma, which relies on the definition of C, states that even if some normal player $i \in I_*$ deviates and continues forever, the play terminates with high probability before the end of the K'th kiloblock.

Lemma 3.9 *If $\widehat{r}^i \notin \mathbb{R}^n_{\geq 0}$ for every $i \in I_*$, then for every player $j \in I_*$ we have*

$$\mathbf{P}_{(C_j,\xi^*_{-j})}(t_* \textit{ is smaller than the stage in which kiloblock } K+1 \textit{ starts}) \geq 1-\varepsilon.$$

Proof. Set $L := \frac{\binom{n}{2}}{\varepsilon}$. By the choice of C and Condition (A.1"), there are $1 = k_1 < k_2 < \cdots < k_L < k_{L+1} = K$ such that

$$\sum_{k=k_l}^{k_{l+1}} d(y^k, w(y^k)) > \tfrac{2}{\varepsilon}, \quad \forall l \in [L]. \tag{17}$$

Call the collection of kiloblocks $\{k \colon k_l \leq k < k_{l+1}\}$ the *l'th megablock*. Eqs. (14) and (17) imply that under the strategy profile ξ^* the probability that the play terminates during each megablock is at least $1-\varepsilon$. We claim that there are at least two normal players who, under strategy profile ξ^*, quit during each megablock with probability at least ε. Indeed, consider the l'th megablock and assume by way of contradiction that there is a unique player $i \in I_*$ who quits with probability larger than $1-\varepsilon$ during this megablock. Then $\|w(y^{K-k_l+1}) - \widehat{r}^i\|_\infty \leq n\varepsilon$. However, $w(y^{K-k_l+1}) \in D_0 \subseteq \mathbb{R}^n_{\geq 0}$, while $\widehat{r}^i \notin \mathbb{R}^n_{\geq 0}$, a contradiction when ε is sufficiently small.

Since there are $\binom{n}{2}$ pairs of players, the choice of L implies that there is a pair of players who quit with probability at least ε in at least $\frac{1}{\varepsilon}$ megablocks. The result follows. ■

The next result completes the proof that ξ^* is a sunspot 7ε-equilibrium.

Lemma 3.10 *For every player $i \in I$ and every pure strategy $\xi_i \in X_i$ we have $\gamma_i(\xi_i, \xi^*_{-i}) \le w_i(y^K) + 5\varepsilon$.*

Proof. Consider first a normal player $i \in I_*$. We define a stochastic process $\eta_i = (\eta_i^t)_{t\in\mathbb{N}}$, which approximates the expected continuation payoff of player i from stage t and on. Unlike in the proof of Lemma 3.8, where the process was defined for kiloblocks, here it is defined for stages. For each stage t that lies in kiloblock k:

- If the play was terminated at some stage $t_* < t$, set
$$\eta_i^t := \widehat{r}_i^{S_*} - \sum_{l<k} \|y^{K-l+1} - w(y^{K-l})\|_\infty,$$
where S_* is the set of players who quit at stage t_*.
- If t lies in a block of type 0 in the k'th kiloblock, we set
$$\eta_i^t := y_i^{K-k+1} - \sum_{l<k} \|y^{K-l+1} - w(y^{K-l})\|_\infty.$$
- If k is the first stage of a block of type i in the k'th kiloblock, we set
$$\eta_i^t := w_i(y^{K-k+1}) - \sum_{l<k} \|y^{K-l+1} - w(y^{K-l})\|_\infty.$$
- If k is the l'th stage of a block of type i in the k'th kiloblock, we set
$$\eta_i^t := \delta_* w_i(y^{K-k+1}) + (1-\delta_*)\widehat{r}_i^i - \sum_{l<k} \|y^{K-l+1} - w(y^{K-l})\|_\infty,$$
where $\delta_* = (1-\delta_i(y^{K-k+1}))^{(C_{K-k+1}-l+1)/C_{K-k+1}}$.

By Conditions (F.3) and (A.2"), the process $(\eta_i^t)_{t\in\mathbb{N}}$ is a supermartingale under the strategy profile ξ^*. Whenever player i quits under ξ_i^* with positive probability, he is indifferent between quitting and continuing. Hence, the process η_i is a supermartingale under the strategy profile (C_i, ξ^*_{-i}) as well. Whenever a player other than player i quits with positive probability, he does so with probability smaller than ε. By Assumption 2.1 and since $y^k \in D_0 \subset \mathbb{R}^n_{\ge 0}$ for every $k \in [K]$, it follows that for every strategy ξ_i of player i,
$$w_i(y^K) = \eta_i^1 \ge \mathbf{E}_{(\xi_i,\xi^*_{-i})}[\eta_i^{K+1}] \ge \gamma(\xi_i, \xi^*_{-i}) - 5\varepsilon.$$

Consider now an abnormal player $i \notin I_*$. Under the strategy profile ξ^*, whenever a normal player quits, he does so with probability smaller than ε. Consequently, if player i quits at some stage, his expected terminal payoff is bounded by ε. Let η_i^t be the expected payoff of player i from stage t and on, assuming that if the game is not terminated by the end of the K'th kiloblock, the continuation payoff is 0. The process $(\eta_i^t)_{t\in\mathbb{N}}$ is a martingale that attains nonnegative values before the play terminates at stage t_*. Condition (A.1") and Eq. (14) imply that the probability that the game is not terminated by the end of the K'th kiloblock is smaller than ε, hence as above

$$w_i(y^K) = \eta_i^1 \geq \mathbf{E}_{(\xi_i,\xi^*_{-i})}[\eta_i^{K+1}] \geq \gamma(\xi_i, \xi^*_{-i}) - 2\varepsilon,$$

and the desired result follows. ■

4 Characterizing the Set of Sunspot Equilibrium Payoffs

A vector $x \in \mathbb{R}^N$ is a *sunspot equilibrium payoff* if it is the limit of payoffs that correspond to sunspot ε-equilibria, as ε goes to 0. Theorem 2.13 proves that if the matrix $\widehat{R}$ is a Q-matrix, then there is a sunspot equilibrium payoff in the set D_0. A complete characterization of the set of sunspot equilibrium payoffs seems to be at present out of reach, yet it may be possible to characterize the set of sunspot equilibrium payoffs that can be generated by quittings of single players. In this section we identify one case in which the set of these sunspot equilibrium payoffs can be characterized.

Definition 4.1 *A Q-matrix is called an M-*matrix *if in every row and every column of the matrix there is exactly one positive entry.*

Remark 4.2 *In the literature of linear complementarity problems, a Q-matrix is called an M-*matrix *if its diagonal entries are positive and all other entries are nonpositive, see Murty (1988). Since we require that $r_i^i = 0$ for every $i \in I$, our concept of M-matrix differs from the usual one.*

Theorem 4.3 *If the matrix $\widehat{R}$ is an M-matrix, then any payoff vector in $\widetilde{D}$ is a sunspot equilibrium payoff, where $\widetilde{D} := \mathrm{conv}\{r^1, \cdots, r^n\} \cap \mathbb{R}^N_{\geq 0}$.*

Proof. It is well known that any M-matrix $\widehat{R}$ is inverse positive, that is, its inverse $\widehat{R}^{-1}$ is a nonnegative matrix (see, e.g., Fujimoto and Ranade, 2004). Fix $i \in [n]$. Since $\widehat{R}^{-1}$ is a nonnegative matrix, $\widehat{R}^{-1}e^i \in \mathbb{R}^n_{\geq 0}$, where $e^i = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the i'th unit vector in $\mathbb{R}^n$. Therefore we can write

$$\widehat{R}^{-1}e^i = \sum_{j=1}^{n} \lambda^i_j e^j, \tag{18}$$

where $(\lambda^i_j)_{j=1}^n$ are nonnegative numbers, not all of them zero. Set

$$\widehat{\lambda}^i_j := \frac{\lambda^i_j}{\sum_{k=1}^n \lambda^i_k}.$$

Multiplying both sides of Eq. (18) from the left by $\widehat{R}$ we get

$$e^i = \sum_{j=1}^{n} \lambda^i_j \widehat{R} e^j = \sum_{j=1}^{n} \lambda^i_j \widehat{r}^j,$$

so that $w^i := \frac{1}{\|\lambda^i\|_1} e^i \in D$, for each $i \in [n]$. Both convex hulls $\text{conv}(w^1, \cdots, w^n)$ and $\text{conv}(\widehat{r}^1, \cdots, \widehat{r}^n)$ are $(n-1)$-dimensional sets such that $\text{conv}(w^1, \cdots, w^n) \subseteq \text{conv}(\widehat{r}^1, \cdots, \widehat{r}^n)$. Since in every row of $\widehat{R}$ there is a single positive entry, it follows that $D = \text{conv}(w^1, \cdots, w^n)$: every element of D can be presented as a weighted average of $w^1, \cdots, w^n$.

Since $\widehat{R}$ is an M-matrix, there is a unique index $j_i \in [n]$ such that $\widehat{r}^{j_i}_i > 0$. Since w^i is a convex combination of $\widehat{r}^1, \cdots, \widehat{r}^n$, since $w^i_i > 0$, and since the unique index k such that $\widehat{r}^k_i > 0$ is $k = j_i$, it follows that $\lambda^i_{j_i} > 0$. Since for all coordinates $k \neq i$ we have $\widehat{r}^{j_i}_k \leq 0$, for $\alpha_i > 0$ sufficiently small we have $w^i - \alpha_i \widehat{r}^{j_i} \in \mathbb{R}^n_{\geq 0}$, which implies that $y^{[i]} := \frac{w^i - \alpha_i \widehat{r}^{j_i}}{1-\alpha_i} \in \text{conv}(w^1, \cdots, w^n)$. Consequently $w^i = \alpha_i \widehat{r}^{j_i} + (1-\alpha_i) y^{[i]}$ and there is a probability distribution $\beta^i = (\beta^i_j)_{j \in [n]}$ such that $y^{[i]} = \sum_{j=1}^n \beta^i_j w^j$.

The reader can verify that for every $i \in I_*$, the vectors $(y^{[i]})_{i \in [n]}$, $(w^j)_{j \in [n]}$, and $(\beta^i_j)_{i,j \in [n]}$ satisfy the following conditions, which are analogous to Conditions (F.1)–(F.5):

(F.1') $w^i \in \text{conv}(y^{[i]}, \widehat{r}^i) \setminus \{y^{[i]}\}$ for every $i \in [n]$.

(F.2') $w^i_j \geq 0$ for every $i, j \in [n]$.

(F.3') $y^{[i]} = \sum_{j=1}^n \beta^i_j w^i$, for every $i \in [n]$.

(F.4') If $i \in [n]$ and $\beta^i_j > 0$, then $w^i_i = 0$.

This implies that we can repeat the construction in Section 3.3, yet now the sequence $(y^\alpha)_{\alpha<\alpha_*}$ has a finite range, namely $(y^{[i]})_{i\in[n]}$: to implement $y^{[i]}$ as a sunspot equilibrium payoff, nature chooses $j \in I_*$ according to the probability distribution $(\beta^i_j)_{j\in[n]}$ and the players implement w^j. To implement w^j, player j quits along a sufficiently long block with a total probability of δ_j, where δ_j satisfies $w^i = \delta_j \widehat{r}^j + (1-\delta_j) y^{[j]}$, and, if he did not quit, the players implement the vector $y^{[j]}$.

Since $D = \mathrm{conv}(w^1, \cdots, w^n)$, by adding an initial stage in which Nature chooses which of the vectors $(w^i)_{i\in[n]}$ the players implement as a sunspot ε-equilibrium payoff, we can implement every vector in $\widetilde{D}$ as a the payoff of a sunspot ε-equilibrium. ■

5 Discussion and Open Problems

5.1 On ε-Equilibria and Stationary ε-Equilibria

Theorem 2.13(1) provides a condition, which ensures that the quitting game has an ε-equilibrium: if the matrix $\widehat{R}$ is not a Q-matrix, then an ε-equilibrium exists for every $\varepsilon > 0$. This condition is equivalent to requiring that the function mapping each pair $(w, z) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+$ that satisfies the complementarity condition to $w - \widehat{R}z$ is not onto. In fact, this condition can be weakened to require that the index of this function is not ± 1. The proof is standard, hence omitted.

By changing the definition of normal players, one can provide a condition that ensures the existence of a *stationary* ε-equilibrium for every $\varepsilon > 0$. We describe this result in this section. Define inductively

$$\begin{aligned} I_0 &:= I, \\ I_{l+1} &:= \{i \in I_l \colon \text{ there exists } j \in I_l \setminus \{i\} \text{ such that } r^j_i \le 0\}, \quad \forall l \ge 0. \end{aligned}$$

For each player $i \in I_{l+1}$ there is a player $j \in I_l \setminus \{i\}$ who gives, by quitting alone, a nonpositive payoff to player i. The sequence $(I_l)_{l\in\mathbb{N}}$ is a decreasing sequence of sets, hence the set $I_{**} := \cap_{l\in\mathbb{N}} I_l$ is well defined. Note that

$$i \in I_l, j \notin I_l \quad \Rightarrow \quad r^i_j > 0.$$

and consequently we obtain the analogous statement to Lemma 2.6:

$$i \in I_{**}, j \notin I_{**} \quad \Rightarrow \quad r^i_j > 0. \tag{19}$$

Players in the set I_{**} are called *α-players*, and players not in this set are called *β-players*. Every α-player is normal, yet the converse does not necessarily hold.

It can be shown that if there are no α-players, then a stationary ε-equilibrium exists for every $\varepsilon > 0$. Denote by $n' := |I_{**}|$ the number of α-players, and assume that the α-players are players $\{1, 2, \cdots, n'\}$. For each player $i \in I_{**}$ denote by $\widehat{r}^i_{**} \in \mathbb{R}^{n'}$ the vector r^i restricted to the coordinates of α-players, and by $\widehat{R}_{**}$ the $n' \times n'$ matrix whose i'th column is the vector $\widehat{r}^i_{**}$. Similarly to Lemma 2.12, it can be shown that if the linear complementarity problem $\mathrm{LCP}(\widehat{R}_{**}, \vec{0})$ has a nontrivial solution then a stationary ε-equilibrium exists for every $\varepsilon > 0$. The following result is analogous to Theorem 2.13.

Theorem 5.1 *Suppose that $I_{**} \neq \emptyset$ and the linear complementarity problem $\mathrm{LCP}(\widehat{R}_{**}, \vec{0})$ does not have a nontrivial solution.*

1. *If the matrix $\widehat{R}_{**}$ is not a Q-matrix, then the quitting game $\Gamma((r^S)_{S \subseteq I})$ has a stationary ε-equilibrium, for every $\varepsilon > 0$.*

2. *If the matrix $\widehat{R}_{**}$ is a Q-matrix, then for every $\varepsilon > 0$ the quitting game $\Gamma((r^S)_{S \subseteq I})$ has a sunspot ε-equilibrium in which at every stage at most one player quits with positive probability.*

The proof is similar to the proof of Theorem 2.13. Indeed, the properties of normal and abnormal players that were used in that proof are the following:

- When a normal player quits alone, the payoff of all abnormal players is positive. This property is used to ensure that in ε-equilibria and sunspot ε-equilibria that consist of unilateral quittings of normal players, abnormal players prefer to continue. The analogous property for α-players and β-players holds due to Eq. (19).

- A normal player who has to quit alone and does not do it, can be punished. This property is used in Step 2 of Section 3.2 to construct an ε-equilibrium when the matrix $\widehat{R}$ is not a Q-matrix and a single player quits under the limit of stationary discounted equilibria of the auxiliary game $\Gamma((r^S)_{\emptyset \neq S \subseteq I}, q)$. When the matrix $\widehat{R}_{**}$ is not a Q-matrix, we let $\widehat{q}_{**} \in \mathbb{R}^{n'}$ be a vector such that the linear complementarity problem LCP$(\widehat{R}_{**}, \widehat{q}_{**})$ has no solution, we extend $\widehat{q}_{**}$ to a vector $q_{**} \in \mathbb{R}^N$ by setting the coordinates that correspond to β-players to 1, and we look at the limit x^0 of the stationary discounted equilibria of the auxiliary game $\Gamma((r^S)_{\emptyset \neq S \subseteq I}, q_{**})$. As in Step 1 of Section 3.2, x^0 is absorbing. If under x^0 a single player quits, then the definition of α-players implies that there exists some player $j \neq i$ such that $r_i^j \leq 0$. The following stationary strategy profile x is then an ε-equilibrium:

$$x_k = \begin{cases} C_k & k \neq \{i, j\}, \\ x_i^0 & k = i, \\ (1-\varepsilon^2)C_j + \varepsilon^2 Q_j & k = j. \end{cases}$$

5.2 Extensions

Our main result holds with the same proof if each player has several quitting actions. The extension to the case where each player has several continue actions appears in Solan and Solan (2018).

Quitting games are a subclass of absorbing games, which are repeated games in which each player may have more than two actions and there are several nonabsorbing entries (see Vrieze and Thuijsman, 1989). Absorbing games are a subclass of stochastic games. The next step in the research is to extend our result to absorbing games, and then to stochastic games.

Quitting games are also stopping games in which the payoff processes are constant that are independent of time. Shmaya and Solan (2004) developed a technique that allows reducing the question of existence of ε-equilibrium in stopping games with integrable payoff processes to the question of existence of ε-equilibrium in quitting games or absorbing games. Heller (2012) and Mashiah-Yaakovi (2014) used this technique to prove the existence of normal-form correlated ε-equilibrium in multiplayer stopping games and of subgame-perfect ε-equilibrium in multiplayer stopping games with perfect information, respectively. This approach can be used together with our result to show that every multiplayer stopping game with integrable payoffs admits a sunspot ε-equilibrium. The proof is analogous to the proofs in Shmaya and Solan (2004), Heller (2012), and Mashiah-Yaakovi (2014).

5.3 The Transfinite Construction

To construct a sunspot ε-equilibrium from the solution to the auxiliary game with geometric length we used a transfinite construction, because we could not prove that the per-stage probability that some player quits in the auxiliary game is uniformly bounded away from 0. If the existence of a positive lower bound could be proven, then the construction of a sunspot ε-equilibrium would simplify.

One way to evade the need of a transfinite induction is to show that there is $y^0 \in D_0$ for which the sequence $(y^\alpha)_{\alpha<\alpha_*}$ contains finitely many distinct values. This is what happens when the matrix $\widehat{R}$ is an M-matrix, where the process $(y^\alpha)_{\alpha<\alpha_*}$ contains n distinct values. Another case where this phenomenon occurs is when each of the vectors $\widehat{r}^i$ contains exactly one negative coordinate. In this case one can show that by properly choosing y^0, the sequence $(y^\alpha)_{\alpha<\alpha_*}$ contains two distinct values. We could not identify an example where the use of a sequence $(y^\alpha)_{\alpha<\alpha_*}$ with infinitely many distinct values is necessary.

Solan and Vieille (2001) proved that an ε-equilibrium exists for every $\varepsilon > 0$ in every quitting game in which each player prefers to quit alone rather than quit with other players. To show that the construction they suggest is an ε-equilibrium, Solan and Vieille (2001) perform a rather long and technical computation of probabilities. This technical calculations can be avoided by using Theorem 3.6.

5.4 The Linear Complementarity Problem

When the matrix $\widehat{R}$ is an M-matrix, the set of sunspot equilibrium payoffs that can be generated by unilateral quittings coincides with the set $\widetilde{D}$. We do not know whether this coincidence holds for other classes of Q-matrices, or whether we can provide a different characterization to the set of sunspot equilibrium payoffs that can be generated by our construction, when the matrix $\widehat{R}$ is a Q-matrix that is not an M-matrix.

In our construction, the expected payoff of a player after some history may be negative, albeit at least $-\varepsilon$, see Condition (F.2) in Theorem 3.3. It would be interesting to know whether one can ensure that the expected payoff of all players after every history is nonnegative.

Our study also raises questions about Q-matrices, a topic that was not extensively studied in the last decades. In the characterization of the set of sunspot equilibrium payoffs in Section 4 we used the fact that M-matrices are inverse positive. Unfortunately, the inverse of a Q-matrix whose diagonal entries are 0 need not be a nonnegative matrix. One such example is the 3×3 matrix whose off-diagonal entries are equal to 1. In our application the matrix $\widehat{R}$ satisfies additional properties, for example, each row and each column contains at least one negative entry. Is it true that every Q-matrix whose diagonal entries are 0 and such that in each row and each column there is a negative entry is inverse positive? If not, can one characterize the set of inverse-positive Q-matrices that satisfy these conditions? Suppose that the matrix $\widehat{R}$ is inverse positive; can one characterize the set of sunspot equilibria payoffs?

One last question that we will mention concerns the columns of the matrix $\widehat{R}$. Suppose that there is a normal player i such that $r^i_j < 0$ for every $j \neq i$. Thus, all players prefer to quit alone rather than having player i quit. Is the matrix $\widehat{R}$ a Q-matrix? If not, then it follows that when $\widehat{R}$ is a Q-matrix, and provided the game admits no stationary ε-equilibrium, every column of the matrix $\widehat{R}$ contains at least one positive entry. This property, if true, may help in future study of quitting games.

References

[1] Aumann R.J. (1974) Subjectivity and Correlation in Randomized Strategies, *Journal of Mathematical Economics*, **1**, 67–96.

[2] Aumann R.J. (1987) Correlated Equilibrium as an Expression of Bayesian Rationality, *Econometrica*, **55**, 1–18.

[3] Berman A. and Plemmons R.J. (1994) *Nonnegative Matrices in the Mathematical Sciences*, Academic Press.

[4] Bewley T. and Kohlberg E. (1976) The Asymptotic Theory of Stochastic Games, *Mathematics of Operations Research*, **1**, 197–208.

[5] Cass D. and Shell K. (1983) Do Sunspots Matter? *Journal of Political Economics*, **91**, 193–227.

[6] Fink A.M. (1964) Equilibrium in a Stochastic n-Person Game, *Journal of Science of the Hiroshima University*,**28**, 89–93.

[7] Flesch J., Thuijsman F., and Vrieze O.J. (1997) Cyclic Markov Equilibria in Stochastic Games. *International Journal of Game Theory*, **26**, 303–314.

[8] Forges F. (1986) An Approach to Communication Equilibria, *Econometrica*, **54**, 1375–1385.

[9] Fudenberg D. and Tirole J. (1991) *Game Theory*, The MIT Press.

[10] Fujimoto T. and Ranade R. (2004), Two Characterizations of Inverse-Positive Matrices: The Hawkins-Simon Condition and the Le Chatelier-Braun Principle, *Electronic Journal of Linear Algebra*, **11**, 59–65.

[11] Heller Y. (2012) Sequential Correlated Equilibrium in Stopping Games, *Operations Research*, **60**, 209–224.

[12] Mashiah-Yaakovi A. (2014) Subgame Perfect Equilibria in Stopping Games, *International Journal of Games Theory*, **43**, 89–135.

[13] Mertens J.F. and Neyman A. (1981) Stochastic Games, *International Journal of Game Theory*, **10**, 53–66.

[14] Murty K.G. (1988) *Linear Complementarity, Linear and Non-Linear Programming*, Heldermann Verlag.

[15] Shapley L.S. (1953) Stochastic Games, *Proceedings of the National Academy of Sciences of the U.S.A.*, **39**, 1095–1100.

[16] Shmaya E. and Solan E. (2004) Two Player Non Zero-Sum Stopping Games in Discrete Time, *The Annals of Probability*, **32**, 2733–2764.

[17] Simon R.S. (2007) The Structure of Non-Zero-Sum Stochastic Games. *Advances in Applied Mathematics*, **38**, 1–26.

[18] Simon R.S. (2012) A Topological Approach to Quitting Games, *Mathematics of Operations Research*, **37**, 180–195.

[19] Solan E. (1999) Three-Person Absorbing Games, *Mathematics of Operations Research*, **24**, 669–698.

[20] Solan E. and Solan O.N. (2018) Sunspot Equilibrium in General Quitting Games. arXiv:1803.00878.

[21] Solan E., Solan O.N., and Solan R. (2018) Jointly Controlled Lotteries with Biased Coins and General Quitting Games. arXiv:1803.00802.

[22] Solan E. and Vieille N. (2001) Quitting Games, *Mathematics of Operations Research*, **26**, 265–285.

[23] Solan E. and Vieille N. (2002) Quitting Games - an Example, *International Journal of Game Theory*, **31**, 365–381.

[24] Solan E. and Vohra R. (2002) Correlated Equilibrium Payoffs and Public Signalling in Absorbing Games, *International Journal of Game Theory*, 31, 91–121.

[25] Takahashi M. (1964) Equilibrium Points of Stochastic Non-Cooperative n-Person Games, *Journal of Science of the Hiroshima University*, Series A-I Math, **28**, 95–99.

[26] Thuijsman F. (1992) Optimality and Equilibria in Stochastic Games, CWI-Tract 82, Center for Mathematics and Computer Science, Amsterdam.

[27] Vieille N. (2000a) Equilibrium in 2-Person Stochastic Games I: A Reduction, *Israel Journal of Mathematics*, **119**, 55–91.

[28] Vieille N. (2000b) Equilibrium in 2-Person Stochastic Games II: The Case of Recursive Games, *Israel Journal of Mathematics*, **119**, 93–126.

[29] Vrieze O.J. and Thuijsman F. (1989) On Equilibria in Repeated Games With Absorbing States, *International Journal of Game Theory*, **18**, 293–310.