SHAPLEY MEETS LAKOFF: CONSTRUCTION COST ALLOCATION ALIGNS WITH SOCIAL STATUS IN THE CASE OF SKYSCRAPERS

(Tentative Title)

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March 2005

Abstract

We examine the fair allocation of the construction costs among the buyers of units located on different stories of a building. The rationale for the diverse allocation of the costs is two fold: from a supply perspective, the higher the story is, the greater the inflexibility it imposes on the entrepreneur due to the accompanying commitment to build all stories underneath; from a demand viewpoint, it follows from Lakoff (1980) that, because of inherent cognitive motives, agents commonly prefer higher stories to lower ones, *ceteris paribus*. Relying on cooperative game theory analysis, we thus propose an applied theoretical mechanism that fairly allocates the construction costs among the stories of the building. This mechanism is based on the Shapley value approach. We further develop a relative social status function and show that the latter both conforms to a series of reasonable axioms and is consistent with the attained Shapley solution. Essentially, under the attained solution, each story's share in the total construction cost rises in a particular way with its vertical location in the building. We further derive closed-form and simulated properties of the suggested cost allocation.

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#We would like to thank Chen Alon and the participants of the Economics Department seminar at the Chinese University of Hong Kong for helpful comments.

1 Introduction

Consider a real estate entrepreneur who plans to construct an *N*-story building to be later sold to consumers. Before any construction occurs, potential buyers begin to bid for stories in the future structure.

One of the problems that arise in this standard real world scenario concerns the allocation of the construction costs among the buyers. This study proposes and examines a fair allocation of these costs that, as will soon be clarified, is contingent upon the vertical location of the purchased unit within the building. The proposed allocation mechanism is based on the Shapley value approach [see Shapley (1953)].

Our analysis emerges from the observation that, at the pre-construction stage, buyers of higher stories in the building impose on the entrepreneur harsher constraints than those imposed by buyer of lower stories: while by selling the first floor, for example, the entrepreneur assumes no obligation with respect to building additional stories on top of it, selling higher stories forces the entrepreneur to complete construction at least up to the highest floor sold.[1] In other words, while the actual construction of the lower stories does not require the construction of the higher ones, the construction of the higher stories is utterly contingent upon the construction of the lower ones.

Hence, the construction of high and low stories may be differentiated by the dimension of flexibility it maintains for the entrepreneur: committing to a one-story structure undoubtedly maintains the greatest level of flexibility.[2] Put differently, the higher the story that is sold prior to the completion of construction, the greater the inflexibility it entails on the entrepreneur due to the accompanying commitment to build all stories underneath, even if current demand is weak.[3]

[1] In this argument, we, of course, focus on the entrepreneur's obligation to the buyer as opposed to her potential obligation to the public authorities. It should be noticed that an entrepreneur's unilateral breach of the construction contract with a buyer may lead to substantial penalties.

[2] With the exception that taller buildings might require to place stronger foundations in advance. However, given that the foundations for a tall building are already in place, committing in advance to a high story imposes more inflexibility than the equivalent commitment to a low story. Furthermore, it is important to emphasize that studying the flexibility dimension *per se*, often requires a dynamic setting in addition to the presence of uncertainty. The latter, however, is beyond the focus of this study, which concentrates on the fair allocation of the construction cost within a static framework.

[3] Holding everything else equal, it follows that, *a priori*, choosing to construct a given number of one-story structures on a lot is more flexible, from the entrepreneur's perspective, than constructing only one structure with the equivalent number of stories. That is, building horizontally (vertically) is more (in)flexible, *ceteris paribus*. Of course, horizontal construction generally requires more land, while vertical

From the perspective of fair allocation of the construction cost, it, therefore, intuitively follows that the cost borne by buyers of higher stories should be greater than that borne by buyers of lower stories, *ceteris paribus*.

From a demand perspective, a complementary rationale for the fair allocation of the construction cost follows from Lakoff (1980).[4] By analyzing orientational metaphors, Lakoff (1980) argues and demonstrates that, normally, "up" (that is, higher stories in our case) associates with all that is positive, while "down" (lower stories in our case) connotes with everything, which is negative.

To name a few of the numerous examples provided by Lakoff (1980): happy is up and sad is down (as, for example, in "I'm feeling *up*," "He is *low* these days"); conscious is up and unconscious is down ("Wake *up*," "He *fell* asleep"); health and life are up and sickness and death are down ("Lazarus *rose* from the dead," "He *fell* ill"); having control or force is up and being subject to control of force is down ("I'm on *top* of the situation," "He's *under* my control"); more is up and less is down ("My income *rose* last year," "The number of errors he made is *low*"); high status is up and low status is down ("He has little *upward* mobility," "He's at the *bottom* of social hierarchy"); good is up and bad is down ("Things are looking *up*," "We hit a *peak* last year, but it's been *downhill* ever since"); virtue is up and depravity is down ("She is *high*-minded," "That would be *beneath* me"); and, finally, rational is up and emotional is down ("I *raised* the discussion back *up* to the rational plane," "He couldn't *rise above* his emotions").[5]

It follows from Lakoff (1980) that, *ceteris paribus*, from a cognitive perspective (which may be either conscious or unconscious) occupying a higher story in a building is generally preferred to occupying a lower one. In other words, the higher the floor one

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construction often occupies the resources more efficiently. Also, note that the question of inflexibility might be even more complex in circumstances in which it is not only impossible to decrease the number of floors, but also impossible to increase the number of floors due to the foundations and infrastructure that are already in place and which may only suffice a pre-determined number of stories. We omit these aspects from our analysis, however, in order to focus on the other issues.

[4] Lakoff (1980) is regarded as an imperative contribution to the topic of metaphors within the area of linguistics.

[5] In the context of our framework and following Lakoff (1980), it is not surprising that, for example, the management level of many organizations frequently resides on the top floor of the building or, alternatively, if the organization does not have access to the top floor, then it is the highest available story that populates management members (as commonly heard among peers: "I'm going *up* to the management level…"). While it is commonly believed that management acquires the highest possible floor due to the

occupies, the more one is (or ones are) perceived (by oneself and others), often unconsciously, as happy, healthy, lively, good, rational, having control and force, having more, being in high status, etc.

From an economic perspective, Lakoff (1980) thus generates the underlying motivation for occupying higher stories: namely, that the relative social status associated with a given story rises (falls) with the number of stories below (above) it.[6]

Recall, that according to the existing economic literature, prices generally increase with the story's distance from the ground because of the better visibility of the landscape, the lesser noise, etc..[7] While these arguments are undoubtedly valid, we provide here an additional (normative) aspect that demonstrates the price difference of different stories in a building. More importantly, our approach further proposes a resolution to puzzles regarding, for example, the willingness to pay different prices for identical units located on the same story of buildings of distinct heights or for identical units located on different stories, however, with the same vertical distance from the top.

We thus seek to compute the fair allocation of the construction cost among buyers of units located on different stories in a given building, where the fairness principle aligns with both the level of (in)flexibility maintained by the entrepreneur (a supply side argument) and the relative social status generated by buyers of different stories (a demand side argument).[8]

In the general economic literature and specifically within the cooperative game theory literature, several methods have been proposed for allocating costs (utilities) among players within coalitions.[9] Notably, Shapley (1953) presents a solution to the allocation problem that both conforms to conditions of fairness and is unique. Interestingly, the setup of vertical construction matches the general framework of

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more attractive landscape, it follows from Lakoff (1980) that a significant reason also corresponds to the cognitive association that accompanies the terms "up" an "down."

[6] Of course, the high (low) story receives its positive (negative) connotation from being higher (lower) than the stories underneath (above). This implies that, for example, occupying a top floor of a building with lower stories underneath is more valuable than occupying a floor with the same height, however, with no stories below.

[7] See, for example, …

[8] The solution we propose to the fair allocation problem, in effect, also determines the relative prices of the different stories. In this respect, the analysis here also concerns pricing.

Shapley (1953). We thus adopt the Shapley value approach, by which we propose a mechanism for fairly allocating the construction cost among the buyers of units in a building.[10]

According to the Shapley value approach, then, the construction cost that is distributed to each buyer is contingent upon the vertical location of the purchased story in the building, such that the cost rises with the story's distance from the ground. In fact, the buyer of each floor bears only the floor's true pro rata share in the total cost associated with its construction.

For example, the occupant of the first floor participates only in the total fixed cost of construction (which includes, for example, the cost of laying the foundations of the structure, the cost of setting up the common infrastructure, etc.) and in the incremental cost of constructing the first floor. The buyer of the second floor, however, shares the fixed cost as well as the marginal cost of constructing both the first and the second floor—all of which are necessary for the construction of the second floor. Similarly, the occupant of the n-th story participates in the fixed cost, all marginal costs associated with the construction of the stories underneath, and, of course, the marginal cost of constructing the n-th story.

It follows that, according to the Shapley value approach, the cost of those elements in the building that commonly and equally serve all occupants (all element that embrace the fixed construction cost) and the marginal construction cost of the first floor are equally divided among all occupants of the building. However, the marginal cost of the second floor is equally allocated only among those buyers who occupy the floors beyond the first, and more generally the marginal cost of the n-th floor is distributed among all buyers occupying the stories beyond the $(n–1)$-th floor.

We further show that the behavior of the cost allocation *a la* Sahpley corresponds to that of the relative social status function. Particularly, the difference in the cost allocated to any two distinct stories incorporates almost the same properties exhibited by

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[9] A coalition in this context is any subset that can be formed from a given set. Formally, if the set is denoted by $N=\{1,\ldots,N\}$—where N, may, for example, represent the number of stories in a building—then $S,T \subset N$ are referred to as coalitions.

[10] Note that Moulin and Shenker (1992) also propose a cost sharing mechanism that somewhat resembles the Shapley solution. In their model, however, the agents share a one-input and one-output technology with decreasing returns.

the relative social status function, thereby, providing further support to both the properness and reasonableness of the proposed allocation.

Underlying the various interpretations and extensions of the Shapley solution that were developed over the years,[11] the original analysis of Shapley (1953) shows that the proposed solution conforms to the following properties: a) efficiency, i.e. the sum of the individual cost allocated to each buyer equals the total cost; b) symmetry, i.e. buyers who impose the same level of inflexibility on the entrepreneur or that occupy identical units on the same floor bear the same cost; c) dummy player, i.e. those who do not impose any inflexibility on the entrepreneur bear no cost (that is, only actual buyers share the costs); and d) additivity, i.e. for a given set of players, the Shapley value that attains for a sum of m games, m=1,2,…,M, is equal to the sum of the Shapley values separately solved for each game (that is, for a given set of buyers, the cost allocated to each buyer under an m-building framework is equal to the sum of the cost allocation separately solved for each building).

While property (a) guarantees efficiency, such that all costs are allocated among all buyers, properties (b) and (c) ensure that the solution conforms to standards of fairness. Property (d), which relates the Shapley values of different games, guarantees the uniqueness of the Shapley solution. Thus, the particular power of the Shapely solution is in being the only solution that simultaneously obeys to the above reasonable set of axioms.[12]

The solution proposed and analyzed here relates to both real estate economic and general economic literatures. In real estate economics, there are several studies that focus on the various aspects of construction. For example, Gat (1995) shows that the cost of capital (the construction rate) is inversely (positively) related to the optimal constructed area. Unlike our framework, Gat (1995) assumes that the entire building is instantly sold with complete certainty upon completion. Sullivan (1989) examines the optimal choice between horizontal and vertical construction. Specifically, concentrating on intra-building

[11] For an extensive survey of the Shapley value literature, see, for example, Roth (1988).
[12] More on the interpretations of the Shapley axioms, see Dubey (1992).

travel costs, he claims that if the unit cost of horizontal travel is sufficiently greater than that of vertical travel, then tall buildings are more efficient even if the land price is low.[13]

The Shapley solution has also been applied to several practical problems in economics where allocation among players in coalitions is involved. For example, Littlechild and Thompson (1977) and Littlechild and Owen (1973) show that the Shapley value approach to the allocation of the common costs of runway construction and landing fees at Birmingham Airport highly resembles the cost allocation that actually occurred during the examined period.[14]

In Section 2 we develop the relative social status function, derive the Shapley value mechanism to allocating the construction cost, and examine their properties both in closed-form and by simulation. We summarize in Section 3.

2 The Model

In this section we formulate first the relative social status associated with occupying different stories and then the corresponding construction cost allocation. We follow by examining the properties of the proposed cost allocation and its relationship to the social status function.

The Relative Social Status Function

Consider an N-Story building in which, without loss of generality, all units are identical and each story contains one unit. Define the relative social status associated with occupying the j-th story compared to occupying the i-th story in an N-Story building by $S(j,i,N)$.

[13] More on the economics of construction, see, for example, Vandell and Lane (1989) and Hysom and Crawford (1997).

[14] Our framework also relates to the literature on the preference for flexibility. The latter generally adopts three major directions: the axiomatic approach [see, for example, Kreps (1989) and Koopmans (1964)]; the applied approach—associates flexibility with liquidity and, alternatively, with irreversible investments [see, for example, Goldman (1974) and Henry (1974)]; and the information approach—relates the value of information for flexibility [see, for example, Epstein (1980) and Jones and Ostroy (1984)]. Our study is consistent with the information approach: in our model, the entrepreneur, who decides on the construction of an N-story structure, must base her decision on incomplete information in the form of uncertain future demand.

Without loss of generality, let us consider the case where $N>j>i$. We then reasonably require that the relative social status function $S(\cdot)$ conforms to the following five axioms:

Axiom 1: "*The relatively higher I am, the better I am*." That is,

$$S(j+k,i,N)>S(j,i,N) \text{ for all } k>0;$$

Axiom 2: "*The relatively higher the other, the worse I am*." That is

$$S(j,i,N)<S(j,i+k,N) \text{ for all } k>0;$$

Axiom 3: "*The taller the building, the worse I am*." That is,

$$S(j,i,N)<S(j,i,N+k) \text{ for all } k>0;$$

Axiom 4: "*Relative* s*tatus path independence*." That is,

$$S(j,l,N)=S(j,i,N)+S(i,l,N) \text{ for all } j,i,l \text{ such that } j>i>l.$$

Axiom 5: "*Increasing marginal relative status*." That is,

$$S(j,i,N)<S(j+k,i+k,N) \text{ for all } k>0.$$

Following Lakoff (1980), the requirements conveyed by Axioms 1 and 2 are immediate: the relatively higher is the j-th story compared to the i-th story, the greater (worse) the relative status experienced by the occupant of the j-th (i-th) story.

Axiom 3 further requires that, holding j and i fixed, the relative status function $S(j,i,N)$ diminishes with the number of stories in the building. In other words, while j always generates a greater social status than i, the relative status drops as additional stories are constructed on top of the former. This follows from the sensitivity of the relative social status function to "irrelevant alternatives:" while the choice between any two given stories is independent of the total number of stories in the building, the attained relative social status if clearly affected.[15]

Axiom 4 requires that the relative status function exhibits the additivity property. That is, that reallocating from a lower story to a higher one supplements the same level of social status independently whether the shift is performed directly from one floor to another or indirectly via an intermediate floor.

[15] That is, the number of stories above and below a given floor directly affects the relative social status it generates. This, of course, does not imply that the preference between any two stories in the building depends on its total number of stories.

Finally, Axiom 5 requires that the relative status function is convex. That is, that the marginal social status rises as a given floor reallocation begins at a higher story in the building.

We then claim

Proposition 1: *The relative status function* $S(j,i,N)=\ln(\frac{N-i}{N-j})$ *(and its positive monotone transformations) conforms to Axioms 1-5.*

Proof: It is immediate to see that $S(j,i,N)=\ln(\frac{N-i}{N-j})$ increases with *j* and decreases with both *i* and *N* (hence, Axiom 1-3 are satisfied). Furthermore, note that $\ln(\frac{N-l}{N-j})=\ln(\frac{N-i}{N-j})+\ln(\frac{N-l}{N-i})$ (hence, Axiom 4 is satisfied). Finally, note that $\ln(\frac{N-i}{N-j})<\ln(\frac{N-i-k}{N-j-k})$ for all *k*>0. □

We will return to the relative status function in Proposition 10. We now turn to the Shapley value solution to the allocation of the construction cost.

A Shapley Value Solution to the Construction Cost Allocation

Consider once again an *N*-story building. Let *FC* be the fixed cost associated with the construction of the entire structure. This cost consists of the development of the foundations, the infrastructure, and all other elements that equally serve all the units in the building, independently of their specific location within the building.[16] Further,

[16] Other than the cost of placing the foundations and of setting up the common infrastructure, the fixed cost function, *FC*, might also include, for example, the cost of building various facilities for the common and equal service of all occupants. Note, however, that the fixed cost function may, in general, be discontinuous. E.g., while one stairway often suffices a 10-story building, additional floors might eventually necessitate the construction of another stairway. Similarly, the foundations laid for a 10-story building are likely to be less costly than those required for a 50-story structure. Therefore, in *FC* we only include those costs that involve the construction of elements that commonly and equally serve all stories and units in the structure. Other costs are to be included in the marginal cost component.

denote the marginal cost that corresponds to the construction of story i by $mc(i)$, $i=1,\ldots,N$.[17]

It follows that for any N-story building, the total construction cost to be allocated among the different floors (and thereby among the units on each floor), $TC(N)$, is[18]

(1)

$$TC\{N\} = FC + \sum_{j=1}^{N} mc(j),$$

where we also assume that

(2)

$$TC\{\phi\} = 0,$$

(3)

$$TC\{S\} = \underset{i \in S}{Max}(TC\{i\}),$$

and

(4)

$$TC\{i\} = FC + \sum_{j=1}^{i} mc(j),$$

where S is any possible "coalition" of stories in the building and i, $i=1,\ldots,N$, denotes a specific story in the building.

Equation (1) implies that the total construction cost of a building simply equals the combination of fixed cost (cost of foundations, infrastructures, etc.) and marginal cost (incremental cost associated with each additional story). Equation (2) guarantees that zero floors convey zero costs, and Equation (3) together with Equation (4) state that the total cost of constructing any subset of floors is equal to the total cost of constructing the highest floor within the subset.[19]

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[17] Following the previous footnote, $mc(i)$ thus represents both the specific marginal cost associated with the construction of the individual story as well as the average cost that corresponds to the construction of the additional group of stories to which floor i belongs. If, for example, any additional group of 10 stories requires the construction of another stairway, then $mc(i)$ equals both the particular marginal cost of constructing the i-th floor as well as one-tenth of the total cost of constructing another stairway.

[18] We focus in the analysis on the construction cost allocation among stories and ignore any intra-story distribution.

[19] Alternatively, S may be viewed as the current demand for the different stories, in which case Equation (3) represents the total construction cost, given that demand.

Following Shapley (1953), the Shapley value assigned to story j, φ_j, in an N-story building is defined by

(5)

$$\varphi_j = \sum_{j \in S; S \subset N} \frac{(N-s)!(s-1)!}{N!}[TC(S) - TC(S - \{j\})] \text{ for } j=1,\ldots N,$$

where s is the number of stories in subset S and φ_j is the Shapley value associated with story j.

The Shapley value that corresponds to story j, φ_j, can be interpreted as the expectation of the marginal contribution of the j-th story to the total construction cost, where the distribution of coalitions (any possible demanded sub-group of stories) arises in a particular way. To get a sense for φ_j, one can assume that the demand for floors may appear in any arbitrary order and that all N! orderings are equally likely. Then, φ_j is the expected value, across all possible orderings, of story j's marginal contribution to the total construction cost.

We now use the solution of Littlechild and Owen (1973) to re-write Equation (5). It should be noted, however, that our cost structure includes a fixed cost element, which is omitted from the setup of Littlechild and Owen (1973). Yet, since the fixed cost in our framework is used for factors that equally serve all occupants in the building, we get that for the cost function presented in Equations (1)-(4), the Shapley value in Equation (5) can be simplified into

(6)

$$\varphi_j = \sum_{k=1}^{j} \frac{TC(k) - TC(k-1)}{N-k+1} \text{ for } j=1,\ldots,N,$$

which can be further simplified into

(7)

$$\varphi_j = \varphi_{j-1} + \frac{mc(j)}{N-j+1}.$$

Now, let i and j be two arbitrary stories in an N-story building, where $j>i$, (j=2,…,N, and i=1,…,N–1). Following Equations (1), (6), and (7), we then argue

Proposition 2: *According to the Shapley value approach, if* $mc(i)\neq 0$ *for all* $i=1,...,N$*, then the ratio between the cost allocated to the j-th story and that allocated to the i-th story increases with j, ceteris paribus.*

Proposition 3: *According to the Shapley value approach, if* $mc(i)\neq 0$ *for all* $i=1,...,N$*, then the ratio between the cost allocated to the j-th story and that allocated to the i-th story decreases with i, ceteris paribus.*

Proof: Following Equations (1), (6), and (7), the Shapley value associated with any floor j, $j=1,\ldots,N$, is

(8)

$$\varphi_j = \frac{FC}{N} + \frac{mc(1)}{N} + \frac{mc(2)}{N-1} + \cdots + \frac{mc(j-1)}{N-j+2} + \frac{mc(j)}{N-j+1},$$

which may be re-expressed as

(9)

$$\varphi_j = \frac{FC}{N} + \sum_{k=N-j+1}^{N} \frac{mc(N-k+1)}{k}.$$

Hence, the ratio between the cost allocated to the j-th story and that allocated to the i-th story, denoted by $R(i,j,N)$, is

(10)

$$R(i,j,N) = \frac{\frac{FC}{N} + \sum_{k=N-j+1}^{N} \frac{mc(N-k+1)}{k}}{\frac{FC}{N} + \sum_{k=N-i+1}^{N} \frac{mc(N-k+1)}{k}}.$$

From Equation (10), however, it is straightforward to see that if $mc(i)\neq 0$ for all $i=1,\ldots,N$, then $R(i,j,N)$ increases (decreases) with j (i), *ceteris paribus*. □

Propositions 2 and 3 imply that, according to the Shapley value approach to allocating the construction cost, the total cost of an N-story building is simply allocated to each story according to its vertical location in the building: the higher the location of the floor in the building, the greater the allocated cost per floor.

More explicitly, it follows from Equation (7) that each floor bears the true pro rata cost associated with its construction: the costs associated with constructing the first floor are simply FC and $mc(1)$. However, since the construction of the second floor relies on the construction of the first floor, the costs associated with the second floor are FC, $mc(1)$, and $mc(2)$. More generally, the costs associated with the construction of the j-th floor are FC, $mc(1)$, $mc(2)$, and up to $mc(j)$.

It thus turns out that the costs FC and $mc(1)$ are equally allocated among all floors in the building, $mc(2)$ is allocated among all floors from the second and beyond, and $mc(j)$ is allocated among all floors from the j-th and beyond.

For example, suppose N=3. Then, according to the Shapley value in Equations (6) and (7), the construction cost allocated to the first floor, φ_1, is

$$\varphi_1 = \frac{FC + mc(1)}{3},$$

the cost allocated to the second floor, φ_2, is

$$\varphi_2 = \frac{FC + mc(1)}{3} + \frac{mc(2)}{2},$$

and the cost allocated to the third and last floor, φ_3, is

$$\varphi_3 = \frac{FC + mc(1)}{3} + \frac{mc(2)}{2} + mc(3),$$

where note that $\varphi_1 + \varphi_2 + \varphi_3 = TC(3)$.[20]

In Figure 1 in the appendix, we demonstrate the costs allocated among the stories in a 50-story building. We compare the cost allocation for different shares of the fixed cost, FC, in the total construction cost. Without loss of generality, we assume that the total cost of construction of the 50-story building is fixed at 100 million dollars.

One can see that while the allocated cost curve always rises with the story level, the steepness of the slope of the curve drops as the share of the fixed cost in the total

[20] As previously discussed, an alternative approach to attain this cost allocation is to average story j's marginal contribution to the total construction cost across all possible orderings:

Possible Ordering	Cost Allocated to Story:		
	1	2	3
1,2,3	$FC+mc(1)$	$mc(2)$	$mc(3)$
1,3,2	$FC+mc(1)$	0	$mc(2)+mc(3)$
2,1,3	0	$FC+mc(1)+mc(2)$	$mc(3)$
2,3,1	0	$FC+mc(1)+mc(2)$	$mc(3)$
3,1,2	0	0	$FC+mc(1)+mc(2)+mc(3)$
3,2,1	0	0	$FC+mc(1)+mc(2)+mc(3)$
Average	$[FC+mc(1)]/3$	$[FC+mc(1)]/3+mc(2)/2$	$[FC+mc(1)]/3+mc(2)/2+mc(3)$

construction cost increases. That is, given, for example, a total construction cost of 100 million dollars, the cost allocated to the first floor rises from \$236,000 up to \$8,298,570 for the top floor (50-th floor), when the share of the fixed cost is only 10% of the total construction cost. However, when the share of the fixed cost is increased to 50% of the total cost, then the cost allocated to the first floor rises to \$1,020,000 and the cost allocated to the 50-th floor drops to \$5,499,205.

While it follows from Propositions 2 and 3 that the ratio between the cost allocated to the j-th floor and that allocated to the i-th floor, $j>i$, increases (decreases) with j (i), the question yet remains whether the ratio between the costs allocated to any two consecutive stories rises or falls as we ascend along the building. For the rest of the analysis, we assume that $mc(i)=mc(j)=mc$ for all $i,j=1,\ldots,N$.[21] We then claim

Proposition 4: *According to the Shapley value approach, if* $\sum_{k=N-j+1}^{N}\frac{1}{k}>1+\frac{FC}{N\times mc}$ *(*$\sum_{k=N-j+1}^{N}\frac{1}{k}<1+\frac{FC}{N\times mc}$*), then the ratio between the cost allocated to the (*j+1*)-th floor and that allocated to the j-th floor increases (decreases) with j, ceteris paribus.*

Proof: Note that when $mc(i)=mc(j)=mc$ for all $i,j=1,\ldots,N$, then, it follows from Equation (10) that the ratio between the cost allocated to the j-th floor and that allocated to the i-th floor, denoted by $R(i,j,N)$, (where $j>i$; $i=1,\ldots,N-1$; $j=2,\ldots,N$), is

(11)

$$R(i,j,N)=\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}.$$

Following Equation (11), one can see that the ratio between the cost allocated to the (j+1)-th floor and that allocated to the j-th floor increases with j, *ceteris paribus*, if and only if

(12)

$$\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-j+2}^{N}\frac{1}{k}}<\frac{\frac{FC}{N\times mc}+\sum_{k=N-j}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}.$$

However, the latter inequality yields

(13)

$$1+\frac{\frac{1}{N-j+1}}{\frac{FC}{mc\times N}+\sum_{k=N-j+2}^{N}\frac{1}{k}}<1+\frac{\frac{1}{N-j}}{\frac{FC}{mc\times N}+\sum_{k=N-j+1}^{N}\frac{1}{k}},$$

which may, in turn, be re-expressed as

(14)

$$\frac{FC}{N\times mc}\left(\frac{1}{N-j+1}-\frac{1}{N-j}\right)<\frac{1}{N-j}\sum_{k=N-j+2}^{N}\frac{1}{k}-\frac{1}{N-j+1}\sum_{k=N-j+1}^{N}\frac{1}{k}.$$

Inequality (14) may be further simplified into

(15)

$$\frac{FC}{N\times mc}\left[\frac{-1}{(N-j+1)(N-j)}\right]<\sum_{k=N-j+2}^{N}\frac{1}{k}\left(\frac{1}{N-j}-\frac{1}{N-j+1}\right)-\frac{1}{(N-j+1)^2}.$$

Multiplying both sides of (15) by $(N-j+1)(N-j)$ produces

(16)

$$\frac{-FC}{N\times mc}<\sum_{k=N-j+2}^{N}\frac{1}{k}-\frac{N-j}{(N-j+1)},$$

implying that

(17)

$$\frac{N-j}{N-j+1}-\frac{FC}{N\times mc}<\sum_{k=N-j+1}^{N}\frac{1}{k}-\frac{1}{N-j+1},$$

and therefore that

(18)

$$1-\frac{FC}{N\times mc}<\sum_{k=N-j+1}^{N}\frac{1}{k}.$$

[21] Although some of the results may hold even if $mc(i)\neq mc(j)$ for some $i,j=1,2,\ldots N$, the derivations under this assumption are neater and easier to follow.

Hence, if $\sum_{k=N-j+1}^{N}\frac{1}{k}>1-\frac{FC}{N\times mc}$ ($\sum_{k=N-j+1}^{N}\frac{1}{k}<1-\frac{FC}{N\times mc}$), then the ratio between the cost allocated to the (j+1)-th floor and that allocated to the j-th floor increases (decreases) with j, *ceteris paribus*. □

Interestingly, Proposition 4 entails that despite the fact that the Shapley value approach monotonically allocates greater construction costs to higher stories, the percentage change between the costs allocated to any two consecutive floors is not monotonic. In fact, in the general case, the ratio between the costs allocated to two successive stories first falls and then rises as we climb along within a given structure. Moreover, the minimum of the ratio function between the costs allocated to any two successive stories is attained at the lowest j-th floor that sustains $\sum_{k=N-j+1}^{N}\frac{1}{k}>1-\frac{FC}{N\times mc}$.

In Figure 2 in the appendix, we demonstrate the additional cost allocated to any floor compared to that allocated to the preceding one (measured in percentage of the cost allocated to the preceding floor) for a 50-story building. The results are simulated and presented for a fixed cost share of 10%, 30%, and 50% of the total construction cost.

One can see that the percent change in the cost allocated to any floor relative to the previous one is contingent upon the share of the fixed cost in the total construction cost. For example, while the function monotonically increases when the fixed cost share equals 50%, its value first falls and then rises for a fixed cost share of both 10% and 30%.

We further argue

Proposition 5: *According to the Shapley value approach, when N is large (i.e. for tall buildings), then the ratio between the cost allocated to the j-th story and that allocated to the i-th story converges to* $\frac{j+FC/mc}{i+FC/mc}$.

Proof: It follows from Equation (11) that

(19)

$$\frac{\frac{FC}{N\times mc}+\frac{j}{N-j+1}}{\frac{FC}{N\times mc}+\frac{i}{N}} \geq R(i,j,N) \geq \frac{\frac{FC}{N\times mc}+\frac{j}{N}}{\frac{FC}{N\times mc}+\frac{i}{N-i+1}},$$

where the left-hand side (right-hand side) of the first (second) inequality in (19) is the maximal (minimal) value of the right-hand side of Equation (11).

However, (19) implies that

(20)

$$\frac{\frac{FC/mc+j}{N-j+1}}{\frac{FC/mc+i}{N}} \geq R(i,j,N) \geq \frac{\frac{FC/mc+j}{N}}{\frac{FC/mc+i}{N-i+1}}.$$

Finally, note that for a large *N*, the terms on both the left-hand side and the right-hand side of (20) converge to $\frac{j+FC/mc}{i+FC/mc}$. □

That is, for skyscrapers, it turns out that, according to the Shapley value approach, the construction cost allocated to each story is linear in the story's vertical location in the building.

Furthermore,

Proposition 6: *According to the Shapley value approach, the construction cost allocated to the j-th story converges to* $\frac{FC}{N}+\ln(\frac{N}{N-j})^{mc}$.

Proof: Note that when $mc(i)=mc(j)=mc$ for all $i,j=1,\ldots,N$, then Equation (9), may be re-expressed as

(21)

$$\varphi_j = \frac{FC}{N}+mc\sum\nolimits_{k=N-j+1}^{N}\frac{1}{k},$$

which may, in turn, be re-expressed as

(22)

$$\varphi_j = \frac{FC}{N}+mc[H(N)-H(N-j)],$$

where $H(l)$ is the sum of the harmonic series $\sum_{k=1}^{l} \frac{1}{k}$.

It is known, however, that H(l)≅ln(l). And thus Equation (22) yields

(23)

$$\varphi_j = \frac{FC}{N} + mc[\ln(N) - \ln(N-j)],$$

which further implies that

(24)

$$\varphi_j = \frac{FC}{N} + \ln(\frac{N}{N-j})^{mc}. \square$$

Proposition 6 thus provides a quick and simple method to compute the construction cost to be allocated to each story according to the Shapley value approach.

Several conclusions can now be drawn following the expression $H(N)–H(N–j)$ that appears in Equation (22). First, note that this difference increases with j at a relatively slow pace, implying that while the construction cost allocated to each story monotonically rises with the story's vertical location, this growth is relatively slow.

Also, note that it follows from Equation (22) that the cost allocated to the floor j that sustains $H(N) - H(N-j) = 1 - \frac{FC}{N \times mc}$ is equal to mc. In other words, story j that sustains $\sum_{k=N-j+1}^{N} \frac{1}{k} = 1 - \frac{FC}{N \times mc}$ is the only story in the building for which the construction cost allocated according to the Shapley value approach also equals the marginal cost of construction (that is, $\varphi_j=mc$).[22]

We further examine the effect of the number of stories in the building on the derived Shapley solution. We claim

Proposition 7: *According to the Shapley value approach, the construction cost allocated to any j-th story drops with the number of stories in the building.*

[22] Note, however, that since the stories are always represented by whole numbers, it might be the case that no story will actually maintain this condition, in which case none will be allocated a cost equal to *mc*.

Proof: Focusing on the expression that appears on the right-hand side of Equation (22), note that both $\frac{FC}{N}$ and the difference $H(N)-H(N-j)$ drop with the value of N (where, once again, $H(l)$ is the sum of the harmonic series $\sum_{k=1}^{l}\frac{1}{k}$). □

That is, *ceteris paribus*, the cost of any given story falls with the total number of stories in the building. Put differently, holding the marginal construction cost constant, the cost allocated to any j-th floor drops with the height of the building.

In Figure 3 in the appendix, we show the cost allocated to each of the 1-50 stories when we alter the total number of stories in the building and assume that the share of the fixed cost is 30% of the total construction cost. In conducting the simulation, we once again assume that the total cost of construction of a 50-story building is fixed at 100 million dollars and, further, that the total construction cost of any K-story building is equal to K/N of the total cost of construction of an N-story building (in other words, both fixed and marginal costs rise linearly with the number of stories in the building).

One can see that the cost allocated to any given story drops with the number of stories in the building. Focusing, for example, on the first floor: the allocated cost is 1.28 million dollars (out of a total construction cost of 20 million dollars) in a 10-story building, 1.23 million dollars (out of a total construction cost of 60 million dollars) in a 30-story building, and 1.22 million (out of a total construction cost of 100 million dollars) in a 50-story building. One can also observe that the discrepancy between the cost allocated to any given story in any two buildings of different heights increases as we ascend to a higher floor in the buildings. For example, the cost allocated to the 10-th floor is 3.54 million dollars, 1.52 million, and 1.38 million, in a 10-, 30-, and 50-story building, respectively.

Figure 3 also implies that the cost allocated to a given floor drops with the number of stories that are constructed on top of it. That is, in comparing two stories that are equally distant from the top in two distinct buildings, the cost allocated to the floor that corresponds to the taller building is always greater, *ceteris paribus*.

We further argue

Proposition 8: *According to the Shapley value approach, the ratio between the cost allocated to the j-th story and that allocated to the i-th story* ($j>i$; i=1,…,N–1; j=2,…,N, $j>i$) *drops with the number of stories in the building.*

Proof: It follows from Equation (11) that the ratio between the cost allocated to the j-th floor and that allocated to the i-th floor, $R(i,j,N)$ ($j>i$; i=1,…,N–1; j=2,…,N, $j>i$), drops with the number of stories, N, if and only if

(25)

$$\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}>\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+2}^{N+1}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+2}^{N+1}\frac{1}{k}}.$$

However, note that the right-hand side of Inequality (25) can be developed into

(26)

$$\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+2}^{N+1}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+2}^{N+1}\frac{1}{k}}=\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}+(\frac{1}{N+1}-\frac{1}{N-j+1})}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}+(\frac{1}{N+1}-\frac{1}{N-i+1})}.$$

Substituting the right-hand side of Equation (26) with the right-hand side of Inequality (25) produces after reduction

(27)

$$\frac{j(N-i+1)}{i(N-j+1)}>\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}.$$

Given that $j>i$, we can re-write the right-hand side of (27) to generate

(28)

$$\frac{j(N-i+1)}{i(N-j+1)}>\frac{\frac{FC}{N\times mc}+\sum_{k=N-j+1}^{N-i}\frac{1}{k}+\sum_{k=N-i+1}^{N}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}},$$

and thus

(29)

$$\frac{j(N-i+1)}{i(N-j+1)}-1>\frac{\sum_{k=N-j+1}^{N-i}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}},$$

which can then be developed into

(30)

$$\frac{(N+1)(j-i)}{i(N-j+1)}>\frac{\sum_{k=N-j+1}^{N-i}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}.$$

Focusing on the right-hand side Equation (30), one can see that

(31)

$$\frac{\sum_{k=N-j+1}^{N-i}\frac{1}{k}}{\frac{FC}{N\times mc}+\sum_{k=N-i+1}^{N}\frac{1}{k}}\leq\frac{\frac{(j-i)}{(N-j+1)}}{\frac{FC}{N\times mc}+\frac{i}{N}},$$

where the right-hand side of (31) is the maximal value of the right-hand side of (30).

Finally, note that the right-hand side of (31) is smaller than the left-hand side of (30), which thus implies that Inequality (25) holds for all N and $j>i$, and hence that the ratio between the cost allocated to the j-th floor and that allocated to the i-th floor ($j>i$; $i=1,\ldots,N-1$; $j=2,\ldots,N$, $j>i$) drops with the number of stories. □

Proposition 8 essentially argues that, when focusing on any two stories j and i, where $j>i$, then, according to the Shapley value approach, the ratio between the cost allocated to the j-th floor and that allocated to the i-th floor drops as the building becomes taller.

We also claim

Proposition 9: *According to the Shapley value approach, the ratio between the cost allocated to the j-th story and that allocated to the i-th story* ($i=1,\ldots,N-1$, $j=2,\ldots,N$, $j>i$) *drops with the share of the fixed cost within the total construction cost.*

Proof: It follows from Equation (11) that the ratio $R(i,j,N) > 1$ for all i=1,…,N–1, j=2,…,N, and j>i. Hence, as the share of FC increases, we get that $\frac{FC}{N \times mc}$ rises, and thus that $R(i,j,N)$ falls. □

Proposition 9 essentially states that as the share of *FC* within the total construction cost increases, the discrepancies in the costs allocated to different stories drops. Put differently, the average percent change in the cost allocated to all couplets of succeeding stories within a given building falls with the share of *FC*, *ceteris paribus*.

Figure 4 depicts the average percent change in the cost allocated to every two succeeding stories as a function of the number of stories in a given building. We replicate this simulation for varying shares of the fixed cost.

One can see that, independently of the share of the fixed cost in the total construction cost, the average percent change in the cost allocated to all couplets of successive stories falls with the number of stories in the building. For example, for the case where the fixed cost is 10% of the total construction cost, the average percent change in the cost allocated to every two successive floors drops from about 17% to about 5% as we increase the number of floors from 20 to 80.

One can also see from Figure 4 that as we increase the share of the fixed cost within the total construction cost, the average percent change in the cost allocated to all couplets of successive floors falls for any given building height. For example, for the case where the share of the fixed cost is 30%, the average percent change in the cost allocated to every two successive floors drops from about 11% to about 3% as we increase the number of floors from 20 to 80. However, for the case where the share of the fixed cost rises to 50%, the spoken average percent change falls from approximately 8% to about 2% as we, once again, raise the number of floors from 20 to 80.

The Status Function and the Shapley Value Solution

Following Propositions 1 and 6, we argue

Proposition 10: *According to the Shapley value approach, for any two stories j and i (j>i), the difference between their allocated construction costs converges to the product of the value of the relative status function and the marginal cost.*

Proof: From Proposition 6 we have that φ_k converges to $\frac{FC}{N} + \ln(\frac{N}{N-k})^{mc}$ for all *k*. Then, subtracting φ_i from φ_j generates after reduction $mc \times \ln(\frac{N-i}{N-j})$, which, following Proposition 1 is further equal to $mcS(j,i,N)$. □

Proposition 10 thus provides further support to the Shapley value approach to allocating the construction cost. It turns out that the cost allocation derived by supply considerations match a major demand characteristic, namely, the relative social status function.

3 Summary

We apply the Shapley value approach to the allocation of the construction cost among the buyers of units located on different stories of a building. According to the proposed mechanism each story's share in the total construction cost rises with its vertical location.

The rationale underlying the attained construction cost allocation is two fold: from a supply perspective, the higher the story that is sold prior to the completion of the construction, the greater the inflexibility it entails on the entrepreneur due to the accompanying commitment to build all the floors underneath. From a demand viewpoint, it follows from Lakoff (1980) that, due to inherent cognitive motives (which may be either conscious or unconscious), the higher the location of the unit in the building, the relatively greater the corresponding social status. Interestingly, it turns out that the difference in the costs allocated to any two stories converges to the value of the relative status function multiplied by the fixed marginal cost.

We further examine the properties of the derived cost allocation mechanism and show, among other things, the conditions under which the ratio between the costs allocated to any two successive stories either rises or falls with the ascent to higher stories. We further show that the construction cost allocated to any given story drops with the number of stories in the building and that the ratio between the cost allocated to any given higher story and that allocated to any given lower story always drops with the number of stories in the building.

The results presented here incorporate both normative and positive economic implications. On the normative aspect, we provide a benchmark for the fair allocation of the construction cost as a function of the various construction parameters (such as the share of the fixed cost, the number of stories in the building, etc.). While, undoubtedly, the cost allocation in some construction projects closely conforms to the results derived in this study (especially, in those cases where the share of the fixed cost ranges at the 30%-40% of the total construction cost), for other projects, our framework allows a straightforward computation of the discrepancy between the actual cost allocation and the potential fair allocation.

On the positive aspect, our framework proposes a procedure for allocating costs in an organization (in both the public and private sectors) that shares diverse profit centers, however, where the profit centers occupy different stories within a common building. The suggested allocation mechanism thus incorporates imperative implications for the organization from both the accounting and the economic aspects.

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Appendix

Figure 1: The costs allocated to each story in a 50-story building. The results are shown for a fixed cost share of 10%, 30%, and 50% of the total construction and constant marginal costs.

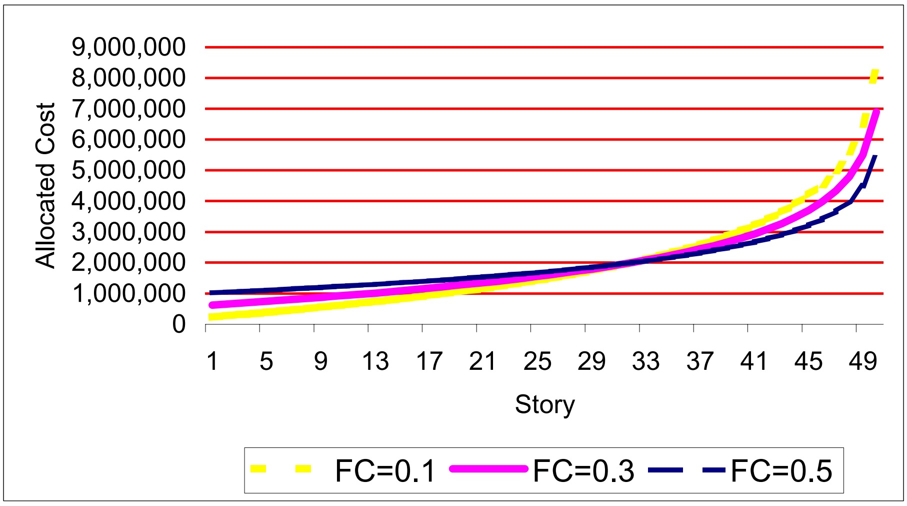

Figure 2: The additional cost allocated to any floor compared to the cost allocated to the preceding floor (measured in percentage of the cost allocated to the preceding floor) for a 50-story building. The results are shown for constant marginal costs and a fixed cost share of 10%, 30%, and 50% of the total construction cost.

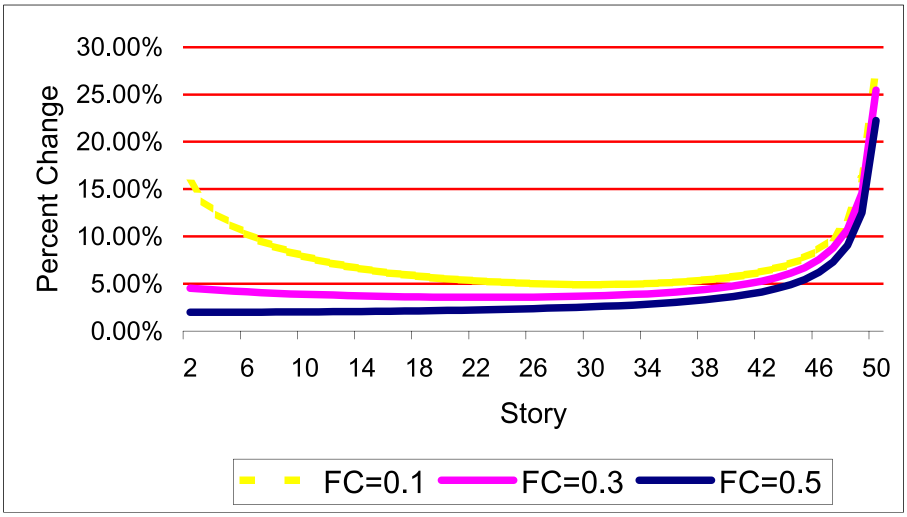

Figure 3: the cost allocated to each story when the share of the fixed cost is fixed at 30% of the total construction cost and marginal costs are constant. The results are shown for a 10-,30-, and 50-story building.

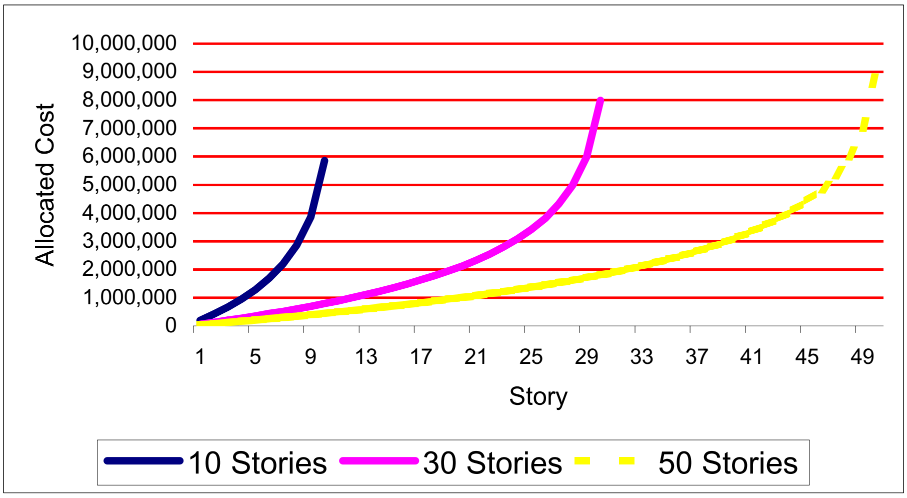

Figure 4: Average percent change in the cost allocated to any two succeeding stories as a function of the number of stories in the building. The results are shown for constant marginal costs and a fixed cost share of 10%, 30%, and 50% of the total construction cost.