Characterizing Truthful Market Design

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Abstract

This paper characterizes the family of truthful double-sided auctions. Despite the importance of double-sided auctions to market design, to date no characterization of truthful double-sided auctions was made. This paper characterizes truthful mechanisms for double-sided auctions by generalizing Roberts classic result [29], to show that truthful double-sided auctions must "almost" be affine maximizers.

Our main result of characterizing double-sided auctions required the creation of a new set of tools, reductions that preserve economic properties. This paper utilizes two such reductions; a *truth-preserving reduction* and a *non-affine preserving reduction*. The truth-preserving reduction is used to reduce the double-sided auction to a special case of a combinatorial auction to make use of the impossibility result proved in [20]. Intuitively, our proof shows that truthful double-sided auctions are as hard to design as truthful combinatorial auctions.

Two important concepts are developed in addition to the main result. First, the form of reduction used in this paper is of independent interest as it provides a means for comparing mechanism design problems by design difficulty. Second, we define the notion of *extension of payments*; which given a set of payments for some players finds payments for the remaining players. The extension payments maintain the truthful and affine maximization properties.

1 Introduction

This paper characterizes the class of truthful double-sided auctions. In recent years a large body of research has focused on designing algorithms for environments where the input to the algorithm is distributed among players. Each player attempts to maximize its output function (utility) without considering the environment as a whole. Such environments are increasingly common e.g. the Internet and communication networks. One of the main approaches to designing such auctions is to design *truthful mechanisms* which motivate the players to reveal their true input to the algorithm.

In this paper we look at a double-sided auction which is a market that consists of multiple buyers and sellers who wish to exchange goods. The market's main objective is to produce an allocation of sellers' goods to buyers that maximizes the total gain from trade (i.e., the total value associated with an allocation).

A commonly studied model of participant behavior is taken from the field of economic mechanism design in such papers as e.g., [1, 11, 13, 17, 21, 25, 26, 28]. In this model each player has a private valuation function that assigns real (non-negative) values to each possible allocation. The auction algorithm uses the payments to the mechanism in order to motivate players to truthfully reveal their functions. Each player is a utility maximizing rational agent, i.e., the player maximizes the difference between his valuation of the algorithm's allocation and his payment. The couplet consisting of an algorithm and a payment is called a mechanism. Mechanisms for which reporting the truth is a dominant strategy for each player are called truthful. Although, truthful mechanisms are the central paradigm of the literature most work has focused on the model consisting of a single seller and multiple buyers[1]. Double-sided auctions remain an important open question.

In a double-sided auction mechanism, there are n sellers each offering a unique good. Each seller s_i, $1 \leq i \leq n$ has a valuation function v_i that assigns a real value $v_i(g_i)$ for his good g_i and each buyer b_j, $1 \leq j \leq m$ has a valuation function v_j that assigns a real value $v_j(g_i)$ for every good g_i, $1 \leq i \leq n$. The goal is to find a match M between buyers and sellers such that the total gain from trade $\sum_{i,j|(s_i,b_j)\in M} v_j(g_i) - v_i(g_i)$ is maximized. The problem of a double-sided auctions where all sellers' goods are identical has been exten-

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[1] Or equivalently a single buyer and multiple sellers

sively studied in the literature (see e.g. [22]). Relatively little work, e.g., [3] attempts to look at the more general case, where sellers may sell different goods.

Our goal in this paper, is to characterize the set of truthful mechanisms for the double-sided auction. Our proof is partially based on a *truthful preserving reduction* to a special case of a combinatorial auction. In a combinatorial auction, n unique goods are auctioned among m players. Players value bundles of goods in a way that may depend on the combination they win, i.e., each player has a valuation function v_i that assigns a real value $v_i(\lambda)$ for each possible subset of goods. The goal is to find a partition $\lambda_1 ... \lambda_m$ of the goods that maximizes the total social welfare $\sum_i v_i(\lambda_i)$. The combinatorial auction problem is NP-complete and has been extensively studied as it is an important instance of the interplay between computational difficulty and game theoretic difficulty. A thorough study of combinatorial auctions can be found in e.g., [4, 5, 8, 9, 15, 19, 27].

To better explain the background to the problem of characterizing the class of truthful double-sided auction mechanisms, we formalize the basic model slightly more. Let A denote the range of all mechanism's possible outcomes and let $v_i : A \to \mathbb{R}$ be player i's valuation function that specifies his value $v_i(a)$ for each possible outcome $a \in A$ where v_i is chosen out of a valuations domain V_i. Given the valuations $v = (v_1, ..., v_n)$ the mechanism computes the function $\phi(v)$ that is referred to as the *social choice function*. In the context of double-sided auction mechanisms A is all the possible matchings of buyers and sellers and V_i depends only on a_i (as we assume "no externalities"[2]) The mechanism computes $\phi(v)$ and payments $\vec{p}$ to the players. We say that a social choice function ϕ is *implementable* if there exists payments supporting ϕ such that the pair $(\phi, \vec{p})$ yields a truthful mechanism. So the basic question is what are the implementable social choice functions?

The well known VCG payment scheme insures the truthfulness of a welfare maximizing social choice function $\phi(v) \in \arg\max_{a \in A} \sum_i v_i(a)$ [31, 16, 7][3]. The VCG payment scheme can be generalized in three ways: (1) The range can be restricted to $A' \subseteq A$; (2) Non-negative weights can be assigned to the players; (3) weights can be added to different outcomes. When applying those generalizations to the VCG payment scheme we obtain an implementation for any social choice function that is an affine maximizer.

Definition: *A social choice function ϕ is an affine maximizer if for some $A' \subset A$, non-negative $\{\omega_i\}$, and $\{\gamma_a\}$, for all $v_1 \in V_1, ..., v_n \in V_n$ the function $\phi(v_1, ..., v_n) \in \arg\max_{a \in A'} (\sum_i \omega_i v_i(a) + \gamma_a)$.*

Are there other implementable social choice functions that are not affine maximizers? The answer to that question was given by Roberts [29] in his classic negative result showing that if the players' valuation domain is unrestricted and the outcome range is nontrivial then there does not exist an implementable social choice function that is not an affine maximizer.

Theorem (Roberts, 1979): *If there are at least 3 possible outcomes, and players' valuations are unrestricted ($V_i = R^{|A|}$), then any implementable social choice function is an affine maximizer.*

The requirement for players' valuations domain to be unrestricted is very strong as it implies that players have a value for every possible outcome of the social choice function. In most realistic and practical applications this is not the case. For example for double-sided auctions, players' valuations are restricted by the demand that there are no externalities as mentioned above and for combinatorial auctions, players' valuations are restricted in two ways: free disposal (i's valuation is monotone by inclusion in a_i) and no externalities. On the other hand restrictions on the players' valuation domain sometimes simplify the problem in a way that does not capture problems of interest. For instance in single dimensional valuation spaces (single value), implementable social choice functions do not imply affine maximization. Such is the case with single minded bidders in combinatorial auctions where the valuation function is given by a single positive value v_i which is offered for a single set of items. For instance, [19] present a computationally efficient truthful approximation that is not affine maximizing. Additional mechanisms for the single-minded case were presented e.g., in [2, 24].

However, most interesting problems (computationally and practically) lie somewhere between the two extremes of unrestricted domains and single dimensional domains. This intermediate range includes double-sided auctions with heterogenous goods (which is the model we focus on), a number of single sided auctions (multiple buyers, single seller), non-single-minded combinatorial auctions, and multi-unit (homogeneous) auctions. Little is known about the intermediate range. The only

[2]For simplicity of the analysis, we make throughout this paper the standard assumption of *free disposal*, i.e., that the functions are monotone non decreasing.

[3]It also insures truthfulness for the welfare maximizing extension for double-sided mechanisms, i.e., the gain from trade social choice function $\phi(v) \in \arg\max_{a \in A} \sum_j v_j(a) - \sum_i v_i(a)$

positive example of a non-VCG mechanism for non-single-dimensional domains is for multi-unit combinatorial auctions where each bidder is restricted to demand at most a fraction of the number of units of each type [4][4]. On the negative side [20] showed that for multi-minded bidders a truthful combinatorial auction essentially implies affine maximization.

In all of the mechanisms discussed so far we assume players have quasi-linear utility. Interestingly in the non-quasi-linear case the classic Gibbard-Satterthwaite result [16, 30] shows that no non-trivial social choice function over an unrestricted domain is implementable. However, for restricted single peaked domains (which is a single dimension domain) [6, 23] implement a non-trivial social choice function.

1.1 Our Results

In this paper we characterize the implementable social choice functions of the double-sided auction mechanism over restricted domains in quasi-linear environments. The work follows the initiative of [20] to extend Roberts's impossibility result to multi dimensional but restricted domains. [20]'s work extends Roberts work for multi-minded combinatorial auctions while our work extends Roberts work for double-sided auctions. The double-sided auction is the principle mechanism for many real life markets (such as the stock market) and therefore is fundamental to mechanism design.

To prove their main theorem [20] characterize conditions over the domain in which implementable social choice functions implies affine maximization. The basic conditions require that the domain is an *order based domain* in which valuations over different possible outcomes in the domain can be compared, and that the best outcome for one player is the worst outcome for the other players, i.e. "conflicting preferences". Those domain conditions capture combinatorial auctions and multi-unit auctions. However, since matching problems do not have conflicting preferences [20] left open the question of whether implementable social choice functions implies affine maximization for matching. Our work answers that question positively by showing that the implementable social choice double-sided auctions imply affine maximization.

Our work builds on Roberts results and the results achieved by [20] integrating a classic tool of computer science theory: reductions. Although reduction is a widely used tool in proving the hardness of problems by reducing them to other hard problems this tool has not been used before in the context of mechanism design. Moreover the existing literature does not try to classify the difficulty of different mechanism design problems in terms of game theory but rather classifies difficulty in the context of computational complexity. This work makes use of the negative result in [20] for combinatorial auctions to show the same negative result for double-sided auctions by the means of a reduction. We believe that the use of reduction in the context of mechanism design is of independent interest.

The task of building a reduction between the combinatorial auction and the double-sided auction is not as straightforward as it may sound. Since our main theorem shows that: a mechanism with the property of truthfulness implies affine maximization, we need to construct a reduction that maintains the truthfulness property and the non affine property. In order to use a reduction in the context of mechanism design we define the new concepts of *truth-preserving reduction* and *non-affine preserving reduction*. These new concepts are inspired by the well established concept of gap-preserving reductions [18, 12] which expand the concept of a reduction.

Definition: *A social choice function ϕ is* reducible *to a social choice function $\bar{\phi}$, namely, $\phi \leq \bar{\phi}$, if ϕ's input can be reduced to $\bar{\phi}$'s input such that the target function of ϕ is optimum if and only if the target function of $\bar{\phi}$ on the reduced input is optimum.*

Definition: Truth preserving reduction: *Given mechanisms $\alpha = (\phi, p_1, ..., p_n)$ and $\beta = (\bar{\phi}, \bar{p}_1, ..., \bar{p}_m)$, a reduction $\alpha \leq \beta$ is a* truth preserving reduction *if there exists a function $h : \phi \rightarrow \bar{\phi}$ such that $\phi \leq \bar{\phi}$ and for every $1 \leq i \leq m$ there exists a function $h_i : \{p_1, ..., p_n\} \rightarrow \bar{p}_i$ s.t. if $(\phi, p_1, \ldots, p_n)$ is truthful then $(h(\phi), h_1(p_1 \ldots p_n), \ldots, h_m(p_1 \ldots p_n))$ is truthful.*

Definition: Non-affine maximizing preserving reduction: *Given social choice functions ϕ and $\bar{\phi}$, a reduction $\phi \leq \bar{\phi}$ is a* non-affine maximizing preserving reduction *if the following holds: if ϕ is a non-affine maximizing social choice function then $\bar{\phi}$ is a non-affine maximizing social choice function.*

Our proof that any implementable double-sided auction's (DSA) social choice function is affine maximizing utilizes the main theorem of [20] for a special case of combinatorial auction which we call the *combinatorial*

[4]Other examples are known for relaxations of the deterministic dominant strategy model such as random algorithms and implementations in undominated strategies [10, 5]

auction product space (CAPS)[5]. The first stage in our proof is then:

Lemma: *The social choice function of any truthful CAPS mechanism is an almost affine-maximizer.*

We then utilize a reduction from a special case of DSA (where the sellers all have value zero for their good) which we call *double-sided auction cost 0 problem (DSAC0)* to CAPS. This reduction preserves the truthfulness and the non-affine properties. This reduction will then yield the following theorem:

Lemma: *The social choice function of any truthful DSAC0 mechanism is an almost affine maximizer.*

Once any implementable DSAC0 is shown to be affine maximizing a reduction from DSA to DSAC0 is constructed which again preserves the truthfulness and non-affine properties[6]. As the sellers' values (and therefore the sellers' prices) in the DSAC0 are zero, the reduction DSA $\leq$ DSAC0 preserves the non-max affine property only for the DSA buyers. To prove the max-affine maximizing property for the sellers as well, we define and perform a *price expansion* of the induced buyers' prices and define critical value prices for sellers. The critical prices are shown to be truthful affine-maximizing prices. To complete the structure we prove that the integration of two price vectors that are truthful and affine maximizing is also truthful and affine maximizing.

This then yields our main result:

Main Theorem: *The social choice function of any truthful DSA mechanism is an almost affine maximizer.*

The "almost" in the above theorems is an artifact from the proof of [20]'s main theorem and any improvement in the proof of [20] will benefit our theorem.

[20]'s theorem only shows that the social choice function must be an affine maximizer for large enough input valuations.

Definition ([20]): Almost Affine maximizer: *A social choice function ϕ is an almost affine maximizer if there exists a threshold H s.t. the function is an affine maximizer if $v_z(a) \geq H$ for all a and z.*

[5]Although CAPS is defined as a special case of single-minded combinatorial auction it is no wonder that the following lemma holds as we show later on in the paper that the CAPS problem is equivalent to a special case of multi-minded combinatorial auction.

[6]Although DSAC0 is a special case of DSA we show that all cases of DSA can be expressed as an instance of DSAC0

[20] believe that this restriction is a technical artifact of their current proof.

Our proof differs significantly from both Roberts' proof and from [20]'s proof. The definitions that we need to get our main result naturally yields the question of whether it is possible to classify mechanism design problems into classes of problems. Such classification can be either into equivalency classes or into a hierarchal relationship. This classification is a refinement of the standard computational complexity classes. For instance, our reduction shows that the double-sided auction is at least *mechanism design hard* as a particular case of combinatorial auction. This despite the fact that from a computational complexity point of view, double-sided auctions can be solved in polynomial time (if the input is given truthfully). This observation emphasizes that the difficulty of mechanism design does not necessarily require us to focus on computationally hard problems but rather, a simple polynomial problem such as the double-sided auction can already capture the essence of the mechanism's design difficulty

Organization: The rest of the paper is organized as follows. The next section gives notations, defines the new concepts of a truth preserving reduction and non-affine preserving reduction and defines the CAPS, DSAC0 and DSA problems. Section 4 and 5 gives the characterization of the CAPS and DSAC0 problems respectively. Sections 6 and 7 give the characterization of the DSA problem for buyers and for sellers, respectively. Finally, in section 8 we prove the main theorem. We conclude with Section 9.

2 Setting, Notations and Problems Definitions

We use the standard setting and notations. The formal setting can be found in appendix B. In this section we informally define the problems that we use. The formal definitions can be found in Appendix C.

The first problem is the combinatorial auction product space (CAPS). The CAPS has nm players and $m+n$ different goods where each player is interested in a subset (of size two) of the goods. The objective is to maximize the players' social welfare. To convince the sceptical reader that the CAPS problem can follow [20]'s theorem although it is single minded, we show in section C and 4 that the CAPS problem is equivalent to a multi-minded CA special case which has $2n*m/(n+m)$

players and $m+n$ different goods where each player has a preference over $m+n$ different bundles each of which is of size two. We denote the multi-minded CA special case CAPS-MM.

Thus the CAPS and the CAPS-MM are maximizing essentially the same objective function.

The second problem is the double-sided auction (DSA) in which there is a set of buyers and a set of sellers and the objective is to maximize the gain from trade between buyers and sellers. Finally, the third problem is the double-sided auction cost 0 (DSAC0) is the DSA in which all of the sellers costs are set to 0.

The intuition behind our reduction construction is that each of the CAPS' players corresponds to a possible trade between a pair of players in the DSA and the goods are used as indicator functions to ensure that no player buys more than one good and no good is sold more than once. By maximizing the welfare in CAPS we maximize the gain from trade in the original DSA.

3 Road Map

In order to prove our main theorem we prove that the social choice function of any truthful CAPS mechanism is an almost affine maximizer. By showing that the DSAC0 is reducible to CAPS, and that the reduction is truthful non-affine maximizing preserving, it follows that the social choice function of any truthful DSAC0 mechanism is almost affine maximizer. We move on to prove that DSA is reducible to DSAC0, and that the reduction is truthful non-affine maximizing preserving for buyers. It follows that the social choice function of any truthful DSA mechanism is an almost affine maximizer for buyers. The solution is expanded to sellers by defining critical value prices. We then prove that any mechanism when applying critical price payment scheme to the sellers in the DSA problem is a truthful mechanism for sellers. In addition we show that the social choice expansion of any truthful DSA mechanism with critical value prices to sellers is an affine maximizer for sellers. It follows that the social choice function of any truthful DSA mechanism is an almost affine maximizer for sellers. Thus our main theorem is concluded.

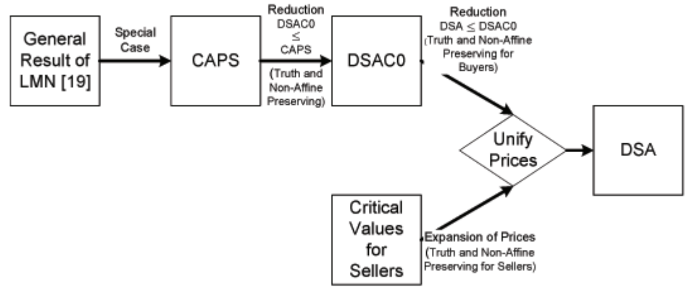

Figure 1: Roadmap of our proof

4 Combinatorial Auction Product Space (CAPS) Characterization

In this section we prove that the social choice function of any truthful CAPS mechanism is an almost affine-maximizer, using the result of [20]:

[20]'s Main Theorem: *Every social choice function over an order-based domain with conflicting preferences and onto non-degenerate range, that is player decisive and satisfies S-MON, must be an almost affine maximizer.*

In appendix E we prove that the CAPS mechanism maintains the properties required by [20]. Therefore:

LEMMA 4.1. *The social choice function of any truthful CAPS mechanism is an almost affine-maximizer.*

REMARK 4.1. *One might be concerned that the results of lemma 4.1 are not aligned with the existing literature as CAPS is a known single-minded combinatorial auction and therefore should have many truthful mechanisms (e.g. [19, 24]) which are not affine. However when one is aware of the fact that CAPS can be easily translated to CAPS-MM and maximize essentially the same objective function as CAPS-MM, then actually lemma 4.1 is not surprising and is fully aligned with the literature.*

Moreover, lemma 4.1 illustrates the delicate analysis of the affine property. Although the CAPS is a known single-minded combinatorial auction (KSM) the problem is defined such that the players preferences are completely symmetric and in addition the players'required bundles have intersections with other players' bundles only in part of the outcome space. The above structure allows for a reach enough alternative space and therefore maintains the properties of a multi-minded combinatorial auction setting problem. And in fact CAPS-MM is the equivalent multi-minded problem in that struc-

ture of preferences.[7]

Also note that the KSM combinatorial auction is a special case of all possible preferences [29] (the general case GC) which must be affine. Schematically, $CAPS \subset KSM \subset GC$ here both CAPS and the general case must be affine maximizers but KSM does not require affineness.

To insure the reader that lemma 4.1 is fully aligned with the literature. We prove the following Theorem which claims that CAPS and CAPS-MM are equivalent problems:

THEOREM 4.1. *$CAPS \leq CAPS-MM$, $CAPS-MM \leq CAPS$.*

The proof of Theorem 4.1 is in Appendix H.

5 DSAC0 Characterization

In this section we prove that the social choice function of any DSAC0 truthful mechanism is (almost) an affine-maximizer by showing that DSAC0 is reducible to CAPS, using a truthful and non-affine-maximizer preserving reduction.

The intuition behind the DSAC0 $\leq$ CAPS reduction construction is to turn each pair of seller and buyer in DSAC0 to a player in the CAPS, to turn each seller and each buyer in the DSAC0 to a good in CAPS, such that the valuation of each player in the CAPS for a pair of goods defined by a seller and a buyer in the DSAC0 is the valuation of the buyer for the seller's good. Due to lack of space the technical details of the reduction construction, reduction proofs and the properties preserving proofs are defer to appendix F.

6 DSA Characterization for Buyers

In this section we proof that the social choice function of any DSA truthful mechanism is an affine-maximizer for buyers by showing that DSA is reducible to DSAC0, using a truthful and non-affine-maximizer preserving reduction.

The intuition behind the DSA $\leq$ DSAC0 reduction construction is to turn every seller s_i with valuation greater than 0 for his good g_i' in the DSA to both a buyer with the same valuation as s_i for g_i', and to a seller who has the good g_i' but who has a valuation of 0 for the good g_i' in the DSAC0.

Due to lack of space the technical details of the reduction construction, reduction proofs and the properties preserving proofs are defer to appendix G.

[7]Interestingly, when restricting the truthful non-affine mechanisms for the known single-minded combinatorial auction to a known single-minded with all bundles of size 2 case, both [19, 24] are affine. The above observation should not necessarily be true for any truthful non-affine mechanism unless the problem is of the CAPS's symmetric form.

7 DSA Characterization for Sellers

In this section we sketch the prove that the social choice function of any truthful DSA mechanism for sellers is an almost affine maximizer.

Our sellers are a special case of a more general class of players:

DEFINITION 7.1. *(Single Value Player) A single value player is a player that a single value determines his valuation. I.e., for a single value player j, good g_i and allocation a where $g_i \in a_j$, for all allocations e, $v_j(a) = v_j(e)$ if $g_i \in e_j$ and $v_j(e) = 0$ otherwise.*

We first prove a generalization of the folk theorem that states that critical price payment scheme used for single value players is truthful. We then conclude that a critical price payment scheme for sellers in the DSA problem is truthful as sellers are single value players.

It is important to note that sellers are inherently single value players (their cost). Even if our DSA model allowed seller to sell multiple goods, the sellers are still single value player as there are no dependencies between the different goods. Therefore, a seller with multiple goods for sale can be viewed as multiple sellers with a single good for sale. On the other hand the buyers in the DSA are *not* single value players as they are interested in only one good out of number of goods which they value. Due to lack of space the technical details of this section are defer to appendix J.

8 Proof of Main Theorem

In this section we conclude the main theorem using Lemma G.1 and Corollary J.2 from previous sections, and prove two new Lemmas that integrate the truthful and non-affine properties maintained for buyer and sellers separately into truthful and non-affine properties maintained for all players. Due to lack of space the technical details of the main theorem proof is defer to appendix I.

9 Conclusions and future work

Our conclusions and future work can be found in appendix A.

References

[1] A. Archer, and E. Tardos. Truthful Mechanisms for one-parameter agents. *Proceeding of 42th FOCS 2001.*

[2] A. Archer, C. Papadimitriou, K. Talwar, and E. Tardos. An Approximate Truthful Mechanism for Combinatorial Auctions with Single Parameter Agents. *Proceeding of 14th SODA 2003.*

[3] Y. Bartal, R. Gonen, and P. LaMura. Negotiation-Range Mechanisms:Exploring the Limits of Truthful Efficient Markets. *Proceeding of 5th EC 2004.*

[4] Y. Bartal, R. Gonen, and N. Nisan. Incentive Compatible Multi-Unit Combinatorial Auctions. *Proc. of TARK 2003*, pp. 72-87, June 2003.

[5] M. Babaioff, R. Lavi, and E. Pavlov. Single-Value Combinatorial Auctions and Implementation in Undominated Strategies. *Proc. of SODA 2006.*

[6] M.L. Breton, and J. Weymark. Arrovian Social Choice Theory on Economic Domains. *In K. Arrow, A.Sen, and K.Suzumura, editors, Handbook of Social Choice and Welfare: Volume 2. North-Holland:.*

[7] E. H. Clarke Multipart Pricing of Public Goods. *In journal Public Choice 1971*, vol. 2, pp. 17-33.

[8] V. Conitzer, and T. Sandholm. Self-Interested Automated Mechanism Design and Implications for Optimal Combinatorial Auctions. *Proc. of 5th EC 2004.*

[9] P. Cramton, Y. Shoham, and R. Steinberg. Combinatorial Auctions. *Book MIT press 2006.*

[10] S. Dobzinski, N. Nisan, and M. Schapira. Truthful Randomized Mechanisms for Combinatorial Auctions. *Proceeding of 38th STOC 2006.*

[11] A. Fiat, A. Goldberg, J. Hartline, and A. Karlin. Competitive Generalized Auctions. *Proceeding of 34th STOC 2002.*

[12] U. Feige, S. Goldwasser, L. Lovasz, S. Safra, M. Szegedy. Interactive Proofs and the Hardness of Approximating Cliques. *Journal of the ACM, 43:268-292, 1996.*

[13] J. Feigenbaum, C. Papadimitriou and S. Shenker. Sharing the Cost of Multicaset Transmissions. *Journal of Computer and System Sciences 63(1), 2001.*

[14] A. Gibbard. Manipulation of Voting Schemes:A General Reslt. *Journal of Econometrica, 41(4):587-601, 1973.*

[15] R. Gonen, and D. Lehmann Optimal Solutions for Multi-Unit Combinatorial Auctions: Branch and Bound Heuristics. *Proc. of EC'00*, pp. 13-20, October 2000.

[16] T. Groves Incentives in teams. *In journal Econometrica 1973*, vol. 41, pp. 617-631.

[17] M. Hajiaghayi, R. Kleinberg, M. Mahdian and D. Parkes. Online Auctions with Re-usable Goods. *Proceeding of 6th EC 2005.*

[18] D. Hochbaum. Approximation Algorithms for NP-hard Problems. *Book PWS Publishing Company 1995.*

[19] D. Lehmann, L. I. O'Callaghan, and Y. Shoham. Truth Revelation in Approximately Efficient Combinatorial Auctions. In *Journal of ACM 2002*, vol. 49(5), pp. 577-602.

[20] R. Lavi, A. Mu'alem and N. Nisan. Towards a Characterization of Truthful Combinatorial Auctions. *Proceeding of 44th FOCS 2003.*

[21] R. Lavi, and N. Nisan. Competetitive Analysis of Incentive Compatible On-Line Auctions. *Proceeding of 2nd EC 2000.*

[22] R. McAfee. A Dominant Strategy Double Auction. *Journal of economic Theory, 56:434-450 1992.*

[23] H. Moulin. The Strategy of Social Choice. *North-Holland 1983.*

[24] A. Mu'alem, and N. Nisan. Truthful Approximation Mechanisms for Restricted Combinatorial Auctions. *Proceeding of AAAI 2002.*

[25] N. Nisan. Algorithms for Selfish Agents. *Proceeding of 16th STACS 1999.*

[26] N. Nisan, A Ronen. Algorithmic Mechanism Design. *Proceeding of 31st STOC 1999 and Journal of Games and Economic Behavior, 35:166-196, 2001.*

[27] N. Nisan, A Ronen. Computationally Feasible VCG mechanisms. *Proceeding of 2nd EC 2000 and Journal of Games and Economic Behavior, 35:166-196, 2001.*

[28] C. Papadimitriou. Algorithms, Games, and the Internet. *Proceeding of 33rd STOC 2001.*

[29] K. Roberts. The Characterization of Implementable Choice Rules. *In J.-J. Laffont, editor, Aggregation and Revelation Of Preferences. Papers presented at the first European Summer Workshop of the Econometric Society, North-Holland 1979.*

[30] M. Satterthwaite. Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorem for Voting Procedures and social Welfare Functions. *Journal of Economic Theory, 10:187-217, 1975.*

[31] W. Vickrey Counterspeculation, Auctions and Competitive Sealed Tenders. *In Journal of Finance 1961*, vol. 16, pp. 8-37.

A Conclusions and future work

In this paper we generalized Roberts theorem to the case of a double-sided auctions. The generalization is achieved by reducing the double-sided auction problem to a variation of a combinatorial auction problem. As a building block for our work we define the new notions of a truth-preserving reduction and a non-affine preserving reduction. These notions introduce the possibility of creating a taxonomy of mechanism design problems similar to the polynomial hierarchy of computational complexity.

We also introduced the notion of payments' extensions which appears to be of independent interest and proved several properties of such extensions.

Our main theorem shows that for every truthful social choice function ϕ there exists a vector of prices supporting the allocation such that ϕ is affine maximizing. The main open question is: Are there other price vectors supporting ϕ that are not of affine maximizing form? If so can they be characterized?. Another interesting question is expanding the open question which we answered from [20] to further generalize Roberts theorem to additional classes of mechanism design problems.

B Setting and Notations

In this paper we characterize the properties of the double sided auction mechanism (DSA). In the DSA problem there are n sellers each willing to sell a single good of a unique type and m buyers potentially interested in every good but willing to buy only one of them. To characterize the properties of the DSA problem we characterize a different problem: the combinatorial auction product space (CAPS). The CAPS has $n*m$ players and $m+n$ different goods where each player is interested in a subset (of size two) of the goods[8].

For clarity we present our basic notations and definitions for mechanisms with n players and later on make use of the basic notations and definitions for different numbers of players and goods. We assume that all of our mechanisms have a finite set of possible outcomes (range of alternatives to choose from) and denote this set by A, where $|A| = l$. Each player z, $1 \leq z \leq n$, assigns a real value $v_z(a)$ to each possible alternative a from A. Namely, $v_z(a)$ is the valuation of player z on an output a. The vector $v_z \in \mathbb{R}^l$ specifies z's preferences on all possible $a \in A$. The set $V_z \subseteq \mathbb{R}^l$ is the set of all possible valuations v_z on all possible $a \in A$ we refer to V_z also as z's domain. The set of all possible valuations of all the players is denoted by $V = V_1 \times ... \times V_n$. Let $v(a) = (v_1(a), ..., v_n(a)) \in \mathbb{R}^n$ be the vector of valuations of all the players on outcome a. Let $v_{-z} = (v_1, ..., v_{z-1}, v_{z+1}, ..., v_n)$ be the vector of valuations of all the players besides player z, and let V_{-z} be the set of all possible vectors v_{-z}.

In this paper we assume players have quasi linear utility namely that player z's utility is $v_z(a) - p_z(v)$ where $p_z(v)$ is the price player z is charged by the mechanism when alternative a is chosen given v as the valuation vector.

As our main theorem extends the main result in [20] we present our allocations in terms of social choice functions as in [20]. In appendix D the necessary definitions from [20] used in our paper can be found.

In order to prove our main theorem we reduce the

[8]To convince the sceptical reader that the CAPS problem can follow [20]'s theorem although it is single minded, we show in section C and 4 that the CAPS problem is equivalent to a multi-minded CA special case which has $2n * m/(n+m)$ players and $m+n$ different goods where each player has a preference over $m+n$ different bundles each of which is of size two.

DSA mechanism to (a variation of) a combinatorial auction. Moreover, we prove that the reduction maintains the necessary properties (in order to use [20]'s main theorem) of the combinatorial auction, i.e., truthfulness and non-affine maximization. To prove that our reduction preserves the desired properties, we need to define new concepts of reduction:

DEFINITION B.1. *A social choice function ϕ is* reducible *to a choice function $\bar{\phi}$, namely, $\phi \le \bar{\phi}$, if ϕ's input can be reduced to $\bar{\phi}$'s input such that the target function of ϕ is optimum if and only if the target function of $\bar{\phi}$ on the reduced input is optimum.*

DEFINITION B.2. Truth Preserving Reduction: *Given mechanisms $\alpha = (\phi, p_1, ..., p_n)$ and $\beta = (\bar{\phi}, \bar{p}_1, ..., \bar{p}_m)$, a reduction $\alpha \le \beta$ is a* truth preserving reduction *if there exist a function $h : \phi \to \bar{\phi}$ such that $\phi \le \bar{\phi}$ and for every $1 \le i \le m$ there exists a function $g_i : \{p_1, ..., p_n\} \to \bar{p}_i$ s.t. if $(\phi, p_1, \ldots, p_n)$ is truthful then $(h(\phi), g_1(p_1 \ldots p_n), \ldots, g_m(p_1 \ldots p_n))$ is truthful.*

DEFINITION B.3. Non-affine maximizing preserving reduction: *Given social choice functions ϕ and $\bar{\phi}$, a reduction $\phi \le \bar{\phi}$ is a* non-affine maximizing preserving reduction *if the following holds: if ϕ is a non-affine maximizing social choice function then $\bar{\phi}$ is a non-affine maximizing social choice function.*

C Problem Definition

In this section we give the formal definitions of the problems we use in our reductions.

Problem 1. Combinatorial Auction product space (CAPS): *Let D be a set of $m * n$ players, and let G be a set of goods of size $m + n$. Namely, $D = \{d_{ji} | 1 \le j \le m, 1 \le i \le n\}$ and $G = \{g_k | 1 \le k \le n + m\}$. For every player $d_{ji} \in D$ there is mapping $f_{ji} : 2^G \to \mathbb{R}^+$. We assume that for all $1 \le i \le n, 1 \le j \le m$ and for all sets $Q \in 2^G$ it holds that if $g_i, g_{j+n} \in Q$ then $f_{ji}(Q) \ge 0$, and if $g_i \notin Q$ or $g_{j+n} \notin Q$ then $f_{ji}(Q) = 0$. We also assume free disposal, i.e., for every $Q \in 2^G$ and every i, j such that $g_i, g_{j+n} \in Q$ it holds that $f_{ji}(Q) = f_{ji}(\{g_i, g_{j+n}\})$.*

Intuitively, each player d_{ji} corresponds to a possible trade between players j, i and the goods are used as indicator functions to ensure that no player buys more than one good and no good is sold more than once. By maximizing the welfare in this auction we will maximize the gain from trade in the original auction.

The $m + n$ goods are allocated to the mn players, namely, players $d_{ji} \in D$ receives a set of goods $G_{ji} \in 2^G$, such that $\sum_{j,i} f_{ji}(G_{ji})$ is maximum.

Since we assume free disposal we can assume w.l.o.g. that either $G_{ji} = \{g_i, g_{j+n}\}$, or $G_{ji} = \phi$.

Note that for each player $d_{ji} \in D$ and any allocation a $v_{ji}(a) = f_{ji}(Q)$, where Q is the set of goods allocated in a to players d_{ji}.

The CAPS problem can be easily translated into a multi-minded version: CAPS-MM.

Problem 2. CAPS-MM:
Let $\tilde{D}$ be a set of m players, and let $\tilde{G}$ be a set of goods of size $m + n$. Namely, $\tilde{D} = \{\tilde{d}_j | 1 \le j \le m\}$ and $\tilde{G} = \{g_k | 1 \le k \le n + m\}$. For every player $\tilde{d}_j \in \tilde{D}$ there is a vector of mappings $F_j = (\tilde{f}_{j1}, ..., \tilde{f}_{jn})$ s.t. $\tilde{f}_{ji} : 2^{\tilde{G}} \to \mathbb{R}^+$ and the following hold: for all sets $Q \in 2^{\tilde{G}}$ if $\tilde{g}_i, \tilde{g}_{j+n} \in Q$ then $\tilde{f}_{ji}(Q) \ge 0$, if $\tilde{g}_i \notin Q$ or $\tilde{g}_{j+n} \notin Q$ then $\tilde{f}_{ji}(Q) = 0$. We also assume free disposal.

The $m + n$ goods are allocated to the m players, namely, player $\tilde{d}_j \in \tilde{D}$ receives a set of goods $\tilde{G}_{ji} \in 2^{\tilde{G}}$ s.t. $\sum_{j,i} \tilde{f}_{ji}(\tilde{G}_{ji}) x_{ji}$ is maximum, under the following constraints:

1. *for all $1 \le i \le n$, $1 \le j \le m$ $x_{ji} \in \{0, 1\}$.*
2. *for all $1 \le j \le m$ $\sum_{i=1}^{n} x_{ji} \le 1$.*

Intuitively, each multi minded player $\tilde{d}_j$ in CAPS-MM corresponds to n players d_{ji} where $1 \le i \le n$ in CAPS. Every bundle player $\tilde{d}_j$ in CAPS-MM values (more than zero) has an equivalent bundle that is valued (more than zero) identically by one of the players d_{ji} where $1 \le i \le n$ in CAPS. The objective function maximize welfare under the constraint that every player can be allocated at most single bundle out of all his desired bundles.

Thus the CAPS and the CAPS-MM are maximizing essentially the same objective function.

Problem 3. DSA:
Let $S = \{s_1, ..., s_n\}$ be a set of sellers each having a single good. Let $G' = \{g'_1, ..., g'_n\}$ be the set of goods, where g'_i denotes the good of seller s_i. For each seller s_i $1 \le i \le n$ there is mapping $f''_i : \{g'_i\} \to \mathbb{R}^+$. Let $B = \{b_1, ..., b_m\}$ be a set of buyers. For each $1 \le j \le m$ there is mapping $f'_j : G' \to \mathbb{R}^+$ meaning that every buyer has a value for every good. Let M be a set of pairs (s_i, b_j), where $s_i \in S$ and $b_j \in B$, such that M is a matching between S and B.

We want to match buyers to sellers such that the gain from trade (social welfare) of the allocation is maximized, i.e., our target function is

$$\max \sum_{i,j | (s_i, b_j) \in M} f'_j(g'_i) - f''_i(g'_i).$$

Note that for all matching M and $1 \leq i \leq n$ $v_i(a) = -f''_i(g'_i)$, where a is the allocation immersed from M. Moreover, for all matching M and $1 \leq j \leq m$ $v_{j+n}(a) = f'_j(g'_i)$, where g'_i is the good of seller s_i that is matched to buyer b_j.

The *DSAC0 problem* is a special case of the DSA problem for which $f'' = 0$.

We will refer to the prices charged by the DSA mechanism to players in set B as buyers' prices and the prices charged by the DSA mechanism to players in set S as sellers' prices.

The intuition behind our reduction construction is that each of the CAPS' players corresponds to a possible trade between a pair of players in the DSA and the goods are used as indicator functions to ensure that no player buys more than one good and no good is sold more than once. By maximizing the welfare in CAPS we maximize the gain from trade in the original DSA.

D Necessary Definitions from LMN:03

Definition D.1. *Social choice function: Social choice function $\phi : V_1 \times ... \times V_n \Rightarrow A$ is a function that gets as input a vector of players' preferences and chooses an alternative among a finite set of possible alternatives A. We assume w.l.o.g that ϕ is onto A.*

Definition D.2. *([20]) Truthfulness: A mechanism $(\phi, p_1, .., p_n)$, where $\phi : V \to A$ and $p_z : V \to \mathbb{R}$ is called truthful if for any player z, any $v_{-z} \in V_{-z}$, and any $v_z, u_z \in V_z$ it holds that $v_z(\phi(v)) - p_z(v) \geq v_z(\phi(u_z, v_{-z})) - p_z(u_z, v_{-z})$. The social choice function ϕ is implementable or simply truthful if there exist some mechanism that implements it.*

Definition D.3. *([20]) Affine maximization: A social choice function ϕ is an affine maximizer if there exist constants $\omega_1, ..., \omega_n \geq 0$ and $\{\gamma_a\}_{a \in A}$ such that for any $v \in V : \phi(v) \in \arg\max_{a \in A}\{\sum_{z=1}^{n} \omega_z v_z(a) + \gamma_a\}$. In this case ϕ is implemented by the prices $p_z = -\omega_z^{-1}(\sum_{k \neq z}^{n} \omega_k v_k(a) + \gamma_a)$.*

Our main theorem shows that the social choice function of any truthful DSA mechanism is an almost affine maximizer, using the same social choice function. The almost affine maximization is an artifact of our use of [20]'s main theorem. [20]'s theorem only shows that the social choice function must be an affine maximizer for large enough input valuations.

Definition D.4. *([20]) Almost Affine maximization: A social choice function ϕ is an almost affine maximizer if there exists a threshold H s.t. the function is an affine maximizer if $v_z(a) \geq H$ for all a and z*

[20] believe that this restriction is a technical artifact of their current proof.

Definition D.5. *Critical price: A mechanism uses a critical price payment scheme if given an allocation it charges players the minimum value they need to report to the mechanism in order to receive the same allocation.*

Definition D.6. *([20]) Order-based domains: Domains where each V_z is defined by a (finite) family of inequalities and equalities of the form $v_z(a) \leq v_z(e)$, $v_z(a) < v_z(e)$, $v_z(a) = v_z(e)$ or $v_z(a) = 0$*

We denote by $R_z(a, e)$ the relation of player z between alternatives a, e, and use $0_z = \{a \in A | v_z(a) = 0\}$.

Definition D.7. *([20]) Top and Bottom Alternatives of Player z: Suppose V_z is order based. The alternative $a \in A \backslash 0_z$ is a top alternative if its value is never smaller than the value of any other alternative. I.e., if for all other $e \in A$, $R_z(a, e) \in \{>, \geq, null\}$. Similarly, the alternative $a \in A$ is a bottom alternative if for all other $e \in A$, $R_z(a, e) \notin \{>, \geq\}$.*

Definition D.8. *([20]) Conflicting preferences: An order based domain has conflicting preferences if:*

1. *Any player z has at least one top alternative (denoted C^z).*
2. *For all z and $k \neq z$, c^k is a bottom alternative for player z, and $C^k \in 0_z$.*

Definition D.9. *([20] definition 8) Non-degenerate range: A is non-degenerate if for any player $z > 1$ there exist $a \in A$ such that $a \notin 0_1$ and $a \notin 0_z$.*

Definition D.10. *([20]) Player decisiveness: ϕ is player decisive if for any $v \in V$ and any player z there exist $u_z(C^z) = v_z(C^z) + \delta$, and for all $e \neq C^z$, $u_z(e) = v_z(e)$ for some $\delta > 0$ such that $\phi(u_z, v_{-z}) = C^z$*

Definition D.11. *([20]) S-MON: A social choice function ϕ satisfies S-MON if for any $v \in V$, player z, and $u_z \in V_z$: $\phi(v) = a$ and $\phi(u_z, v_{-z}) = e \neq a$ imply that $u_z(e) - v_z(e) > u_z(a) - v_z(a)$.*

E CAPS Characterization

Lemma E.1. *The CAPS is an order-based domain*

Proof. Define the CAPS domain by the following set of inequalities: for all $a, e \in A$ such that $a_{ji} = e_{ji}$: $v_{ji}(a) = v_{ji}(e)$ (no externalities); for all $a, e \in A$ such that a_{ji} allocates goods g_i and g_{j+n} to players d_{ji} and $a_{ji} \subseteq e_{ji}$: $v_{ji}(a) = v_{ji}(e)$ (free disposal);for all $a \in A$ such that $a_{ji} = \emptyset$: $v_{ji}(a) = 0$. It follows by the definition of the order-based domains that the CAPS is an order-based domain. □

Lemma E.2. *The CAPS has conflicting preferences*

Proof. According to lemma E.1 CAPS is an order-based domain. The following holds for CAPS:

- Any players d_{ji} in CAPS has a top alternative denoted C^{ji} where all goods for k, t $1 \leq k \leq n$ and $n+1 \leq t \leq n+m$ g_k and g_t are allocated to d_{ji}.
- For all d_{ji} and $j \neq t$, $k \neq i$, C^{tk} is a bottom alternative for players d_{ji}, and $C^{tk} \in 0_{ji}$.

□

Lemma E.3. *The CAPS has non-degenerate range*

At first glance it looks like we can not prove lemma E.3 as the definition of non-degenerate range (definition D.9) taken from [20] requires that the players d_{j1} for *all* $j > 1$ and d_{1i} for all $i > 1$ be allocated their preferred goods when d_{11} is allocated his preferred goods. Since there is only one good g_i and one good g_{j+n} this can not be done. We can easily overcome the problem by adding a dummy good $\tilde{g}_1$ and a dummy player $\tilde{d}_{11}$ where the only good he values more than 0 is $\tilde{g}_1$. The addition of such dummy goods and dummy players does not change any of our other claims. So we prove lemma E.3 by requiring that both j and i are greater than 1.

Proof. As every player d_{ji} has a value greater than 0 only for the pair of goods g_i and g_{j+n}, there exist an allocation a that would allocate the pair of goods g_1 and g_{1+n} to player d_{11} and the pair of g_i and g_{j+n} to some other player d_{ji} where $j \neq 1$ and $i \neq 1$. Thus for any player d_{ji} $j > 1$ and $i > 1$ there exist $a \in A$ such that $a \notin 0_{11}$ and $a \notin 0_{ji}$. □

Lemma E.4. *The CAPS is player decisive*

Proof. Let Q be the set of all goods in CAPS. Any player d_{ji} where $f_{ji}(Q) > \sum_{t,k} f_{tk}(Q) + f_{jk}(Q) + f_{ti}(Q)$ for every $k \neq i$ and $t \neq j$ will be allocated all goods as $f_{ji}(Q)$ must be in any allocation that maximizes $\sum_{ji} fji(G)$ for all j, i. □

Lemma E.5. *The CAPS satisfies S-MON*

Proof. There are two cases to check where the allocation of the CAPS could have been changed by player d_{ji} valuation change:

- Player d_{ji} was allocated his goods g_i and g_{j+n} in allocation a and not allocated any goods in allocation e. It follows that $u_i(a) < v_i(a)$. As player d_{ji} is not allocated goods in e it follows that $u_i(e) = v_i(e) = 0$. Thus $u_i(e) - v_i(e) > u_i(a) - v_i(a)$.
- Player d_{ji} was not allocated goods in allocation a and allocated goods g_i and g_{j+n} in allocation e. It follows that $u_i(a) > v_i(a)$. As player d_{ji} is not allocated goods in a it follows that $u_i(a) = v_i(a) = 0$. Thus $u_i(e) - v_i(e) > u_i(a) - v_i(a)$.

□

F DSAC0 Characterization

In this section we prove that the social choice function of any DSAC0 truthful mechanism is (almost) an affine-maximizer by showing that DSAC0 is reducible to CAPS, using a truthful and non-affine-maximizer preserving reduction.

F.1 Reduction: DSAC0 $\leq$ CAPS

In this subsection we proof that the DSAC0 problem is reducible to the CAPS problem.

Construction 1. *Let n, m in CAPS be the same n, m as in DSAC0. Let $D = \{d_{ji} | s_i \in S, b_j \in B\}$, and let $G = \{g_i | s_i \in S\} \cup \{g_{j+n} | b_j \in B\}$. For each $1 \leq j \leq m$, $1 \leq i \leq n$ let $f_{ji}(\{g_i, g_{j+n}\}) = f'_j(g'_i)$, and for every $Q \subset G$ s.t. $g_i \notin Q$ or $g_{j+n} \notin Q$ it holds that $f_{ji}(Q) = \phi$.*

Definition F.1. *Let $a = \{G_{ji}\}$ be an allocation of the CAPS problem. Then the* induced matching M_0 *of the DSAC0 problem is the following matching:*

$$(s_i, b_j) \in M_0 \Leftrightarrow G_{ji} = \{g_i, g_{j+n}\}.$$

The induced allocation a_0 of DSAC0 is the allocation invokes from the induced matching M_0. In a similar manner, if M_0 is a matching of the DSAC0 problem, then the induced allocation $a = \{G_{ji} | 1 \leq i \leq n, 1 \leq j \leq m\}$ of the CAPS problem is

$$G_{ji} = \{g_i, g_{j+n}\}$$

if $(s_i, b_j) \in M_0$. Otherwise

$$G_{ji} = \phi.$$

Let M_0 be a matching of the DSAC0 problem, then:

Claim F.1. *The matching M_0 maximizes $\sum_{(s_i,b_j)\in M_0} f'_j(g'_i)$ if and only if the induced allocation $a = \{G_{ji} | 1 \leq i \leq n, 1 \leq j \leq m\}$ maximizes $\sum_{j,i} f_{ji}(G_{ji})$.*

Intuitively claim F.1 is true as the allocation of DSAC0, i.e., the matches of buyers and sellers are translated exactly to the equivalent allocation in CAPS. As the sellers valuations is zero and $f'_j(g'_i) = f_{ji}(\{g_i, g_{j+n}\})$ the DSAC0 social welfare is equivalent to the CAPS social welfare and thus the maximum is achieved for the same allocation.

Proof. Assume that M_0 maximizes $\sum_{j,i|(s_i,b_j)\in M_0} f'_j(g'_i)$. Let $a = \{G_{ji}\}$ be its induced allocation. Assume to the contrary that the allocation $a' = \{G'_{ji}\}$ that maximizes $\sum_{j,i} f_{ji}(G'_{ji})$ is not the allocation a induced by M_0. Let M'_0 be the matching induced by the allocation a'. Then

$$\begin{aligned}
\sum_{j,i|(s_i,b_j)\in M'_0} f'_j(g'_i) &= \sum_{j,i|(s_i,b_j)\in M'_0} f_{ji}(\{g_i, g_{j+n}\}) \quad (1)\\
&= \sum_{j,i} f_{ji}(G'_{ji}) > \sum_{j,i} f_{ji}(G_{ji})\\
&= \sum_{j,i|(s_i,b_j)\in M_0} f_{ji}(\{g_i, g_{j+n}\})\\
&= \sum_{j,i|(s_i,b_j)\in M_0} f'_j(g'_i).
\end{aligned}$$

This contradicts the fact that M_0 maximizes the target function of DSAC0.

Let $a = \{G_{ji}\}$ be an allocation of the CAPS problem that maximizes $\sum_{j,i} f_{ji}(G_{ji})$, where f_{ji}, D, G are defined as in the construction. Let M_0 be the induced matching. Assume to the contrary that the matching M'_0 that maximizes the target function of DSAC0 is not the induced matching M_0. Let $a' = \{G'_{ji}\}$ be the induced allocation of M'_0. Then we get that

$$\begin{aligned}
\sum_{j,i} f_{ji}(G'_{ji}) &= \sum_{j,i|(s_i,b_j)\in M'_0} f_{ji}(\{g_i, g_{j+n}\}) \quad (2)\\
&= \sum_{j,i|(s_i,b_j)\in M'_0} f'_j(g'_i)\\
&> \sum_{j,i|(s_i,b_j)\in M_0} f'_j(g'_i)\\
&= \sum_{j,i|(s_i,b_j)\in M_0} f_{ji}(\{g_i, g_{j+n}\})\\
&= \sum_{j,i} f_{ji}(G_{ji}).
\end{aligned}$$

This contradicts the fact that the allocation $a = \{G_{ji}\}$ maximizes the target function of the CAPS problem.

□

F.2 Truthful Non-Affine Maximizer Preserving Reduction

In order to prove that the DSAC0 $\leq$ CAPS reduction is a truthful non-affine maximizing preserving reduction it is left to show that:

- the reduction maintains the truthfulness property for buyers.
- the reduction maintains the non-affine maximization property for buyers.

Let a_0 be an allocation in DSAC0, and let a be its induced allocation in CAPS. Denote by $v_{ji}(a)$ the valuation of the player d_{ji} in allocation a in CAPS. Denote the valuation of buyer b_j (seller s_i) in allocation of the DSAC0 a_0 by $v_j(a_0)$ ($v_i(a_0)$) respectively. Then the *reduced valuations* of the players in the CAPS problem are defined as follows:

Definition F.2. Reduced Valuations of players in the CAPS problem: *for all* $1 \leq i \leq n$,$1 \leq j \leq m$ *the* induced valuation *of player* d_{ji} *is* $v_{j,i}(a) = v_j(a_0)$.

Denote by p_{ji} the price computed for player d_{ji} when telling the truth in an allocation a. Denote by p^{C0}_{ji} buyer b_j's price in the DSAC0 problem when matched to seller s_i. Let M_0 be a maximum matching of the DSAC0 problem and a its induced allocation. Then the *induced prices* of the CAPS players from the prices of the DSAC0 buyers is defined as follows:

Definition F.3. Induced Prices of CAPS players from the prices of DSAC0 buyers: *for every* i, j *s.t.* $(s_i, b_j) \in M_0$ $p_{ji} = p^{C0}_{ji}$. *Otherwise* $p_{ji} = 0$.

Claim F.2. *The DSAC0 $\leq$ CAPS reduction is a truth preserving reduction, using the induced prices.*

Proof. Recall that by definition B.2 we have to prove that given a truthful DSAC0 mechanism we can define a truthful CAPS mechanism, using the reduced input of DSAC0, the reduced valuations, and the induced prices. Assume to the contrary that the described CAPS mechanism is not a truthful mechanism. It follows that there exists a player d_{ji} in the CAPS problem that can report a lie value about a pair of goods G_{ji} and improve his utility. Denote the true value reported by player d_{ji} to the mechanism as $f_{ji}(G_{ji})$ and the lie value as $\hat{f}_{ji}(G_{ji})$. Recall that the price computed for d_{ji} when telling the truth is p_{ji}. Let $\hat{p}_{ji}$ denote the computed price when lying. There are three different cases to consider:

- d_{ji} was allocated goods when telling the truth and allocated goods when lying: if $\hat{f}_{ji}(G_{ji}) > f_{ji}(G_{ji})$ then by the reduction construction it means that $\hat{f}'_j(g'_i) > f'_j(g'_i)$. By our contrary assumption we know that $\hat{p}_{ji} < p_{ji}$ in CAPS. This implies that $\hat{p}^{C0}_{ji} < p^{C0}_{ji}$ for buyers in DSAC0 as $p_{ji} = p^{C0}_{ji}$. It follows that buyer b_j in DSAC0 is better off reporting $\hat{f}'_j(g'_i)$ as $f'_j(g'_i) - \hat{p}^{C0}_{ji} > f'_j(g'_i) - p^{C0}_{ji}$ in contradiction to the DSAC0 truthfulness assumption. Similar arguments can be made for the case $\hat{f}_{ji}(G_{ji}) < f_{ji}(G_{ji})$ as we consider in this case only allocations that did not change by the lie report.

- d_{ji} was allocated goods when telling the truth and not allocated goods when lying. As the player was not allocated goods when lying it means that the DSAC0 could not have given a price for buyer b_j on good g'_i as he was not matched to seller s_i and therefore the player in CAPS could not have possibly improved his utility by lying and losing the allocation of goods.

- d_{ji} was not allocated goods when telling the truth and allocated goods when lying. It follows from the reduction that buyer b_j in DSAC0 when reporting $\hat{f}'_j(g'_i)$ was matched to seller s_i and allocated good g'_i. Since it is given that the DSAC0 is a truthful mechanism $f'_j(g'_i) - \hat{p}^{C0}_{ji} < 0$, otherwise buyer b_j is better off reporting $\hat{f}'_j(g'_i) = \hat{f}_{ji}(G_{ji})$ when his true value is $f'_j(g'_i)$. Since according to the reduction $\hat{p}^{C0}_{ji} = \hat{p}_{ji}$, and the fact that $f'_j(g'_i) < \hat{p}^{C0}_{ji}$, it follows that $f_{ji}(G_{ji}) < \hat{p}_{ji}$ which contradicts the assumption that the player d_{ji} improved his utility by lying and reporting the value $\hat{f}_{ji}(G_{ji})$.

□

Claim F.3. *The DSAC0 $\leq$ CAPS reduction maintains the non-affine maximization property.*

Proof. Recall that by definition B.3 we have to show that, given a non-affine maximizer DSAC0 social choice function, the CAPS social choice function received by the reduction and the reduced valuations is not an affine maximizer.

Assume for a contradiction that the described CAPS social choice function is an affine maximizer. By the affine maximizing social choice function definition it follows that there exist $\omega_1, ..., \omega_m \geq 0$ and $\{\gamma_a\}_{a \in A}$ such that for any vector of valuations of players, the CAPS allocation a is such that $a \in \arg\max_{a \in A}\{\sum_{j=1}^{m} \omega_j v_{ji}(a) + \gamma_a\}$. It follows from the DSAC0 $\leq$ CAPS reduction and the reduced valuations definition that $v_j(a_0) = v_{ji}(a)$. This means that the affine prices of the players in CAPS are the prices of the buyers in DSAC0. Moreover, by the definition of DSAC0 it is known that for all $1 \leq i \leq n$ $f''_i(g'_i) = 0$ for all allocations, so for all $1 \leq i \leq n$ $v_i(a_0) = 0$. Therefore it follows that for any $\omega_{m+1}, .., \omega_{mn} \geq 0$, $\{\gamma_a\}_{a \in A}$, and the $\omega_1, ..., \omega_m$ found for CAPS, it holds that for any vector of valuations in the DSAC0 problem $a_0 \in \arg\max_{a \in A}\{\sum_{j=1}^{n+m} \omega_j v_j(a) + \gamma_a\}$. But this means that the described DSAC0 social choice function is an affine maximizer social choice function, contradicting the negation assumption. □

Lemma F.1. *The social choice function of any truthful DSAC0 mechanism is an almost affine maximizer.*

The Lemma immediately follows from Lemma 4.1, Claim F.2, and Claim F.3: Let $\alpha = (\phi, \vec{p}^{C0}_{ji})$ be a truthful mechanism for DSAC0. Let $\beta = (\bar{\phi}, \vec{p}_{ji})$ be the mechanism received from the DSAC0 $\leq$ CAPS reduction and the induced prices. From Claim F.2 it follows that β is truthful. Therefore, according to Lemma 4.1, $\bar{\phi}$ is almost affine maximizer. Assume to the contrary that ϕ is an almost non-affine maximizer social choice function. Then according to Claim F.3 $\bar{\phi}$ is almost non-affine maximizer - a contradiction. Thus ϕ is an almost affine maximizer social choice function.

G DSA Characterization for Buyers

In this section we proof that the social choice function of any DSA truthful mechanism is an affine-maximizer for buyers by showing that DSA is reducible to DSAC0, using a truthful and non-affine-maximizer preserving reduction.

G.1 Reduction: DSA $\leq$ DSAC0

In this subsection we prove that DSA is reducible to DSAC0.

Construction 2. *Let S^+ be the set of sellers in the DSA with $f_i''(g_i') > 0$. We use " $-$ " notation for the reduced input parameters. Then $\bar{S} = S$, $\bar{G} = G'$, and $\bar{B} = B \cup \{b_{m+1}, ..., b_{|S^+|+m+1}\}$. In addition, for every $1 \leq i \leq n$, for all $1 \leq j \leq m$ $\bar{f}_j'(g_i') = f_j'(g_i')$, and for all $m+1 \leq j \leq |S^+|+m+1$, $\bar{f}_j'(g_i') = f_i''(g_i')$.*

The intuition behind construction 2 is to turn every seller s_i with valuation greater than 0 for his good g_i' in the DSA to both a buyer with the same valuation for g_i' and to a seller who has the good g_i' but who has a valuation of 0 for the good in the DSAC0.

Definition G.1. Induced Matching: *Let M be a matching of the input of the DSA problem. Then the* induced matching *M_0 of the DSAC0 problem is the following matching:*
$(s_i, b_j) \in M_0 \iff (s_i, b_j) \in M$ *or* ($j \geq m+1$ *and* $\bar{f}_j'(g_i') > 0$)

Let M_0 be a matching of the reduced input of the DSAC0 problem. Then the induced matching M of the DSA problem is defined in the following manner:
$(s_i, b_j) \in M \iff (s_i, b_j) \in M_0$ *and* $j \leq m$

Claim G.1. *M is a maximum matching of the DSA problem if and only if its induced matching M_0 is a maximum matching of the reduced input of the DSAC0 problem.*

Proof. Let M be a matching that maximizes the gain from trade (and hence the welfare) in the DSA, i.e., maximizes $\sum_{(s_i,b_j)\in M} f_j'(g_i') - f_i''(g_i')$. Assume to the contrary that the induced matching M_0 does not maximize
$\sum_{(s_i,b_j)\in M_0} \bar{f}_j'(g_i')$. Let M_0' be a matching of the reduced input of the DSAC0 problem that maximizes $\sum_{(s_i,b_j)\in M_0'} \bar{f}_j'(g_i')$. Let M' be the induced matching of M_0'. Then it holds that

$$\begin{aligned}
\sum_{(s_i,b_j)\in M'} f_j'(g_i') - f_i''(g_i') &\underset{*}{=} \sum_{(s_i,b_j)\in M_0'} \bar{f}_j'(g_i') - \sum_i f_i''(g_i') \\
&> \sum_{(s_i,b_j)\in M_0} \bar{f}_j'(g_i') - \sum_i f_i''(g_i') \\
&= \sum_{(s_i,b_j)\in M} f_j'(g_i') - f_i''(g_i')
\end{aligned}$$

Note that $(*)$ is true since if M is a matching of DSA and M_0 its induced matching then for all s_i s.t $f_i''(g_i') > 0$ there exists j s.t $(s_i, b_j) \in M_0$.

This contradicts the fact that M is a maximum matching.

Let M_0 be a matching s.t $\sum_{(s_i,b_j)\in M_0} \bar{f}_j'(g_i')$ is maximum. Assume to the contrary that the induced matching M of M_0 does not maximize $\sum_{(s_i,b_j)\in M} f_j'(g_i') - f_i''(g_i')$. Let M' be a matching of DSA that does maximize $\sum_{(s_i,b_j)\in M'} f_j'(g_i') - f_i''(g_i')$. Let M_0' be the induced matching of M'. Then it holds that

$$\begin{aligned}
\sum_{(s_i,b_j)\in M_0'} \bar{f}_j'(g_i') &= \sum_{(s_i,b_j)\in M'} \bar{f}_j'(g_i') + \sum_{(s_i,b_j)\in M_0'\setminus M'} \bar{f}_j'(g_i') \\
&= \sum_{(s_i,b_j)\in M'} f_j'(g_i') - f_i''(g_i') + \sum_i f_i''(g_i') \\
&> \sum_{(s_i,b_j)\in M} f_j'(g_i') - f_i''(g_i') + \sum_i f_i''(g_i') \\
&= \sum_{(s_i,b_j)\in M} \bar{f}_j'(g_i') + \sum_{(s_i,b_j)\in M_0\setminus M} \bar{f}_j'(g_i') \\
&= \sum_{(s_i,b_j)\in M_0} \bar{f}_j'(g_i')
\end{aligned}$$

This contradicts the fact that M_0 is a maximum matching.

□

G.2 Truthful Non-Affine Maximizer Preserving Reduction

The DSA $\leq$ DSAC0 reduction defines how to find the maximizing gain from trade allocation for a DSA problem using a DSAC0 mechanism. In order to construct a complete solution we also need to compute the players prices in the DSAC0 given the allocation and buyer prices produced by the DSA.

Let $v^{C0}(a_0)$ be the valuation vector of the DSAC0 mechanism in allocation a_0, and let $v(a)$ be the valuation vector of the DSA mechanism in allocation a. Let M be a maximum matching of the DSA problem and let M_0 be its induced matching. Let a be the allocation immersed from M, and let a_0 be the allocation immersed from M_0.

Definition G.2. Reduced Valuations of DSAC0 buyers from the valuations of DSA buyers: *For all $1 \leq j \leq m$ $v_j^{C0}(a_0) = v_j(a)$.*

Denote by p_{ji}^D buyer b_j's price in the DSA when matched to seller s_i. Recall that p_{ji}^{C0} denotes buyer b_j's price in the DSAC0 when matched to seller s_i.

Definition G.3. Induced Prices of DSAC0 buyers from the prices of DSA buyers: *for every buyer* $b_j \in B$ *s.t there exists* i *s.t.*$(s_i, b_j) \in M$ *we define* $p^{C0}_{ji} = p^{D}_{ji}$. *For all others buyers* $b_j \in B$ $p^{C0}_{ji} = 0$.

The claims showing the truthfulness property and the non-affine property maintaining by the reduction follow:

Claim G.2. *The DSA* $\leq$ *DSAC0 is a truth preserving reduction for buyers.*

Proof. Recall that by definition B.2 we have to prove that given a truthful DSA mechanism for buyers we can define a truthful DSAC0 mechanism for buyers, using the reduced input of DSA, the reduced valuations, and the induced prices.

Assume to the contrary that the reduction does not maintain truthfulness for buyers. It follows that there exists a buyer b_j in the DSAC0 that, given the induced prices, can report a false value for one of the goods and improve his utility. Assume that $\bar{f}'_j(g'_i)$ is the true value reported by buyer b_j on good g'_i, and denote by $\hat{\bar{f}}'_j(g'_i)$ the false value. Denote by p^{C0}_{ji} the induced price of buyer b_j when *telling the truth* and winning good g'_i, and denote by $\hat{p}^{C0}_{ji}$ the induced price of buyer b_j when *lying* and winning good g'_i. There are four different cases to consider:

- Buyer b_j is matched with seller s_i when telling the truth and when lying: assume that $\hat{\bar{f}}'_j(g'_i) > \bar{f}'_j(g'_i)$ in DSAC0. Then by the reduction construction for $1 \leq j \leq m$ $\hat{f}'_j(g'_i) > f'_j(g'_i)$ in DSA . By our contrary assumption we know that $\hat{p}^{C0}_{ji} < p^{C0}_{ji}$ in DSAC0. This implies that $\hat{p}^{D}_{ji} < p^{D}_{ji}$ as $p^{C0}_{ji} = p^{D}_{ji}$. It follows that buyer b_j in DSA is better off reporting $\hat{f}'_j(g'_i)$ as $f'_j(g'_i) - \hat{p}^{D}_{ji} > f'_j(g'_i) - p^{D}_{ji}$ in contradiction to the DSA truthfulness assumption. A similar argument can be made for the case $\hat{\bar{f}}'_j(g'_i) < \bar{f}'_j(g'_i)$ as we consider in this case only allocations that are not changed by the false report. For the case where $m+1 \leq j \leq |S^+|+m+1$ even if buyer b_j is lying he can not change his price as $p^{C0}_{ji} = 0$ by the reduction construction and therefore can not improve his utility in DSAC0.

- Buyer b_j is matched with seller s_i when telling the truth and with seller s_k, $k \neq i$ when lying: it must be the case that $\hat{\bar{f}}'_j(g'_i) < \bar{f}'_j(g'_i)$ in DSA and that $1 \leq j \leq m$ (buyers j where $m+1 \leq j \leq |S^+|+m+1$ all values the other goods 0). Therefore, by the reduction construction, for $1 \leq j \leq m$ $\hat{f}'_j(g'_i) < f'_j(g'_i)$ in DSA. By our contrary assumption we know that $\bar{f}'_j(g'_k) - \hat{p}^{C0}_{jk} > \bar{f}'_j(g'_i) - p^{C0}_{ji}$ in DSAC0. This implies that $f'_j(g'_k) - \hat{p}^{D}_{jk} > f'_j(g'_i) - p^{D}_{ji}$ as $p^{C0}_{jk} = p^{D}_{jk}$ and $p^{C0}_{ji} = p^{D}_{ji}$. It follows that buyer b_j in DSA is better off reporting $\hat{f}'_j(g'_i)$ in contradiction to the DSA truthfulness assumption.

- Buyer b_j is matched with seller s_i when telling the truth and not matched when lying: as buyer b_j is not matched when lying it means that the DSA could not have given a price for buyer b_j on good g'_i. Therefore buyer b_j in DSAC0 could not have possibly improved his utility by lying and losing the match.

- Buyer b_j is not matched when telling the truth and matched when lying: if $1 \leq j \leq m$ it follows from the reduction that buyer b_j in DSA when reporting $\hat{f}'_j(g'_i)$ was matched to seller s_i and allocated good g'_i. Since it is given that the DSA is a truthful mechanism $f'_j(g'_i) - \hat{p}^{D}_{ji} < 0$, Otherwise buyer b_j is better off reporting $\hat{f}'_j(g'_i) = \hat{\bar{f}}'_j(g'_i)$ when his true value is $f'_j(g'_i)$. Since according to the reduction $\hat{p}^{D}_{ji} = \hat{p}^{C0}_{ji}$ and the fact that $f'_j(g'_i) < \hat{p}^{D}_{ji}$, it follows that $\bar{f}'_j(g'_i) < \hat{p}^{C0}_{ji}$ which contradict the counter assumption that buyer b_j improved his utility by lying and reporting the value $\hat{\bar{f}}'_j(g'_i)$. For the case where $m+1 \leq j \leq |S^+|+m+1$ even if buyer b_j is lying he can not change his price as $p^{C0}_{ji} = 0$ by the reduction construction and therefore can not improve his utility in DSAC0.

□

Claim G.3. *The DSA* $\leq$ *DSAC0 reduction maintains the non-affine maximization property for buyers.*

Proof. Recall that by definition B.3 we have to show that, given a non-affine maximizer DSA social choice function, the DSAC0 social choice function received by the reduction and the reduced valuations is non-affine maximizer.

Assume to the contrary that the described DSAC0 social choice function is an affine maximizer for buyers. Let $v = \{v_1, ..., v_m\}$ be a set of valuations in the DSA mechanism for an allocation a that is immersed from a maximum matching M. We have to prove that there exist $\omega_1, ..., \omega_m \geq 0$ and $\{\gamma_a\}_{a \in A}$ such that $a \in \arg\max_{a \in A}\{\sum_{j=1}^{m} \omega_j v_j(a) + \gamma_a\}$. Let $\bar{a}$ be the allocation immersed from the induced matching M_0. Then, since the DSAC0 mechanism is an affine maximizer mechanism for buyers, there exist $\omega_1, ..., \omega_{m+|S^+|} \geq 0$ and $\{\gamma_a\}_{a \in A}$ such that $\bar{a} \in \arg\max_{a \in A}\{\sum_{j=1}^{m+|S^+|} \omega_j v_j^{C0}(a) + \gamma_a\}$. Since $v_j(a) = v_j^{C0}(\bar{a})$, using the same $\omega_1, ..., \omega_m$ and $\{\gamma_a\}_{a \in A}$ we get that $a \in \arg\max_{a \in A}\{\sum_{j=1}^{m} \omega_j v_j(a) + \gamma_a\}$. (Otherwise there exist $\bar{\omega}_1, ..., \bar{\omega}_m$, $\{\bar{\gamma}_a\}_{a \in A}$, and a' such that $a' \in \arg\max_{a \in A}\{\sum_{j=1}^{m} \bar{\omega}_j v_j(a) + \bar{\gamma}_a\}$. Therefore $\sum_{j=1}^{m} \bar{\omega}_j v_j^{C0}(a) + \bar{\gamma}_a + \sum_{j=m+1}^{m+|S^+|} \omega_j v_j^{C0}(a) + \gamma_a > \sum_{j=1}^{m+|S^+|} \omega_j v_j^{C0}(a) + \gamma_a$ contradicting the fact that $a \in \arg\max_{a \in A}\{\sum_{j=1}^{m+|S^+|} \omega_j v_j^{C0}(a) + \gamma_a\}$). This contradicts the fact that our DSA social choice function is a non-affine maximizer for buyers. □

Lemma G.1. *The social choice function of any truthful DSA mechanism for buyers is an almost affine maximizer.*

The Lemma immediately follows from Lemma F.1, Claim G.2, and Claim G.3: Let $\alpha = (\phi, \vec{p}_{ji}^{D})$ be a truthful mechanism for DSA. Let $\beta = (\bar{\phi}, \vec{p}_{ji}^{C0})$ be the mechanism received from the DSA $\leq$ DSAC0 reduction and the induced prices. From Claim G.2 it follows that β is truthful. Therefore, according to Lemma F.1, $\bar{\phi}$ is an almost affine maximizer. Assume to the contrary that ϕ is an almost non-affine maximizer social choice function. Then according to Claim G.3 $\bar{\phi}$ is an almost non-affine maximizer - a contradiction. Thus ϕ is an almost affine maximizer social choice function.

H Proof of Theorem 4.1

Construction 3. *Let m, n in CAPS-MM be the same as in CAPS. Let $\tilde{D} = \{\tilde{d}_j | 1 \leq j \leq m\}$. Let $\tilde{G} = G$. For all $1 \leq i \leq n$, $1 \leq j \leq m$ let $\tilde{f}_{ji} = f_{ji}$.*

Definition H.1. *Let $a = \{G_{ji}\}$ be an allocation of the CAPS problem. Then the* induced *allocation $\tilde{a}$ of CAPS-MM is the following allocation:*

$$\tilde{a} = \{\tilde{G}_{ji} | G_{ji} \in a\}.$$

Obviously, by the reduction's construction we get the following claim:

Claim H.1. *An allocation a maximizes $\sum_{j,i} f_{ji}(G_{ji})$ if and only if its induced allocation $\tilde{a}$ maximizes $\sum_{j,i} \tilde{f}_{ji}(\tilde{G}_{ji}) x_{ji}$.*

Construction 4. *Let m, n in CAPS be the same as in CAPS-MM. Let $D = \{d_{ji} | 1 \leq j \leq m, 1 \leq i \leq n\}$. Let $G = \tilde{G}$. For all $1 \leq i \leq n$, $1 \leq j \leq m$ let $f_{ji} = \tilde{f}_{ji}$.*

Definition H.2. *Let $\tilde{a} = \{\tilde{G}_{ji}\}$ be an allocation of the CAPS-MM problem. Then the* induced *allocation a of CAPS is the following allocation:*

$$a = \{G_{ji} | \tilde{G}_{ji} \in \tilde{a}\}.$$

Obviously, by the reduction's construction we get the following claim:

Claim H.2. *An allocation $\tilde{a}$ maximizes $\sum_{j,i} \tilde{f}_{ji}(\tilde{G}_{ji}) x_{ji}$ if and only if its induced allocation a maximizes $\sum_{j,i} f_{ji}(G_{ji})$.*

Theorem 4.1 immediately follows from claims H.2 and H.1.

I Proof of Main Theorem

In this section we conclude the main theorem using Lemma G.1 and Corollary J.2 from previous sections, and prove two new Lemmas that integrate the truthful and non-affine properties maintained for buyer and sellers separately into truthful and non-affine properties maintained for all players.

Lemma I.1. *Given two disjoint sets of players B and S and an allocation a where each set has a price vector $\vec{p}^B$ and $\vec{p}^S$ supporting allocation a such that mechanism μ^B with allocation a is truthful for players in B under $\vec{p}^B$ and mechanism μ^S with allocation a is truthful for players in S under $\vec{p}^S$, then the mechanism μ with allocation a is truthful for players in $B \cup S$ under the price vector which is the concatenation of the two price's vectors.*

Proof. Assume to the contrary that the mechanism μ is not truthful for players in $B \cup S$ under the union of the two price vectors. Then there exist a player in $B \cup S$ denoted i that is better off reporting a lie value to the mechanism. Assume w.l.o.g that $i \in B$. I.e., there exist allocation such that $v_i(e) - \hat{p}_i^B \geq v_i(a) - p_i^B$, thus it means that i was better of lying also in μ^B contradicting our assumption that i is telling the truth in μ^B.

□

Lemma I.2. *Given two disjoint sets of players B and S and two affine maximizing social choice functions $\phi^B : V_B \Rightarrow A$, $\phi^S : V_S \Rightarrow A$, then the social choice function $\phi : V_B \times V_S \Rightarrow A$ is affine maximizing for players in $B \cup S$.*

Proof. Since ϕ^B is affine maximizing then there exist constants $\omega_1^B, ..., \omega_m^B$ and $\{\gamma_a^B\}_{a \in A}$ s.t for all $v^B \in V^B$

$\phi^B(v^B) \in \arg\max_{a\in A}\{\sum_{z=1}^{m} \omega_z^B v_z^B(a) + \gamma_a^B\}$. Since ϕ^S is affine maximizing then there exist constants $\omega_1^S, ..., \omega_n^S$ and $\{\gamma_a^S\}_{a\in A}$ s.t for all $v^S \in V^S$,
$\phi^S(v^S) \in \arg\max_{a\in A}\{\sum_{z=1}^{n} \omega_z^S v_z^S(a) + \gamma_a^S\}$. Therefore if we choose $\omega_z = \omega_z^S$ for $1 \le z \le n$, and $\omega_{z+n} = \omega_z^B$ for $1 \le z \le m$, and $\{\gamma_a\}_{a\in A} = \{\gamma_a^B + \gamma_a^S\}_{a\in A}$, then for any $v \in V_B \times V_S : \phi(v) \in \arg\max_{a\in A}\{\sum_{z=1}^{n+m} \omega_z v_z(a) + \gamma_a\}$. Thus ϕ is affine maximizing. □

THEOREM I.1. *Main theorem: the social choice function of any truthful DSA mechanism is an almost affine maximizer.*

Proof. of main theorem: The proof immediately follows from Corollary J.2, Lemma G.1, Lemma I.1, and Lemma I.2.

□

J DSA Characterization for Sellers

In this section we sketch the proof that the social choice function of any truthful DSA mechanism for sellers is an almost affine maximizer.

Our sellers are a special case of a more general class of players:

Definition: 7.1 *single-value-player:* A single value player is a player that a single value determines his valuation. I.e., for a single value player j, good g_i and allocation a where $g_i \in a_j$, for all allocations e, $v_j(a) = v_j(e)$ if $g_i \in e_j$ and $v_j(e) = 0$ otherwise.

We first prove a generalization of the folk theorem that states that critical price payment scheme used for single value players is truthful. We then conclude that a critical price payment scheme for sellers in the DSA problem is truthful as sellers are single value players.

It is important to note that sellers are inherently single value players (their cost). Even if our DSA model allowed seller to sell multiple goods, the sellers are still single value player as there are no dependencies between the different goods. Therefore, a seller with multiple goods for sale can be viewed as multiple sellers with a single good for sale. On the other hand the buyers in the DSA are *not* single value players as they are interested in only one good out of number of goods which they value.

LEMMA J.1. *Any monotonic mechanism when applying critical price payment scheme to single value players is a truthful mechanism.*

Proof. Given a mechanism μ with n single value players each with valuation function $v_j(g_i)$ where g_i is some good. Assume to the contrary that when applying critical price payment scheme to the players in μ one of the players j can benefit by reporting a false value for his desired good g_i. Denote player j's true value for good g_i as $v_j(g_i)$ and his reported lie value for the good as $\hat{v}_j(g_i)$[9]. Let the critical price of player j charged by the mechanism μ on allocating good g_i denoted as p_j. There are several cases to consider.

- player j is allocated g_i when reporting the truth $v_j(g_i)$ and when reporting a lie $\hat{v}_j(g_i)$. We need to show that $v_j(g_i) - p_j \ge v_j(g_i) - \hat{p}_j$. As the mechanism μ allocates to player j a good g_i[10] both when reporting the true value and when reporting a lie $p_j = \hat{p}_j$ as p_j is a critical value.

- player j is allocated g_i when reporting the truth meaning $v_j(g_i)$ and is not allocated any good when reporting $\hat{v}_j(g_i)$. It needs to be shown that $v_j(g_i) - p_j \ge 0$ which is true by the critical price definition.

- player j is not allocated any good when reporting the truth meaning $v_j(g_i)$ and is allocated g_i when reporting $\hat{v}_j(g_i)$. We need to show that $0 \ge v_j(g_i) - \hat{p}_j$. Since $\hat{p}_j$ is the minimum value player j had to report in order to be allocated good g_i by the critical price definition and since player j was not allocated any good when reporting $v_j(g_i)$ it follows that $\hat{p}_j > v_j(g_i)$.

□

COROLLARY J.1. *Any monotonic DSA mechanism when applying critical price payment scheme to the sellers is a truthful mechanism for sellers.*

Proof. Recall that the sellers of the DSA problem are single value players, so the Corollary immediately follows from Lemma J.1.

LEMMA J.2. *The social choice function of any truthful DSA mechanism with critical value prices to sellers is an affine maximizer for sellers.*

Proof. Fix seller s_i and his reported value $f_i''(g_i')$. First we want to show that a critical value exists for $f_i''(g_i')$. Since truthful mechanisms are monotone nondecreasing in the reported value of s_i. Meaning that if s_i is

[9] Note that if there are multiple goods valued at the same price we can look at the good allocated
[10] or another good of same value

matched in the optimal allocation a when reporting $f_i''(g_i')$, then by reporting $\hat{f}_i''(g_i')$ where $\hat{f}_i''(g_i') < f_i''(g_i')$ s_i does not decrease his chance to be matched in a. Thus there exists a critical value $\tilde{f}_i''(g_i')$ from which s_i is matched in a and any value $\hat{f}_i''(g_i') > \tilde{f}_i''(g_i')$ will result in s_i not being matched in a. It is left to prove that there exist constants $\omega_1, ..., \omega_{m+n} \geq 0$ and $\{\gamma_a\}_{a \in A}$ such that $\tilde{f}_i''(g_i') = -\omega_i^{-1}(\sum_{k \neq i}^{n} \omega_k f_k''(g_k') + \sum_{(s_k, b_t \in a)} \omega_t f_t'(g_k') + \gamma_a)$.

Let a^* be the optimal allocation such that the matched pair $(s_i, b_j) \in a^*$. Let a_1 and a_2 be two allocations of the mechanism achieving W_{a_1} and W_{a_2} gain from trade respectively. Let a_1 be such that it is identical to a^* but without the pair (s_i, b_j), and let b_j's value $f_j'(g_i')$. Let a_2 be such that s_i is not matched in it and $W_{a_1} - \tilde{f}_i''(g_i') + f_j'(g_i') = W_{a_2}$. It follows that $\tilde{f}_i''(g_i') = W_{a_1} - W_{a_2} + f_j'(g_i')$. All is left to be shown is that $W_{a_1} - W_{a_2} + f_j'(g_i')$ is affine maximizing. As for all $1 \leq i \leq n + m$ according to definition D.3 $\omega_i \geq 0$ we need to show that there are no positive values in W_{a_2} that are not in W_{a_1} and no negative values in W_{a_1} that are not in W_{a_2} to insure affine maximization in $W_{a_1} - W_{a_2}$. Since we consider gain from trade of a DSA problem, W_{a_1} and W_{a_2} are composed of sellers' values in a negative sign and buyers' values in a positive sign. So we need to insure that there does not exist buyers that are matched in a_2 that are not matched in a_1 and sellers that are matched in a_1 that are not matched in a_2.

There are two cases to consider:

- if buyer b_j is not matched in allocation a_2: In that case both a_1 and a_2 do not match buyer b_j and do not match seller s_i. Assume to the contrary that a_1 and a_2 matches different pairs. As a_1 is identical to the optimal allocation a^* except for not matching buyer b_j and seller s_i there exist a value $\tilde{\tilde{f}}_i''(g_i') < \tilde{f}_i''(g_i')$ and an allocation a_3 that does not matches buyer b_j and does not matches seller s_i such that $W_{a_1} - \tilde{\tilde{f}}_i''(g_i') + f_j'(g_i') = W_{a_3} > W_{a_2}$. It follows that $\tilde{\tilde{f}}_i''(g_i')$ is s_i's critical value contradicting the value $\tilde{f}_i''(g_i')$ as s_i critical value. Thus a_1 and a_2 match the same pairs and all buyers matched in a_2 are matched in a_1 and all sellers matched in a_1 are matched in a_2.

- if buyer b_j is matched in allocation a_2: Let G be the bipartite graph where the players on one side of G represent the buyers and the players on the other side represent the sellers. The potential allocations represented by bipartite matchings in G. It will be convenient for the proof to think of the edges that belong to each of the matchings as colored with a specific color representing this matching.

  Assign color 1 to the edges in the allocation matching a_1 and assign color 2 to the edges in the allocation matching a_2.

  Define an alternating path P starting at b_j. Let s_1 be the seller matched to b_j in a_2. Let b_1 be the buyer matched to s_1 in a_1, s_2 be the seller matched to b_1 in a_2, b_2 be the buyer matched to s_2 in a_1, and so on. This defines an alternating path P, starting at b_j, whose edges' colors alternate between colors 1 and 2 (starting with 2). This path ends either with a seller who is not matched in a_1 or with a buyer who is not matched in a_2. So all buyers on the path P are matched in a_1 except for b_j and all sellers on the path P are matched in a_2. As $\tilde{f}_i''(g_i') = W_{a_1} - W_{a_2} + f_j'(g_i')$ the negative value of b_j, $f_j'(g_i')$ in W_{a_2} is reduced. It is also concluded that on the path there are no negative values of sellers in a_1 that are not in a_2. Similarly to the first case any matching outside of path P should be identical between a_1 and a_2 otherwise there exist matching $W_{a_3} > W_{a_2}$ such that a_3 matching is identical to a_1 matching outside path P and s_i has a lower critical value.

Thus for $\omega_i = 1$ $\gamma_{a_1} = 0$, γ_{a_2} and any $\omega_k \geq 0$ $k \neq i$ the critical value of player s_i can be expressed as $-\omega_i^{-1}(\sum_{k \neq i}^{n} \omega_k f_k''(g_k') + \sum_{(s_k, b_t) \in a_1} \omega_t f_t'(g_k')$
$+ \sum_{(s_k, b_t) \in a_2} \omega_t f_t'(g_k'))$. □

COROLLARY J.2. *The social choice function of any truthful monotonic DSA mechanism is an affine maximizer for sellers.*

Proof. The proof immediately follows from Corollary J.1 and Lemma J.2.