Simple Approximate Equilibria in Large Games

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We prove that in every normal form n-player game with m actions for each player, there exists an approximate Nash equilibrium in which each player randomizes uniformly among a set of $O(\log m + \log n)$ pure actions. This result induces an $O(N^{\log \log N})$-time algorithm for computing an approximate Nash equilibrium in games where the number of actions is polynomial in the number of players ($m = poly(n)$); here $N = nm^n$ is the size of the game (the input size). Furthermore, when the number of actions is a fixed constant ($m = O(1)$) the same algorithm runs in $O(N^{\log \log \log N})$ time. In addition, we establish an inverse connection between the entropy of Nash equilibria in the game, and the time it takes to find such an approximate Nash equilibrium using the random sampling method.

We also consider other relevant notions of equilibria. Specifically, we prove the existence of approximate correlated equilibrium of support size polylogarithmic in the number of players, n, and the number of actions per player, m. In particular, using the probabilistic method, we show that there exists a multiset of action profiles of polylogarithmic size such that the uniform distribution over this multiset forms an approximate correlated equilibrium. Along similar lines, we establish the existence of approximate coarse correlated equilibrium with logarithmic support. We complement these results by considering the computational complexity of determining small-support approximate equilibria. We show that random sampling can be used to efficiently determine an approximate coarse correlated equilibrium with logarithmic support. But, such a tight result does not hold for correlated equilibrium, i.e., sampling might generate an approximate correlated equilibrium of support size $\Omega(m)$ where m is the number of actions per player. Finally, we show that finding an *exact* correlated equilibrium with smallest possible support is NP-hard under Cook reductions, even in the case of two-player zero-sum games.

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1. INTRODUCTION

Equilibria are central solution concepts in the theory of strategic games. Arguably the most prominent examples of such notions of rationality are Nash equilibrium [Nash 1951], correlated equilibrium [Aumann 1974], and coarse correlated equilibrium [Hannan 1957]. At a high level, these concepts denote distributions over players' action profiles where no player can benefit, in expectation, by unilateral deviation. Equilibria are used to model the outcomes of interaction between strategic human players, and between organizations run by human agents. Hence, if a solution concept is too complicated (say, on account of the fact that it requires randomization over a large set of action profiles) then its applicability is debatable, simply because it is hard to imagine that human players would adopt highly intricate strategies. Such concerns have been raised in the context of bounded rationality, see, e.g., [Simon 1982] and [Rubinstein 1998]. Therefore, studying the simplicity of these solution concepts is of fundamental importance.

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The relevance of simple approximate[1] equilibria was pointed out early on by [Lipton et al. 2003], who generalized the results of [Althöfer 1994] and [Lipton and Young 1994] . Specifically, [Lipton et al. 2003] considers a very natural notion of simplicity that deems an approximate equilibrium to be simple if the equilibrium is a uniform distribution on a set of small size. The primary contribution of [Lipton et al. 2003] is to prove the existence of simple approximate *Nash* equilibria in *two-player games*. In this paper we extend this line of work and *establish the existence of simple (defined to be uniform distributions over multisets of small size) approximate Nash, correlated, and coarse correlated equilibria in large games*; specifically, in games with n players and m actions per player (see Theorems 3.1, 5.1, and 4.1). Our result on the existence of simple approximate Nash equilibrium has notable computational consequences as well. In particular, we improve upon the running time of the previously best known algorithm for computing an approximate Nash equilibrium in large games (see Corollaries 3.2 and 3.3).

Simple Approximate Nash Equilibrium

Our results are built upon the sampling method, which has been used in prior work to establish the existence of simple approximate Nash equilibria [Althöfer 1994; Lipton and Young 1994; Lipton et al. 2003]. In this method, a (possibly complicated) mixed strategy x_i of player i is replaced by k i.i.d. samples (of pure actions) from the distribution x_i. These k samples are each chosen at random with probability $1/k$, and together they form a simple *k-uniform strategy* s_i. Equivalently, k-uniform strategies are mixed strategies that assign to each pure action a rational probability with denominator k. The main advantage of the k-uniform strategy s_i over the original strategy x_i is that there are at most m^k such strategies (actually $\binom{m+k-1}{k}$), where m is the number of actions of player i. Therefore, in the case where we do not know the original strategy x_i (and thus we cannot produce the strategy s_i from x_i), we can *search* for the strategy s_i over a relatively small set of size m^k.

The sampling method has a very important consequence for the computation of approximate Nash equilibria. If we prove existence of a k-uniform approximate Nash equilibrium $(s_i)_{i=1}^n$ for small k, then we need only search exhaustively for an approximate Nash equilibrium over all the possible n-tuples of k-uniform strategies. Although this method seems naive, it provides the best upper bound that is known today for computing an approximate Nash equilibrium. In this paper we develop a novel concentration inequality that reduces the dependence of k on the number of players from $O(n)$ to $O(\log n)$, which yields improvements on the upper bounds for computing an approximate Nash equilibrium.

[Althöfer 1994] was the first to introduce the sampling method, when he studied two-player zero-sum games and showed existence of k-uniform approximately optimal strategies with $k = O(\log m)$. Althöfer [Althöfer 1994] also showed that the order of $\log m$ is optimal (for two-player games). Lipton, Markakis, and Mehta [Lipton et al. 2003] generalized this result to all two-player games; i.e., they proved existence of a k-uniform approximate Nash equilibrium for $k = O(\log m)$. For n-player games, [Lipton et al. 2003] proved the existence of a k-uniform approximate Nash equilibrium for $k = O(n^2 \log m)$. [Daskalakis and Papadimitriou 2008] gave another upper bound $k = O(nm)$ improving the dependence on n. [Hémon et al. 2008] obtained the best previously known upper bound of $k = O(n \log m)$.

In the present paper, we improve upon this previously known upper bound and prove the existence of a k-uniform approximate Nash equilibrium for $k = O(\log n + \log m)$ (see Theorem 3.1). The results in [Lipton et al. 2003] and [Hémon et al. 2008] induce a $poly(N^{\log N})$ algorithm for computing an approximate Nash equilibrium (see [Nitzan 2005]), where $N = nm^n$ is the input size. Daskalakis and Papadimitriou point out that the same algorithm runs in $poly(N^{\log \log N})$ time in the special case of games with a constant number of actions ($m = O(1)$). To our knowledge, $poly(N^{\log N})$ (of [Lipton et al. 2003]) for general games and $poly(N^{\log \log N})$ (of [Daskalakis and Papadimitriou 2008]) for

[1]An ε-approximate equilibrium, where $\varepsilon > 0$, is a distribution over action profiles at which no player has more than an ε incentive to deviate.

games with constant number of actions was the best previously known upper bound. Our result improves those bounds. Our result yields a $poly(N^{\log\log N})$ algorithm for games where $m = poly(n)$ (the previously known bound was $poly(N^{\log N})$), and $poly(N^{\log\log\log N})$ for games with a constant number of actions (the previously known bound was $poly(N^{\log\log N})$); see Corollary 3.2 and 3.3.

A key technical contribution of this work is a novel concentration inequality for product distributions (see Lemma 3.4). Given that this inequality holds for arbitrary product distributions (and not just Nash equilibria), it may be of independent interest with applications in other contexts as well.

Simple Approximate Correlated and Coarse Correlated Equilibrium

Moving on to correlated and coarse correlated equilibrium, we note that these are probability distributions (not necessarily product) over the action profiles in a game.[2] Hence, in a game with n players and m actions per player, the supports of correlated and coarse correlated equilibria are subsets of the m^n action profiles. In other words, the support size of an equilibrium can be as large as m^n. But, both *exact* coarse correlated equilibria and *exact* correlated equilibria are relatively simple solution concepts in terms of their support size. Since correlated equilibria can be specified by $nm(m-1)$ linear inequalities (see Section 2 for details; specifically, consider the definition in Remark 2.3 with $\varepsilon = 0$ in inequality (1)) there exists a correlated equilibrium with support of size $O(nm^2)$ (this support-size bound for exact correlated equilibrium appears in [Germano and Lugosi 2007]). Using similar arguments, we can show that there exists a coarse correlated equilibrium with support size $O(nm)$, because they are defined by nm linear inequalities (see Definition 2.1). Examples A.1, A.2, and A.4 in Appendix A show that these bounds are tight.

But what if we are interested in *approximate* correlated equilibrium or *approximate* coarse correlated equilibrium? Can the tight $\text{poly}(n, m)$ bounds be significantly improved? In this paper we show that the answer is yes. For both coarse correlated equilibrium (see Theorem 4.1) and correlated equilibrium (see Theorem 5.1) we prove that in any n-player m-action game, for any fixed ε, there exists an ε-approximate equilibrium with support size $\text{poly}(\log m, \log n)$. In fact, the small-support equilibria, whose existence we establish, are just uniform distributions (over multisets of size $\text{poly}(\log m, \log n)$), and hence they are simple.

It is important to note that our result for approximate Nash equilibrium (specifically, Theorem 3.1) bounds the support size of the mixed strategies of the players. In comparison, the support-size results we have for approximate correlated and coarse correlated equilibrium (i.e., Theorem 4.1 and 5.1) bound the number of *action profiles* that are played with positive probability. Naturally, the difference in this support-size specification (and our consideration of when an equilibrium is simple) stems from the fact that Nash equilibria are product distributions, but correlated and coarse correlated equilibria are general (not necessarily product) distributions.

A relevant observation is that Theorem 3.1 in itself implies the existence of an approximate correlated and coarse correlated equilibrium with *overall* support size $O((\log n + \log m)^n)$; since a Nash equilibrium is a correlated and coarse correlated equilibrium as well. Theorem 4.1 and 5.1 show that for coarse correlated equilibrium and correlated equilibrium in games with more than two players, this bound can in fact be improved significantly. See Table I for a summary of our results that establish the existence of simple equilibria.

Complementary Results

To complement the sampling method in this context, we also establish an inverse connection between the entropy of Nash equilibria in the game and the time that it takes the sampling method algorithm to find an approximate Nash equilibrium (see Theorem 3.5). In particular, this result generalizes the result of [Daskalakis and Papadimitriou 2009] on existence of a polynomial algorithm for an approximate Nash equilibrium in *small probability games*, which are a sub-class of the games where the entropy of a Nash equilibrium is very high. [Daskalakis and Papadimitriou 2009] proved

[2]This is unlike Nash equilibrium, which is defined to be a *product* of independent distributions, one for each player.

Table I: Bounds on the support size of ε-approximate equilibrium in n-player m-action games. For approximate Nash equilibrium the corresponding entry in the second column bounds the support size of the mixed strategies of the every player. For approximate correlated and coarse correlated equilibrium the entry bounds the support size of the overall distribution, i.e., it bounds the number of action profiles that are played with positive probability.

ε-Approximate Equilibrium	Support-Size Upper Bound
Nash	$O\left(\left(\frac{\log n+\log m-\log\varepsilon}{\varepsilon^2}\right)\right)$ [Theorem 3.1]
Correlated	$O\left(\frac{\log m(\log m+\log n-\log\varepsilon)}{\varepsilon^4}\right)$ [Theorem 5.1]
Coarse Correlated	$O\left(\frac{\log m+\log n}{\varepsilon^2}\right)$ [Theorem 4.1]

this result for two-player games. A corollary of our result (see Corollary 3.7) is that an appropriate generalization of that statement holds for any number of players n.

Beyond existence, we also consider computational issues related to small-support approximate equilibria. For any fixed ε, we present polynomial-time algorithms for computing ε-approximate coarse correlated equilibrium of support size $O(\log m + \log n)$ and approximate correlated equilibrium of support size $O(m \log m + \log n)$. We also prove that finding an *exact* correlated equilibrium with smallest possible support is NP-hard under Cook reductions (see Section 6).

Further discussion on the complexity of finding approximate Nash equilibria can be found in [Chen et al. 2009], and [Daskalakis 2013]. The sampling method has been applied in other settings in game theory, as well. For example, [Azrieli and Shmaya 2013] study the existence of pure approximate equilibria in Lipschitz games; and [Kalai 2004] studies the existence of ex-post Bayesian equilibria in semi-anonymous games.

2. PRELIMINARIES

We consider *n-player m-action games*, i.e., games with n players and m actions per player.[3] The *size of the game* is denoted by $N := nm^n$.

We use the following standard notation. The set of players is $[n] = \{1, 2, ..., n\}$ and the set of actions for any player $i \in [n]$ is $A_i = [m] = \{1, 2, ..., m\}$. The set of action profiles is $A = [m]^n$. Let (a_i, a_{-i}) denote an action profile in which a_i is the action of the ith player and a_{-i} denotes the actions chosen by players other than i. Players' utilities are normalized between 0 and 1; in particular, the payoff function of player i is $u_i : A \to [0, 1]$. The payoff function profile is denoted by $u = (u_i)_{i\in[n]}$.

The set of probability distributions over a set B is denoted by $\Delta(B)$. The payoff function can be multilinearly extended to $u_i : \Delta(A) \to [0, 1]$. That is, for probability distribution $x \in \Delta(A)$, write $u_i(x)$ to denote the expected payoff of player i under x.

A mixed action profile $x = (x_i)_{i\in[n]}$, where $x_i \in \Delta(A_i)$ is an *ε-Nash equilibrium* if no player can gain more than ε by a unilateral deviation; i.e., $u_i(x) \geq u_i(a_i, x_{-i}) - \varepsilon$, for every player i and every action $a_i \in [m]$, where x_{-i} denotes the action profile of all players other than i. A 0-equilibrium is called an *exact* or *Nash* equilibrium.

At a high level, the idea behind the notions of *correlated equilibrium* (CE) and *coarse correlated equilibrium* (CCE) is the following. Players implement some distribution $x \in \Delta(A)$, which is not necessarily a product distribution. We can interpret such a correlated implementation in terms of a *mediator* that randomizes according to the distribution x, i.e., draws an action profile $a = (a_i)_{i\in[n]}$

[3]All the results in the paper also generalize to the case where each player has a different number of actions, i.e., player i has m_i actions. For ease of exposition, we assume throughout that all the players have the same number of actions m.

from x. Then the mediator (privately) tells to every player i the corresponding action a_i. We will call the drawn action a_i *the recommendation to player i*.

A distribution $x \in \Delta(A)$ is an *ε-coarse correlated equilibrium*[4] if no player can gain more than ε by switching to a single pure action $j \in A_i$ instead of following the recommendation of the mediator.

In addition, we say that a distribution $x \in \Delta(A)$ is an *ε-correlated equilibrium* if no player can gain more than ε by following any switching rule $f : A_i \to A_i$ (i.e., by switching from the recommended action a_i to some other action $f(a_i)$).

More formally, we have the following definitions.

Definition 2.1. Write $R^i_j(a) := u_i(j, a_{-i}) - u_i(a)$ to denote the regret of player i for not playing j at action profile a. A distribution $x \in \Delta(A)$ is an *ε-coarse correlated equilibrium* (ε-CCE) if $\mathbb{E}_{a\sim x}[R^i_j(a)] \le \varepsilon$ for every player i and every action $j \in A_i$.

Definition 2.2. Write $R^i_f(a) := u_i(f(a_i), a_{-i}) - u_i(a)$ to denote the regret of player i for not implementing the switching rule f at action profile a. A distribution $x \in \Delta(A)$ is an *ε-correlated equilibrium* (ε-CE) if $\mathbb{E}_{a\sim x}[R^i_f(a)] \le \varepsilon$ for every player i and every mapping $f : A_i \to A_i$.

If in the above definitions we set $\varepsilon = 0$, then we obtain the concepts of (exact) coarse correlated and correlated equilibrium.

Remark 2.3. There is another common definition of ε-correlated equilibrium (see, e.g., [Hart and Mas-Colell 2000]) which requires that no player can gain more than ε by changing *a single* recommendation, a_i, to another action j. Formally,

$$\sum_{a_{-i}\in A_{-i}} (u_i(j, a_{-i}) - u_i(a_i, a_{-i}))\, x(a_i, a_{-i}) \le \varepsilon \tag{1}$$

for every player i and every pair of actions $a_i, j \in A_i$.

For an exact correlated equilibrium (i.e., with $\varepsilon = 0$) these inequalities are satisfied if and only if Definition 2.2 holds (again with $\varepsilon = 0$). But, for $\varepsilon > 0$, this definition is *not* equivalent to Definition 2.2 of ε-CE. We argue that this definition is vacuous when the number of actions per player, m, is large and ε is a constant. To see this consider, for example, the uniform distribution x over the k actions $\{(j, j, ..., j)_{j\in[k]}\}$, where $1/\varepsilon \le k \le m$. According to this definition x is a $1/k$-correlated equilibrium, *irrespective* of the payoff function. This is because the marginal probability of every action a_i (i.e., $\sum_{a_{-i}} x(a_i, a_{-i})$) is at most $1/k$ and the utilities are between 0 and 1.

A mixed strategy $x_i \in \Delta(A_i)$ is called k-uniform strategy if it is a uniform distribution over a multiset of k pure actions from A_i. A mixed-strategy profile (i.e., a product distribution) $x = (x_i)_{i\in[n]}$ will be called *k-uniform* if every x_i is k-uniform. Along these lines, a general distribution (not necessarily product) $x \in \Delta(A)$ is called *k-uniform* if it is the uniform distribution over a size-k multiset of *action profiles* from A. Note that the size of the support of any k-uniform distribution is at most k.

Note that different notions of equilibria will be called k-uniform under slightly different conditions. In particular, a (approximate) Nash equilibrium $(x_i)_{i\in[n]}$ is said to be k-uniform if every x_i is a uniform distribution over a size-k multiset of A_i. In contrast, an (approximate) correlated or coarse correlated equilibrium $x \in \Delta(A)$ is said to be k-uniform if x is a uniform distribution over a size-k multiset of A.

3. NASH EQUILIBRIUM

Our Main Theorem states the following:

[4]The set of coarse correlated equilibria is sometimes called the Hannan set, see, e.g., [Hart 2005; Young 2004].

THEOREM 3.1. *Every n-players m-actions game admits a k-uniform ε-Nash equilibrium for every*

$$k > \frac{8(\ln m + \ln n - \ln \varepsilon + \ln 8)}{\varepsilon^2}.$$

COROLLARY 3.2. *Let* $m = poly(n)$ *and* $N := nm^n$ *be the input size of an n-player m-action normal form game. For every constant* $\varepsilon > 0$ *there exists an algorithm that computes an ε-Nash equilibrium of the given game in* $O(poly(N^{\log\log N}))$ *time.*

PROOF OF COROLLARY 3.2. The number of all the possible k-uniform profiles is at most m^{nk}. Note that

$$m^{nk} = poly(m^{n\log n}) = poly((m^n)^{\log\log(m^n)}) = poly(N^{\log\log N}).$$

Therefore the exhaustive search algorithm that searches for an ε-Nash equilibrium over all possible k-uniform profiles finds such an ε-Nash equilibrium after at most $poly(N^{\log\log N})$ iterations. □

COROLLARY 3.3. *For constant m and constant* $\varepsilon > 0$ *there exists an algorithm for computing an ε-Nash equilibrium in* $poly(N^{\log\log\log N})$ *steps in every n-players m-actions game.*

PROOF OF COROLLARY 3.3. The number of all the possible k-uniform profiles is at most $(k+1)^{nm}$. This follow from the fact that the probability mass on every action of every player is from the set $\{\frac{c}{k} : c \in 0, 1, ..., k\}$. Note that

$$(k+1)^{mn} = poly((\log n)^n) = poly(2^{n\log\log n}) = poly(N^{\log\log\log N}).$$

Therefore the exhaustive search algorithm that searches for an ε-Nash equilibrium over all possible k-uniform profiles finds such an ε-Nash equilibrium after at most $poly(N^{\log\log\log N})$ iterations. □

The proof of Theorem 3.1 is based on the following lemma. This lemma is a key technical contribution of this work that proves a novel concentration inequality for product distributions. Even though we apply the sampling method as in [Lipton et al. 2003] and [Hémon et al. 2008], the fact that we use Lemma 3.4 instead of some standard concentration inequality essentially enables us to significantly improve upon the previously-best-know bound of [Hémon et al. 2008].

Assume that players are playing according to a product distribution $x = (x_i)_{i\in[n]}$. We observe k i.i.d. *samples* from x that are denoted by $(a(t))_{t\in[k]}$ where $a(t) \in A$. We denote by s_i^k the *empirical distribution of player i* defined to be the empirical distribution of the samples $(a_i(t))_{t\in[k]}$. Namely, $s_i^k(a_i) = \frac{1}{k}|\{t : a_i(t) = a_i\}|$. The *product empirical distribution of play*, s^k, is the product distribution $\Pi_i\, s_i^k$. Also, write s_{-i}^k to denote $\Pi_{j\neq i} s_j^k$.

LEMMA 3.4. *For every n-player m-action game, every player* $i \in [n]$, *every action* $a_i \in A_i = [m]$, *and every product distribution of the opponents* $x_{-i} = (x_j)_{j\neq i}$ *we have*

$$\mathbb{P}\left(|u_i(a_i, s_{-i}^k) - u_i(a_i, x_{-i})| \geq \varepsilon\right) \leq \frac{4e^{-\frac{\varepsilon^2}{2}k}}{\varepsilon}.$$

In other words, this lemma states that with probability that is exponentially (in k) close to 1, player i is almost indifferent between the case where her opponents are playing the original distribution x_{-i} or the product empirical distribution s_{-i}^k.

PROOF OF LEMMA 3.4. Assume without loss of generality that $i = 1$ and $a_i = 1$. We begin by rewriting the payoff of player 1. For every $l \in [k]$, we can write

$$u_1(1, s_{-1}^k) = \frac{1}{k^{n-1}} \sum_{j_2, j_3, ..., j_n \in [k]} u_1(1, a_2(j_2 + l), a_3(j_3 + l), ..., a_n(j_n + l))$$

where the indexes $j_i + l$ are taken modulo k. If we take the average over all possible l we have

$$u_1(1, s_{-1}^k) = \frac{1}{k^{n-1}} \sum_{j_2, j_3, ..., j_n \in [k]} \frac{1}{k} \sum_{l\in[k]} u_1(1, a_2(j_2 + l), a_3(j_3 + l), ..., a_n(j_n + l)). \quad (2)$$

For every initial profile of indices $j_* = (j_2, j_3, ..., j_n) \in [k]^{n-1}$ and every $l \in [k]$, we denote $a_{-1}(j_* + l) := (a_2(j_2 + l), a_3(j_3 + l), ..., a_n(j_n + l)) \in A_{-1}$, and we define the random variable

$$d(j_*) := \begin{cases} 0 & \text{if } \left| \frac{1}{k} \sum_{l \in [k]} u_1(1, a_{-1}(j_* + l)) - u_1(1, x_{-1}) \right| \leq \dfrac{\varepsilon}{2} \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

By the definition of $d(j_*)$, we have

$$d(j_*) + \frac{\varepsilon}{2} \geq \left| \frac{1}{k} \sum_{l \in [k]} u_1(1, a_{-1}(j_* + l)) - u_1(1, x_{-1}) \right|. \tag{4}$$

Note also that for any fixed j_* the random action profiles $a_{-1}(j_* + 1), a_{-1}(j_* + 2), \ldots, a_{-1}(j_* + k)$ are independent. Therefore by Hoeffding's inequality (see [Hoeffding 1963]) we have

$$\mathbb{E}[d(j_*)] \leq 2e^{-\frac{\varepsilon^2}{2}k}. \tag{5}$$

Using representation (2) of the payoffs and inequalities (4) and (5), we get

$$\begin{aligned}
\mathbb{P}\left(|u_i(1, s^k_{-1}) - u_i(1, x_{-1})| \geq \varepsilon\right) &= \mathbb{P}\left(\left| \frac{1}{k^{n-1}} \sum_{j_* \in [k]^{n-1}} \frac{1}{k} \sum_{l \in [k]} u_1(1, a_{-1}(j_* + l)) - u_1(1, x_{-1}) \right| \geq \varepsilon \right) \\
&\leq \mathbb{P}\left(\frac{1}{k^{n-1}} \sum_{j_* \in [k]^{n-1}} \left| \frac{1}{k} \sum_{l \in [k]} u_1(1, a_{-1}(j_* + l)) - u_1(1, x_{-1}) \right| \geq \varepsilon \right) && (6) \\
&\leq \mathbb{P}\left(\frac{1}{k^{n-1}} \sum_{j_* \in [k]^{n-1}} d(j_*) \geq \frac{\varepsilon}{2} \right) && (7) \\
&\leq \frac{4e^{-\frac{\varepsilon^2}{2}k}}{\varepsilon}
\end{aligned}$$

where the last inequality follows from Markov's inequality. □

Proof of Theorem 3.1. Let $x = (x_i)_{i \in [n]}$ be a Nash equilibrium of the given game and s^k be the product empirical distribution of play with respect to x. Lemma 3.4 and the choice of k guarantees that

$$\mathbb{P}(|u_i(a_i, s^k_{-i}) - u_i(a_i, x_{-i})| \geq \frac{\varepsilon}{2}) \leq \frac{8e^{-\frac{\varepsilon^2}{8}k}}{\varepsilon} < \frac{1}{2mn}$$

for every player i and every action $a_i \in [m]$. Using the union bound, we get that with probability greater than $1/2$ we have $|u_i(a_i, s^k_{-i}) - u_i(a_i, x_{-i})| < \frac{\varepsilon}{2}$ for *all* players $i \in [n]$ and *all* actions $a_i \in [m]$. In such a case $(s^k_i)_{i \in [n]}$ is an ε-Nash equilibrium because:

$$\begin{aligned}
u_i(a_i, s^k_{-i}) &\leq u_i(a_i, x_{-i}) + \frac{\varepsilon}{2} \\
&\leq \sum_{a'_i \in A_i} s^k_i(a'_i) u_i(a'_i, x_{-i}) + \frac{\varepsilon}{2} \\
&\leq \sum_{a'_i \in A_i} s^k_i(a'_i) u_i(a'_i, s^k_{-i}) + \varepsilon \\
&= u_i(s^k_i, s^k_{-i}) + \varepsilon,
\end{aligned}$$

where the second inequality holds because all the strategies in the support of s_i^k are in the support of x_i, which contains only best replies to x_{-i}. We get the stated claim via the probabilistic method. $\square$

3.1. Games with a High-Entropy Nash Equilibrium

In the sequel it will be convenient to consider the set of k-uniform strategies as the set of *ordered k-tuples* of pure actions. To avoid ambiguity we will call those strategies *k-uniform ordered strategies*.[5] Now the number of k-uniform ordered profiles is exactly m^{nk}.

The algorithm of Corollaries 3.2 and 3.3 suggests that we should search over all the possible k-uniform profiles (or k-uniform ordered profiles), one by one, until we find an approximate equilibrium. Consider now the case where a large fraction of the k-uniform ordered strategies form an approximate equilibrium, say a fraction of $1/r$. In such a case we can pick k-uniform ordered profiles *at random*, and then we will find the approximate equilibrium in expected time r.

Define the *k-uniform random sampling algorithm* (k-URS) to be the algorithm described above; i.e., it samples uniformly at random n-tuples of k-uniform ordered strategies and checks whether this profile forms an ε-Nash equilibrium.[6]

An interesting question arises: For which games does the k-URS algorithm find an approximate equilibrium fast? [Daskalakis and Papadimitriou 2009] focused on two-player games with m actions, and they showed that the k-URS algorithm finds an approximate equilibrium after $poly(m)$ samples for *small-probability games*. A *small-probability game* is a game that admits a Nash equilibrium where each pure action is played with probability at most c/m for some constant c.

Here we generalize the result of Daskalakis and Papadimitriou to n-player games. Instead of focusing on the specific class of small-probability games we establish a general connection between the entropy of equilibria in the game and the expected number of samples of the k-URS algorithm until an approximate Nash equilibrium is found.

THEOREM 3.5. *Let u be an n-players m-actions game with a Nash equilibrium $x = (x_i)$. Let $k \geq \max\{\frac{16}{\varepsilon^2}(\ln n + \ln m - \ln \varepsilon + 2), e^{16/\varepsilon^2}\} = O(\log m + \log n)$; then the k-uniform random sampling algorithm finds an ε-Nash equilibrium after at most $4 \cdot 2^{k(n \log_2 m - H(x))}$ samples in expectation, where $H(x)$ is Shannon's entropy of the Nash equilibrium x.*

The following corollary of this theorem is straightforward.

COROLLARY 3.6. *Families of games where $n \log_2 m - \max_{x \in \mathrm{NE}} H(x)$ is bounded admit a $poly(m, n)$ probabilistic algorithm for computing an approximate Nash equilibrium.*

The corollary follows from the fact that $k = O(\log m + \log n)$, and therefore $4 \cdot 2^{kO(1)} = poly(n, m)$.

A special case where $n \log_2 m - H(x)$ is constant is that of small-probability games with a *constant* number of players n.

COROLLARY 3.7. *Let $c \geq 1$, and let u be an n-player m-action game with a Nash equilibrium $x = (x_i)_{i \in [n]}$, where $x_i(a_i) \leq \frac{c}{m}$ for players i and all actions $a_i \in A_i$. Let $k = O(\log m)$, as defined in Theorem 3.5. Then the expected number of samples of the k-URS algorithm is at most $4 \cdot 2^{kn \log c} = poly(m)$.*

The corollary follows from the fact that the entropy of the Nash equilibrium x is $H(x) = \sum_{i \in [n]} H(x_i) \geq n(\log_2 m - \log_2 c)$.

The following example demonstrates that even in the case of two-player games, the class of games that have PTAS according to Corollary 3.6 is slightly wider than the class of small-probability games.

[5] Many k-uniform ordered strategies correspond to the same mixed strategy of the player in the game.

[6] Checking whether a strategy profile forms an approximate equilibrium can always be done in $poly(N)$ time. Actually, it can even be done by using only $poly(n, m)$ samples from the mixed profile. Using the samples, the answer will be correct with a probability that is exponential (in n and m) close to 1 (see, e.g., [Babichenko 2014], proof of Theorem 2).

Example 3.8. Consider a two-player m-action game where the equilibrium is $x = (x_1, x_2)$, where x_1 is the uniform distribution over all actions $x_1 = (\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m})$, and $x_2 = (\frac{1}{\sqrt{m}}, \frac{1}{m+\sqrt{m}}, \frac{1}{m+\sqrt{m}}, ..., \frac{1}{m+\sqrt{m}})$. This game is not a small-probability game, but it does satisfy $n \log_2 m - H(x) = o(1)$:

$$\begin{aligned} 2\log_2 m - H(x) &\leq \log_2 m - \frac{m-1}{m+\sqrt{m}} \log_2(m+\sqrt{m}) \\ &\leq \frac{1}{\sqrt{m}+1} \log m = o(1). \end{aligned}$$

In the proof of Theorem 3.5 we use the following lemma from information theory.

Lemma 3.9. *Let y be a random variable that assumes values in a finite set M. Let $S \subset M$ such that $\mathbb{P}(y \in S) \geq 1 - \frac{1}{\log_2 |M|}$; then $|S| \geq \frac{1}{4} 2^{H(y)}$.*

Proof.

$$\begin{aligned} H(y) &= \mathbb{P}(y \in S) H(y|y \in S) + \mathbb{P}(y \notin S) H(y|y \notin S) + H(\mathbb{1}_{\{y \in S\}}) \\ &\leq \log_2 |S| + \mathbb{P}(y \notin S) \log_2 |M| + 1 \leq \log_2 |S| + 2. \end{aligned}$$

□

Proof of Theorem 3.5. Note that $k \geq \max\{\frac{16}{\varepsilon^2}(\ln n + \ln m - \ln \varepsilon + 2), e^{16/\varepsilon^2}\}$ guarantees that

$$\frac{8e^{-\frac{k\varepsilon^2}{8}}}{\varepsilon} \leq \frac{1}{mn} \frac{1}{nklog_2 m}.$$

By considering inequality (6) in the proof of Theorem 3.1, we can see that the above choice of k implies that $\mathbb{P}(E_{1,1}) \leq \frac{1}{mn} \frac{1}{nk \log_2 m}$, which implies that $\mathbb{P}(s \in \cup_{i,j} E_{i,j}) \leq \frac{1}{nk \log_2 m}$. This means that if we sample k-uniform ordered strategy profiles according to the Nash equilibrium x, then the resulting k-uniform ordered strategies form an ε-Nash equilibrium with a probability of at least $1 - \frac{1}{nk \log_2 m} = 1 - \frac{1}{\log_2(m^{nk})}$.

Next, using Lemma 3.9, we provide a lower bound on the number of k-uniform profiles that form an ε-Nash equilibrium. The random k-uniform profiles are elements of a set of size m^{nk}. The entropy of the random k-uniform profile is $kH(x)$. The probability that the random profile will form an ε-Nash equilibrium is at least $1 - \frac{1}{\log_2(m^{nk})}$. Therefore, by Lemma 3.9, we get that there are at least $\frac{1}{4} 2^{kH(x)}$ different k-uniform profiles that are ε-equilibria.

To conclude, the fraction of the k-uniform profiles that form an ε-Nash equilibrium (among all the k-uniform profiles) is at least:

$$\frac{\frac{1}{4} 2^{kH(x)}}{m^{nk}} = \frac{1}{4} 2^{k(H(x) - n \log_2 m)}.$$

Therefore, the expected time for finding an ε-Nash equilibrium is at most $4 \cdot 2^{k(n \log_2 m - H(x))}$. □

4. COARSE CORRELATED EQUILIBRIUM

As a warm-up to the correlated equilibrium case, we first prove the existence of small-support ε-CCE.

4.1. Existence

Theorem 4.1. *Every n-player m-action game admits a k-uniform ε-coarse correlated equilibrium for all*

$$k > \frac{2(\ln m + \ln n)}{\varepsilon^2} \tag{8}$$

Proof. The proof is based on the probabilistic method. Let $\sigma \in \Delta(A)$ be an exact coarse correlated equilibrium of the game; i.e., we have $\mathbb{E}_{a\sim\sigma}[R^i_j(a)] \leq 0$ for every player i and every action $j \in A_i$. We sample k action profiles $a(1), a(2), ..., a(k) \in A$ independently at random according to the distribution σ. Denote by s the uniform distribution over $a(1), a(2), ..., a(k)$.

Note that, for any player i and action j, $R^i_j(a)$ with $a \sim \sigma$ is a random variable that takes a value between -1 and 1 (since the utilities of players are between 0 and 1), and $\mathbb{E}_{a\sim\sigma}[R^i_j(a)] \leq 0$. Therefore by Hoeffding's inequality (see [Hoeffding 1963]) we have

$$\Pr_{a(1),...,a(k)\sim\sigma}\left(\mathbb{E}_{a\sim s}[R^i_j(a)] \geq \varepsilon\right) = \Pr\left(\frac{1}{k}\sum_{\ell\in[k]} R^i_j(a(\ell)) \geq \varepsilon\right) \leq e^{-\frac{k\varepsilon^2}{2}}. \tag{9}$$

For player i and action j, write $\mathcal{E}_{i,j}$ to denote the event: $\mathbb{E}_{a\sim s}[R^i_j(a)] \geq \varepsilon$ (equivalently, $\frac{1}{k}\sum_{\ell\in[k]} R^i_j(a(\ell)) \geq \varepsilon$). Inequality (9) implies that for $k > \frac{2(\log m+\log n)}{\varepsilon^2}$ we have $\Pr(\mathcal{E}_{i,j}) < \frac{1}{nm}$. Note that there are nm such events, one for every player $i \in [n]$ and action $j \in [m]$. Therefore, via the union bound, we get that with positive probability none of these events will happen; implying that the sampled distribution s is an ε-CCE. □

We note that a support-size bound similar to Theorem 4.1 can potentially be obtained via regret-minimizing dynamics as well (e.g., though regret matching [Hart 2005]). In particular, fast convergence rates of dynamics imply small supports. But, we do not follow this direction for coarse correlated equibliibrium since our aim in this section is to emphasize the applicability of the sampling method and develop some intuition for the correlated-equilibria case. Moreover, standard regret-minimizing dynamics do lead to polylogarithmic support-size bounds for approximate correlated equilibrium, and hence they provide weaker bounds than the one developed in Theorem 5.1. The result of [Goldberg and Roth 2013] does imply existence of approximate coarse correlated equilibria of support size polylogarithmic in n, but [Goldberg and Roth 2013] only addresses games in which the number of actions per player is a fixed constant, i.e., $m = O(1)$.

4.2. Computation

The following proposition shows that not only can we prove the existence an ε-CCE with logarithmic support size, but we can efficiently determine it as well.

Proposition 4.2. *There exists a polynomial (in n and m) time randomized algorithm for computing k-uniform ε-coarse correlated equilibrium for*

$$k > \frac{2(\ln m + \ln n + \ln 2)}{\varepsilon^2} \tag{10}$$

Proof. If we set $k > \frac{2(\ln m+\ln n+\ln 2)}{\varepsilon^2}$ in inequality (9) then the following holds: $\Pr(\mathbb{E}_{a\sim s}[R^i_j(a)] \geq \varepsilon) < \frac{1}{2nm}$. This implies that the sampled distribution is an ε-CCE with probability at least $1/2$.

Note that in multi-player games a coarse correlated equilibrium can be efficiently determined. In particular, we can compute a correlated equilibrium in polynomial time using the algorithm[7] of [Jiang and Leyton-Brown 2013] (see also [Papadimitriou and Roughgarden 2008]). We can then treat the computed correlated equilibrium as a coarse correlated equilibrium.

Now our randomized algorithm is direct, it starts with a coarse correlated equilibrium σ and then it samples k pure action profiles according to σ. Finally, the algorithm checks whether the uniform

[7]This algorithm requires the game to be *succinct*—equivalently, a black box that can compute expected utilities under given product distributions (see [Jiang and Leyton-Brown 2013] and [Papadimitriou and Roughgarden 2008] for details). If the game is not succinct (i.e., such a black box does not exist), then as an alternative we can use regret-minimizing dynamics, e.g., regret matching [Hart 2005], to efficiently compute an approximate CE. Starting form an approximate CE, say with approximation guarantee $\varepsilon/2$, instead of an exact CE will worsen the support-size guarantee by at most a constant factor.

distribution over the samples forms an ε-CCE or not. If not, then it samples (k action profiles) again. In expectation, after two sampling iterations the algorithm will find an ε-CCE. $\square$

5. CORRELATED EQUILIBRIUM

In this section we establish the existence of an ε-CE with polylogarithmic support size. Note that in Definition 2.2, an ε-CE is specified via nm^m inequalities of the form $\mathbb{E}_{a\sim x}[R^i_f(a)] \leq \varepsilon$, one for every $i \in [n]$ and $f : A_i \to A_i$. Hence simply applying the probabilistic method, as in the case of coarse correlated equilibrium, will not give us the desired polylogarithmic bound. In particular, sampling from an *arbitrary* correlated equilibrium leads to a support-size guarantee of about $\log(nm^m) = m \log m + \log n$. [8] We get around this issue (see Proof of Theorem 5.1 for details) by sampling from a particular approximate Nash equilibrium (which, obviously, is an approximate correlated equilibrium as well) for which we only have to consider $nm^{(\log n + \log m)}$ inequalities.

5.1. Existence

Theorem 5.1. *Every n-player m-action game admits a k-uniform ε-correlated equilibrium for all*

$$k > \frac{264 \ln m(\ln m + \ln n - \ln \varepsilon + \ln 16)}{\varepsilon^4} = O\left(\frac{\log m(\log m + \log n - \log \varepsilon)}{\varepsilon^4}\right) \tag{11}$$

Proof. Let $\sigma \in \Delta(A)$ be a distribution in which every player i plays actions only from a subset $B_i \subseteq A_i$; i.e., $\sigma(a) > 0$ implies $a_i \in B_i$. Then σ is an ε-CE iff $\mathbb{E}_{a\sim\sigma}[R^i_f(a)] \leq \varepsilon$ for every switching rule $f : B_i \to A_i$. In other words, we can consider only switching rules $f : B_i \to A_i$ instead of $f : A_i \to A_i$, because all the recommendations to player i will be in the set B_i. Note that, given a player i and subset $B_i \subseteq A_i$, there are $m^{|B_i|}$ switching rules of the form $f : B_i \to A_i$.

In Definition 2.2 there are nm^m inequalities ($\mathbb{E}_{a\sim\sigma}[R^i_f(a)] \leq \varepsilon$), one for every $i \in [n]$ and mapping $f : A_i \to A_i$. To avoid dealing with all these nm^m inequalities, we start with an approximate correlated equilibrium σ in which every player plays actions from a small subset, i.e., the sets B_i for σ are of small cardinality. Then, by the above argument, the number of switching rules (in other words, the number of inequalities of the form $\mathbb{E}_{a\sim\sigma}[R^i_f(a)] \leq \varepsilon$) that we need to consider will be significantly smaller than nm^m. Existence of an approximate correlated equilibrium wherein each player uses only a small subset of her actions follows from Theorem 3.1.

Theorem 3.1 shows that in any n-player m-action game there exists an $(\varepsilon/2)$-approximate Nash equilibrium $\sigma = \Pi_i \sigma_i$ in which each player uses a mixed strategy with support size at most b where $b = \left\lceil \frac{32(\ln n + \ln m - ln\varepsilon + \ln 16)}{\varepsilon^2} \right\rceil$. That is, $|\text{supp}(\sigma_i)| \leq b$ for all i. Since σ is an $(\varepsilon/2)$-Nash equilibrium it is an $(\varepsilon/2)$-CE as well. In addition, here, the set B_i is equal to the support of player i's mixed strategy. Therefore, we have $|B_i| \leq b$ for all i.

We now apply the probabilistic method. We sample k action profiles $a(1), a(2), ..., a(k) \in A$ independently at random according to the distribution σ and denote by s the uniform distribution over the samples. For every player i and a switching rule $f : B_i \to A_i$, the regret $R^i_f(a)$, with $a \sim \sigma$, is a random variable that takes a value in $[-1, 1]$ and satisfies: $\mathbb{E}_{a\sim\sigma}[R^i_f(a)] \leq \varepsilon/2$.

Therefore by Hoeffding's inequality (see [Hoeffding 1963]) we have

$$\Pr\left(\mathbb{E}_{a\sim s}[R^i_f(a)] \geq \varepsilon\right) = \Pr\left(\frac{1}{k}\sum_{\ell\in[k]} R^i_f(a(\ell)) \geq \varepsilon\right) \leq e^{-\frac{k\varepsilon^2}{8}}. \tag{12}$$

Setting $k > \frac{264 \ln m(\ln n + \ln m - \ln \varepsilon + \ln 16)}{\varepsilon^4}$ guarantees that $\Pr(\mathbb{E}_{a\sim s}[R^i_f(a)] \geq \varepsilon) < \frac{1}{nm^b}$. Since we have at most nm^b such events (one for every player $i \in [n]$ and every switching rule $f : B_i \to [m]$), union

[8] We show in Example 5.3 that for particular correlated equilibria a sublinear (in m) number of samples do not generate an approximate correlated equilibrium.

bound implies that with positive probability none of them will happen. Therefore, with positive probability, s is an ε-CE. □

5.2. Computation

Unlike the coarse correlated equilibrium case, there is no guarantee that sampling an arbitrary correlated equilibrium polylogarithmic many times will generate an ε-CE with small support. This is because in the proof of Theorem 5.1 we sampled from a very specific approximate equilibrium; in particular, an approximate correlated equilibrium in which every player uses at most $O\left(\frac{\log n+\log m-\log\varepsilon}{\varepsilon^4}\right)$ actions. We do not know whether such an approximate correlated equilibrium can be computed in polynomial time. Nevertheless, we are able to compute ε-CE with support size $O\left(\frac{m\log m+\log n}{\varepsilon^2}\right)$, because $O\left(\frac{m\log m+\log n}{\varepsilon^2}\right)$ samples from *any* correlated equilibrium are enough to form an approximate correlated equilibrium. This algorithm improves upon the know results of [Jiang and Leyton-Brown 2013] and [Hart and Mas-Colell 2000] that respectively generate an exact correlated equilibrium with support size $O(nm^2)$.

Proposition 5.2. *There exists a polynomial (in n and m) time randomized algorithm for computing k-uniform ε-correlated equilibrium for*

$$k > \frac{2(m\ln m + \ln n + \ln 2)}{\varepsilon^2}. \tag{13}$$

Proof. Let σ be a correlated equilibrium. If we sample k pure action profiles according to the distribution σ then we have

$$\Pr(\mathbb{E}_{a\sim s}[R^i_f(a)] \geq \varepsilon) \leq e^{-\frac{k\varepsilon^2}{2}} \leq \frac{1}{2nm^m}. \tag{14}$$

for every switching rule $f : A_i \to A_i$ of every player i. Therefore, with probability at least $1/2$ the uniform distribution over the k samples forms an ε-CE.

Now the algorithm is straightforward. First compute a correlated equilibrium (for example, using the algorithm from [Jiang and Leyton-Brown 2013]), then sample $O\left(\frac{m\log m+\log n}{\varepsilon^2}\right)$ actions until the empirical distribution forms an ε-CE.[9]

□

The following example demonstrates that if we sample from an arbitrary correlated equilibrium, then in fact we may need $O(m)$ (and not logarithmic) samples to obtain an ε-CE.

Example 5.3. Consider two-player matching-pennies game where in addition to the standard real actions, $r_i \in \{-1, 1\}$, the two players also choose a dummy number $d_i \in [m]$ that is *irrelevant for the payoffs*. Formally, the action set of each player $i \in [2]$ is $\{(r_i, d_i) : r_i \in \{-1, 1\} \text{ and } d_i \in [m]\}$. The payoffs are given by $u_1((r_i, d_i)_{i=1,2}) = r_1 r_2 = -u_2((r_i, d_i)_{i=1,2})$.

Consider the following correlated equilibrium of the game. First we select a $d \in [m]$ uniformly at random, and then set r_1, and independently r_2, to be 1 or -1 with equal probability. Note that, for any $d \in [m]$, if we sample from this distribution then the probability that drawn action profile contains d—i.e., the drawn action profile is of the form $((r_i, d)_{i=1,2})$)—is equal to $1/m$.

Now, if we sample m action profiles from this distribution, then for any $d \in [m]$ the probability that it is picked *exactly* once during the sampling is $m\frac{1}{m}\cdot\left(1-\frac{1}{m}\right)^{m-1} \approx \frac{1}{e}$. If a certain d was picked exactly once then both players can deduce from d which action their opponent will play. Note that the expected number of $d \in [m]$ that are sampled exactly once is $\frac{m}{e}$. Moreover, the probability

[9] We can test in polynomial time whether a distribution with polynomial support size, say x, is an ε-CE or not. Specifically, for every player i and each action $a_i \in [m]$ we can first determine $a'_i \in [m]$ that maximizes $\sum_{a_{-i}:x(a_i,a_{-i})>0}(u_i(a'_i, a_{-i}) - u_i(a_i, a_{-i}))x(a_i, a_{-i})$. Then, set $f(a_i) = a'_i$ and verify that $\mathbb{E}_{a\sim x}[R^i_f(a)] \leq \varepsilon$.

that the number of exactly-once-sampled d's will be smaller than $\frac{m}{2e}$ is exponentially small in m (see , e.g., Lemma 4 in [Farach-Colton and Mosteiro 2007]). So, with probability exponentially close to 1, in the resulting uniform distribution at least one player may increase her payoff by at least $\frac{1}{4e}$ by reacting optimally to the opponent's known strategy in all cases in which she got the recommendation (r_i, d), where d was chosen exactly once. Therefore with exponentially high probability the samples does not induce an ε-CE for $\varepsilon < \frac{1}{4e}$.

6. HARDNESS RESULT

We prove that finding a correlated equilibrium with smallest possible support is NP-hard under Cook reductions, even in two-player zero-sum games. To accomplish this we first show that in a two-player zero-sum game finding a Nash equilibrium with minimum support size is NP-Hard. Then, we use a correspondence between correlated equilibria and Nash equilibria in two-player zero-sum games to obtain the result.

A sparsest Nash equilibrium is a Nash equilibrium with minimum support size. In the following lemma we reduce exact cover by 3 sets, a problem known to be NP-hard (see [Gary and Johnson 1979]), to the problem of finding a sparsest Nash equilibrium.

LEMMA 6.1. *Given a two-player zero-sum game, finding a sparsest Nash equilibrium is* NP-hard under Cook reductions.

PROOF. In the exact cover by 3 sets problem (X3C) we are given a universe of elements J and a collection, $\mathcal{I} = \{S_i\}_{i \in [m]}$, of 3-element subsets of J and the goal is to determine if there an exact cover of J, i.e., a subcollection $\mathcal{I}' \subseteq \mathcal{I}$ such that every element of J is contained in exactly one member of $\mathcal{I}'$. At a high level, given an X3C instance, we construct a two-player zero-sum game in which the first player picks a set in the collection $\mathcal{I}$ and the second player picks an element from J. The goal of the first player is to select a set that covers the second player's element and, since it is a zero-sum game, the second player wants to avoid getting covered. Formally, the action sets of the players are $[m]$ and $[n]$ respectively, where $n = |J|$ and the utilities are as follows: $u_1(i, j) = -u_2(i, j) = 1$ if $j \in S_i$, else if $j \notin S_i$, $u_1(i, j) = -u_2(i, j) = -1$.

Write (σ_1^*, σ_2^*) to denote a sparsest Nash equilibrium of the game. Here σ_1^* and σ_2^* are the mixed strategies of the first and second player respectively. Below we prove that $|\text{supp}(\sigma_1^*)| = n/3$ iff the given X3C instance has an exact cover. Hence if we are given a sparsest Nash equilibrium we can efficiently determine (by looking at the support size of the mixed strategy of the first player) whether the given X3C instance has an exact cover or not. This completes the reduction.

We assume that for all $j \in J$ there exists a set $S_i \in \mathcal{I}$ such that $j \in S_i$, else the problem is trivial. Therefore, the value of the game is positive: player one can guarantee a payoff of at least $1/m$ by playing the uniform distribution over $[m]$. Since the value of the game is positive, the second player receives a negative payoff at any Nash equilibrium. Using this we can show that for *any* Nash equilibrium (σ_1, σ_2) the sets whose index is in the support of σ_1 cover J: $\cup_{i \in \text{supp}(\sigma_1)} S_i = J$. If this is not the case, i.e., there exists an element $j \in J$ that is not covered by $\cup_{i \in \text{supp}(\sigma_1)} S_i$, then the second player can play the pure action corresponding to j and get a payoff of 1, which contradicts the fact that the second player receives a negative payoff at any Nash equilibrium.

Note that the subsets in $\mathcal{I}$ are of cardinality three, therefore any cover of J must contain at least $n/3$ subsets. This implies that $|\text{supp}(\sigma_1^*)| \geq n/3$. This inequality holds regardless of the existence of an exact cover. In particular, if the X3C instance does not have an exact cover then $|\text{supp}(\sigma_1^*)| > n/3$.

On the other hand, if the given X3C instance has an exact cover then $|\text{supp}(\sigma_1^*)| = n/3$. To show this we first consider the following Nash equilibrium: the mixed strategy of the first player is the uniform distribution over the exact cover and the mixed strategy of the second player is the uniform distribution over $[n]$. Since in two-player zero-sum games mixed strategies of Nash equilibria are interchangeable, we get that $|\text{supp}(\sigma_1^*)| = n/3$. Overall, this establishes the desired claim that $|\text{supp}(\sigma_1^*)| = n/3$ iff the X3C instance has an exact cover. □

Note that in a two-player zero-sum game, (σ_1, σ_2) is a Nash equilibrium iff the mixed strategies σ_1 and σ_2 are optimal strategies[10] of the two players. Hence, Lemma 6.1 implies that finding a sparsest optimal strategy, say for the first player, is NP-hard.

We now prove the hardness of finding a sparsest (i.e., one with minimum support size) correlated equilibrium.

Theorem 6.2. *Given a two-player zero-sum game, finding a sparsest correlated equilibrium is* NP-hard under Cook reductions.

Proof. Let π be a correlated equilibrium of a two-player zero-sum game. It is shown in [Forges 1990] that for any action a_2 of the second player such that $\pi(a_2) > 0$ (i.e., a_2 is played with positive probability), the conditional probability distribution $\pi \mid a_2$ over the first player's actions is an optimal strategy for the first player. We have the same result for actions a_1 of the first player (with $\pi(a_1) > 0$) and conditional probability distribution $\pi \mid a_1$. Therefore, $(\pi \mid a_2, \pi \mid a_1)$ is a Nash equilibrium.

Write π^* to denote a sparsest correlated equilibrium of the given game. Also, let $\sigma^* = (\sigma_1^*, \sigma_2^*)$ be a sparsest Nash equilibrium of the game and $r_i = |\mathrm{supp}(\sigma_i^*)|$ for $i \in \{1, 2\}$. Since σ^* is a correlated equilibrium as well, we have

$$|\mathrm{supp}(\pi^*)| \leq r_1 r_2. \tag{15}$$

In two-player zero-sum games mixed strategies of Nash equilibria are interchangeable; therefore, for any Nash equilibria of the game, (σ_1, σ_2), we have $|\mathrm{supp}(\sigma_1)| \geq r_1$ and $|\mathrm{supp}(\sigma_2)| \geq r_2$. In particular, $|\mathrm{supp}(\pi^* \mid a_2)| \geq r_1$ and $|\mathrm{supp}(\pi^* \mid a_1)| \geq r_2$ for any two actions a_1 and a_2 that are played with positive probability. Therefore, inequality (15) is tight and we have $|\mathrm{supp}(\pi^* \mid a_2)| = r_1$ along with $|\mathrm{supp}(\pi^* \mid a_1)| = r_2$.

The above stated property implies that, given π^*, we can efficiently determine a sparsest Nash equilibrium. In particular, let a_1 (a_2) be an action of the first (second) player such that $\pi^*(a_1) > 0$ ($\pi^*(a_2) > 0$), then $(\pi^* \mid a_2, \pi^* \mid a_1)$ is a sparsest Nash equilibrium. Overall, using Lemma 6.1 we get the desired result. □

7. DISCUSSION

Having established polylogarithmic upper bounds on the simplicity of approximate equilibria (Nash, correlated, and coarse-correlated) with respect to k-uniformity (i.e., existence of k-uniform equilibrium for $k = poly(\log m, \log n)$), it is natural to ask whether these bounds are tight. [Althöfer 1994] provides an example of a two-players m-action zero-sum game where for one of the players the support size of every approximate optimal strategy is $\Omega(\log m)$. By considering the same game in the context of Nash, correlated, or coarse-correlated equilibrium we can deduce that the support size for every approximate equilibrium (Nash, correlated, or coarse-correlated) in this game is $\Omega(\log m)$. Therefore, in the upper bound of Theorem 3.1 the $\log(m)$ term is tight, in the upper bound of Theorem 4.1 the $\log(m)$ term is tight, and in the upper bound of Theorem 5.1 the $\log^2(m)$ term is almost tight.

On the other hand, the exact dependence of the simplest approximate equilibrium (i.e., the smallest k such that a k-uniform approximate equilibrium exists) on the number of players, n, remains open for all three types of equilibria (Nash, correlated, or coarse-correlated). Theorems 3.1, 5.1 and 4.1 prove that this dependence is at most logarithmic (i.e., $O(\log n)$-uniform approximate equilibrium exists for fixed m). But establishing whether this logarithmic in n dependence is tight remains an interesting open question. Actually, to the best of our knowledge examples that demonstrate that k cannot be a constant that dependents only on ϵ and not on n (formally, an example of a sequence of n-player games $\Gamma(n)$ where $k(n)$-uniform equilibrium exist only for $k(n) = \omega(1)$) are not known either. To pinpoint this open question, we consider n-player 2-action games (where the set of ap-

[10]We work with the standard maxmin definition of optimal strategies. Specifically, σ_i is said to be an optimal strategy for player i if it satisfies: $\sigma_i \in \arg\max_{x_i \in \Delta([m])} \min_{x_{-i} \in \Delta([m])} u_i(x_i, x_{-i})$.

proximate correlated equilibria coincides with the set of approximate coarse correlated equilibria) and ask the following questions:

For the case of Nash equilibrium:

Open Question 1: Is there a $k = k(\varepsilon)$ that is *independent* of n, such that in every n-player 2-action game there exists k-uniform ε-Nash equilibrium?

For the case of correlated equilibrium (and coarse-correlated equilibrium):

Open Question 2: Is there a $k = k(\varepsilon)$ that is *independent* of n, such that in every n-player 2-action game there exists k-uniform ε-correlated equilibrium?

Although the phrasing of the two questions is similar, we emphasize that they are different; because the notion of k-uniformity for product-distribution strategies is different from the notion of k-uniformity for correlated strategies. Specifically, Question 1 asks whether there exists approximate Nash equilibrium where each player uses mixed strategy of the form $(\frac{c}{k}, 1 - \frac{c}{k})$ for $c \in \mathbb{N}$. Question 2 asks whether there exists approximate correlated equilibrium where the distribution over the action profiles is a uniform distribution over k action profiles.

A positive answer to Question 1 will, in particular, prove the existence of a $O(poly(N))$-time algorithm for computing an approximate Nash equilibrium where N is the size of the game (the input size).

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A. TIGHTNESS OF THE BOUND BY [Germano and Lugosi 2007]

In a n-player m-action game with action space A, correlated equilibria are specified by $nm(m-1)$ linear inequalities in the affine space $\Delta(A)$. Using this fact, [Germano and Lugosi 2007] proved the existence of a correlated equilibrium with support of size $nm(m-1)+1$. Along similar lines we can show that the exists a of coarse correlated equilibrium of support size $nm+1$.

The following example demonstrates that an m^2 term is unavoidable in the support-size bound for exact correlated equilibrium.

Example A.1. There exists a two-player m-actions game with unique correlated equilibrium where each player randomizes uniformly over all her m actions. This game is a m-action generalization of the rock-paper-scissors game, and it is presented in [Nitzan 2005] and [Viossat 2008]. The support of the unique correlated equilibrium in this game is m^2.

The following example demonstrates that a factor of m is unavoidable in the $O(nm)$ bound for exact coarse correlated equilibrium.

Example A.2. Consider the following two-player m-actions zero sum game where player 1 tries to match player 2, and player 2 tries to miss match.

$$u_1(a,b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

The payoff of player 2 is defined by $u_2(a,b) = -u_1(a,b)$. In this game player 1 can guarantee the value $1/m$ by playing uniformly over all her actions. For every distribution over A with support of size less than m player 2 can get a payoff of 0 by playing pure action. Therefore, a coarse correlated equilibrium of size less than m does not exist.

Next we construct a game in which the support size of any coarse correlated, and hence correlated, equilibrium is at least n. This shows that a factor of n is unavoidable in the support-size bound for correlated and coarse correlated equilibrium. We will need the following proposition in order to establish the claim.

Proposition A.3. *Let $P = \{p_j\}_{j\in[k]}$ be a set of positive reals ($p_j > 0$) that can generate all the values in $\{2^{-i}\}_{i\in[n]}$ as partial sums; i.e., there exist subsets of indexes $\{B_i\}_{i\in[n]}$ such that $\sum_{j\in B_i} p_j = 2^{-i}$. Then $k \geq n$.*

Proof. By multiplying all the elements in P by 2^n, we have the following equivalent claim that we establish below. Let $P = \{p_j\}_{j\in[k]}$ be a set of positive reals that can generate all the values in $\{2^{i-1}\}_{i\in[n]}$ as partial sums, then $k \geq n$.

We prove by induction that for $i \leq n$ there must be at least i elements in P that are less than or equal to 2^{i-1}. This statement for i completes the proof.

For $i = 1$, we need a partial sum equal to 1. Therefore, in P ,there must be an element $p \leq 1$.

For $i+1$, by induction we know that there exist i elements, say $p_1, p_2, ..., p_i$, that satisfy $p_j \leq 2^{j-1}$. If we sum these elements we have $\sum_{j\in[i]} p_j \leq 2^i - 1 < 2^i$. Therefore, in order to have 2^i as a partial sum, there must be at least one additional element that satisfies $p_{i+1} < 2^i$. □

Example A.4. We construct a $2n$-player 2-action game in which the support size of any correlated equilibrium is at least n. In 2-action games the set of correlated equilibria and coarse correlated equilibria coincide. Therefore the example holds for both solution concepts.

	1	2
1	$v, -v$	$0, 0$
2	$0, 0$	$1, -1$

Fig. 1: A 2-player 2-action game with unique correlated equilibrium

First note the 2-player 2-action zero-sum game, with $v > 0$, shown in Figure 1 has a unique correlated equilibrium, which is the Nash equilibrium, wherein both players play the mixed strategy $(\frac{1}{v+1}, \frac{v}{v+1})$.

Now consider n pairs of players $(R_i, C_i)_{i\in[n]}$ who play the above game with *different* parameters v_i, i.e., for all i, we replace v in the above game by v_i. Here, the payoffs of players R_i and C_i do not depend on the actions of players R_j and C_j for $j \neq i$.

Correlated equilibria of this game can be characterized as follows: x is a correlated equilibrium iff, for all i, the marginals of the pairs of strategies of players (R_i, C_i) is exactly

	1	2
1	$\frac{1}{(v_i+1)^2}$	$\frac{v_i}{(v_i+1)^2}$
2	$\frac{v_i}{(v_i+1)^2}$	$\frac{v_i^2}{(v_i+1)^2}$

That is, the marginals form the unique correlated equilibrium of the game between players R_i and C_i. In particular, the marginals over the strategies of player R_i are $\left(\frac{1}{v_i+1}, \frac{v_i}{v_i+1}\right)$.

Note that, by definition, marginals are (partial) sums of probabilities $\sum_{a\in B} x(a)$ with $B \subset \mathrm{supp}(x)$. Set $v_i = 2^i - 1$, then for any correlated equilibrium, x, we can take partial sums of the probabilities of strategy profiles in supp(x) to generate all values in $\left\{\frac{1}{v_i+1}\right\}_{i\in[n]} = \{2^{-i}\}_{i\in[n]}$. Therefore, by Proposition A.3, supp(x) must contain at least n elements.