“Strategic supplements” in games with polylinear interactions*

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Abstract

Strategic games are considered where: every player chooses from a compact subset of the real line; the partners' choices affect each player's utility only through their scalar aggregate, which is affine in every single partner's choice; if the choices of all players but two are fixed, then both functions expressing the dependence of one player's aggregate on the other's choice have the same slope; the best response correspondence of each player is nonempty-valued and increases in the aggregate. Every such game admits a “Cournot potential,” i.e., Nash equilibria exist and all best response improvement paths, in a sense, lead to them. *Journal of Economic Literature* Classification Number: C 72.

Key words: Strategic complements; Strategic substitutes; Best response dynamics; Potential games.

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1 Introduction

The recognition of the significance of monotonicity properties in economic models has been steadily growing in recent decades (Tirole, 1988; Fudenberg and Tirole, 1991; Topkis, 1998). Bulow et al. (1985) coined the terms "strategic complements" and "strategic substitutes." From the viewpoint of economics, both properties are equally natural and important; from the technical viewpoint, there is a big difference.

Under strategic complements, i.e., increasing best responses, the existence of Nash equilibria can be derived, under reasonable assumptions, from the famous Tarski (1955) fixed point theorem; the idea was introduced into game-theoretic literature by Topkis (1979), see also Topkis (1998) and references therein. There are also nice comparative statics properties.

On the other hand, the straightforward analogue of Tarski's theorem for decreasing mappings is just wrong, and examples of games with strategic substitutes, but without equilibria, are easy to produce. No comparative statics results in this context have been obtained so far.

Nonetheless, Novshek (1985) showed that decreasing best responses ensure the existence of a Nash equilibrium provided the strategy sets are closed intervals on the real line and the partners' choices affect each player's utility only through their sum. Doubts seem legitimate as to whether the original text contains a "proof" in the true sense of the word, but the mysterious beauty of the construction is above such trifles. Kukushkin (1994) modified Novshek's argument, obtaining a rigorous proof fit for discrete models as well (actually, even better).

This paper concerns with conditions for more than the mere existence of a Nash equilibrium, viz. for nice best response improvement dynamics. A systematic investigation of games where the convergence of unilateral improvement dynamics is ensured was started by Monderer and Shapely (1996). Milchtaich (1996) suggested similar treatment of best response improvements. Kukushkin (1999) showed the usefulness of the language of binary relations in the studies.

In the context of games with strategic complements, certain convergence results concerning best response improvements were established by Topkis (1979) and Vives (1990), but those results lacked a universal character. Kandori and Rob (1995) proved the convergence to a Nash equilibrium of all best response paths in every finite, symmetric, and strictly supermodular game with scalar strategies. Kukushkin (2004) established the convergence to equilibria of all best response improvement paths in every finite game with strategic complements or strategic substitutes and with additive aggregation. Dubey et al. (2004) suggested an alternative approach to the last situation, building on an idea developed first by Huang (2002) for somewhat different purposes. A very interesting feature of the approach was a perfect symmetry between strategic complements and strategic substitutes.

This paper has originated from a surprise at this unusual symmetry. The explanation found is that there is just one general theorem rather than two. The key condition is that each player's best responses should increase in an aggregate of other players' choices, which is affine in every one of them. The slopes may be either upward or downward, but there must be reciprocity in them. Every game from the class admits a "Cournot potential," i.e., Nash equilibria exist and all best response improvement paths, in a sense, lead towards them. In particular, every best response improvement path in every finite game reaches an equilibrium in a finite number of steps.

A striking feature of the result is that strategic substitutability and complementarity may be both present in the same game from the class; moreover, optimal response of player i to player j's strategy may be increasing, decreasing, or constant, depending on the choices of the other players. Hence the tentative term "*strategic supplements*" in the title of the paper. Games with additive aggregation and strategic complements or substitutes (Kukushkin, 2004, Theorems 1 and 2) are just two tiny islands in a vast sea of newly discovered (or rather, tamed) models.

In comparison with Dubey et al. (2004), much more general aggregates are allowed (see Examples 1 and 2 below); the existence of a potential (rather than "pseudo-potential") is established; all superfluous technical restrictions are dropped.

The next section contains definitions related to strategic games and best response dynamics. Our basic model as well as the main result are formulated in Section 3. A simpler version of the theorem, assuming upper hemicontinuous best responses, is proven in Section 4. A proof for the general case is given in Section 5. Several extensions of our basic model, broadening the scope of potential applications considerably, are presented in Sections 6 and 7. A discussion of several related questions in Section 8 concludes the paper.

2 Basic Notions

A *strategic game* Γ is defined by a finite set of players N (we denote $n = \#N$), and strategy sets X_i and utility functions u_i on $X = \prod_{i \in N} X_i$ for all $i \in N$. The *best response correspondence* $\mathcal{R}_i : X_{-i} \to 2^{X_i}$ for each $i \in N$ is defined in the usual way:

$$\mathcal{R}_i(x_{-i}) = \operatorname*{Argmax}_{x_i \in X_i} u_i(x_i, x_{-i}).$$

We always assume that each X_i, hence X too, is a compact metric space; we do not assume the continuity of utilities, but require that $\mathcal{R}_i(x_{-i}) \neq \emptyset$ for all $i \in N$ and $x_{-i} \in X_{-i}$ (the upper semicontinuity of u_i in own strategy x_i is sufficient though by no means indispensable).

We introduce the *Cournot relation* $\triangleright$ on X as in Kukushkin (2004) ($y, x \in X$, $i \in N$):

$$y \triangleright_i x \iff [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i];$$

$$y \triangleright x \iff \exists i \in N\, [y \triangleright_i x].$$

A strategy profile $x \in X$ is a Nash equilibrium if and only if x is a *maximizer* for $\triangleright$, i.e., if $y \triangleright x$ does not hold for any $y \in X$; the assumed existence of the best replies is crucial here.

For a finite game, the acyclicity of the Cournot relation obviously means that every best response improvement path, if continued whenever possible, ends at a Nash equilibrium, the existence of which is thus implied. In an infinite game, the role of acyclicity can, to some extent, be played by an order potential as defined in Kukushkin (1999, 2000).

A *Cournot potential* of a strategic game is an irreflexive and transitive binary relation $\succ$ on X such that:

$$y \rhd x \Rightarrow y \succ x; \tag{1}$$

$$\left[x^{\omega} = \lim_{k\to\infty} x^k \ \& \ \forall k \in I\!N[x^{k+1} \succ x^k]\right] \Rightarrow x^{\omega} \succ x^0. \tag{2}$$

It is essential that (2) implies $x^{\omega} \succ x^k$ for all $k = 0, 1, \dots$ as well. The property seems to have been considered first by Smith (1974), but only for *preference relations*, i.e., complete orderings. In Kukushkin (2003), it was called "ω-transitivity."

The main theorem of Kukushkin (1999) implies that every game admitting a Cournot potential possesses a Nash equilibrium (provided the strategy sets are compact and the best responses exist everywhere). For a finite game, (2) holds by default, so the presence of a Cournot potential is equivalent to the acyclicity of the Cournot relation. In the general case, if we consider best response improvement paths parameterized with transfinite numbers, where best response improvement steps are combined with taking limit points, then the presence of a Cournot potential prevents us from ever coming back; it seems intuitively plausible that, on a compact set X, we will reach an equilibrium eventually. A formalization of the idea and a rigorous proof of "transfinite convergence" can be found in Kukushkin (2003).

3 Main Theorem

Our first subject are *games with reciprocal polylinear interactions* (*RPLI games*). Such games are characterized by the following properties: all strategies x_i are scalar; the partners' choices affect each player's utility only through their scalar aggregate, $\sigma_i(x_{-i})$, which is affine in every single partner's choice x_j; if the choices of all players but two are fixed, then both functions expressing the dependence of one player's aggregate on the other's strategy have the same slope.

To be more formal and exact, we impose these assumptions:

1. $X_i \subset I\!R$ for every $i \in N$;
2. $u_i(x) = U_i(\sigma_i(x_{-i}), x_i)$ for all $i \in N$ and $x \in X$, where

$$\sigma_i(x_{-i}) = \sum_{m=1}^{n-1} \sum_{\substack{j_1,\dots,j_m \in N\setminus\{i\} \\ j_h \neq j_{h'} (h \neq h')}} \alpha^{(m)}_{ij_1\dots j_m} \times x_{j_1} \times \cdots \times x_{j_m} \tag{3}$$

   (the addition of $\alpha_i^{(0)}$ would not change anything);

3. each $\alpha^{(m)}_{i_0 i_1 \dots i_m}$ is invariant under all permutations of $i_0, i_1, \dots, i_m$ (invariance under all permutations of $i_1, \dots, i_m$ could be assumed without restricting generality).

Considering an RPLI game, we denote $S_i = \sigma_i(X_{-i})$ for each $i \in N$; clearly, S_i is compact too. We redefine the best response correspondence:

$$R_i(s_i) = \operatorname*{Argmax}_{x_i \in X_i} U_i(s_i, x_i);$$

our assumption $\mathcal{R}_i(x_{-i}) \neq \emptyset$ implies $R_i(s_i) \neq \emptyset$ for each $s_i \in S_i$.

We also assume that every player's best responses are increasing in s_i:

$$[s_i' > s_i \;\&\; x_i' \in R_i(s_i') \;\&\; x_i \in R_i(s_i)] \Rightarrow x_i' \geq x_i \tag{4}$$

for all $i \in N$ and $s_i', s_i \in S_i$. The standard argument (Milgrom and Shannon, 1994; Topkis, 1998) shows that the following *strict single crossing* condition is sufficient for (4):

$$[x_i' > x_i \;\&\; s_i' > s_i \;\&\; U_i(s_i, x_i') \geq U_i(s_i, x_i)] \Rightarrow U_i(s_i', x_i') > U_i(s_i', x_i) \tag{5}$$

for all $i \in N$, $x_i', x_i \in X_i$, and $s_i', s_i \in S_i$. The conditions (4) or (5) cannot be called either strategic substitutes or strategic complements because $s_i = \sigma_i(x_{-i})$ can be either decreasing or increasing in each x_j, depending on α's and perhaps on the other players' choices.

Remark. Dubey et al. (2004) showed awareness of the fact that even their results are valid for some games that are not characterized by strategic complements or substitutes (p. 9, top). However, they did not pay proper attention to the matter.

Theorem 1. *Every RPLI game satisfying* (4) *admits a Cournot potential.*

The proof is deferred to Sections 4 and 5.

If $\alpha^{(m)}_{i_0 i_1 \ldots i_m} = 0$ for $m > 1$ and $\alpha^{(1)}_{ij} = 1$, we obtain a game with strict strategic complements and additive aggregation; for finite games from the class, the acyclicity of best response improvements was established in Kukushkin (2004, Theorem 1), under even weaker monotonicity conditions. If $\alpha^{(m)}_{i_0 i_1 \ldots i_m} = 0$ for $m > 1$, while $\alpha^{(1)}_{ij} = -1$, we obtain a game with strict strategic substitutes and additive aggregation; for finite games from the class, our Theorem 1 is equivalent to Theorem 2 from Kukushkin (2004).

Example 1. Each player owns a small commercial parking lot in an area. The decision problem for each of them is how much lighting, x_i, to provide at her lot at night. The higher x_i, the higher expenses; on the other hand, the more light, the lower insurance costs. There is a positive externality effect: each player's lamps add something to the light at other lots. It seems reasonable to assume that insurance costs decrease in $x_i + \sum_{j \neq i} \alpha_{ij} x_j$, where $0 \leq \alpha_{ij} < 1$. Each coefficient α_{ij} depending primarily on the distance between i's and j's lots, the reciprocity condition, $\alpha_{ij} = \alpha_{ji}$, seems natural. If we assume the insurance-cost-reduction effect of light to be subject to strictly diminishing returns, then (5) becomes valid, for $\sigma_i(x_{-i}) = -\sum_{j \neq i} \alpha_{ij} x_j$, regardless of the production costs.

Theorem 1 implies that the game possesses a Nash equilibrium and the behaviour of best response improvements is nice enough. In particular, if we assume that only a finite number of x_i's are technologically feasible, then every best response improvement path reaches an equilibrium in a finite number of steps. It is impossible to derive either statement from the previous literature.

Example 2. The players are music fans living in the same apartment block. Each player chooses the volume x_i of his own music, the others providing a negative externality, noise, $\sum_{j \neq i} \alpha_{ij} x_j$ $(0 \leq \alpha_{ij} < 1)$. It seems reasonable to assume $\alpha_{ij} = \alpha_{ji}$ and that each player's optimal volume increases in the outside noise. The existence of an equilibrium, certainly, follows from Tarski's fixed point theorem, but the acyclicity of best response improvements can only be derived from our Theorem 1.

This author is yet unprepared to produce specific models with more general aggregates allowed by Theorem 1, but such aggregates do not seem redundant. For instance, $\alpha^{(1)}_{ij}$ of different signs could appear in a monopolistic competition model if x_i describes the level of advertising by firm i. It seems natural to expect strategic complementarity, $\alpha_{ij}^{(1)} > 0$ when the products of the two firms are substitutes, strategic substitutability, $\alpha_{ij}^{(1)} < 0$ when the products are complements, and "strategic indifference," $\alpha_{ij}^{(1)} = 0$ when they are independent.

The possibility to include nonlinear terms increases the scope of potential applications.

4 Best Responses with Closed Graphs

Here we formulate and prove a less general statement, sufficient for many purposes. In particular, it is equivalent to Theorem 1 for finite games.

Proposition 4.1. *Every RPLI game satisfying* (4) *admits a Cournot potential if the best response correspondences have closed graphs.*

Proof. For every $i \in N$, we define $X_i^0 = \bigcup_{s_i \in S_i} R_i(s_i)$; the compactness of S_i and upper hemicontinuity of R_i imply that X_i^0 is closed in X_i, hence compact too. We denote $s_i^- = \min S_i$ and $s_i^+ = \max S_i$. For each $s_i \in S_i$, we define $r_i^-(s_i) = \min R_i(s_i)$ and $r_i^+(s_i) = \max R_i(s_i)$; by (4), $s_i' > s_i \Rightarrow r_i^-(s_i') \geq r_i^+(s_i)$ and $r_i^+(s_i) = r_i^-(s_i)$ for all $s_i \in S_i$ except for a countable subset. Then we extend r_i^+ to the whole $[s_i^-, s_i^+]$ with the following construction. For every $s_i \in [s_i^-, s_i^+]$ we define $\xi_i^+(s_i) = \min\{\xi_i \in S_i |\ \xi_i \geq s_i\}$ and $\xi_i^-(s_i) = \max\{\xi_i \in S_i |\ \xi_i \leq s_i\}$. Obviously, $\xi_i^+(s_i) = \xi_i^-(s_i) = s_i$ if and only if $s_i \in S_i$; otherwise, $\xi_i^-(s_i) < s_i < \xi_i^+(s_i)$. Now for every $s_i \in [s_i^-, s_i^+] \setminus S_i$ we define $r_i^+(s_i) = r_i^+(\xi_i^-(s_i))$ if $s_i - \xi_i^-(s_i) \leq \xi_i^+(s_i) - s_i$, and $r_i^+(s_i) = r_i^-(\xi_i^+(s_i))$ otherwise.

For each $x_i \in X_i$, we define a function

$$F_i(x_i) = \int_{s_i^-}^{s_i^+} \min\{x_i, r_i^+(s_i)\}\, ds_i.$$

For each $x \in X$, we define a set $N^0(x) = \{i \in N |\ x_i \in X_i^0\}$ and a function

$$P(x) = \sum_{m=1}^{n-1} \Big[\sum_{\substack{i_0, i_1, \ldots, i_m \in N \\ i_h \neq i_{h'} (h \neq h')}} \frac{1}{m+1}\, \alpha_{i_0 i_1 \ldots i_m}^{(m)} \times x_{i_0} \times x_{i_1} \times \cdots \times x_{i_m} \Big] + \sum_{i \in N} \big[F_i(x_i) - s_i^+ \cdot x_i \big]. \quad (6)$$

Finally, we define a binary relation on X (the potential):

$$y \succ x \iff \big[N^0(y) \supset N^0(x) \ \text{ or } \ [N^0(y) = N^0(x) \ \&\ P(y) > P(x)] \big].$$

Obviously, $\succ$ is irreflexive and transitive. Checking (2) is straightforward: the situation $N^0(x^{k+1}) \supset N^0(x^k)$ can only happen for a finite number of k; without restricting generality, $N^0(x^{k+1}) = N^0(x^k)$ for all k, hence $P(x^{k+1}) > P(x^k)$; since each X_i^0 is closed, $N^0(x^\omega) = N^0(x^0)$; since P is continuous, $P(x^\omega) > P(x^0)$.

Let us check (1); let $y \rhd_i x$. Denoting $\bar{s}_i = \sigma_i(x_{-i})$, we have $y_i \in R_i(\bar{s}_i)$ by definition, hence $y_i \in X_i^0$; therefore, $N^0(y) \supseteq N^0(x)$. If the inclusion is strict, we are home. Let $x_i \in R_i(s_i^*)$ for $s_i^* \in S_i^*$; then $s_i^* \neq \bar{s}_i$ because $x_i \notin R_i(\bar{s}_i)$. We have to show $P(y) > P(x)$.

Combining the terms containing x_i (respectively, y_i), we obtain:

$$P(x) = -x_i \cdot (s_i^+ - \bar{s}_i) + F_i(x_i) + C;$$

$$P(y) = -y_i \cdot (s_i^+ - \bar{s}_i) + F_i(y_i) + C;$$

where C only depends on $x_j = y_j$ for $j \neq i$.

Now $y_i \in R_i(\bar{s}_i)$ implies $r_i^+(s_i) \geq y_i$ for all $s_i \geq \bar{s}_i$ and $r_i^+(s_i) \leq y_i$ for all $s_i < \bar{s}_i$, hence $F_i(y_i) = \int_{s_i^-}^{\bar{s}_i} \min\{y_i, r_i^+(s_i)\}\, ds_i + \int_{\bar{s}_i}^{s_i^+} \min\{y_i, r_i^+(s_i)\}\, ds_i = \int_{s_i^-}^{\bar{s}_i} r_i^+(s_i)\, ds_i \;+\; y_i \cdot (s_i^+ - \bar{s}_i)$, hence $P(y) = \int_{s_i^-}^{\bar{s}_i} r_i^+(s_i)\, ds_i + C$.

We have assumed $x_i \in R_i(s_i^*)$; arguing exactly as in the previous paragraph, we see that $F_i(x_i) = \int_{s_i^-}^{s_i^*} r_i^+(s_i)\, ds_i + x_i \cdot (s_i^+ - s_i^*)$. Therefore, $P(x) = x_i \cdot (\bar{s}_i - s_i^*) + \int_{s_i^-}^{\bar{s}_i} r_i^+(s_i)\, ds_i + \int_{\bar{s}_i}^{s_i^*} r_i^+(s_i)\, ds_i \;+\; C$, hence

$$P(y) - P(x) = \int_{\bar{s}_i}^{s_i^*} [x_i - r_i^+(s_i)]\, ds_i.$$

If $s_i^* > \bar{s}_i$, then the integrand is nonnegative on the whole interval and strictly positive in an open neighbourhood of $\bar{s}_i$, because $x_i \notin R_i(\bar{s}_i)$ and the graph of R_i is closed. If $s_i^* < \bar{s}_i$, then the integrand is nonpositive on the whole interval and strictly negative in an open neighbourhood of $\bar{s}_i$, but $ds_i < 0$ (the lower limit is greater than the upper one). In either case, $P(y) > P(x)$, hence $y \succ x$. $\square$

5 General Proof

It is worthwhile to ponder on exactly what makes the above proof unfit for the general situation. If we try to apply it "as is," our proof of (2) will fail because now X_i^0 need not be closed. This obstacle can be overcome by replacing X_i^0 in the definition of $N^0(x)$ with its closure (which coincides with the projection to X_i of the closure of the graph of R_i); however, we shall be unable to assert that $s_i^* \neq \bar{s}_i$.

Example 3. Let $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and the utilities be

$$u_i(x_i, x_j) = \min\{x_i + \beta_i^1(x_j), -x_i + \beta_i^2(x_j)\},$$

where $\beta_1^1(x_2) \equiv 0$, $\beta_2^2(x_1) \equiv 1$,

$$\beta_1^2(x_2) = \begin{cases} 2, & x_2 = 0, \\ 2 - 1/2^{k-1}, & 1/2^{k+1} < x_2 \leq 1/2^k \ (k = 1, 2, \dots), \\ 1 - 1/2^k, & 1/2 + 1/2^{k+1} < x_2 \leq 1/2 + 1/2^k \ (k = 1, 2, \dots), \end{cases}$$

and

$$\beta_2^1(x_1) = \begin{cases} -1/2^{k-1}, & 1/2 - 1/2^k \leq x_1 < 1/2 - 1/2^{k+1} \ (k = 1, 2, \dots), \\ 1 - 1/2^k, & 1 - 1/2^k \leq x_1 < 1 - 1/2^{k+1} \ (k = 1, 2, \dots), \\ 1, & x_1 = 1. \end{cases}$$

Assuming $\sigma_i(x_j) = -x_j$, we easily check (5), hence (4); therefore, Theorem 1 applies. Actually, the existence of a Cournot potential in the game follows from Theorem 5.3 of

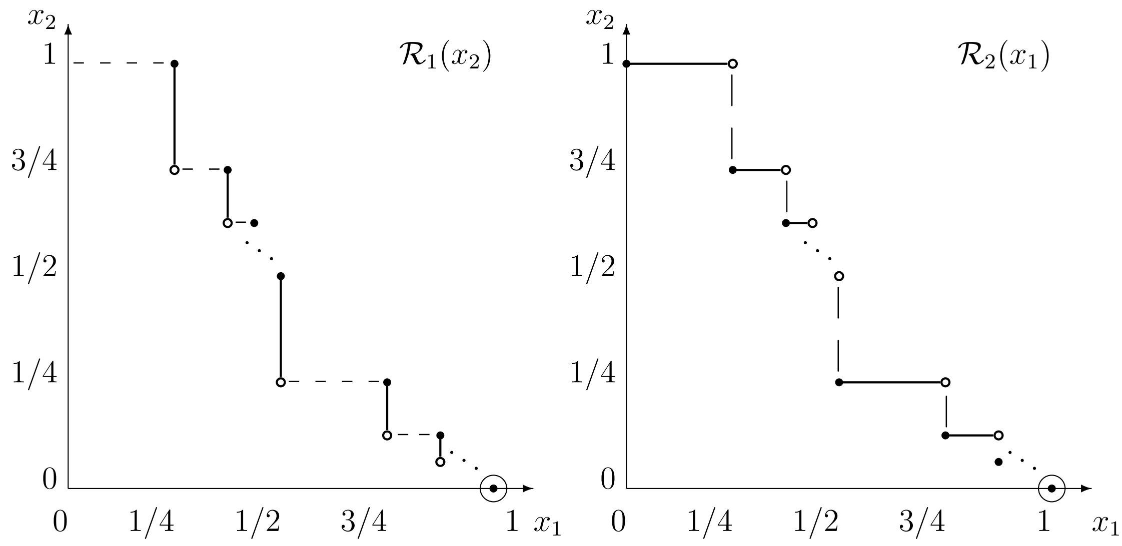

Figure 1: Best responses in Example 3

Kukushkin (2000) since $n = 2$. The existence of an equilibrium even follows straight from Tarski's theorem as Vives (1990, p. 310) has noted.

The best response correspondences are these (see Fig. 1):

$$\mathcal{R}_1(x_2) = \begin{cases} \{1\}, & x_2 = 0, \\ \{1 - 1/2^k\}, & 1/2^{k+1} < x_2 \leq 1/2^k \ (k = 1, 2, \ldots), \\ \{1/2 - 1/2^{k+1}\}, & 1/2 + 1/2^{k+1} < x_2 \leq 1/2 + 1/2^k \ (k = 1, 2, \ldots); \end{cases}$$

$$\mathcal{R}_2(x_1) = \begin{cases} \{1/2 + 1/2^k\}, & 1/2 - 1/2^k \leq x_1 < 1/2 - 1/2^{k+1} \ (k = 1, 2, \ldots), \\ \{1/2^{k+1}\}, & 1 - 1/2^k \leq x_1 < 1 - 1/2^{k+1} \ (k = 1, 2, \ldots), \\ \{0\}, & x_1 = 1. \end{cases}$$

Since they are singletons, we denote them by r_i in the following rather than $\mathcal{R}_i$.

The convergence to the unique equilibrium, $(1, 0)$, may require taking a limit twice: Suppose the players start at $x^0 = (0, 1)$; then they switch to $(r_1(1) = 1/4, 1)$; then to $(1/4, r_2(1/4) = 3/4)$; ... to $(1/2 - 1/2^{k+1}, 1/2 + 1/2^k)$; then to $(1/2 - 1/2^{k+1}, 1/2 + 1/2^{k+1})$; ... in the limit, to $(1/2, 1/2)$; then to $(1/2, r_2(1/2) = 1/4)$; then to $(r_1(1/4) = 3/4, 1/4)$; ... to $(1 - 1/2^k, 1/2^k)$; then to $(1 - 1/2^k, 1/2^{k+1})$; ... in the limit, to $(1, 0)$. On the first step of the process, we have $N^0(x^0) = \{2\} \subset N = N^0(x^1)$. Later on, neither $N^0(x) = N$ nor $P(x) = 1/2 = \max_{x' \in X} P(x')$ change. This fact shows the inadequacy of the potential from Section 4 in the general situation. It also shows that an equilibrium here could hardly be produced by local modifications of an arbitrary maximizer of P as in the proof of Theorem 3 of Dubey et al. (2004). By the way, that theorem is inapplicable: the reaction functions are not strictly decreasing; r_1 is left continuous; r_2 is right continuous.

Remark. For every countable ordinal number α (Natanson, 1974, Chapter XIV), a similar construction provides an example where the convergence to an equilibrium may require "α" steps, cf. Theorem 4.3 of Kukushkin (2000).

To start a proof fit for the general situation, we define $\bar{R}_i(s_i) = \{x_i \in X_i | \ (s_i, x_i) \in \mathrm{cl}(\mathrm{graph}\, R_i)\}$ and replace R_i with $\bar{R}_i$ in the definitions of X_i^0, r_i^-, r_i^+, $\xi_i^-(s_i)$, and $\xi_i^+(s_i)$. Each X_i^0 is still compact, and monotonicity properties of r_i^- and r_i^+ remain intact. Then we reproduce the same definitions of $N^0(x)$ and $P(x)$, which now have a new meaning (actually, P remains the same).

We add a third lexicographic component in the definition of $\succ$ so that the movement from the northwestern to the southeastern corner of Figure 1 be accompanied with increases in the component.

For every $i \in N$, we define binary relations on X_i:

$$y_i \rhd\!\!\rhd_i x_i \iff \exists \bar{s}_i \in S_i \,[y_i \in R_i(\bar{s}_i) \;\&\; x_i \in \bar{R}_i(\bar{s}_i) \setminus R_i(\bar{s}_i)]$$

(in the following, we say "$y_i \rhd\!\!\rhd_i x_i$ holds with $s_i = \bar{s}_i$");

$$y_i \rhd\!\!\rhd_i^+ x_i \iff [y_i \rhd\!\!\rhd_i x_i \;\&\; y_i > x_i];$$

$$y_i \rhd\!\!\rhd_i^- x_i \iff [y_i \rhd\!\!\rhd_i x_i \;\&\; y_i < x_i].$$

An *i-singular upward chain* is a well ordered subset $\Delta \subseteq X_i$ (inevitably countable) such that (1) $y_i \rhd\!\!\rhd_i^+ x_i$ whenever $y_i \in \Delta$ and $x_i = \max\{x_i' \in \Delta |\; y_i > x_i'\}$ (then $y_i = \min\{y_i' \in \Delta |\; y_i' > x_i\}$), and (2) $y_i = \sup\{x_i \in \Delta |\; y_i > x_i\}$ whenever $y_i \in \Delta$ and $\forall x_i \in \Delta\, [y_i > x_i \Rightarrow \exists z_i \in \Delta (y_i > z_i > x_i)]$.

An *i-singular downward chain* is defined dually as a subset $\Delta \subseteq X_i$, well ordered in the reversed order on $I\!R$ (i.e., where every subset contains a greatest point) and such that (1) $y_i \rhd\!\!\rhd_i^- x_i$ whenever $y_i \in \Delta$ and $x_i = \min\{x_i' \in \Delta |\; y_i < x_i'\}$ (then $y_i = \max\{y_i' \in \Delta |\; y_i' < x_i\}$), and (2) $y_i = \inf\{x_i \in \Delta |\; y_i < x_i\}$ whenever $y_i \in \Delta$ and $\forall x_i \in \Delta\, [y_i < x_i \Rightarrow \exists z_i \in \Delta (y_i < z_i < x_i)]$.

It is worth noting that $\Delta \cap [a_i, b_i]$, for $a_i, b_i \in I\!R$, remains an i-singular upward (downward) chain if so was Δ.

We define two more relations on X_i: $y_i \succ\!\!\succ_i^+ x_i$ ($y_i \succ\!\!\succ_i^- x_i$) iff $y_i > x_i$ ($y_i < x_i$) and there is an i-singular upward (downward) chain containing both y_i and x_i. Then, we define

$$y_i \succ\!\!\succ x_i \iff y_i \succ\!\!\succ_i^+ x_i \text{ or } y_i \succ\!\!\succ_i^- x_i.$$

Clearly, all the three relations are irreflexive. Checking transitivity and (2) for the last relation needs some effort.

Step 1. *Both relations $\succ\!\!\succ_i^+$ and $\succ\!\!\succ_i^-$ are transitive and satisfy* (2).

Proof. It is obviously sufficient to consider one of the relations, say, $\succ\!\!\succ_i^+$. Let $z_i \succ\!\!\succ_i^+ y_i \succ\!\!\succ_i^+ x_i$. By definition, there are two i-singular upward chains, Δ' and Δ'', such that $y_i, z_i \in \Delta''$ and $x_i, y_i \in \Delta'$. Defining $\Delta = \big([x_i, y_i] \cap \Delta'\big) \cup \big([y_i, z_i] \cap \Delta''\big)$, we see that Δ is an i-singular upward chain – when checking each condition in the definition, we will find ourselves either totally inside $[x_i, y_i] \cap \Delta'$ or totally inside $[y_i, z_i] \cap \Delta''$. Since $x_i, z_i \in \Delta$, we have $z_i \succ\!\!\succ_i^+ x_i$.

The proof of (2) is quite similar. Let $x_i^k \to x_i^\omega$ and $x_i^{k+1} \succ\!\!\succ_i^+ x_i^k$ for all k; let Δ^k be an i-singular upward chain containing both x_i^k and x_i^{k+1} ($k = 0, 1, \dots$). Denoting $\Delta = \{x_i^\omega\} \cup \bigcup_{k \in I\!N} \big([x_i^k, x_i^{k+1}] \cap \Delta^k\big)$, we again obtain that Δ is an i-singular upward chain (the condition $x_i^\omega = \sup_{k \in I\!N} x_i^k$ is essential here) containing both x_i^0 and x_i^ω. $\square$

Step 2. *Let $z_i \rhd\!\!\rhd_i^+ y_i$ hold with $s_i = \bar{s}_i$, and $y_i' \in [y_i, z_i[$; then $y_i' \rhd\!\!\rhd_i^- x_i$ is only possible, for any $x_i \in X_i$, with $s_i = \bar{s}_i$ (in particular, $y_i \rhd\!\!\rhd_i^- x_i$ is impossible for any $x_i \in X_i$).*

Proof. By definition, $y_i' \rhd\!\!\rhd_i^- x_i$ would imply $y_i' < x_i$. For $s_i > \bar{s}_i$, we have $r_i^-(s_i) \geq r_i^+(\bar{s}_i) \geq z_i > y_i'$, hence $y_i' \notin R_i(s_i)$; for $s_i < \bar{s}_i$, $r_i^+(s_i) \leq r_i^-(\bar{s}_i) \leq y_i \leq y_i' < x_i$, hence $x_i \notin R_i(s_i)$. Therefore, both conditions in the definition of $y_i' \rhd\!\!\rhd_i x_i$ could only be satisfied with $s_i = \bar{s}_i$. □

Step 3. *If $z_i \rhd\!\!\rhd_i^+ y_i$, then $y_i \succ\!\!\succ_i^- x_i$ is impossible for any $x_i \in X_i$.*

Proof. Supposing the contrary, let $z_i \rhd\!\!\rhd_i^+ y_i$ hold with $s_i = \bar{s}_i$ and let Δ be an i-singular downward chain containing both y_i and $x_i > y_i$. By Step 2, $y_i \rhd\!\!\rhd_i^- x_i'$ cannot hold for any x_i'; therefore, condition (1) from the definition of an i-singular downward chain cannot be applicable to y_i, hence condition (2) must hold, implying $\Delta \cap]y_i, z_i[\neq \emptyset$. Let $z_i' = \max\{x_i' \in \Delta |\ x_i' < z_i\}$, $y_i' = \max\{x_i' \in \Delta |\ x_i' < z_i'\}$, and $y_i'' = \max\{x_i' \in \Delta |\ x_i' < y_i'\}$, the maxima existing because Δ is well ordered in the reversed order. By the definition of an i-singular downward chain [condition (1)], we have $y_i'' \rhd\!\!\rhd_i^- y_i' \rhd\!\!\rhd_i^- z_i'$. By the findings of Step 2, both relation must hold with $s_i = \bar{s}_i$, i.e., we must have $y_i' \in R_i(\bar{s}_i)$ and $y_i' \notin R_i(\bar{s}_i)$ simultaneously. □

Step 4. *If $z_i \succ\!\!\succ_i^+ y_i$, then $y_i \succ\!\!\succ_i^- x_i$ is impossible for any $x_i \in X_i$.*

Proof. Let Δ be an i-singular upward chain containing both z_i and y_i. Denoting $y_i' = \min\{x_i' \in \Delta |\ x_i' > y_i\}$, we see that condition (1) from the definition of an i-singular upward chain holds for y_i' and y_i, hence $y_i' \rhd\!\!\rhd_i^+ y_i$. Now the previous step applies. □

Step 5. *If $z_i \succ\!\!\succ_i^- y_i$, then $y_i \succ\!\!\succ_i^+ x_i$ is impossible for any $x_i \in X_i$.*

Proof. The proof is exactly dual to the proofs of Steps 2, 3, and 4. □

Step 6. *The relation $\succ\!\!\succ_i$ is transitive and satisfies* (2).

Proof. Taking into account Steps 4 and 5, Step 1 immediately implies the statement. □

Remark. Returning to the path from the northwestern to the southeastern corner of Figure 1, it is easy to see that the projection of the path to X_1 (X_2) is an i-singular upward (downward) chain. Therefore, all the changes in x_1 (x_2) are upwards in the sense of $\succ\!\!\succ_1^+$ ($\succ\!\!\succ_2^-$) [except for the first step, where $N^0(x^1) \supset N^0(x^0)$].

Finally, we define the potential:

$$y \succ x \iff \big[N^0(y) \supset N^0(x) \text{ or } [N^0(y) = N^0(x) \mathbin{\&} P(y) > P(x)] \text{ or } \big(N^0(y) = N^0(x) \mathbin{\&} P(y) = P(x) \mathbin{\&} \forall i \in N\, [y_i = x_i \text{ or } y_i \succ\!\!\succ_i x_i] \mathbin{\&} \exists i \in N\, [y_i \succ\!\!\succ_i x_i]\big)\big].$$

Step 7. *The relation $\succ$ is irreflexive and transitive, and satisfies* (2).

Proof. The irreflexivity of $\succ$ is obvious; checking transitivity is very easy. Checking (2) is done similarly to Section 4: without restricting generality, $N^0(x^{k+1}) = N^0(x^k)$ for all k; if $P(x^{k+1}) > P(x^k)$ for a single k, then $P(x^\omega) > P(x^0)$ and we are home. If $P(x^{k+1}) = P(x^k)$ for all k, then, for each $i \in N$, either $x_i^{k+1} \succ\!\!\succ_i x_i^k$ for some k, or $x_i^{k+1} = x_i^k$ for all k. In the first case, Step 6 applies, producing $x_i^\omega \succ\!\!\succ_i x_i^0$; in the second, $x_i^\omega = x_i^0$. □

Step 8. *If $y \rhd x$, then $y \succ x$.*

Proof. Let $y \triangleright_i x$ and $\bar{s}_i = \sigma_i(x_{-i})$. Exactly as in Section 4, we obtain $y \succ x$ if $x_i \notin \bar{R}_i(\bar{s}_i)$.

Let $x_i \in \bar{R}_i(\bar{s}_i) \setminus R_i(\bar{s}_i)$; then $N^0(y) = N^0(x)$. We have $r_i^-(\bar{s}_i) \leq x_i, y_i \leq r_i^+(\bar{s}_i)$, hence $F_i(y_i) = \int_{s_i^-}^{\bar{s}_i} r_i^+(s_i)\, ds_i \;+\; y_i \cdot (s_i^+ - \bar{s}_i)$ and $F_i(x_i) = \int_{s_i^-}^{\bar{s}_i} r_i^+(s_i)\, ds_i \;+\; x_i \cdot (s_i^+ - \bar{s}_i)$ exactly as in Section 4; therefore, $P(y) = P(x)$.

By definition, we have $y_i \rhd\!\!\!\rhd_i x_i$. If $y_i > x_i$ ($y_i < x_i$), we have $y_i \rhd\!\!\!\rhd_i^+ x_i$ ($y_i \rhd\!\!\!\rhd_i^- x_i$). Picking $\Delta = \{x_i, y_i\}$, we obtain $y_i \succ\!\!\succ_i^+ x_i$ ($y_i \succ\!\!\succ_i^- x_i$), hence $y_i \succ\!\!\succ_i x_i$. Since $x_j = y_j$ for all $j \neq i$, we have $y \succ x$. □

Thus, $\succ$ is a Cournot potential and Theorem 1 is proved.

6 Abstract Reactions

The proper subject of this paper are "systems of reactions" (Kukushkin, 2000) rather than games as such; Vives (1990) called virtually the same objects "abstract games." There is no big difference between the best response correspondences of a strategic game and abstract reactions, but the latter concept provides a greater flexibility.

A *system of reactions* $\mathcal{S}$ is defined by a finite set of indices N, and sets X_i and mappings $\mathcal{R}_i : X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ for all $i \in N$. A point $x^0 \in X = \prod_{i \in N} X_i$ is called a *fixed point* of $\mathcal{S}$ if $x_i^0 \in \mathcal{R}_i(x_{-i}^0)$ for all $i \in N$.

With every system $\mathcal{S}$, one can associate binary relations on X: $y \triangleright_i^{\mathcal{S}} x \iff [y_{-i} = x_{-i} \;\&\; x_i \notin \mathcal{R}_i(x_{-i}) \ni y_i]$, $y \triangleright^{\mathcal{S}} x \iff \exists i \in N\, [y \triangleright_i^{\mathcal{S}} x]$. Obviously, $x \in X$ is a maximizer for $\triangleright^{\mathcal{S}}$ if and only if x is a fixed point of $\mathcal{S}$. We omit the superscript $^{\mathcal{S}}$ at $\triangleright$ when the system is clear from the context.

A *potential* for a system of reactions is an irreflexive and transitive binary relation $\succ$ on X such that (1) and (2) hold (with the new interpretation of $\triangleright$). As in Section 2, the main theorem of Kukushkin (1999) implies that every system of reactions admitting a potential possesses a fixed point (provided the sets X_i are compact). And again, the iteration of reactions leads towards fixed points, reaching one in a finite number of steps if all X_i are finite.

A *system of reactions with reciprocal polylinear aggregates* (an *RPLA system*) is characterized by these assumptions: $X_i \subset I\!R$ and $\mathcal{R}_i = R_i \circ \sigma_i$ for every $i \in N$, where $\sigma_i : X_{-i} \to I\!R$ is defined by (3), and R_i is a correspondence from $S_i = \sigma_i(X_{-i})$ to X_i; $\alpha_{i_0 i_1 \ldots i_m}^{(m)}$ is invariant under all permutations of $i_0, i_1, \ldots, i_m$.

Theorem 2. *Every RPLA system where every mapping R_i satisfies* (4) *admits a potential.*

There are two ways to prove the theorem. The first is to repeat Sections 4 and 5: the utility functions were never mentioned there (Example 3 can easily be reformulated without them). The second is to define an RPLI game by $U_i(s_i, x_i) = 1$ if $x_i \in R_i(s_i)$, and $U_i(s_i, x_i) = 0$ otherwise. The same R_i become the best response correspondences (actually, even (5) holds), so Theorem 1 implies Theorem 2.

The following obvious statement shows an advantage of the new formulation.

Corollary. *Let, in an RPLA system, there exist correspondences $R_i' : X_{-i} \to 2^{X_i} \setminus \{\emptyset\}$ satisfying* (4) *and such that $R_i'(x_{-i}) \subseteq R_i(x_{-i})$ for all $i \in N$ and $x_{-i} \in X_{-i}$. Then the system has a fixed point.*

Remark. Actually, we have a "restricted potential" in this situation, which is more than the mere existence of a fixed point (Nash equilibrium), cf. Kukushkin (2004, Sections 6 and 7.7).

The conditions of the Corollary hold if the original reactions R_i are monotonic in a weaker sense than (4). For instance,

$$[s'_i \geq s_i \;\&\; x_i \in R_i(s_i) \;\&\; x'_i \in R_i(s'_i)] \Rightarrow x'_i \vee x_i \in R_i(s'_i), \tag{7a}$$

or

$$[s'_i \geq s_i \;\&\; x_i \in R_i(s_i) \;\&\; x'_i \in R_i(s'_i)] \Rightarrow x'_i \wedge x_i \in R_i(s_i). \tag{7b}$$

Proposition 6.1. *Let, for every $i \in N$ in an RPLA system, each $R_i(s_i)$ be closed and either* (7a) *be satisfied for all $x'_i, x_i \in X_i$ and $s'_i, s_i \in S_i$, or* (7b) *be satisfied for all $x'_i, x_i \in X_i$ and $s'_i, s_i \in S_i$. Then there is a fixed point.*

Proof. If (7a) is satisfied for a given $i \in N$, we define $R'_i(s_i) = \{\max R_i(s_i)\}$; otherwise, we define $R'_i(s_i) = \{\min R_i(s_i)\}$. The standard argument (Topkis, 1998) shows that R'_i satisfies (4). Now Corollary to Theorem 2 applies. □

Conditions (7) for the best response correspondences in a strategic game are ensured by the *weak single crossing* conditions:

$$[x'_i \geq x_i \;\&\; s'_i \geq s_i \;\&\; U_i(s_i, x'_i) \geq U_i(s_i, x_i)] \Rightarrow U_i(s'_i, x'_i) \geq U_i(s'_i, x_i); \tag{8a}$$

$$[x'_i \geq x_i \;\&\; s'_i \geq s_i \;\&\; U_i(s'_i, x_i) \geq U_i(s'_i, x'_i)] \Rightarrow U_i(s_i, x_i) \geq U_i(s_i, x'_i). \tag{8b}$$

As is well known, (8a) implies (7a), whereas (8b) implies (7b).

Corollary. *Let, for every $i \in N$ in an RPLI game, each $R_i(s_i)$ be closed (which holds, e.g., if U_i is upper semicontinuous in own strategy x_i) and either* (8a) *be satisfied for all $x'_i, x_i \in X_i$ and $s'_i, s_i \in S_i$, or* (8b) *be satisfied for all $x'_i, x_i \in X_i$ and $s'_i, s_i \in S_i$. Then the game possesses a Nash equilibrium.*

The theorem on monotone selections ascribed by Milgrom and Shannon (1994, Theorem A2) to A. Veinott implies that the monotonicity conditions in Proposition 6.1 could be weakened even further; however, the weaker condition admits no clear interpretation in terms of utility functions.

If the existence of neither greatest nor least best response is ensured, the existence of a Nash equilibrium can be obtained if we assume (8a) and (8b) together, which is Milgrom and Shannon's (1994) *single crossing* condition. The condition ensures that the best response correspondence is *ascending*, i.e.,

$$[s'_i \geq s_i \;\&\; x_i \in R_i(s_i) \;\&\; x'_i \in R_i(s'_i)] \Rightarrow [x'_i \vee x_i \in R_i(s'_i) \;\&\; x'_i \wedge x_i \in R_i(s_i)]. \tag{9}$$

The use of the property is based on the following technical result.

Lemma 6.2. *Let R be a mapping $S \to 2^X \setminus \{\emptyset\}$, where S is a partially ordered set and X is a chain; let R satisfy* (9)*. Then there exists a monotone selection from R, i.e., a mapping $r : S \to X$ such that (1) $r(s) \in R(s)$ and (2) $s' \geq s \Rightarrow r(s') \geq r(s)$, for all $s, s' \in S$.*

Proof. We use the Axiom of Choice to the full extent. The set S can be well ordered; to avoid considering two independent orders on the same set, we assume that there is a bijection $\lambda : \mathrm{A} \to S$, where A is a well ordered set (of the same cardinality as S). We define $r(\lambda(\alpha))$ by (transfinite) induction in $\alpha \in \mathrm{A}$. First, we pick $r(\lambda(0)) \in R(\lambda(0))$ arbitrarily.

Let $r(\lambda(\beta))$ be defined for all $\beta < \alpha$. We define $\mathrm{B}(\alpha) = \{\beta < \alpha |\ r(\lambda(\beta)) \in R(\lambda(\alpha))\}$. If $\mathrm{B}(\alpha) = \emptyset$, we pick $r(\lambda(\alpha)) \in R(\lambda(\alpha))$ arbitrarily. Otherwise, we define $r(\lambda(\alpha)) = r(\lambda(\min \mathrm{B}(\alpha)))$, the minimum existing because A is well ordered.

Since there is no possibility that $r(\lambda(\alpha))$ could be left undefined, we obtain $r(\lambda(\alpha))$ for all $\alpha \in \mathrm{A}$ eventually. Clearly, $r(\lambda(\alpha)) \in R(\lambda(\alpha))$ for all $\alpha \in \mathrm{A}$, so we only have to check monotonicity.

Suppose to the contrary that $\lambda(\alpha') < \lambda(\alpha)$ whereas $r(\lambda(\alpha')) > r(\lambda(\alpha))$; the assumption that X is a chain is essential here. By (9), $r(\lambda(\alpha)) \in R(\lambda(\alpha'))$ and $r(\lambda(\alpha')) \in R(\lambda(\alpha))$. Without restricting generality, $\alpha' < \alpha$, hence $\alpha' \in \mathrm{B}(\alpha) \neq \emptyset$. The assumption that $r(\lambda(\alpha')) \neq r(\lambda(\alpha))$ implies that $\min \mathrm{B}(\alpha) = \beta < \alpha'$ and $r(\lambda(\alpha)) = r(\lambda(\beta))$. Now $\beta \in \mathrm{B}(\alpha') \neq \emptyset$, so the assumption that $r(\lambda(\alpha')) \neq r(\lambda(\beta))$ implies that $\min \mathrm{B}(\alpha') = \beta' < \beta$ and $r(\lambda(\alpha')) = r(\lambda(\beta'))$. However, now we have $\beta' \in \mathrm{B}(\alpha)$, hence $\beta \leq \beta'$. The contradiction proves the monotonicity of r. □

Remark. If $S \subseteq I\!R$, there exists a countable subset order dense in S; then the transfinite induction can be replaced with ordinary one where parameters are natural numbers.

Proposition 6.3. *Every RPLA system where every mapping R_i satisfies* (9) *has a fixed point.*

Proof. By Lemma 6.2, each R_i admits a monotone selection, which satisfies (4). Now Corollary to Theorem 2 applies. □

Proposition 6.3 is applicable to RPLI games where the best response correspondences are ascending (e.g., both conditions (8) hold), but the sets $R_i(s_i)$ need not be closed.

7 Further Extensions

7.1. The reciprocity condition can be replaced with a hierarchy of players; moreover, even polylinearity can be weakened considerably in this case. In this subsection we introduce the concept of a *system of reactions with hierarchic-reciprocal polylinear aggregates* (an *HRPLA system*). Such a system satisfies the first two conditions from the definitions of an RPLA system, viz. $X_i \subset I\!R$ and $\mathcal{R}_i = R_i \circ \sigma_i$ for every $i \in N$, but the restrictions on $\sigma_i : X_{-i} \to I\!R$ are different.

We assume that N is partitioned into a number of subsets, $N = \bigcup_{k=1}^{q} N_k$ with $N_k \cap N_h = \emptyset$ whenever $h \neq k$. For each $k = 1, \dots, q$, we denote $M_k = \bigcup_{h=k+1}^{q} N_h$ (so $M_q = \emptyset$); we define the *rank* $\rho(i)$ of a player i by $i \in N_{\rho(i)}$. Now we assume that

$$\sigma_i(x_{-i}) = \sum_{m=1}^{\#N_{\rho(i)}-1} \sum_{\substack{j_1,\dots,j_m \in N_{\rho(i)} \setminus \{i\} \\ j_h \neq j_{h'} (h \neq h')}} \alpha^{(m)}_{ij_1 \dots j_m}(x_{M_{\rho(i)}}) \times x_{j_1} \times \cdots \times x_{j_m},$$

where $\alpha^{(m)}_{i_0 i_1 \dots i_m}(x_{M_k})$ is invariant under all permutations of $i_0, i_1, \dots, i_m \in N_k$ for each $k = 1, \dots, q$, $1 \le m < \# N_k$, and $x_{M_k} \in X_{M_k}$.

In other words, every player is indifferent to the choices of partners with lower ranks, there is reciprocity between peers, and players with higher ranks may affect their inferiors in an arbitrary way.

Proposition 7.1. *Every HRPLA system where every mapping R_i satisfies* (4) *admits a potential.*

Proof. If $q = 1$, Theorem 2 applies, providing the basis for an induction process. Assuming the statement valid for some q, we have to prove it for $q+1$. We denote $I = N_{q+1}$ and $J = N \setminus I$ $(= \bigcup_{k=1}^{q} N_k)$.

The correspondences R_i with $i \in I$ form an RPLA system; by Theorem 2, it admits a potential $\succ_I$ defined on $X_I = \prod_{i \in I} X_i$. For every $x_I \in X_I$, the correspondences R_i with $i \in J$ form an HRPLA system; by the induction hypothesis, it admits a potential $\succ^{x_I}$ defined on $X_J = \prod_{i \in J} X_i$. Now we define our (global) potential as

$$y \succ x \iff [y_I \succ_I x_I \ \text{ or } \ [y_I = x_I \ \& \ y_J \succ^{x_I} x_J]]$$

Let $y \rhd_i x$; then $y \succ x$ by the first lexicographic component if $i \in I$ and by the second if $i \in J$. Condition (2) holds for $\succ$ because it holds for both $\succ_I$ and every $\succ^{x_I}$. □

7.2. Essentially the same aggregates may work when strategy sets are not necessarily subsets of the real line. In this subsection we introduce the concept of a *system of reactions with reciprocal quasi-polylinear aggregates* (an *RQPLA system*). Such a system is characterized by these assumptions: each X_i is a metric compact with a partial order; there is a continuous and increasing mapping $\nu_i : X_i \to \mathbb{R}$ for each $i \in N$; $\mathcal{R}_i = R_i \circ \sigma_i$ for every $i \in N$, where

$$\sigma_i(x_{-i}) = \sum_{m=1}^{n-1} \sum_{\substack{j_1,\dots,j_m \in N \setminus \{i\} \\ j_h \ne j_{h'} (h \ne h')}} \alpha^{(m)}_{i j_1 \dots j_m} \times \nu_{j_1}(x_{j_1}) \times \cdots \times \nu_{j_m}(x_{j_m}), \tag{10}$$

and $R_i : S_i \to 2^{X_i} \setminus \{\emptyset\}$ for $S_i = \sigma_i(X_{-i})$; each $\alpha^{(m)}_{i_0 i_1 \dots i_m}$ is invariant under all permutations of $i_0, i_1, \dots, i_m$.

Proposition 7.2. *Every RQPLA system where every mapping R_i satisfies* (4) *admits a potential.*

Remark. The only requirement on concord between topology and order on each X_i is the existence of a continuous and increasing function.

Proof. For every $i \in N$, we define $\Xi_i = \nu_i(X_i) \subset \mathbb{R}$ and $\sigma_i^*(\xi_{-i}) = \sigma_i(x_{-i})$, where $\xi_j = \nu_j(x_j)$ for all $j \ne i$; equality (10) implies that the choice of x_j in the pre-image of ξ_j does not matter. It is easy to see that $\sigma_i^*(\Xi_{-i}) = S_i$. Then we define $R_i^* = \nu_i \circ R_i$. Since R_i satisfies (4) and ν is increasing, R_i^* satisfies (4) too.

Now the sets Ξ_i and correspondences R_i^* define an RPLA system; by Theorem 2, it admits a potential $\succ^*$ defined on $\Xi = \prod_{i \in N} \Xi_i \subset \mathbb{R}^N$. For every $x \in X$, we define

$N^0(x) = \{i \in N|\ x_i \in R_i(\sigma_i(x_{-i}))\}$. We denote $\nu : X \to \Xi$ the product of all ν_i. Finally, we define our potential as

$$y \succ x \iff \left[\nu(y) \succ^* \nu(x) \ \text{ or } \ [\nu(y) = \nu(x) \ \& \ N^0(y) \supset N^0(x)]\right]$$

Checking (1), we suppose that $y \triangleright_i x$; then $y_j = x_j$, hence $\nu_j(y_j) = \nu_j(x_j)$ for all $j \neq i$. We consider two alternatives. If $\nu_i(y_i) \neq \nu_i(x_i)$, we have $\nu(y) \succ^* \nu(x)$, hence $y \succ x$. If $\nu_i(y_i) = \nu_i(x_i)$, we have $\nu(y) = \nu(x)$ and $i \in N^0(y) \setminus N^0(x)$; moreover, $\nu_{-j}(y_{-j}) = \nu_{-j}(x_{-j})$ for each $j \neq i$, hence $N^0(y) \cap (N \setminus \{i\} = N^0(x) \cap (N \setminus \{i\}$. Thus, $y \succ x$.

Condition (2) holds for $\succ$ because it holds for $\succ^*$ and ν is continuous. □

This extension is useful when the players also have decision variables free of any external effect. For instance, in Example 1, the players could also decide on how much to spend on signalization, pavement, drainage, etc., and these variables could enter into any restrictions together with the original x_i. However, the applicability of our approach cannot survive another external effect, e.g., price competition.

8 Concluding Remarks

8.1. The Cournot relation is purely ordinal, i.e., invariant under any strictly increasing transformation of the utility function. The same is true of the definition of an RPLI game and of our conditions (5) and (4). Therefore, this paper belongs to the ordinal strand in the theory of potential games.

8.2. The class of RPLI games (as well as its generalizations) can be viewed as a natural extension of the class of games with additive aggregation, considered in Kukushkin (2004): we just defined $\sigma_i(x_{-i})$ in a more general way. There is a principal difference, however: In a game from the latter class, there is a single aggregate characteristic, $\sum_{i \in N} x_i$, which enters into each player's utility. In this paper, each player is affected by his "personal" aggregate, and there is no analogue of the total sum. In a sense, aggregation here is not separable.

8.3. As was noted in Section 3, for finite games with additive aggregation Theorem 2 of Kukushkin (2004) requires the same monotonicity condition as our Theorem 1. Meanwhile, Theorem 1 from the same paper only requires (8a) above, which is much weaker than (5). The difference is not due to any technical shortcomings.

Example 4. Let $N = \{1, 2, 3\}$, $X_1 = \{0, 1, 2, 3, 4\}$, $X_2 = \{0, 1, 2, 3, 4, 5\}$, $X_3 = \{0, 1\}$, $\alpha^{(m)}_{i_0 i_1 \ldots i_m} = 0$ for $m > 1$, and $\alpha^{(1)}_{ij} = -1$ (i.e., we actually have a game with strategic substitutes and additive aggregation), hence $S_1 = \{0, -1, \ldots, -6\}$, $S_2 = \{0, -1, \ldots, -5\}$, $S_3 = \{0, -1, \ldots, -9\}$. Let the utilities be:

$$U_1: \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 4 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} \quad U_2: \begin{bmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad U_3: \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

where own choice, x_i, is on the ordinates axis, and s_i (= minus the sum of the partners' choices), on the abscissae axis.

The best responses are easily seen in the matrices. Condition (4) holds for players 2 and 3, but not for player 1, so our Theorem 1 does not apply. Actually, every utility function satisfies Topkis's (1979) increasing differences condition in x_i and s_i, hence the best responses satisfy (9). Nonetheless, there is a best response improvement cycle:

$$\begin{array}{ccccccc}
(3,0,0) & \xrightarrow{1} & (4,0,0) & \xrightarrow{3} & (4,0,1) & \xrightarrow{1} & (1,0,1) \\
\uparrow{\scriptstyle 2} & & & & & & \downarrow{\scriptstyle 2} \\
(3,5,0) & \xleftarrow{1} & (0,5,0) & \xleftarrow{3} & (0,5,1) & \xleftarrow{1} & (1,5,1)
\end{array} .$$

Therefore, the strict condition (4) in our Theorem 1 cannot be replaced with (9), to say nothing of (8a). This could be possible under some restrictions (all $\alpha^{(m)}_{i_0 i_1 \ldots i_m} \geq 0$ or all $\alpha^{(m)}_{i_0 i_1 \ldots i_m} > 0$), but so far there is no result to the effect beyond Theorem 1 of Kukushkin (2004).

There is a very peculiar manifestation of this asymmetry between strategic complements and substitutes. Consider finite three person RPLI games with linear aggregates $\sigma_i(x_j, x_k) = \alpha_{ij} x_j + \alpha_{ik} x_k$. The reciprocity condition implies that we have three independent parameters α_{ij}; let none of them be zero. Rescaling (and perhaps reversing) the axes, we can transform the game into that with additive aggregation and strategic complements or substitutes (i.e., with all $\alpha_{ij} = 1$ or all $\alpha_{ij} = -1$). Thus we come to the conclusion that if the number of negative coefficients α_{ij} in the original game is even (0 or 2), then condition (8a) ensures acyclicity; if the number is odd (1 or 3), a stricter version, (5) above, must be imposed. How could such a conclusion have been expected?

8.4. It goes without saying that the extensions from Sections 6 and 7 can be combined together.

8.5. Proposition 7.1 does not imply the possibility to replace $\alpha^{(m)}_{i_0 i_1 \ldots i_m}$ with zeros in an arbitrary way.

Example 5. Let $N = \{1,2,3\}$, $X_i = \{0, i\}$ for all $i \in N$, $\sigma_1(x_2, x_3) = -x_2 - x_3$, $\sigma_2(x_1, x_3) = -x_1$, $\sigma_3(x_1, x_2) = -x_1 - x_2$, $R_1(s_1) = \{0\}$ for $s_1 \leq -3$, $R_1(s_1) = \{1\}$ for $s_1 \geq -2$, $R_2(-1) = \{0\}$, $R_2(0) = \{2\}$, $R_3(s_3) = \{0\}$ for $s_3 \leq -2$, and $R_3(s_3) = \{3\}$ for $s_3 \geq -1$. A fixed point x^0 would have to satisfy the equations:

$$\operatorname{sign}(x^0_1) = 1 - \operatorname{sign}(x^0_3), \quad \operatorname{sign}(x^0_2) = 1 - \operatorname{sign}(x^0_1), \quad \operatorname{sign}(x^0_3) = 1 - \operatorname{sign}(x^0_2),$$

which is clearly impossible.

8.6. Example 3 raises a natural question: Is a transfinite best response improvement path possible in an RPLI game with strict strategic supplements, in other words, can an infinite path fail to find an equilibrium in the limit, if all best response correspondences have closed graphs (e.g., if the utilities are continuous)? If the path converges, a negative answer is obvious: the limit must belong to each graph. If $n = 2$ and $x^0, x^1, \ldots, x^k, \ldots$ is an infinite best response improvement path, then each sequence $x^0_i, x^1_i, \ldots, x^k_i, \ldots$ is either increasing or decreasing; in either case, it converges. For $n > 2$, there is neither proof nor counterexample to the statement that every infinite best response improvement path has a Nash equilibrium among its limit points, nor even counterexample to the hypothesis that every limit point of every infinite best response improvement path is a Nash equilibrium.

8.7. S. Takahashi (personal communication) has discovered another asymmetry between strategic complements and substitutes: The exact analogue of Theorem 2 from Kandori and Rob (1995), establishing the acyclicity of best responses in symmetric strictly supermodular games, does not hold for submodular games. Interestingly, no situation is known where best response improvements would behave nicer under strategic substitutes than under strategic complements. At a first glance, this seems quite natural; however, if one starts looking for more or less formal arguments why it should be so, none presents itself.

8.8. A fastidious reader may feel dissatisfaction with the use, in the definition of F_i in Section 4, of the extension of r_i^+ beyond S_i, as if events in the real world are supposed to depend on what happens in purely imaginary worlds. Actually, the extension was only needed to simplify notations. For a compact subset $S_i \subset I\!R$, we can define a measure $\mu_i = \mu_i^{\mathrm{L}} + \mu_i^{\mathrm{S}}$ on S_i, where μ_i^{L} is the usual Lesbegue measure on S_i (induced from $I\!R$), while μ_i^{S} assigns one half of the length of each constituent interval of $[s_i^-, s_i^+] \setminus S_i$ to either endpoint (thus, $\mathrm{supp}(\mu_i^{\mathrm{S}})$ is the set of the endpoints). It is a simple exercise to show that $\mu_i(S_i) = s_i^+ - s_i^-$ and $F_i(x_i) = \int_{S_i} \min\{x_i, r_i^+(s_i)\}\, \mu_i(ds_i)$ for all $i \in N$ and $x_i \in X_i$. The piecemeal linear interpolation used by Dubey et al. (2004) seems to admit no similar interpretation of F_i.

8.9. Among the proofs of the acyclicity of best response improvements in the existing literature, a certain number are "relatively straightforward," i.e., given the assumptions, one can more or less convincingly explain the choice of arguments: Theorem 2 of Kandori and Rob (1995); Theorems 1 and 3 of Kukushkin (2004); Theorems 7 and 8 of Kukushkin (2003). There are also two tricks defying any explanation: Novshek's construction used in the proof of Theorem 2 from Kukushkin (2004) and the Huang–Dubey–Haimanko–Zapechelnyuk definition of the function P, somewhat modified in (6) above. Both are logically independent in the sense that there is a situation where one works but the other does not: finite games with separable, but non-additive, aggregation rules for Novshek's trick; e.g., linear aggregates with arbitrary coefficients as in Examples 1 and 2 for the function P. However, if one takes into account the relative importance of the domain of applicability of either approach, the latter appears a clear winner.

If the principle *tres faciunt collegium* can be relied upon, we should expect a third trick to spring up; perhaps it will prove acyclicity wherever it holds.

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