Static Stability in Games

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Static stability of equilibrium in strategic games differs from dynamic stability in not being linked to any particular dynamical system. In other words, it does not make any assumptions about off-equilibrium behavior. Examples of static notions of stability include evolutionarily stable strategy (ESS) and continuously stable strategy (CSS), both of which are meaningful or justifiable only for particular classes of games, namely, symmetric multilinear games or symmetric games with a unidimensional strategy space, respectively. This paper presents a novel, general notion of local static stability, of which the above two are essentially special cases. It is applicable to virtually all n-person strategic games, both symmetric and asymmetric, with "continuous" (rather than discrete) strategy spaces.

Keywords: Stability of equilibrium, static stability

1. Introduction

Strategic games, markets and other economic systems are said to be in equilibrium when the participating agents do not have any incentives to act differently than they do. Stability of the equilibrium refers to the effects that perturbations, or shocks, would have on the agents' incentives or actual actions. Since any change in actions in turn creates new incentives, an initial perturbation may set the system in motion, which may eventually either bring it close to the original equilibrium state or to states further away from it. Thus stability of equilibrium can be defined in terms of the trajectories the system would take following a perturbation. However, such a dynamic definition is arguably less basic than a static one, which involves only incentives. In particular, it requires making specific assumptions about off-equilibrium behavior, i.e., the translation of incentives (e.g., to increase or lower output) into actions (actual production adjustments).

A physical analogy illustrates this point. A body is in equilibrium at point x if the resultant force acting on it there is zero. If the body is slightly displaced, to point y, the force may become nonzero. The equilibrium is stable if the new force vector points approximately in the direction of the body's original location x, and it is unstable if the vector points in the opposite direction (Figure 1). This definition only involves forces. It makes no mention of motion, and hence has no use of such concepts as the body's inertia and Newton's second law.

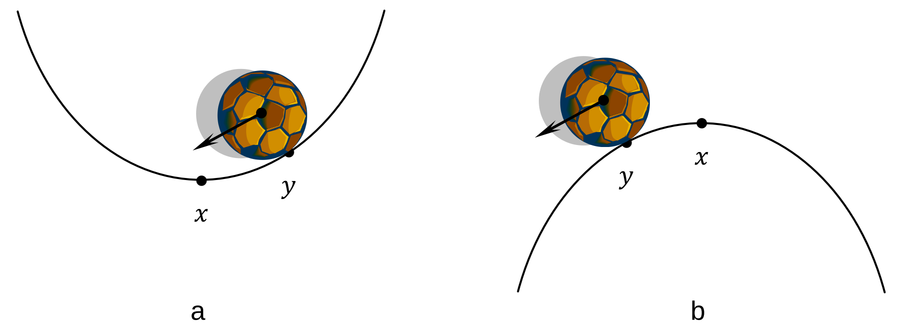

Figure 1. a. A stable equilibrium. b. An unstable equilibrium.

Forces are analogous to incentives in economic settings. Translation of the above definition of stability into game-theoretic language gives the requirement that, at any state y close to the equilibrium one x, a unilateral deviation in the direction of x is more profitable (or less costly) to the deviating player than a deviation in the opposite direction. To make this a meaningful definition, 'direction' has to be defined. It also has to be specified, how large the unilateral deviations considered are. If a player's strategy space is ordered, e.g., if strategies are numbers, then the direction of a strategy change has a well-defined meaning. If, in addition, the strategy space is endowed with a metric, then there is at least a well defined sense in which deviations in opposite directions are of equal magnitude. Such equality may suffice, since incentives can then be compared using the notion of marginal payoffs. This, however, essentially restricts the analysis to a special class of games, namely, games in which the strategy spaces are unidimensional (i.e., subsets of the real line) and the payoff functions are differentiable. This leaves out many important games, most notably, bimatrix games and, more generally, multilinear games that are the mixed extensions of finite games. In a multilinear game, each player's strategy space is the unit simplex in a Euclidean space $\mathfrak{R}^n$ (where n is the number of pure strategies for the player), i.e., strategies are probability vectors. The unit simplex is not ordered and it has *several* natural notions of distance, such as the L_1 and L_2 distances. However, it has another structure that may be useful in the present context, namely, it is a convex set. Thus a natural interpretation of a unilateral deviation of player i from y, where his strategy is y^i, in the direction of x, where the strategy is x^i, is that the player's strategy changes to

$$\alpha x^i + (1 - \alpha) y^i, \tag{1}$$

for some $0 < \alpha \leq 1$. A deviation of the same magnitude in the opposite direction brings the player to

$$z^i = -\alpha x^i + (1 + \alpha) y^i, \tag{2}$$

which is well defined if y^i is sufficiently close to x^i. This leaves one degree of freedom, namely, the choice of α. Choosing, for example, $\alpha = 1$ gives stability the following meaning:

Following a slight perturbation of strategies,[1] each player i would gain more (or lose less) from unilaterally reverting to his equilibrium strategy x^i than from a deviation of the same magnitude in the opposite direction.[2] This still leaves open the question of how stability can be defined for strategy spaces without an underlying linear structure.

A time-honored way of defining convex combinations like (1) in games without an underlying linear structure is using mixed strategies, i.e., interpreting α as the *probability* that player i plays x^i. However, this interpretation is not applicable to (2), where x^i has a negative coefficient. A key to overcoming this obstacle is the observation that, regardless of α,

$$y^i = \frac{1}{2}x^i + \frac{1}{2}z^i.$$

Thus, suppose that the perturbation is probabilistic to begin with: each of the players plays according to x with probability $1/2$ and according to z with probability $1/2$. Two possible unilateral deviations for player i are playing according to x or according to z with probability 1. The former represents a unilateral reversion to x and the latter may be interpreted as completing the move to z. If, for each of the players, reversion is the better alternative,[3] and this holds for all z close to x, than the latter will be defined as stable.[4]

The following sections flesh out this idea, starting with symmetric two-player games and gradually extending the ground coverage all the way to asymmetric multiplayer games. Special attention is given to the two kinds of games mentioned above: those with a unidimensional strategy space and multilinear games. This reflects the importance of these kinds of games in both the theory of strategic games and in applications. In addition, in both cases there is at least one established notion of static stability, which can be compared with the one proposed here. For symmetric multilinear games, this is the notion of an evolutionarily stable strategy (ESS), and for symmetric games with a unidimensional strategy space, the notion of a continuously stable strategy (CSS), which can be described also in terms of the slope of the reaction curve, or the graph of the best-response function. It is shown that, for two-player games, these two seemingly unrelated notions of static stability are in fact essentially special cases of the general one proposed in this paper. The situation is

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[1] The requirement that the perturbation is small, i.e., that the players' strategies do not change much, implicitly refers to the topological structure of the simplex, i.e., to the notion of a neighborhood of a probability vector. L_1 and L_2, and every other metric derived from a norm on $\mathfrak{R}^n$, give the same metric topology. Thus *closeness* of points in the simplex has a well defined meaning even if no particular notion of *distance* is specified.

[2] An alternative to choosing a particular value of α is to require that a similar condition holds for all $0 < \alpha \leq 1$. Another alternative is to require this only for α sufficiently close to 0.

[3] Note that reversion is not required to be the best among all possible unilateral deviations.

[4] The definition of stability would not be affected if 'probability 1' were replaced by 'probability β', for any $0.5 < \beta < 1$. Since a player's payoff is linear in the probabilities of his own mixed strategy, this change would not affect the relative merits of the two deviations from a half-half mixture of x and z. Compare this with the remark in footnote 2.

more complex for multiplayer games, for which more than one generalization of (two-player) ESS exists.

For asymmetric games, definitions of stability are customarily derived from dynamic considerations. Specifically, they refer to asymptotic stability with respect to specified dynamics. These dynamics (e.g., asymmetric replicator dynamics) are applicable only to a particular class of games, and even for a single class, different dynamics may give different notions of stability. The conditions for dynamic stability can sometimes be put in a form reminiscent of static stability, e.g., as a condition on the relative slopes of the players' reaction curves. (This is somewhat similar to the stability condition for market equilibrium, which is determined by the relative slopes of the supply and the demand curves; see Samuelson, 1983.) As it turns out, none of the familiar notions of dynamic stability in the classes of asymmetric games considered in this paper is equivalent to static stability. However, dynamic and static stability do essentially coincide in the special case of inessentially asymmetric games, which differ from symmetric games only in that the players are distinguished as player 1, player 2, etc. Although seemingly minor, this difference is in fact highly consequential for stability analysis, mainly because it allows for more perturbations of the equilibrium state than in a truly symmetric game.

Static and dynamic stability are not the only kinds of stability considered in the game-theoretic literature. Another kind of stability refers to the effects of perturbations of the players' *strategy spaces*, e.g., allowing only completely mixed strategies, or a combination of perturbations of strategy spaces and of the strategies themselves. The requirement that a strategy profile in a strategic, or normal form, game be stable against such perturbations gives the notions of (trembling-hand) perfect equilibrium (Selten, 1975), proper equilibrium (Myerson, 1978), strict perfection (Okada, 1981) and (strategic[5]) stability and full stability (Kohlberg and Mertens, 1986). Stability may also refer to the effects on a given equilibrium of perturbations of the *payoff functions*, i.e., of the game itself. Essentiality (Wu and Jiang, 1962) and strong stability (Kojima et al., 1985) are examples of this kind of stability, which is known to have interesting connections with some of the other kinds. For example, in multilinear games, every essential equilibrium is strictly perfect (van Damme, 1991, Theorem 2.4.3), and in symmetric $n \times n$ games, every regular ESS is essential (Selten, 1983). Another, striking example of the connection between different kinds of stability is the finding that, in several classes of games, the (local) degree of an equilibrium (or of a connected component of equilibria) is equal to its index (Govindan and Wilson, 1997; Demichelis and Germano, 2000). The index of an equilibrium is connected with its asymptotic stability or instability with respect to a large class of natural dynamics, which determine how strategies *in the game* change over time. The degree, by contrast, expresses a topological property of the same equilibrium when viewed as a point on a manifold that includes the various equilibria of *different games* (Ritzberger, 2002).

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[5] 'Strategic stability' is also used, informally, to describe the equilibrium, or self-enforcement, condition that no player ever has an incentive to deviate (Kohlberg and Mertens, 1986, p. 1004). This refers to extensive as well as normal form games.

The notion of static stability proposed in this paper is thus just one of several possible interpretations of 'stability' in strategic games. It has the distinction of not being tied to any particular structure of the strategy spaces or the payoff functions, which makes it uniquely general and widely applicable. The only formal requirement is that the strategy spaces are endowed with some (completely general) topology. However, since this notion of stability is a local one, meaning that it only refers to small perturbations of the state, it is nontrivial only in the case of infinite strategy spaces. If a game has only finitely many strategy profiles, each of them is trivially stable. However, these (pure) strategies do not necessarily remain stable if mixed strategies are allowed. In the mixed extension of the game, players have infinitely many strategies.

2. Symmetric Two-Player Games

A symmetric two-player game is a function[6] $g: X \times X \to \mathfrak{R}$, where X is a topological space, called the *strategy space*, and $\mathfrak{R}$ is the real line. If one player uses strategy x and the other uses y, their *payoffs* are $g(x, y)$ and $g(y, x)$, respectively. The topology on X, which defines a neighborhood system for each strategy x (Kelly, 1955), is totally unrestricted (e.g., metrizability is not assumed). In principle, it should be specified as part of the definition of the game. However, in many cases it is clear from the context that X is a subspace of some standard topological space, most commonly a Euclidean space; its topology is thus the relative one. For example, if strategies are numbers, X is by default viewed as a subspace of $\mathfrak{R}$, with the usual topology, so that a set of strategies is a neighborhood of a strategy x if and only if, for some $\varepsilon > 0$, every $y \in X$ with $|x - y| < \varepsilon$ is in the set. In (the mixed extension of) a symmetric $n \times n$ game (where n is the number of pure strategies), the strategy space, which is the unit simplex, is viewed as a subspace of $\mathfrak{R}^n$.

A strategy x in a symmetric two-player game g is a (symmetric Nash) *equilibrium strategy* if the strategy profile (x, x) is a symmetric equilibrium, i.e., for every strategy y,

$$g(y, x) \leq g(x, x). \tag{3}$$

The corresponding *equilibrium payoff* is $g(x, x)$. The following definition plays a central role in this paper.

Definition 1. A strategy x in a symmetric two-player game g is *stable*, *weakly stable* or *definitely unstable* if it has neighborhood where, for all $y \neq x$, the inequality

$$g(y, x) - g(x, x) + g(y, y) - g(x, y) < 0, \tag{4}$$

a similar weak inequality, or the reverse (strict) inequality, respectively, holds.

In general, the stability condition does not imply the equilibrium condition (but see Section 2.1), and vice versa. Unlike equilibrium, stability is a local condition: only small

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[6] This is usually called the *payoff function*. In this paper, the payoff function and the game itself are identified.

deviations are considered. Hence, it trivially holds for every isolated strategy.[7] A more substantial difference is that the equilibrium condition requires x to be at least as good an alternative as any other strategy y, assuming that the opponent sticks with x. The stability condition, by contrast, implicitly assumes that the opponent may deviate. Specifically, with probability $p = 1/2$, the opponent plays y instead of x. Inequality (4), which is equivalent to

$$pg(x,x) + (1-p)g(x,y) > pg(y,x) + (1-p)g(y,y), \tag{5}$$

requires that, against such an opponent, x yields a higher expected payoff than y. If x is a *stable equilibrium strategies*, i.e., it satisfies both conditions, then it follows from (3) that (5) actually holds for all $0 < p \leq 1/2$. This property is somewhat in the spirit of risk dominance of pure equilibrium strategies in 2×2 games (Harsanyi and Selten, 1988).[8] However, it is in fact more closely related to some other familiar notions of stability of equilibrium, which, as shown below, are essentially special cases of the one in Definition 1. Unlike the latter, these notions are only applicable to specific classes of symmetric two-player games: either $n \times n$ games or games with a unidimensional strategy space.

1.1. Symmetric $n \times n$ games

A symmetric $n \times n$ game is given by a square *payoff matrix* A, with these dimensions. The strategy space X is the unit simplex in $\mathfrak{R}^n$. Its elements, which by default are considered column vectors, are usually referred to as mixed strategies. The interpretation is that there are n possible actions, and a strategy $x = (x_1, x_2, \dots, x_n)$ is a probability vector specifying the probability x_i with which each action i is used ($i = 1,2, \dots, n$). The set of all actions i with $x_i > 0$ is the *support* (or *carrier*) of x. A strategy is *pure* if its support includes only one action i (in which case the strategy itself may also be denoted by i) and *completely mixed* if the support includes all n actions. The game (i.e., the payoff function) $g: X \times X \to \mathfrak{R}$ is defined by

$$g(x,y) = x^{\mathrm{T}}Ay.$$

Thus, g is bilinear, and $A = \big(g(i,j)\big)_{i,j=1}^{n}$.

A standard notion of stability for symmetric $n \times n$ games is evolutionary stability, which can be defined as follows.

Definition 2. In a game g, a strategy y can *invade* another strategy x if either (i) $g(y,x) > g(x,x)$ or (ii) $g(y,x) = g(x,x)$ and $g(y,y) > g(x,y)$, and it can *weakly invade* x if a similar condition holds with the strict inequality in (ii) replaced by a weak one. Strategy x is an *evolutionarily stable strategy* (ESS; Maynard Smith, 1982) in g if there is no strategy $y \neq x$ that can weakly invade it, and it is a *neutrally stable strategy* (NSS) if there is no strategy that can invade it.

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[7] The discussion in this paper is therefore irrelevant to games in which the strategy space has the discrete topology, i.e., all singletons are open sets. It is only relevant to games with infinite and "continuous" strategy spaces.

[8] This similarity does not extent to asymmetric 2×2 games (Section 3).

As the following proposition shows, these notions of stability are in fact equivalent to stability and weak stability in the sense of Definition 1. Since every ESS is clearly an NSS and every NSS is an equilibrium strategy, this implies that in the class of symmetric $n \times n$ games, every stable or (even) weakly stable strategy is automatically an equilibrium strategy.

Proposition 1. A strategy in a symmetric $n \times n$ game g is an ESS or an NSS if and only if it is stable or weakly stable, respectively.

Proof. A strategy x is an ESS or an NSS if and only if it has a neighborhood where

$$g(y,y) < g(x,y) \tag{6}$$

or a similar weak inequality, respectively, holds for all strategies $y \neq x$ (Weibull, 1995, Propositions 2.6 and 2.7). As indicated, in both cases x is an equilibrium strategy. The above strict or weak inequality and the equilibrium condition (3) together imply that (4) or the corresponding weak inequality, respectively, holds. This proves that every ESS or NSS is stable or weakly stable, respectively.

To prove the converse, consider a stable strategy x. If follows from Definition 1 that for *every* strategy $y \neq x$ the following inequality holds for sufficiently small $\varepsilon > 0$:

$$\begin{aligned} &g((1-\varepsilon)x + \varepsilon y, x) - g(x,x) \\ &+ g((1-\varepsilon)x + \varepsilon y, (1-\varepsilon)x + \varepsilon y) - g(x, (1-\varepsilon)x + \varepsilon y) < 0. \end{aligned} \tag{7}$$

It follows from the bilinearity of g that this inequality is equivalent to

$$(2-\varepsilon)(g(y,x) - g(x,x)) + \varepsilon(g(y,y) - g(x,y)) < 0. \tag{8}$$

Inequality (8) holds for sufficiently small $\varepsilon > 0$ if and only if either (i) $g(y,x) < g(x,x)$ or (ii) $g(y,x) = g(x,x)$ and $g(y,y) < g(x,y)$. It follows that x is an ESS. Similar arguments show that a weakly stable strategy is an NSS; the only difference is that the strict inequalities in (7), (8) and (ii) are replaced by weak ones. ■

A completely mixed equilibrium strategy x in a symmetric $n \times n$ game g is said to be *definitely evolutionarily unstable* (Weissing, 1991) if every strategy $y \neq x$ can invade it. Similar arguments to those in the proof of Proposition 1 show that this condition holds if and only if x is definitely unstable in the sense of Definition 1.

2.2. Symmetric games with a unidimensional strategy space

In a symmetric two-player game g in which the strategy space is a subset of $\Re$, i.e., strategies are real numbers, the stability of an equilibrium strategy has a simple, intuitive interpretation. As shown below, if g is twice continuously differentiable, and with the possible exception of certain borderline cases, the equilibrium strategy is stable or definitely unstable if, at the equilibrium point, the graph of the best-response function intersects the forty-five degree line from above or below, respectively. Stability is also very close to the notion of continuously stable strategy (Eshel and Motro, 1981; Eshel, 1983).

Definition 3. In a symmetric two-player game g with a strategy space that is a subset of the real line, a (symmetric) equilibrium strategy x is a *continuously stable strategy* (CSS) if it has a neighborhood where for every other strategy y, for sufficiently small $\epsilon > 0$

$$g((1-\epsilon)y + \epsilon x, y) > g(y, y), \tag{9}$$

and a similar inequality does not hold with ϵ replaced by $-\epsilon$.

In other words, a strategy x that satisfies the "global" condition of being an equilibrium strategy[9] is a CSS if it also satisfies the "local" condition that, if both players use a strategy y that is close to x, a small unilateral deviation from y is advantageous to the deviating player if and only if it brings him closer to x rather than further away from it. This local condition, known as *m*-stability or convergence stability (Taylor, 1989; Christiansen, 1991), is very similar to the informal description of stable strategy in the Introduction. However, the actual definition of stability in this paper is based on the more generally-applicable probabilistic formulation. It would therefore be reassuring to know that this does not substantially alter the meaning of stability. The following proposition establishes this. The proposition and subsequent discussion concern an *interior* equilibrium strategy, i.e., one lying in the interior of the strategy space.

Proposition 2. Let x an interior equilibrium strategy in a symmetric two-player game g with a strategy space that is a subset of the real line, such that g has continuous second-order partial derivatives[10] in a neighborhood of the equilibrium point (x, x). If

$$g_{11}(x,x) + g_{12}(x,x) < 0, \tag{10}$$

then x is stable and a CSS. If the reverse inequality holds, then x is definitely unstable and not a CSS.

Proof. Using Taylor's theorem, it is easy to show that, for y tending to x, the left-hand side of (4) can be expressed as

$$2(y-x)g_1(x,x) + (y-x)^2(g_{11}(x,x) + g_{12}(x,x)) + o((y-x)^2). \tag{11}$$

Since x is an interior equilibrium strategy, the first term in (11) must be zero. Therefore, a sufficient condition for the left-hand side of (4) to be negative or positive for all $y \neq x$ in some neighborhood of x (and, hence, for x to be stable or definitely unstable, respectively) is that $g_{11}(x,x) + g_{12}(x,x)$ has that sign.

Dropping the factor 2 from (the first term in) (11) gives an expression for

$$(y-x)g_1(y,y), \tag{12}$$

[9] The original definition of CSS differs slightly from the version given here in requiring a somewhat stronger global condition.

[10] The partial derivatives of (the payoff function) g are denoted by subscripts. For example, g_{12} is the mixed partial derivative.

for y tending to x. Therefore, if (10) or the reverse inequality holds, then (12) is negative or positive, respectively, for all $y \neq x$ in some neighborhood of x. For every such y, for (positive or negative) ϵ tending to zero the left-hand side of (9) can be expressed as

$$g(y,y) - \epsilon(y-x)g_1(y,y) + o(\epsilon).$$

Therefore, (10) or the reverse inequality implies that x is or is not a CSS, respectively. ■

The connection between inequality (10) and the slope of the best-response function can be established as follows (Eshel, 1983). If x is as in Proposition 2, then it follows from the equilibrium condition (3), which holds for all strategies y, that $g_1(x,x) = 0$ and $g_{11}(x,x) \leq 0$. If the inequality is in fact strict, then by the implicit function theorem there is a continuously differentiable function ϕ from some neighborhood of x to the strategy space that assigns to each strategy y in the neighborhood a strategy $\phi(y)$ such that $g_1(\phi(y),y) = 0$ and $g_{11}(\phi(y),y) < 0$. Thus, strategy $\phi(y)$ is a local best response to y. Moreover, the values of ϕ and its derivative at the point x are given by $\phi(x) = x$ and

$$\phi'(x) = -\frac{g_{12}(x,x)}{g_{11}(x,x)}. \tag{13}$$

This implies that (10) holds at the equilibrium point (x,x) (so that x is stable) if and only if the slope of the function ϕ at the point x is less than 1. In this case, the graph of ϕ, or reaction curve, intersects the forty-five degree line from above, implying that the (local) fix point index (Dold, 1980) is $+1$ (see Figure 2). The reverse inequality holds (so that x is definitely unstable) if and only if the slope of ϕ at x is greater than 1. In this case, the graph of ϕ intersects the forty-five degree line from below and the fix point index is -1. (Compare this with the stability condition at the end of Section 3.3.)

An alternative notion of stability of an equilibrium strategy x in a symmetric two-player game with a unidimensional strategy space, called *neighborhood invader strategy* (NIS; Apaloo, 1997), replaces the "local" CSS condition with the requirement that x is *locally superior* in the sense that (6) holds for all $y \neq x$ in some neighborhood of x. For symmetric $n \times n$ games, this requirement is equivalent to stability (see Section 2.1). However, this is not the case for the games considered here, for which local superiority of an equilibrium strategy is a more demanding condition than stability, and thus more demanding than the CSS condition. An NIS x is a stable equilibrium strategy in the sense of Definition 1, since (6) and the equilibrium condition (3) together imply (4). However, the converse is not true, as can be seen by considering the differential condition for local superiority of an equilibrium strategy x, which differs from (10) in that the second term on the left-hand side is multiplied by 2 (Oechssler and Riedel, 2002).

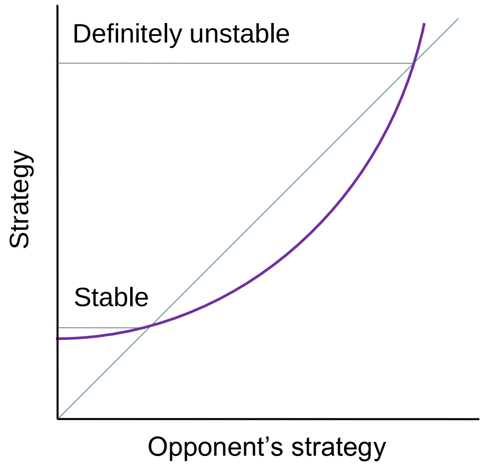

Figure 2. An equilibrium strategy is stable (and a CSS) or definitely unstable (and not a CSS) if, at the equilibrium point, the best-response curve (thick line) intersects the forty-five degree line (thin) from above or below, respectively.

3. Asymmetric Two-Player Games

An asymmetric two-player game is a function $h = (h^1, h^2): X^1 \times X^2 \to \Re^2$, where X^1 and X^2 are the strategy spaces of players 1 and 2, respectively.[11] If player 1 uses strategy x^1 and player 2 uses x^2, their payoffs are $h^1(x^1, x^2)$ and $h^2(x^1, x^2)$, respectively. The strategy profile $x = (x^1, x^2)$ is an *equilibrium* in h if each player's strategy is a best response to that of the other player, i.e., for all strategy profiles $y = (y^1, y^2)$,

$$h^1(y^1, x^2) \leq h^1(x^1, x^2) \text{ and } h^2(x^1, y^2) \leq h^2(x^1, x^2). \quad (14)$$

The equilibrium is *strict* if these best responses are unique, i.e., the first inequality in (14) is strict if $y^1 \neq x^1$ and the second is strict if $y^2 \neq x^2$.

An equilibrium in an asymmetric two-player game h is a different object than an equilibrium strategy in a symmetric game: it is a strategy profile rather than a single strategy. However, it may be identified with an equilibrium strategy in another, symmetric two-player game g. In that game, first one player is assigned to the role of player 1 in h and the other is assigned to the role of player 2, and then the roles are interchanged. A strategy in g is thus a strategy profile in h: it specifies the player's action in each of the two roles. A player's payoff in g is defined as the average of his two payoffs in h – the payoff in player 1's role and that in player 2's role. The following definition presents this notion of symmetrization of an asymmetric two-player game more formally. The proposition that follows the definition, which is proved (for an arbitrary number of players) in Section 5, asserts that the (symmetric) equilibrium strategies in the resulting symmetric game are precisely the equilibria in the original asymmetric one.

[11] In this paper, superscript indices always refer to players in an asymmetric game. Subscript indices have other meanings, which depend on the context.

Definition 4. The game obtained by *symmetrizing* an asymmetric two-player game $h = (h^1, h^2): X^1 \times X^2 \to \Re^2$ is the symmetric game $g: X \times X \to \Re$ in which the strategy space X is the product space[12] $X^1 \times X^2$ and, for all $x = (x^1, x^2), y = (y^1, y^2) \in X$,

$$g(x, y) = \frac{1}{2} h^1(x^1, y^2) + \frac{1}{2} h^2(y^1, x^2).$$

Proposition 3. A strategy profile $x = (x^1, x^2)$ in an asymmetric two-player game h is an equilibrium if and only if it is a (symmetric) equilibrium strategy in the symmetric game g obtained by symmetrizing h. In this case, the equilibrium payoff in g is equal to the players' average equilibrium payoff in h.

This one-to-one correspondence between equilibria in the asymmetric game and equilibrium strategies in the symmetric one naturally leads to the following.

Definition 5. A strategy profile $x = (x^1, x^2)$ in an asymmetric two-player game h is *stable*, *weakly stable* or *definitely unstable* if it has the same property as a strategy in the symmetric game g obtained by symmetrizing h.

The definition means that inequality (4), which defines stability in the symmetric case, is replaced in the asymmetric case by

$$\frac{1}{2}\left(h^1(y^1, x^2) - h^1(x^1, x^2) + h^1(y^1, y^2) - h^1(x^1, y^2)\right) + \frac{1}{2}\left(h^2(x^1, y^2) - h^2(x^1, x^2) + h^2(y^1, y^2) - h^2(y^1, x^2)\right) < 0. \tag{15}$$

In other words, a strategy profile x is stable if it gives a higher expected payoff than any other, nearby profile y when (i) the opponent is equally likely to play according to x or y and (ii) both assignments of the players to the roles in h are equally likely.

A stable strategy profile is not necessarily an equilibrium. And even if it is an equilibrium, it is not necessarily strict. However, as the following proposition shows, every stable strategy profile is in a sense "locally strict".

Proposition 4. If $x = (x^1, x^2)$ is a stable strategy profile in an asymmetric two-player game $h = (h^1, h^2)$, then player 1's strategy x^1 has a neighborhood where it is the player's unique best response to x^2, and similarily for player 2.

Proof. If x is stable in h, then it has a neighborhood where (15) holds for every other strategy profile $y = (y^1, y^2)$. For $y^2 = x^2$, that inequality simplifies to

$$h^1(y^1, x^2) < h^1(x^1, x^2).$$

Therefore, there is neighborhood of x^1 where this inequality holds for all $y^1 \neq x^1$. The argument for player 2 is similar. ■

[12] The topology on X is thus the product topology.

Although the definition of stability in an asymmetric game is based on the stability of a strategy in an auxiliary symmetric game, Proposition 4 suggests that the former notion of stability is more demanding than the latter. This suggestion is supported by the concrete, exact comparison between stability in symmetric two-player games and in asymmetric games in Section 3.3. Additional support is provided by the fact, established in the next subsection, that for a particular, important class of games, stable strategy profiles are not only "locally strict" but are actually strict equilibria.

3.1. Bimatrix games

A bimatrix game is given by a pair of matrices (A, B) of equal dimensions, $m \times n$. The strategy spaces of players 1 and 2 are the unit simplices in $\mathfrak{R}^m$ and $\mathfrak{R}^n$, respectively. Viewing strategies as column vectors, the game $h = (h^1, h^2): X^1 \times X^2 \to \mathfrak{R}$ is defined by (the payoffs functions)

$$h^1(x^1, x^2) = (x^1)^{\mathrm{T}} A x^2 \text{ and } h^2(x^1, x^2) = (x^1)^{\mathrm{T}} B x^2.$$

As the next proposition shows, stability in this class of games has a rather strong meaning. This result, which is essentially due to Selten (1980; see also Hammerstein and Selten, 1994, Result 17; van Damme, 1991, Theorem 9.6.2; and footnote 13 below), is proved (for an arbitrary number of players) in Section 5.1.

Proposition 5. A strategy profile in a bimatrix game $h = (h^1, h^2)$ is stable if and only if it is a strict equilibrium. In particular, every stable equilibrium is pure.

The reason why a strategy profile that is not a strict equilibrium cannot be stable is rather simple. Stability in an asymmetric bimatrix game h is determined by reference to an auxiliary symmetric game g, in which the two players take turns in playing the two rolls in h.[13] If a strategy profile $x = (x^1, x^2)$ in h is not a strict equilibrium, for example, if there is some strategy $y^1 \neq x^1$ that is a best response to x^2 (in which case such a strategy exists in every neighborhood of x^1), then using y^1 instead of x^1 when playing player 1's role in h does not decrease a player's payoff in g. Moreover, playing according to $y = (y^1, x^2)$ rather than x does not decrease the payoff also if the opponent also plays according to y rather than x. This is because a switch from x to y is effective only when the player is assigned to player 1's role in h, which is when the opponent is assigned to player 2's role, for which both x and y prescribe the same strategy (namely, x^2). Therefore, the left-hand side of (4) is nonnegative, so that this inequality does not hold.

The result that only strict equilibria are stable raises the question of whether it only indicates that the present notion of stability is inadequate for analyzing asymmetric bimatrix games. If

[13] Note that g is not a symmetric $k \times k$ game, for any k. A strategy in g is in a sense a *behavior strategy*. By definition of symmetrization, it prescribes one (mixed) strategy for player 1's role in h and another one for player 2's role, but mixing such pairs of (mixed) strategies is not allowed. This implies that the usual notion of ESS would have to be extended somewhat to apply to g (van Damme, 1991, Definition 9.5.2). Here, such an extension is not required since the general notion of stability (Definition 1) is applicable to g, and can be shown (using rather similar arguments to those in the proof of Proposition 1) to coincide with that extension.

only static stability concepts are considered, the answer seems to be negative, since in this framework strict equilibrium is the only obvious extension of ESS to asymmetric bimatrix games (Hammerstein and Selten, 1994, p. 965; Hofbauer and Sigmund, 1998, p. 114). However, strictness is not a necessary condition for dynamic stability, i.e., asymptotic stability with respect to specified dynamics. Whether a given equilibrium is dynamically stable may depend on the choice of dynamics (Demichelis and Germano, 2000, Example 2). However, Samuelson and Zhang (1992, Theorem 4) showed that for a large class of reasonable (continuous) dynamics, every asymptotically stable outcome is, in some well-defined sense, "almost" a strict equilibrium. In particular, if it is pure (which is not necessarily the case), then it must be a strict equilibrium.

3.2. Games in the plane

Another important class of asymmetric games is two-player games in which the players' strategy spaces X^1 and X^2 are intervals or some other subsets of the real line. Strategy profiles are therefore points in the real plane. If the players' payoff functions h^1 and h^2 are differentiable, the stability condition can be expressed in terms of the partial derivatives of these functions. The next proposition, which is proved (for an arbitrary number of players) in Section 5.2, presents such a condition for *interior* equilibria, i.e., equilibria in which the strategy of each player is an interior point in the player's strategy space.

Proposition 6. A sufficient condition for stability or definite instability of an interior equilibrium (x^1, x^2) with a neighborhood in which h^1 and h^2 have continuous second-order derivatives is that the matrix

$$H = \begin{pmatrix} h^1_{11} & h^1_{12} \\ h^2_{21} & h^2_{22} \end{pmatrix}, \tag{16}$$

with the derivatives evaluated at (x^1, x^2), is negative definite or positive definite, respectively. A necessary condition for weak stability is that the matrix is negative semidefinite.[14]

Example 1. The players' strategy spaces are the entire real line, and their payoffs are given by the quadratic functions

$$h^1(x, y) = -x^2 + 3xy \text{ and } h^2(x, y) = -\frac{1}{2}y^2. \tag{17}$$

It is not difficult to check that the origin $(0,0)$ is the unique equilibrium. Since

$$H = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix}, \tag{18}$$

and this matrix is not negative semidefinite, by Proposition 6 the equilibrium is not even

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[14] A square, $k \times k$ (not necessarily symmetric) matrix A is negative definite if $x^{\mathrm{T}} A x < 0$ for all nonzero $x \in \Re^k$. A necessary and sufficient condition for this is that all eigenvalues λ of the symmetric matrix $(1/2)(A + A^{\mathrm{T}})$ satisfy $\lambda < 0$. The definition and characterization of negative semidefiniteness are similar, except that the two strict inequalities are replaced by weak ones.

weakly stable. This can also be seen directly, by considering the game g obtained by symmetrizing $h = (h^1, h^2)$. The payoff from using $(0,0)$ in g is 0 regardless of the opponent's strategy. Using $(3,4)$ gives -8.5 if the opponent uses $(0,0)$ (-9 when playing the first role in h and -8 when playing the second role) and 9.5 if the opponent also uses $(3,4)$. Hence, if the opponent is equality likely to use $(0,0)$ or $(3,4)$, the latter is a better response than the former (since it gives a positive expected payoff, 0.5). The same is true also for any positive multiple of (3,4). This shows that the equilibrium $(0,0)$ is not weakly stable in h. However, the same strategy profile is stable in the game

$$h^1(x,y) = -x^2 + 3xy \text{ and } h^2(x,y) = -\frac{1}{2}y^2 - xy, \tag{19}$$

which differs from (17) only in the second term in h^2, and also has $(0,0)$ as the unique equilibrium. This is because, in this game,

$$H = \begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix},$$

which is a negative definite matrix.

For any interior equilibrium as in Proposition 6, negative definiteness of the matrix H implies that

$$h_{11}^1, h_{22}^2 < 0 \text{ and } h_{11}^1 h_{22}^2 > h_{12}^1 h_{21}^2. \tag{20}$$

However, these inequalities by themselves are not a sufficient condition for stability of the equilibrium, as demonstrated by the fact that they hold for (17) (as well as for (19)). On the other hand, (20) is a sufficient (and almost necessary) condition for *D*-stability of the matrix H (Hofbauer and Sigmund, 1998). As explained in Section 5.2 below, *D*-stability implies that a natural myopic adjustment process in which both players simultaneously and continuously adjust their strategies converges to the equilibrium point if it starts sufficiently close to it. Thus, asymptotic stability with respect to this process is essentially a weaker condition than static stability of the equilibrium as defined in this paper. For example, it holds for both games in Example 1.

The same is not necessarily true for other kinds of adjustment processes. In particular, static stability of the equilibrium does not imply asymptotic stability with respect to another natural adjustment process, in which the players alternate in myopically playing best response to each other's strategy. As seen in Figure 3, staring from any other strategy profile, these dynamics quickly bring the players to the (statically unstable) equilibrium $(0,0)$ in the game (17), but take them increasingly farther away from the same (statically stable) equilibrium point in (19).

This difference between the two kinds of dynamics can be understood by considering the equivalent form of (20) in which the right inequality is replaced by

$$\left(-\frac{h_{21}^2}{h_{22}^2}\right)\left(-\frac{h_{12}^1}{h_{11}^1}\right) < 1. \tag{21}$$

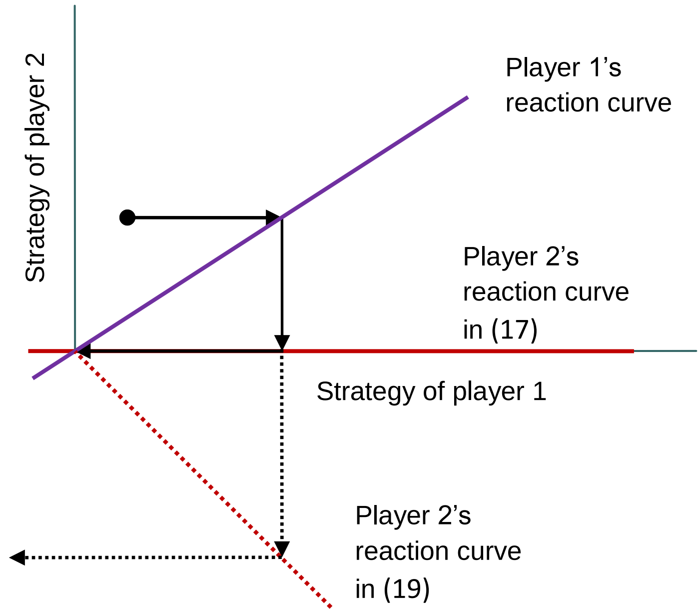

Figure 3. The players' reaction curves in the two games in Example 1. Player 1's reaction curve (upward sloping line) is the same in both games, but those of player 2 (horizontal and downward sloping lines) are different. The arrows show possible trajectories under the alternating-best-responses dynamics, in which player 1 moves first, then player 2, then player 1 again, and so on. For the game given by (17) (solid arrows), the trajectory ends at the equilibrium point $(0, 0)$. For the game in (19) (dotted arrows), it spirals ways.

Evaluated at the equilibrium point, where the players' reactions curves intersect, the left-hand side of (21) is the product of the slope of player 2's curve and the reciprocal of player 1's curve (cf. (13)). The condition for asymptotic stability of the equilibrium with respect to alternating best responses is that the *absolute value* of this product be less than 1 (Fudenberg and Tirole, 1995). This stronger condition, which means that player 1's reaction curve is steeper than 2's, is not implied by (20). The condition is also not implied by, and it does not imply, negative definiteness of H, as demonstrated by the fact that it does not hold for the game in (19) but holds for that in (17).

In summary, for games in the plane, the two notions of dynamic stability considered above are not equivalent, and none of them is equivalent to the notion of static stability introduced in this paper. As shown in the next subsection, this nonequivalence is a consequence of the asymmetry of the games.

3.3. Inessentially asymmetric games

An asymmetric two-player game $h = (h^1, h^2)$ can be essentially symmetric. This is so if the players' strategy spaces are identical and their roles are interchangeable, i.e., $X^1 = X^2$ and

$$h^1(x^1, x^2) = h^2(x^2, x^1) \tag{22}$$

for all strategy profiles (x^1, x^2). This condition holds, for example, for all bimatrix games of the form (A, A^{T}), where A is any $n \times n$ matrix. The condition is often taken to be the *definition* of a symmetric game (von Neumann and Morgenstern, 1953). However, this notion of symmetry is different from that in Section 2. To distinguish symmetric games in the sense of Section 2 from asymmetric games satisfying the above condition, the latter will be referred to as *inessentially asymmetric games*.

The distinction between symmetry and inessentially asymmetry may seem like a mere formality. For example, for most purposes an inessentially asymmetric bimatrix game (A, A^{T}) can be identified with the symmetric $n \times n$ game with payoff matrix A. More generally, for any strategy space X, there is a one-to-one correspondence between the class of all inessentially asymmetric two-player games $h = (h^1, h^2): X \times X \to \mathfrak{R}^2$ (satisfying (22)) and the class of all symmetric two-player games with this strategy space. The correspondence is given by projection on the first coordinate,

$$h \mapsto h^1. \tag{23}$$

Nevertheless, for the purpose of stability analysis, inessential asymmetry is not the same as symmetry. Whereas for symmetric games stability is defined for strategies (Definition 1), for asymmetric games it is defined for strategy profiles (Definition 5). Hence, only the latter covers asymmetric equilibria, i.e., equilibria (x^1, x^2) with $x^1 \neq x^2$. Furthermore, even for symmetric equilibria ($x^1 = x^2$), stability of the equilibrium in the inessentially asymmetric game is not equivalent to stability of the equilibrium strategy in the corresponding symmetric game.[15] In fact, as shown below, stability of the equilibrium is a more stringent requirement. This may seem a bit surprising, seeing that the definition of stability of (symmetric or asymmetric) equilibrium (Definition 5) is based on that of a (symmetric) equilibrium strategy in an auxiliary symmetric game. However, that auxiliary game is obtained from h not by the projection (23) but rather by symmetrization. Compare the next result, which is proved (for an arbitrary number of players) in Section 5.3, with Proposition 3 and Definition 5.

Proposition 7. A symmetric strategy profile (x, x) in an inessentially asymmetric two-player game $h = (h^1, h^2): X \times X \to \mathfrak{R}^2$ is an equilibrium if and only if it is an equilibrium in the corresponding symmetric game $h^1: X \times X \to \mathfrak{R}$. If (x, x) is stable in h, then the strategy x is stable in h^1, but the converse does not hold even if (x, x) is an equilibrium.

The second part of Proposition 7 is illustrated by the example of an equilibrium strategy x in a symmetric $n \times n$ game with (any) payoff matrix A. By Proposition 1, x is stable if and only if it is an ESS. By Proposition 5, the symmetric equilibrium (x, x) is stable in the corresponding inessentially asymmetric bimatrix game (A, A^{T}) if and only if it satisfies the stronger condition of being strict.

The intuitive reason why stability of an equilibrium strategy x in a symmetric game does not imply the same for the equilibrium (x, x) in the corresponding inessentially asymmetric game is that, for (x, x) to be stable, it has to withstand more kinds of perturbations than x. First, it has to resist changes in one coordinate only, which result in asymmetric strategy

[15] For bimatrix games, a related difference holds for the index and degree of the symmetric equilibrium, which may depend on whether it is viewed as an equilibrium in the inessentially asymmetric bimatrix game or in the corresponding symmetric $n \times n$ one (Demichelis and Germano, 2000).

profiles like (y, x), with $y \neq x$. The use of (y, x) by both players in the game obtained by symmetrizing the inessentially asymmetric one is tantamount to a *correlated* deviation from the equilibrium in the original symmetric game: each of the players may use the strategy y, but they do not both use it (and hence the payoff corresponding to the strategy profile (y, y) is irrelevant, unlike in Definition 1). The equilibrium withstands such correlated deviations if and only if it is "locally strict", i.e., x has a neighborhood where it is the unique rest response to itself (see Proposition 4). Second, (x, x) has to withstand perturbations in which the two players change to different strategies, say y^1 and y^2. Whether it satisfies this requirement depends, inter alia, on the payoff of a player using y^1 against an opponent using y^2 (again unlike in Definition 1, where alternatives to x are considered one at a time).

These considerations suggest that the seemingly subtle difference between modeling a pairwise contest as a symmetric game and modeling it as an inessentially asymmetric game may actually translate into widely divergent predications. This, in fact, is not a novel observation but one made long ago in the biological game theory literature, where inessential asymmetry is often referred to by other names such as uncorrelated asymmetry (Maynard Smith and Parker, 1976; the correlation this term refers to is between the players' traits and their payoff functions). A symmetric pairwise contest with identical contestants, such as two equal-size males seeking to obtain a newly vacated territory, is best modeled as a symmetric game such as the Hawk–Dove game (or Chicken). Precedence or other perceivable asymmetries between the contestants, which do not by themselves change the payoffs (i.e., the stakes or the opponents' fighting abilities), makes the contest an inessentially asymmetric game, and, in reality, may significantly affect the contestants' behavior (Maynard Smith, 1982; Riechert, 1998).

Another example of the difference between stability of an equilibrium strategy in a symmetric two-player game and stability of the symmetric equilibrium in the corresponding inessentially asymmetric game is provided by games with a unidimensional strategy space. If the payoff functions h^1 and h^2 in an asymmetric game are twice continuously differentiable in a neighborhood of an interior symmetric equilibrium (x, x), then the inessential asymmetry condition (22) implies that, at that point,

$$h_{11}^1 = h_{22}^2 \text{ and } h_{12}^1 = h_{21}^2. \tag{24}$$

With these equalities, the negative definiteness condition in Proposition 6 is equivalent to (20) (which is the condition for *D*-stability of the matrix H defined in (16)), which in turn is equivalent to the simpler condition

$$|h_{21}^2| < -h_{22}^2.$$

Since $h_{22}^2 \leq 0$ holds automatically at any interior equilibrium, the last condition only adds the requirements that the inequality is strict and that

$$\left|\frac{h_{21}^2}{h_{22}^2}\right| < 1. \tag{25}$$

The last inequality says that, at the equilibrium point, the slope of player 2's reaction curve is less than 1 but greater than -1. This is stronger than the stability condition for symmetric games (Section 2.2), which consists of the former inequality only.

Note that, with (24), the left-hand side of (21) can be replaced by its absolute value; in both cases the inequality is equivalent to (25). This implies that, for inessentially asymmetric games in the plane, unlike for "truly" asymmetric ones (Section 3.2), asymptotic stability of a symmetric equilibrium with respect to the continuous adjustment process is essentially equivalent to stability with respect to alternating best responses. In addition, both are essentially equivalent to static stability (since for inessentially asymmetric games *D*-stability of the matrix H is equivalent to negative definiteness). Thus, dynamic and static stability of an interior symmetric equilibrium are essentially equivalent, and both are stronger than (static) stability of the equilibrium strategy in the corresponding symmetric game.

4. Symmetric Multiplayer Games

A symmetric n-player game ($n \geq 1$) is a real-valued function $g: X \times X \times \cdots \times X \to \mathfrak{R}$, defined on the n-times product of a topological space X, that is invariant to permutations of its second through nth arguments. X is the players' common strategy space. If one player uses strategy x and the others use $y, z, \ldots, w$ (in any order), the first player's payoff is $g(x, y, z, \ldots, w)$. A strategy x is a (symmetric) equilibrium strategy in g if, for all strategies y,

$$g(y, x, \ldots, x) \leq g(x, x, \ldots, x). \tag{26}$$

Generalizing the notion of stable strategy from symmetric two-player games to an arbitrary number of players is not straightforward. The gist of Definition 1 is that a stable strategy x is superior to any other strategy y close to it if the opponent it equally likely to use x or y. In an n-player game, if each of the opponents is equally likely to use x or y then the expected number of opponents using x is equal to the expected number of opponents using y. Denoting by p_j the probability that $n - j$ of the opponents use x (and $j - 1$ use y), the condition of equal expectations can be written as

$$\sum_{j=1}^{n} (j-1)p_j = \frac{n-1}{2}. \tag{27}$$

However, Eq. (27) does not completely specify the probability vector $p = (p_1, p_2, \ldots, p_n)$. One way of completing the definition of stability is to require x to be superior to y for every p satisfying (27). An alternative is to require this only for a particular, "natural" p. As shown below, these two alternatives can differ only if $n \geq 3$. For single- and two-player games they both give the same definition, which in the latter case coincides with that in Definition 1.

Definition 6. For a probability vector $p = (p_1, p_2, \ldots, p_n)$, a strategy x in a symmetric n-player game $g: X \times X \times \cdots \times X \to \mathfrak{R}$ is *p-stable*, *weakly p-stable* or *definitely p-unstable* if it has a neighborhood where

$$\sum_{j=1}^{n} p_j \left(g(y, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}}) - g(x, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}}) \right) < 0, \tag{28}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds for all strategies $y \neq x$. Strategy x is *stable*, *weakly stable* or *definitely unstable* if the corresponding condition holds for all probability vectors p satisfying (27).

The left-hand side of (28) expresses a player's (positive or negative) expected gain from switching from strategy x to y, for a particular distribution of the opponents' strategies. For single-player games ($n = 1$), the latter is of course irrelevant, so that a stable strategy is simply a strict local optimum: switching to any other, nearby strategy reduces the payoff. For $n = 2$, condition (27) reads $p_2 = 1/2$, and thus implies that the left-hand side of (28) is equal to one-half that of (4), so that these two inequalities are equivalent. Thus, stability and p-stability can differ only for $n \geq 3$. Special cases of p-stability are *dependent-stability*, defined by

$$p_j = \begin{cases} \frac{1}{2}, & j = 1, n \\ 0, & j = 2, \dots, n-1 \end{cases}, \tag{29}$$

independent-stability, defined by

$$p_j = \frac{1}{2^{n-1}} \binom{n-1}{j-1}, \qquad j = 1, 2, \dots, n, \tag{30}$$

and *uniform-stability*, defined by

$$p_j = \frac{1}{n}, \qquad j = 1, 2, \dots, n. \tag{31}$$

In each case, the corresponding notions of weak stability and definite instability are similarly defined. Note that (29) describes the distribution of the number of opponents using strategy x (and the number using the alternative strategy y) if either all of them use x or they all use y, and both possibilities have probability $1/2$. By contrast, (30) corresponds to independent randomizations by the players between x and y with half-half probabilities. This symmetry between x and y implies that both (29) and (30) actually satisfy a stronger condition than the equal-expectations condition (27), namely,

$$p_j = p_{n-j+1}, \qquad j = 1, 2, \dots, n. \tag{32}$$

This equality, which obviously holds also for (31), says that the probability that $n - j$ of the opponents use x and $j - 1$ use y is equal to the probability that it is the other way around. In other words, the number of opponents using x and the number using y (which sum up to $n - 1$) have equal *distributions*. Equivalently, their joint distribution is symmetric.

For any $n \geq 2$, (32) implies that the left-hand side of (28) is equal to the more symmetrically-looking expression

$$G_p(y,x) \stackrel{\text{def}}{=} \sum_{j=1}^{n} p_j \left(g(\underbrace{y,\dots,y}_{j \text{ times}},x,\dots,x) - g(\underbrace{x,\dots,x}_{j \text{ times}},y,\dots,y) \right).$$

Thus, for p satisfying (32), a strategy x is p-stable if and only if it has a neighborhood where it is the unique best response to itself in the symmetric two-player zero-sum game $G_p: X \times X \to \Re$.

1.1. Symmetric multilinear games

A special case of symmetric multiplayer games is symmetric multilinear games, which are the n-player generalization of the games in Section 2.1.[16] As in the latter, the strategy space X is the unit simplex in some Euclidean space and the game $g: X \times X \times \cdots \times X \to \Re$ is multilinear, i.e., linear in each of its n arguments.

As shown in Section 2.1, for $n = 2$ stability as defined in this paper coincides with the standard notion of ESS. The latter can easily be generalized to multilinear games with an arbitrary number of players.

Definition 7. A strategy x in a symmetric multilinear game is an *evolutionarily stable strategy* (ESS) if, for every other strategy y, for sufficiently small $\epsilon > 0$

$$g(y, y_\epsilon, y_\epsilon, \dots, y_\epsilon) < g(x, y_\epsilon, y_\epsilon, \dots, y_\epsilon), \tag{33}$$

where $y_\epsilon = (1-\epsilon)x + \epsilon y$.

The condition in Definition 7 is equivalent to the following (Broom et at., 1997): For every $y \neq x$, the finite sequence

$$g(y, \underbrace{x,\dots,x}_{n-j \text{ times}}, \underbrace{y,\dots,y}_{j-1 \text{ times}}) - g(x, \underbrace{x,\dots,x}_{n-j \text{ times}}, \underbrace{y,\dots,y}_{j-1 \text{ times}}), \qquad j = 1, 2, \dots, n, \tag{34}$$

has at least one nonzero entry, and the first such entry is negative. This equivalence shows that Definition 7 is indeed an extension of Definition 2. However, it is not the only possible extension. Another stability notion that is equivalent to ESS if $n = 2$ is local superiority, or strong uninvadability, which refers to the last entry in (34) (Bomze and Pötscher, 1989; van Damme, 1991, Theorem 9.2.8; Weibull, 1995, Propositions 2.6; Bomze and Weibull, 1995).

Definition 8. A strategy x is *locally superior* if it has a neighborhood where, for every other strategy y,

$$g(y, y, \dots, y) < g(x, y, \dots, y). \tag{35}$$

It is easy to see that every locally superior strategy is an ESS. However, as shown below, for $n > 2$ not every ESS is locally superior. Thus, these two notions of stability are not equivalent.[17] This demonstrates a point of general significance. Namely, for some classes of

[16] Note that, here, n denotes the number of players, not the number of pure strategies.

[17] This ostensibly contradicts Theorem 2 of Bukowski and Miekisz (2004), which asserts that local superiority strategy and the ESS condition *are* equivalent even for multiplayer games. However, these

games there is more than one reasonable notion of static stability. This underlines the desirability of deriving such a notion from general principles, like Definition 6, rather than from game-specific considerations. It also raises the question of which of the two definitions in this subsection is equivalent to the restriction of the general notion of stability (Definition 6) to multilinear games. As the following theorem shows, neither of them is equivalent to it. However, local superiority is equivalent to certain kinds of p-stability.

Theorem 1. For symmetric multilinear games, the following implications and equivalences among the possible properties of a strategy x hold:

> stable ⇒ dependently-stable ⇔ independently-stable ⇔ locally superior ⇒ uniformly-stable ⇒ ESS ⇒ equilibrium strategy.

Proof. The first implication holds by definition of stability. To prove the second implication (local superiority ⇒ uniform-stability), consider a locally superior strategy x. Such a strategy has a *convex* neighborhood U where (35) holds for every strategy $y \neq x$. For every such y and any $0 < t \leq 1$,

$$g((1-t)x + ty, (1-t)x + ty, \dots, (1-t)x + ty) \\ < g(x, (1-t)x + ty, \dots, (1-t)x + ty). \tag{36}$$

By the multilinearity of g, (36) is equivalent to

$$\sum_{j=1}^{n} B_{j-1,n-1}(t) \left(g(y, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}}) - g(x, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}}) \right) < 0, \tag{37}$$

where $B_{j-1,n-1}(t) = \binom{n-1}{j-1} t^{j-1} (1-t)^{n-j}$ is Bernstein polynomial. By the identity

$$\int_0^1 B_{j-1,n-1}(t)\, dt = \frac{1}{n}, \qquad j = 1, 2, \dots, n,$$

the inequality obtained by integrating the left-hand side of (37) over t coincides with the special case of (28) (given by (31)) that defines uniform-stability. This proves that x is uniformly-stable.

The penultimate implication in the theorem is a special case of the following result.

Lemma 1. For a probability vector $p = (p_1, p_2, \dots, p_n)$, with $p_n \neq 0$, every p-stable strategy x is an ESS.

Proof. For a p-stable strategy x and any strategy $y \neq x$, for sufficiently small $\epsilon > 0$ the following expression is negative:

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authors' definition of ESS is different from (and more demanding than) Definition 7 in that it interchanges the two logical quantifiers and requires that, for sufficiently small $\epsilon > 0$, (33) holds for all $y \neq x$. (Using standard terminology, this requirement means that there is a *uniform invasion barrier*.)

$$\sum_{j=1}^{n} p_j \left(g(y_\epsilon, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y_\epsilon, \dots, y_\epsilon}_{j-1 \text{ times}}) - g(x, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y_\epsilon, \dots, y_\epsilon}_{j-1 \text{ times}}) \right),$$

where $y_\epsilon = (1-\epsilon)x + \epsilon y$. The expression can also be written as

$$\sum_{j=1}^{n} p_j \sum_{k=1}^{j} \binom{j-1}{k-1} \epsilon^k \hat{g}(y-x, \underbrace{x, \dots, x}_{n-j \text{ times}}, \underbrace{y-x, \dots, y-x}_{k-1 \text{ times}}, \underbrace{x, \dots, x}_{j-k \text{ times}}),$$

where $\hat{g}$ denotes the multilinear extension of g (the domain of which consists of all n-tuples of linear combinations of strategies). This is a polynomial in ϵ, in which the constant term is zero and the coefficients of the higher-order terms are

$$\left(\sum_{j=k}^{n} p_j \binom{j-1}{k-1} \right) \hat{g}(\underbrace{y-x, \dots, y-x}_{k \text{ times}}, x, \dots, x), \qquad k = 1,2, \dots, n. \tag{38}$$

Since the polynomial is negative for sufficiently small $\epsilon > 0$, the sequence of coefficients (38) must have the property that at least one of its entries is not zero, and the first such entry is negative. The expression in parenthesis in (38) is the sum of nonnegative terms, and at least the last term is positive, since $p_n \neq 0$. Therefore, that expression is positive, so that dropping it does not affect the signs of the various entries in (38). This implies that the sequence (34) also has the property described above, for otherwise either

$$g(y, \underbrace{y, \dots, y}_{j-1 \text{ times}}, \underbrace{x, \dots, x}_{n-j \text{ times}}) - g(x, \underbrace{y, \dots, y}_{j-1 \text{ times}}, \underbrace{x, \dots, x}_{n-j \text{ times}}) = 0$$

would hold for all $1 \leq j \leq n$ or

$$g(y, \underbrace{y, \dots, y}_{j-1 \text{ times}}, \underbrace{x, \dots, x}_{n-j \text{ times}}) - g(x, \underbrace{y, \dots, y}_{j-1 \text{ times}}, \underbrace{x, \dots, x}_{n-j \text{ times}}) > 0$$

would hold for the smallest j for which the equality does not hold. Both possibilities contradict the previous finding, since they imply that either

$$\hat{g}(\underbrace{y-x, \dots, y-x}_{j \text{ times}}, x, \dots, x) = 0$$

holds for all $1 \leq j \leq n$ or

$$\hat{g}(\underbrace{y-x, \dots, y-x}_{j \text{ times}}, x, \dots, x) > 0$$

holds for the smallest j for which the equality does not hold. This contradiction proves that x is an ESS. ■

The last implication in the theorem follows from the fact that the ESS condition gives (in the limit $\epsilon \to 0$) an inequality similar to (33) in which y_ϵ is replaced by x and the strict inequality is replaced by a weak one.

It remains to prove the two equivalences in the theorem. For $0 < t \leq 1$, call a strategy x *t-stable* if it has a neighborhood U such that (35) holds for every strategy $y \neq x$ in the set

$$U_t = \{(1-t)x + tz \mid z \in U\}. \tag{39}$$

Local superiority is a special case of t-stability, corresponding to $t = 1$. The following lemma shows that, in fact, it is the *only* case.

Lemma 2. For every $0 < t \leq 1$, a strategy x is t-stable if and only if it is locally superior.

Proof. Fix $0 < t \leq 1$, and let x be a t-stable strategy, with a neighborhood U as in the definition. Since the set U_t defined in (39) is also a neighborhood of x, and every $y \neq x$ in U_t satisfies (35), x is locally superior.

Conversely, suppose that x is locally superior, and let U be a convex neighborhood of x where every strategy $y \neq x$ satisfies (35). The convexity of U implies that the set U_t defined in (39) is a subset of U. Therefore, every $y \neq x$ in U_t satisfies (35), which proves that x is t-stable. ■

Since (36) and (37) are equivalent for $0 < t \leq 1$, and $B_{j-1,n-1}(1/2) = \binom{n-1}{j-1}(1/2)^{n-1}$ ($j = 1,2,\dots,n$), comparison with (30) shows that independent-stability is equivalent to 1/2-stability. Therefore, by Lemma 2, it is also equivalent to local superiority. To prove that local superiority is equivalent also to *dependent*-stability, note, first, that by Lemma 1 every locally superior strategy (which is necessarily independently-stable) is an ESS, and hence also an equilibrium strategy, and the same is true for every dependently-stable strategy. The equivalence of these two conditions is therefore an immediate consequence of the following lemma, which completes the proof of the theorem.

Lemma 3. An equilibrium strategy x has a neighborhood where, for all $y \neq x$,

$$g(y, y, \dots, y) - g(x, y, \dots, y) < 0 \tag{40}$$

if and only if it has a neighborhood where, for all $y \neq x$,

$$\big(g(y, y, \dots, y) - g(x, y, \dots, y)\big) + \big(g(y, x, \dots, x) - g(x, x, \dots, x)\big) < 0. \tag{41}$$

Proof. One direction is trivial: (40) and the equilibrium condition (26) imply (41). To prove the other direction, suppose that the equilibrium strategy x has a neighborhood of the second kind in the lemma but does not have a neighborhood of the first kind. It has to be shown that this assumption leads to a contradiction.

Let $(y_k)_{k\geq 1}$ be a sequence of strategies that converges to x such that, for all k, inequality (41) holds for $y = y_k$ but (40) does not hold, i.e.,

$$g(y_k, y_k, \dots, y_k) - g(x, y_k, \dots, y_k) \geq 0. \tag{42}$$

If all the other players use x, none of the strategies y_k is a best response (for otherwise the left-hand sides of (40) and (41) would be equal for $y = y_k$). Hence, each of them can be presented as

$$y_k = (1 - \alpha_k) w_k + \alpha_k z_k, \tag{43}$$

where z_k is a strategy whose support includes only pure strategies that are not best responses when everyone else uses x, w_k is a strategy whose support includes only pure strategies that *are* best responses, and $0 < \alpha_k \leq 1$. Since there are only finitely many pure strategies, there is some $\delta > 0$ such that, for all k,

$$g(z_k, x, \dots, x) - g(x, x, \dots, x) < -\delta, \tag{44}$$

while

$$g(w_k, x, \dots, x) - g(x, x, \dots, x) = 0. \tag{45}$$

By (42), (43), (44) and (45),

$$\big(g(y_k, y_k, \dots, y_k) - g(y_k, x, \dots, x)\big) - \big(g(x, y_k, \dots, y_k) - g(x, x, \dots, x)\big) > \delta\alpha_k.$$

As $k \to \infty$, the two expressions in parentheses tend to zero, since $y_k \to x$. Therefore, $\alpha_k \to 0$, which by (43) implies that $w_k \to x$. It follows that, for almost all k (i.e., all $k > K$, for some integer K), either $y = w_k$ satisfies (41) or $w_k = x$. By (45), in both cases $g(w_k, w_k, \dots, w_k) - g(x, w_k, \dots, w_k) \leq 0$. Therefore, for almost all k

$$\begin{aligned}
&\alpha_k^{-1}\big(g(w_k, y_k, \dots, y_k) - g(x, y_k, \dots, y_k)\big) \\
&\qquad \leq \alpha_k^{-1}\big(g(w_k, y_k, \dots, y_k) - g(x, y_k, \dots, y_k)\big) \\
&\qquad - \alpha_k^{-1}(1 - \alpha_k)^{n-1}\big(g(w_k, w_k, \dots, w_k) - g(x, w_k, \dots, w_k)\big) \\
&\qquad = \sum_{j=2}^{n} \alpha_k^{-1} B_{j-1,n-1}(\alpha_k)(g(w_k, \underbrace{w_k, \dots, w_k}_{n-j \text{ times}}, z_k, \dots, z_k) \\
&\qquad - g(x, \underbrace{w_k, \dots, w_k}_{n-j \text{ times}}, z_k, \dots, z_k)).
\end{aligned}$$

The right-hand side tends to zero as $k \to \infty$, since $w_k \to x$. Therefore, for almost all k

$$g(w_k, y_k, \dots, y_k) - g(x, y_k, \dots, y_k) < \frac{1}{2}\delta\alpha_k. \tag{46}$$

On the other hand, by (44) and since $y_k \to x$, for almost all k

$$\begin{aligned}
\frac{1}{2}\delta\alpha_k < \alpha_k\big[&\big(g(x, x, \dots, x) - g(z_k, x, \dots, x)\big) + \big(g(z_k, x, \dots, x) - g(z_k, y_k, \dots, y_k)\big) \\
&+ \big(g(w_k, y_k, \dots, y_k) - g(w_k, x, \dots, x)\big)\big].
\end{aligned}$$

By (43) and (45), the right-hand side is equal to $g(w_k, y_k, \dots, y_k) - g(y_k, y_k, \dots, y_k)$, which by (42) is less than or equal to

$$g(w_k, y_k, \dots, y_k) - g(x, y_k, \dots, y_k).$$

This contradicts (46). The contradiction proves that an equilibrium strategy x as above does not exist. ∎

None of the four implications in Theorem 1 holds as equivalence. For the fourth implication, the reverse implication does not hold even in the special case of symmetric 2×2 games: it is well known that not every equilibrium strategy in such a game is an ESS. For the other three implications, the reverse implications do hold for two-player games, but as the following example shows, this is not so in general.

Example 2. A symmetric four-player multilinear game g is defined as follows. There are three possible actions (or pure strategies), so that the strategy space X consists of all probability vectors $x = (x_1, x_2, x_3)$ (with $x_1 + x_2 + x_3 = 1$). The payoff of a player using strategy x against opponents using strategies $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$ is given by

$$g(x, y, z, w) = \sum_{i,j,k,l=1}^{3} g_{ijkl}\, x_i y_j z_k w_l.$$

The coefficients $(g_{ijkl})_{i,j,k,l=1}^{3}$ that define the game satisfy the symmetry condition $g_{ijkl} = g_{ij'k'l'}$, for all (i, j, k, l) and (j', k', l') such that the latter is a permutation of (j, k, l). There are three versions of the game, with different coefficients, as detailed in the following table:

Coefficient	Version 1	Version 2	Version 3
g_{2211}	-2	-18	-4
g_{2221}	0	-16	-4
g_{3221}	4	4	0
g_{2331}	4	20	4
g_{2222}	3	-9	-3
g_{2332}	4	12	2
g_{3333}	-3	-15	-4
g_{2322}	4	4	0

Coefficients that are not listed in the table and cannot be deduced from it by using the above symmetry condition are zero. In all three versions of the game, the strategy $x = (1,0,0)$ is an equilibrium strategy, since if all the other players use x, *any* strategy is a best response. However, the stability properties of x are different for the three versions.

Claim. The equilibrium strategy $x = (1,0,0)$ is an ESS in all three versions of the game, but it is uniformly-stable only in versions 2 and 3, independently-stable (equivalently, dependently-stable, locally superior) only in version 3, and stable in none of them.

In view of Theorem 1, to prove the Claim is suffices to show that x is: (i) an ESS but not uniformly-stable in version 1, (ii) uniformly-stable but not independently-stable in version 2, and (iii) independently-stable but not stable in version 3.

In version 1 of g, (28) reads

$$-2p_2 y_2^2 - 4p_3(y_1 y_2^2 - y_2^2 y_3 - y_2 y_3^2) \\ - 3p_4(2y_1^2 y_2^2 - 4y_1 y_2^2 y_3 - 4y_1 y_2 y_3^2 - y_2^4 - 4y_2^2 y_3^2 + y_3^4 - 4y_2^3 y_3) < 0.$$

Uniform stability corresponds to $p = (1/4, 1/4, 1/4, 1/4)$, for which the above inequality can be simplified to

$$\frac{7}{16} y_2^2 < (y_2 - \frac{3}{8}(1 - y_1)^2)^2.$$

There are strategies $y = (y_1, y_2, y_3)$ arbitrarily close to $(1,0,0)$ for which this inequality does not hold. For example, this is so whenever $y_2 = (3/8)(1 - y_1)^2$. This proves that the equilibrium strategy is not uniformly-stable. To prove that it is nevertheless an ESS, consider (33), which in the present case can be simplified to

$$2y_2^2 < (2y_2 - \epsilon(1 - y_1)^2)^2.$$

For every (fixed) strategy $y = (y_1, y_2, y_3) \neq (1,0,0)$, this inequality holds for sufficiently small $\epsilon > 0$. Therefore, $(1,0,0)$ is an ESS.

In version 2 of the game, for $p = (1/4, 1/4, 1/4, 1/4)$ inequality (28) can be simplified to

$$-\frac{1}{80} y_2^2 < (y_2 - \frac{3}{8}(1 - y_1)^2)^2.$$

This inequality holds for all strategies y other than $(1,0,0)$, and therefore the latter is uniformly-stable. However, it is not independently-stable, since for $p = (1/8, 3/8, 3/8, 1/8)$ inequality (28) can be simplified to

$$\frac{8}{5} y_2^2 < (4y_2 - (1 - y_1)^2)^2.$$

This inequality does not hold for strategies y with $y_2 = (1/4)(1 - y_1)^2$, which exist in every neighborhood of $(1,0,0)$.

Finally, in version 3 of the game, for $p = (1/8\ , 3/8, 3/8, 1/8)$ inequality (28) can be simplified to

$$-y_3^4 < 3(4y_2 - (y_2 + y_3)^2)^2.$$

This inequality holds for all strategies y other than $(1,0,0)$, and therefore the latter is independently-stable. However, it is not stable. There are probability vectors p satisfying (27) (and even (32)) for which (28) does not hold for some strategies y arbitrarily close to $(1,0,0)$. For examples, for $p = (1/20, 9/20, 9/20, 1/20)$ inequality (28) can be simplified to

$$24y_2^2 - \frac{1}{3} y_3^4 < (8y_2 - (1 - y_1)^2)^2.$$

For strategies y with $y_2 = (1/8)(1 - y_1)^2$, this inequality is equivalent to $512 + 2048(1 - y_1) - 384(1 - y_1)^2 + 32(1 - y_1)^3 - (1 - y_1)^4 < 0$. Hence, it does not hold if y_1 is sufficiently close to 1.

4.2. Symmetric multiplayer games with a unidimensional strategy space

The notion of a continuously stable strategy (CSS), originally defined only for two-player games (see Section 2.2), extends in a straightforward way to multiplayer games.

Definition 9. In a multiplayer game g with a strategy space that is a subset of the real line, a (symmetric) equilibrium strategy x is a *continuously stable strategy* (CSS) if it has a neighborhood where for every other strategy y, for sufficiently small $\epsilon > 0$

$$g((1-\epsilon)y + \epsilon x, y, \dots, y) > g(y, y, \dots, y),$$

and a similar inequality does not hold with ϵ replaced by $-\epsilon$.

The following theorem shows that, as in the two-player case, continuous stability is essentially equivalent to stability as defined in this paper (Definition 6). Moreover, stability, dependent-stability, independent-stability and uniform-stability are all essentially equivalent, unlike for multilinear games (Theorem 1).

Theorem 2. Let x be an interior equilibrium strategy in a symmetric n-player game g with a strategy space that is a subset of the real line, such that g has continuous second-order partial derivatives in a neighborhood of the equilibrium point $(x, x, \dots, x)$. If

$$g_{11}(x, x, \dots, x) + (n-1)g_{12}(x, x, \dots, x) \neq 0, \tag{47}$$

then for every probability vector p satisfying (27) the following conditions are equivalent:

- x is a CSS
- the left-hand side of (47) is negative
- x is not definitely p-unstable
- x is p-stable
- x is not definitely unstable
- x is stable.

Proof. Using Taylor's theorem, it is easy to show that, for y tending to x, the left-hand side of (28) can be expressed as

$$\begin{aligned}(y-x)g_1(x, x, \dots, x) + \frac{1}{2}(y-x)^2 g_{11}(x, x, \dots, x) \\ + (y-x)^2 g_{12}(x, x, \dots, x)\sum_{j=1}^{n}(j-1)p_j + o((y-x)^2).\end{aligned} \tag{48}$$

By (27), the sum in (48) is equal to $(n-1)/2$. The rest of the proof is very similar to that of Proposition 2. ■

5. Asymmetric Multiplayer Games

The definitions of asymmetric multiplayer games and symmetrization of such games are conceptually similar to those in the two-player case (Section 3). An asymmetric n-player game is a function $h = (h^1, h^2, \dots, h^n): X^1 \times X^2 \times \cdots \times X^n \to \mathfrak{R}^n$, where, for $1 \le i \le n$, X^i is player i's strategy space. For a strategy profile $(x^1, x^2, \dots, x^n)$, the payoff of player i is $h^i(x^1, x^2, \dots, x^n)$. The strategy profile is an *equilibrium* if, for every player i,

$$h^i(x^1, x^2, \dots, y^i, \dots, x^n) \le h^i(x^1, x^2, \dots, x^i, \dots, x^n)$$

for all $y^i \ne x^i$ in X^i, and it is a *strict* equilibrium if these inequalities are all strict.

An asymmetric game h is symmetrized by allowing the players to take turns playing the different roles in h. Thus, in the symmetric game, each player i has to choose a strategy profile $x_i = (x_i^1, x_i^2, \dots, x_i^n)$ in h. An assignment of the n players to the n roles in h is described by a permutation π of $(1,2,\dots,n)$. Player i is assigned to role $\pi(i)$, and the player assigned to role j is $\pi^{-1}(j)$. Symmetrization involves averaging a player's payoff as π varies over the set Π of all $n!$ permutations.

Definition 10. The game obtained by *symmetrizing* an asymmetric n-player game $h = (h^1, h^2, \dots, h^n): X^1 \times X^2 \times \cdots \times X^n \to \mathfrak{R}^n$ is the symmetric n-player game $g: X \times X \times \cdots \times X \to \mathfrak{R}$, where the strategy space X is the product space $X^1 \times X^2 \times \cdots \times X^n$ and

$$g(x_1, x_2, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \Pi} h^{\pi(1)}\left(x^1_{\pi^{-1}(1)}, x^2_{\pi^{-1}(2)}, \dots, x^n_{\pi^{-1}(n)}\right) \tag{49}$$

for all $x_1 = (x_1^1, x_1^2, \dots, x_1^n), x_2 = (x_2^1, x_2^2, \dots, x_2^n), \dots, x_n = (x_n^1, x_n^2, \dots, x_n^n) \in X$.

Proposition 8. A strategy profile $x = (x^1, x^2, \dots, x^n)$ in an asymmetric n-player game h is an equilibrium if and only if it is a (symmetric) equilibrium strategy in the symmetric game g obtained by symmetrizing h. In this case, the equilibrium payoff in g is equal to the players' average equilibrium payoff in h.

Proof. By definition, x is an equilibrium strategy in g if and only if choosing $x_1 = x$ maximizes the expression obtained by setting $x_2 = x_3 = \cdots = x_n = x$ in the right-hand side of (49). That expression can be simplified by partitioning the set of permutations Π into n parts, each of cardinality $(n-1)!$, according to the value i of $\pi(1)$. Thus, the expression under consideration is equal to

$$\frac{1}{n} \sum_{i=1}^{n} h^i\left(x^1, x^2, \dots, x_1^i, \dots, x^n\right). \tag{50}$$

Clearly, choosing $x_1 = x$ maximizes this sum if and only if, for each i, the ith term is maximized by choosing $x_1^i = x^i$. The latter is also the condition for x to be an equilibrium in h. If it holds, then the maximum (obtained by setting $x_1^i = x^i$ in each of the terms in (50)) is the players' average equilibrium payoff in h. ∎

The last result suggests the following.

Definition 11. A strategy profile $x = (x^1, x^2, \dots, x^n)$ in an asymmetric n-player game h is *stable*, *weakly stable* or *definitely unstable* if it has the same property as a strategy in the symmetric game g obtained by symmetrizing h. Similar definitions apply to p-stability, weak p-stability and definite p-instability, for every probability vector p.

As in the case of Definition 6, dependent-stability, independent-stability, uniform-stability and the related notions of weak stability and definite instability are defined by specifying a particular vector p, i.e., the one given by (29), (30) or (31).

As in the two-player case (Proposition 4), every stable strategy profile is "locally strict." In fact, this is true also with stability replaced by the weaker condition of p-stability.

Proposition 9. For every probability vector $p = (p_1, p_2, \dots, p_n)$ and every p-stable strategy profile $x = (x^1, x^2, \dots, x^n)$ in an asymmetric game h, the strategy x^i of each player i has a neighborhood where it is i's unique best response to x.

Proof. It suffices to prove this for $i = 1$. (The argument for any other player is similar.) Let p, h and x be as in the proposition, and let g be the game obtained by symmetrizing h. It follows from Definition 11 that there is a neighborhood of x^1 where, for every $y^1 \neq x^1$, inequality (28) holds for $y = (y^1, x^2, \dots, x^n)$. The expression in parenthesis in (28) is the difference between the payoff from using y in g and the payoff from using x, when a particular number of other players use x and the rest use y. Recall that, in g, the players are assigned to the different roles in h, and a player's payoff is obtained by summing up his payoffs in all such possible assignments and dividing by their number $n!$. Since x and y prescribe different strategies only for a user taking the role of player 1 in h, only the payoff obtained in that role contributes to the above difference. If follows that it does not matter whether each of the opponents uses x or y: both prescribe the same strategy x^i for a user taking player i's role in h, for all $i \neq 1$. Therefore, the expression in parenthesis in (28) is equal to

$$\frac{1}{n}\left(h^1(y^1, x^2, \dots, x^n) - h^1(x^1, x^2, \dots, x^n)\right).$$

Thus, (28) says that this expression is negative, which shows that y^1 is a worse response to x than x^1. ∎

Working directly with the definition of stability may be possible, as the proof of Proposition 9 demonstrates. However, this is not always very convenient. The following lemma offers a useful alternative characterization.

Lemma 4. For any probability vector $p = (p_1, p_2, \dots, p_n)$, a strategy profile $x = (x^1, x^2, \dots, x^n)$ in an asymmetric n-player game $h = (h^1, h^2, \dots, h^n)$ is p-stable if and only if it has a neighborhood where, for every other strategy profile $y = (y^1, y^2, \dots, y^n)$,

$$\sum_{i=1}^{n} \sum_{S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \left(\chi_S(i) - \chi_{S^c}(i)\right) h^i(\chi_{S^c}(1)x^1 + \chi_S(1)y^1, \chi_{S^c}(2)x^2 + \chi_S(2)y^2, \dots, \chi_{S^c}(n)x^n + \chi_S(n)y^n) < 0, \tag{51}$$

where $N = \{1,2,\ldots,n\}$ and χ_S denotes the characteristic, or indicator, function of a subset $S \subseteq N$, i.e., the function that is 1 on S and 0 on its complement S^c. The characterizations of weak p-stability and definite p-instability are similar, except that the strict inequality in (51) is replaced by a weak one or by the reverse (strict) inequality, respectively.

Proof. Identify the set of players in h with N, and let g be the game obtained by symmetrizing h. Let C be the collection of all characteristic functions of subsets of N, i.e., all functions of the form $\chi: N \to \{0,1\}$. If $\chi \in C$ is the characteristic function of a set $S \subseteq N$ (i.e., $\chi = \chi_S$), then, for every $i \in N$,

$$\chi_i \overset{\text{def}}{=} \sum_{j \neq i} \chi(j)$$

is equal to the cardinality of $S \setminus \{i\}$. Since g is a symmetric game, for any pair of strategies in this game, $x = (x^1, x^2, \ldots, x^n)$ and $y = (y^1, y^2, \ldots, y^n)$, and any $\chi \in C$,

$$\begin{aligned} &g((1-\chi(1))x + \chi(1)y, (1-\chi(2))x + \chi(2)y, \ldots, (1-\chi(n))x + \chi(n)y) \\ &\qquad = (1-\chi(1))g(x, \underbrace{x, \ldots, x}_{n-j\text{ times}}, \underbrace{y, \ldots, y}_{j-1\text{ times}}) + \chi(1)g(y, \underbrace{x, \ldots, x}_{n-j\text{ times}}, \underbrace{y, \ldots, y}_{j-1\text{ times}}), \end{aligned} \tag{52}$$

where $j = \chi_1 + 1$. (Note that the expression on the left-hand side is well defined even if this is not so for general convex combination of strategies in g. This is because each of the arguments is simply either x or y.) For every $1 \leq j \leq n$, the equality $\chi_1 = j - 1$ holds for $\binom{n-1}{j-1}$ elements χ of C with $\chi(1) = 0$ and the same number of elements with $\chi(1) = 1$. Therefore, by (52), for any probability vector p, the left-hand side of (28) can be written as

$$\begin{aligned} \sum_{\chi \in C} \frac{p_{\chi_1+1}}{\binom{n-1}{\chi_1}} (2\chi(1)-1)\, g((1-\chi(1))x + \chi(1)y, (1-\chi(2))x + \chi(2)y, \ldots, (1-\chi(n))x \\ + \chi(n)y). \end{aligned}$$

By Definition 10, this is equal to

$$\begin{aligned} \frac{1}{n!} \sum_{\pi \in \Pi} \sum_{\chi \in C} \frac{p_{\chi_1+1}}{\binom{n-1}{\chi_1}} (2\chi(1)-1)\, h^{\pi(1)}((1-\chi(\pi^{-1}(1)))x^1 \\ + \chi(\pi^{-1}(1))y^1, (1-\chi(\pi^{-1}(2)))x^2 \\ + \chi(\pi^{-1}(2))y^2, \ldots, (1-\chi(\pi^{-1}(n)))x^n + \chi(\pi(n))y^n). \end{aligned}$$

For any permutation $\pi \in \Pi$, changing χ in the summand to the composed function $\chi \circ \pi$ leaves the inner sum unchanged, since as χ varies over all elements of C, so does $\chi \circ \pi$. Since $(\chi \circ \pi)_1 = \chi_{\pi(1)}$, the above double sum is therefore equal to

$$\begin{aligned} \frac{1}{n!} \sum_{\pi \in \Pi} \sum_{\chi \in C} \frac{p_{\chi_{\pi(1)}+1}}{\binom{n-1}{\chi_{\pi(1)}}} (2\chi(\pi(1))-1)\, h^{\pi(1)}((1-\chi(1))x^1 + \chi(1)y^1, (1-\chi(2))x^2 \\ + \chi(2)y^2, \ldots, (1-\chi(n))x^n + \chi(n)y^n). \end{aligned}$$

Note that π now appears in the summand only as part of the expression $\pi(1)$. As π varies over all permutations, this expression returns each of the elements i of N exactly $(n-1)!$ times. Therefore, the double sum is equal to

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{\chi\in C}\frac{p_{\chi_i+1}}{\binom{n-1}{\chi_i}}(2\chi(i)-1)\,h^i((1-\chi(1))x^1+\chi(1)y^1,(1-\chi(2))x^2 + \chi(2)y^2,\dots,(1-\chi(n))x^n+\chi(n)y^n),$$

which is equal to $1/n$ times the expression in (51). ∎

Inequality (51) is a generalization of (15). It may be interpreted as follows. All possible partitions of the set of players in h into a set S and its complement S^c are considered. The players in S play according to the strategy profile $y=(y^1,y^2,\dots,y^n)$ (i.e., each $i\in S$ uses y^i) and those outside it play according to $x=(x^1,x^2,\dots,x^n)$. Each such partition is associated with a particular linear combination of the players' payoffs, which assigns a positive weight to players in S and a negative weight to those outside it. Inequality (51) requires that the sum of all these linear combinations is negative. Roughly, this expresses the requirement that when the players' only choices are playing according to x and playing according to a particular strategy profile that is different from x but close to is, those who choose the former alternative tend to fare better.

1.1. Asymmetric multilinear games

An asymmetric game $h=(h^1,h^2,\dots,h^n)$ is a multilinear game if for each player i the strategy space X^i is the unit simplex in some Euclidean space and h^i is multilinear. As Example 2 shows, for *symmetric* multilinear games there is a real difference between stability of an equilibrium strategy and the various versions of p-stability. The following theorem shows that this is not so for asymmetric games, for which these notions of stability all mean the same.

Theorem 3. For every probability vector $p=(p_1,p_2,\dots,p_n)$, a strategy profile in an asymmetric multilinear game $h=(h^1,h^2,\dots,h^n)$ is p-stable if and only if it is a strict equilibrium. In particular, a p-stable equilibrium is pure.

Proof. Fix p and h as in the proposition. It follows immediately from Proposition 9 and the linearity of each payoff function h^i in the ith argument that every p-stable strategy profile is a strict equilibrium. To prove the converse, fix a strict equilibrium $x=(x^1,x^2,\dots,x^n)$. It has to be shown that x is p-stable.

The strategy x^i of each player i is an element of i's strategy space X^i, that is, a probability vector $(x_1^i,x_2^i,\dots,x_{n_i}^i)$ of some player-specific dimension n_i. For every other strategy $y^i\in X^i$,

$$h^i(x^1,x^2,\dots,y^i,\dots,x^n)-h^i(x^1,x^2,\dots,x^i,\dots,x^n)<0. \tag{53}$$

Consider the collection Z^i of all strategies $z^i=(z_1^i,z_2^i,\dots,z_{n_1}^i)\in X^i$ that satisfy $z_j^i=0$ for some j with $x_j^i>0$. This is a compact subset of X^i that does not include x^i, and therefore the expression on the left-hand side of (53) is bounded away from zero for $y^i\in Z^i$. In other words, there is some $\delta>0$ such that, for all i and $z^i\in Z^i$,

$$h^i(x^1,x^2,\dots,z^i,\dots,x^n)-h^i(x^1,x^2,\dots,x^i,\dots,x^n)<-\delta. \tag{54}$$

Given any strategy y^i of any player i, there is a unique $0 \leq \epsilon_i \leq 1$ (which depends on y^i) such that for some (in fact, unique) $z^i \in Z^i$,

$$y^i = (1 - \epsilon_i)x^i + \epsilon_i z^i. \tag{55}$$

As y^i tends to x^i, $\epsilon_i(z^i - x^i) = y^i - x^i \to 0$. This implies that ϵ_i tends to zero, for otherwise it would be possible to find an example in which ϵ_i is bounded away from zero, and hence $z^i \to x^i$, which is impossible by the compactness of Z^i.

For any strategy profile $y = (y^1, y^2, \dots, y^n) \neq x$, expressing each strategy y^i as in (55) gives the left-hand side of (51) the following form:

$$\sum_{i=1}^{n} \sum_{S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big)\, h^i((1 - \epsilon_1 \chi_S(1))x^1 + \epsilon_1 \chi_S(1) z^1, (1 - \epsilon_2 \chi_S(2))x^2 + \epsilon_2 \chi_S(2) z^2, \dots, (1 - \epsilon_n \chi_S(n))x^n + \epsilon_n \chi_S(n) z^n).$$

By multilinearity, this is equal to

$$\sum_{i=1}^{n} h^i(x^1, x^2, \dots, x^n) \sum_{S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big) + \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_j \Big(h^i(x^1, x^2, \dots, z^j, \dots, x^n) - h^i(x^1, x^2, \dots, x^j, \dots, x^n)\Big) \sum_{S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big)\chi_S(j) + o(\epsilon), \tag{56}$$

where $\epsilon = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ (> 0, since $y \neq x$). In the first term in (56), the inner sum is zero for every i, since

$$\sum_{S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big) = \sum_{S \subseteq N \setminus \{i\}} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \Big[\big(\chi_S(i) - \chi_{S^c}(i)\big) + \Big(\chi_{S \cup \{i\}}(i) - \chi_{(S \cup \{i\})^c}(i)\Big)\Big] = 0. \tag{57}$$

(The expression in square parenthesis is identically zero for i and S with $i \notin S$.) In the second term in (56), the innermost sum is zero for every i and j with $i \neq j$, since for such i and j

$$\sum_{S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big)\chi_S(j) = \sum_{\{j\} \subseteq S \subseteq N} \frac{p_{|S \cup \{i\}|}}{\binom{n-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big) = 0, \tag{58}$$

where the second equality follows from an argument similar to (57). Replacing these two sums in (56) with zeroes and simplifying gives

$$\sum_{i=1}^{n} \epsilon_i \Big(h^i(x^1, x^2, \dots, z^i, \dots, x^n) - h^i(x^1, x^2, \dots, x^i, \dots, x^n)\Big) \sum_{\{i\} \subseteq S \subseteq N} \frac{p_{|S|}}{\binom{n-1}{|S|-1}} + o(\epsilon). \tag{59}$$

The inner sum in (59) clearly has the same value for all i. Therefore, by (54), the first term in (59) is negative and its absolute value is greater than

$$\epsilon\delta \sum_{\{1\}\subseteq S\subseteq N} \frac{p_{|S|}}{\binom{n-1}{|S|-1}}.$$

Hence, if ϵ is sufficiently small (but positive), the whole of (59) is negative. As shown above, ϵ tends to zero as y tends to x. It follows that (59) is negative for all $y \neq x$ in some neighborhood of x. Since (59) is just a different form of (51), it follows from Lemma 4 that x is p-stable. ■

1.1. Asymmetric games with unidimensional strategy spaces

For an asymmetric multiplayer game $h = (h^1, h^2, \dots, h^n)$ in which the strategy space X^i of each player i is a subset of the real line, the stability condition can be presented in a differential form, at least in the case of an interior equilibrium and sufficiently smooth payoff functions. For symmetric games with such strategy spaces, the move from two to an arbitrary number of players proved to be rather easy. A single condition, not much different from that in the two-payer case, essentially characterizes stability, dependent-stability, independent-stability and uniform-stability, which are therefore all essentially equivalent (see Theorem 2). As the next theorem shows, the same is true for asymmetric games.

Theorem 4. A sufficient condition for stability or definite instability of an interior equilibrium $x = (x^1, x^2, \dots, x^n)$ with a neighborhood in which $h^1, h^2, \dots, h^n$ have continuous second-order derivatives is that the matrix

$$H = \begin{pmatrix} h^1_{11} & \cdots & h^1_{1n} \\ \vdots & \ddots & \vdots \\ h^n_{n1} & \cdots & h^n_{nn} \end{pmatrix}, \tag{60}$$

with the derivatives computed at x, is negative definite or positive definite, respectively. A necessary condition for weak stability is that the matrix is negative semidefinite. The same is true with 'stability', 'weak stability' or 'definite instability' replaced by 'p-stability', 'weak p-stability' or 'definite p-instability', respectively, for any probability vector p satisfying (27).

Proof. It suffices to prove the last part of the proposition, since this clearly implies the rest. Thus, fix a probability vector $p = (p_1, p_2, \dots, p_n)$ satisfying (27). For every vector $y = (y^1, y^2, \dots, y^n) \neq x$, the left-hand side of (51) can be written as

$$\sum_{i=1}^{n} \sum_{S\subseteq N} \frac{p_{|S\cup\{i\}|}}{\binom{n-1}{|S\setminus\{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i)\big)\, h^i\big(x^1 + \epsilon_1\chi_S(1), x^2 + \epsilon_2\chi_S(2), \dots, x^n + \epsilon_n\chi_S(n)\big),$$

where

$$\epsilon_i = y^i - x^i, \qquad i = 1, 2, \dots, n.$$

For y tending to x (equivalently, $\epsilon_i \to 0$ for all i), this can be presented as

$$\sum_{i=1}^{n} h^i(x^1,x^2,\dots,x^n) \sum_{S\subseteq N} \frac{p_{|S\cup\{i\}|}}{\binom{n-1}{|S\setminus\{i\}|}} \big(\chi_S(i)-\chi_{S^c}(i)\big)$$
$$+ \sum_{i=1}^{n}\sum_{j=1}^{n} \epsilon_j h_j^i(x^1,x^2,\dots,,x^n) \sum_{S\subseteq N} \frac{p_{|S\cup\{i\}|}}{\binom{n-1}{|S\setminus\{i\}|}} \big(\chi_S(i)-\chi_{S^c}(i)\big)\chi_S(j)$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \epsilon_j \epsilon_k h_{jk}^i(x^1,x^2,\dots,,x^n) \sum_{S\subseteq N} \frac{p_{|S\cup\{i\}|}}{\binom{n-1}{|S\setminus\{i\}|}} \big(\chi_S(i)$$
$$- \chi_{S^c}(i)\big)\chi_S(j)\chi_S(k) + o(\epsilon^2), \tag{61}$$

where $\epsilon = \sqrt{\epsilon_1^2+\epsilon_2^2+\cdots+\epsilon_n^2}$ (> 0, since $y \neq x$). By (57), the first term in (61) is zero. By (58), the innermost sum in the second term is zero for every i and j with $i \neq j$. For $i = j$, $h_j^i(x^1,x^2,\dots,x^n) = 0$, since x is an interior equilibrium. Therefore, the second term in (61) is zero. By an argument similar to (57), the innermost sum in the third term, which can be written also as

$$\sum_{\{j,k\}\subseteq S\subseteq N} \frac{p_{|S\cup\{i\}|}}{\binom{n-1}{|S\setminus\{i\}|}} \big(\chi_S(i)-\chi_{S^c}(i)\big),$$

is zero if $i \notin \{j,k\}$. Therefore, the third term in (61) is equal to $1/2$ times

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \epsilon_j \epsilon_k h_{jk}^i \chi_{\{j,k\}}(i) \sum_{\{j,k\}\subseteq S\subseteq N} \frac{p_{|S|}}{\binom{n-1}{|S|-1}}, \tag{62}$$

where the partial derivatives are computed at x. The innermost sum in (62) has two possible values. If $j = k$, then

$$\sum_{\{j,k\}\subseteq S\subseteq N} \frac{p_{|S|}}{\binom{n-1}{|S|-1}} = \sum_{l=1}^{n} \frac{\binom{n-1}{l-1}}{\binom{n-1}{l-1}} p_l = 1.$$

If $j \neq k$, then by (27)

$$\sum_{\{j,k\}\subseteq S\subseteq N} \frac{p_{|S|}}{\binom{n-1}{|S|-1}} = \sum_{l=2}^{n} \frac{\binom{n-2}{l-2}}{\binom{n-1}{l-1}} p_l = \frac{1}{n-1}\sum_{l=2}^{n}(l-1)p_l = \frac{1}{2}.$$

It follows that (62) is equal to

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{h_{jk}^j + h_{jk}^k}{2} \epsilon_j \epsilon_k. \tag{63}$$

If H is negative definite or positive definite, then (63) is negative or positive, respective, and its absolute value is at least $|\lambda_0|\epsilon^2$, where $\lambda_0 \neq 0$ is the eigenvalue closest to 0 of the matrix $(1/2)(H + H^{\mathrm{T}})$ and ϵ is defined above. This implies that, if H is negative definite, then (61) is negative for $y \neq x$ sufficiently close to x, so that (51) holds, which proves that x is p-stable. Similarly, if H is positive definite, then (61) is positive for $y \neq x$ sufficiently close to x, which proves that x is definitely p-unstable.

If H is not negative semidefinite, then $(1/2)(H + H^{\mathrm{T}})$ has a positive eigenvalue $\lambda > 0$ (see footnote 14). If $(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \neq 0$ is a corresponding eigenvector, then (63) is positive and equal to $\lambda\epsilon^2$. This implies that there are strategy profiles y arbitrarily close to x for which the reverse inequality to that in (51) holds, which proves that x is not weakly p-stable. ■

As indicated, stability as defined in this paper is based on incentives rather than motion. In the previous sections, it is compared mainly with other, special notions of static stability, which are defined only for particular classes of games. However, for the class of games considered in this section, the most well-established notion of stability is a dynamic one, namely, asymptotic stability with respect to the following equations of motion:

$$\frac{dx^i}{dt} = d_i h_i^i(x^1, x^2, \dots, x^n), \qquad i = 1,2, \dots, n, \tag{64}$$

where t is the time variable. This system of differential equations expresses the assumption that, for each player i, the rate of change of strategy x^i is proportional to the marginal payoff h_i^i. The coefficient of proportionality d_i is a positive and (possibly) player-specific parameter. With these dynamics, the condition for asymptotic stability of an equilibrium $x = (x_1, x_2, \dots, x_n)$ is that, at the equilibrium point, the (Jacobian) matrix

$$\begin{pmatrix} d_1 h_{11}^1 & \cdots & d_1 h_{1n}^1 \\ \vdots & \ddots & \vdots \\ d_n h_{n1}^n & \cdots & d_n h_{nn}^n \end{pmatrix}$$

is stable, i.e., all its eigenvalues have negative real parts. This is usually required to hold for all positive adjustment speeds $d_1, d_2, \dots, d_n$ (Dixit, 1986). This requirement is known as *D-stability* of the matrix H (defined in (60)). It is a strictly weaker condition than negative definiteness: every negative definite matrix is *D*-stable (this follows immediately from Lyapunov stability theorem), but not conversely. Unlike negative definiteness, for which a number of useful characterizations are known, necessary and sufficient conditions for *D*-stability are known only for small n (Impram et al., 2005), and they are reasonably simple only for $n = 2$ (see Section 3.2). As Example 1 shows, even in the latter case *D*-stability of H does not imply that the equilibrium is (statically) stable in the sense considered in this paper: H may be *D*-stable but not negative semidefinite.

Negative definiteness and *D*-stability are equivalent in the special case of symmetric matrices. This fact is used in the next subsection.

5.3. Inessentially asymmetric games

An asymmetric n-player game $h = (h^1, h^2, \dots, h^n)$ is inessentially asymmetric if all the players have the same strategy space and for every strategy profile $(x^1, x^2, \dots, x^n)$ and permutation π of $(1,2, \dots, n)$

$$h^i(x^{\pi(1)}, x^{\pi(2)}, \dots, x^{\pi(n)}) = h^{\pi(i)}(x^1, x^2, \dots, x^n), \qquad i = 1,2, \dots, n. \tag{65}$$

This condition says that if the players' strategies are shuffled, such that each player i takes the strategy of some other player $\pi(i)$, then the latter's old payoff becomes player i's new one. In other words, the rules of the game ignore the players' identities, and are therefore completely specified by the payoff function of any single player, and in particular by h^1. The latter can be viewed as a symmetric game in the sense of Section 4. Thus, for fixed strategy space and number of players n, the projection $h \mapsto h^1$, restricted to inessentially asymmetric games, is one-to-one. By means of it, the inessentially asymmetric games can be identified with the symmetric ones. In fact, inessentially asymmetric games are usually referred to simply as symmetric games (von Neumann and Morgenstern, 1953). However, as the two-player case (Section 3.3) already demonstrates, inessential asymmetry and symmetry are actually not the same thing. In particular, they correspond to substantially different conditions for static stability.

Proposition 10. A symmetric strategy profile $(x, x, \dots, x)$ in an inessentially asymmetric n-player game $h = (h^1, h^2, \dots, h^n): X \times X \times \cdots \times X \to \mathfrak{R}^n$ is an equilibrium if and only if it is an equilibrium in the corresponding symmetric game $h^1: X \times X \times \cdots \times X \to \mathfrak{R}$. If $(x, x \dots, x)$ is stable in h, then the strategy x is stable in h^1, but the converse does not hold even if $(x, x, \dots, x)$ is an equilibrium and $n = 2$. The same is true with 'p-stable' instead of 'stable', for every probability vector p.

Proof. The symmetric strategy profile is an equilibrium in h if and only if none of the players can benefit from unilaterally deviating from x to some other strategy y. The inessential asymmetry condition (65) implies that this is so if and only if player 1 cannot benefit from such a deviation, which is the condition for x to be an equilibrium strategy in h^1.

The definitions of stability and p-stability of a strategy profile in the asymmetric n-player game h use an auxiliary symmetric n-player game, namely, the game g obtained by symmetrizing h. The strategies in g are the strategy profiles in h, and according to Definition 11, a strategy profile in h is stable or p-stable, respectively, if and only if it is a stable or p-stable strategy in g. Since h is inessentially asymmetric, (65) and Definition 10 give

$$g(x_1, x_2, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in \Pi} h^1(x_1^{\pi(1)}, x_2^{\pi(2)}, \dots, x_n^{\pi(n)}), \tag{66}$$

for all strategies $x_1 = (x_1^1, x_1^2, \dots, x_1^n), x_2 = (x_2^1, x_2^2, \dots, x_2^n), \dots, x_n = (x_n^1, x_n^2, \dots, x_n^n)$ in g (i.e., strategy profiles in h). This formula shows that g is in a sense a richer game than h^1. The latter can be obtained from the former essentially by restricting the players in g to *symmetric* strategy profiles in h. In particular, it follows from (66) that for any pair of distinct strategies x and y in h, if all the players in g use either $(x, x, \dots, x)$ or $(y, y, \dots, y)$, then for a player using the former, a switch to the latter would change the payoff by

$$h^1(y, \underbrace{x, \dots, x}_{n-j \text{ times}}, y, \dots, y) - h^1(x, \underbrace{x, \dots, x}_{n-j \text{ times}}, y, \dots, y),$$

where $n - j$ is the number of other players using $(x, x, \dots, x)$. If $(x, x, \dots, x)$ is p-stable in g (equivalently, in h), for a particular probability vector $p = (p_1, p_2, \dots, p_n)$, then it follows from the definition of stability for symmetric multiplayer games (Section 4) that there is a neighborhood of x such that, if y belongs to that neighborhood, the expected (with respect to p) change in payoff is negative, i.e.,

$$\sum_{j=1}^{n} p_j \left(h^1(y, \underbrace{x, \dots, x}_{n-j \text{ times}}, y, \dots, y) - h^1(x, \underbrace{x, \dots, x}_{n-j \text{ times}}, y, \dots, y) \right) < 0.$$

By the same definition, this shows that x is a p-stable strategy in the symmetric game h^1. Since p here is arbitrary, this also proves that stability of $(x, x, \dots, x)$ in h implies stability of x in h^1.

To complete the proof of the proposition it suffices to note that, for any stable strategy x in a symmetric 2×2 game (i.e., an ESS) that is not a pure strategy, by Theorem 3 the symmetric equilibrium (x, x) is not p-stable in the corresponding inessentially asymmetric bimatrix game for *any* probability vector p. ■

The following result, which concerns games with unidimensional strategy spaces, provides another example of the difference between stability of a symmetric equilibrium in an inessentially asymmetric game and stability of the equilibrium strategy in the corresponding symmetric game. Comparison with Theorem 2 shows that the latter requires only one of the two inequalities required for the former. The proposition generalizes a result obtained for the two-player case in Section 3.3.

Proposition 11. Let $h = (h^1, h^2, \dots, h^n)$ be an inessentially asymmetric n-player game with a unidimensional strategy space, and $(x, x, \dots, x)$ an interior symmetric equilibrium with a neighborhood in which $h^1, h^2, \dots, h^n$ have continuous second-order derivatives. A sufficient condition for stability of the equilibrium is

$$h_{11}^1(x, x, \dots, x) < h_{12}^1(x, x, \dots, x) < -\frac{h_{11}^1(x, x, \dots, x)}{n-1},$$

and a necessary condition is obtained by replacing the strict inequalities with weak ones.

Proof. It follows from (65) that, at $(x, x, \dots, x)$, all the diagonal entries in the matrix H defined in (60) are equal to $h_{11}^1(x, x, \dots, x)$ and all the off-diagonal entries are equal to $h_{12}^1(x, x, \dots, x)$. Therefore, H is symmetric and has $n-1$ eigenvalues equal to $h_{11}^1(x, x, \dots, x) - h_{12}^1(x, x, \dots, x)$ and one eigenvalue equal to $h_{11}^1(x, x, \dots, x) + (n - 1)h_{12}^1(x, x, \dots, x)$. H is negative definite or negative semidefinite if and only if these eigenvalues are negative or nonpositive, respectively. It remains to use Theorem 4. ■

As the proof shows, at a symmetric equilibrium as in Proposition 11 the matrix H is symmetric, and is therefore negative definite if and only if it is *D*-stable. Thus, the differential condition for stability of the symmetric equilibrium is essentially the same as that for asymptotic stability with respect to the system (64), with adjustment speeds that can take arbitrary values. As Example 1 demonstrates, the same is not true for “truly” asymmetric games.

6. Summary

This paper presents a notion of local stability that is applicable to all strategic games with a finite (but otherwise arbitrary) number of players and “continuous” strategy spaces. This is a static notion of stability, meaning that it is based on incentives rather than motion and therefore does not involve any assumptions about dynamics, or off-equilibrium behavior. Unlike other notions of static stability, the one presented here is not linked to any particular kind of strategy spaces (e.g., subsets of a Euclidean space) or payoff functions (e.g., multilinear ones). Instead, it implicitly introduces a linear structure by considering probabilistic perturbations of the original state. (Nevertheless, in some important classes of games such probabilistic perturbations yield the same notion of static stability that comes out of an analysis based on deterministic perturbations. A continuously stable strategy, or CSS, is an example of this.) A probabilistic perturbation is specified by the joint distribution of the players' deviations from the original state. The marginal distributions, which describe the deviations of individual players, do not give the whole picture, since different players' deviations can be correlated to a lesser or greater degree. In games with three of more players, weaker versions of static stability (i.e., the different kinds of p-stability) are defined by specifying, or constraining, the permissible kinds of correlations.

Stability does not generally imply the equilibrium condition. It is based on a comparison of (only) two possible unilateral deviations from the *perturbed* state: one in the direction of the original state and the other in the opposite direction. The definition of stability has the simplest form for symmetric games. The definition for asymmetric games is based on the latter, and uses a natural notion of symmetrization of an asymmetric game. However, stability for symmetric games is not a special case of that for asymmetric games. Nor should it be. As argued above, symmetric games are not a subset of the asymmetric ones but rather constitute a distinct category. In particular, stability in symmetric games refers to strategies rather than strategy profiles or equilibria. (An evolutionarily stable strategy, or ESS, is an example of this.) The subset of asymmetric games that correspond to the symmetric ones are referred to in this paper as inessentially asymmetric games. The stability condition for these games is more demanding than for their "truly" symmetric kin. For example, in an (inessentially or otherwise) asymmetric multilinear game, stability of a strategy profile means that it is a strict equilibrium.

In some classes of games there are several reasonable non-equivalent notions of static stability. Restriction of the general notion of stability proposed here to such a class singles out one of them, and indicates that this particular notion can be derived from general principles rather than (or in addition to) considerations that are specific to the class of games. For example, there is more than one extension of ESS to symmetric multilinear games with three or more players. The restriction described above gives a notion of stability that is stronger than the other alternatives proposed in the literature. However, one of the latter (namely, local superiority) coincides with the restriction to symmetric multilinear games of one of the weaker versions of static stability (namely, dependent- or independent-stability).

In some classes of asymmetric games there are no well-established notions of static stability, but only dynamic ones. Dynamic stability means asymptotic stability with respect to specified dynamics. Different dynamical systems may yield different notions of stability, which are not necessarily comparable with (i.e., weaker or stronger than) the static one proposed in this paper. A well-known notion of dynamic stability that *is* comparable with static stability is that expressible by the condition of *D*-stability of the Jacobian matrix. This applies to asymmetric games in which strategies are real numbers and the payoff functions are differentiable. This condition is essentially weaker than static stability, for which the differential condition is a negative definite matrix.

This paper does not consider the problem of the existence of stable strategies or equilibria. Any existence result is necessarily specific to a particular structure on the strategy spaces and involves specific assumptions about the payoff functions, which defeats the very idea of a universal notion of stability. Examination of specific examples suggests that the stability condition is satisfiable in many games, but this is far from being always the case. For example, many symmetric 2×2 games have at least one ESS, but many others do not. There are, however, classes of games for which the situation regarding the existence of stable strategies is less clear. For example, this is so for symmetric multilinear games with three or more players.

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