

Bribing and Signaling in Auctions

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Abstract

I study a 2-bidder IPV second-price auction which is preceded by two rounds of alternating bribe offers, where each player can, in his turn, offer his rival a bribe in exchange for the latter's abstention. If both offers are rejected, then the auction is played noncooperatively. I obtain a simple necessary and sufficient condition for the existence of monotonic and bribery-involving equilibria that lead to efficient outcomes. When such equilibria exist, they all give rise to the same path of play. This path can be supported in an *essentially perfect Bayesian equilibrium* (EPBE) when the auction format is first-price. When the first-price format is preceded by a single ("take-it-or-leave-it") round, only trivial monotonic EPBEs exist, where there is no bribing and the auction is played noncooperatively with probability 1. Finally, I obtain a preliminary result for auctions which are preceded by alternating bribe-offers protocols of an arbitrary length. This work develops a line of research which was pioneered by Esö and Schummer (2004), who studied the second-price single-round game.

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1 Introduction

Collusion among participants in auctions is a serious and well-documented problem.¹ In the simplest collusive scenario, the cartel members meet prior to the auction in order to decide on side-payments and on a representative bidder, who will bid in the auction on behalf of the cartel. Thus, the very first question an economist needs to address when modeling such scenarios is: *What is it that the cartel members do before the auction? What is the pre-auction interaction?*

In the theoretical literature, it is typically assumed that the cartel members play a revelation game: each member reports a valuation, and the collusive agreement is determined as a function of the profile of reports (and possibly further information, such as the behavior of non-cartel-members).² This is the natural approach to take if one is interested in the allocations and payoffs that can be achieved, since, by the revelation principle (see Myerson (1981)), anything that can be achieved via some incentive-compatible mechanism can also be achieved through direct revelation.

More precisely, if we let \mathbb{M} denote the set of all possible mechanisms, the revelation principle is stated as follows: Given a mechanism $M \in \mathbb{M}$ and a Bayesian Nash equilibrium (BNE) of this mechanism, e , there exists a direct revelation mechanism, call it $D(M, e)$, in which truth-telling constitutes an equilibrium which is outcome-equivalent to e . In particular, any outcome that can be achieved through a BNE-play of some mechanism, can also be achieved through direct revelation.

The revelation approach, however, suffers from several drawbacks. First, it assumes that the cartel members can employ any mechanism in \mathbb{M} . It may be the case, however, that they are restricted to some strict subset of \mathbb{M} . In fact, there is a wide range of reasons because of which the bidders may be restricted to use a subset of *admissible mechanisms*—social norms, complexity considerations, habits formed through a common history, and more. Consider then a direct revelation mechanism D which has a truthful equilibrium. There is no guarantee that there exists an admissible mechanism with an equilibrium which is outcome-equivalent

¹See, for example, Baldwin et al. (1997), Cassady (1967), and Porter and Zona (1993).

²Leading examples include Graham and Marshall (1987), Mailath and Zemsky (1991), Marshall and Marx (2007), and McAfee and McMillan (1992).

to D 's equilibrium. Consequently, direct revelation loses its bite: there may be outcomes that are achievable through direct revelation, but not achievable in admissible mechanisms.

The second drawback stems from the fact that solution concept utilized in the revelation principle is BNE. To see the problem, consider again the direct revelation mechanism from the previous paragraph, D . Suppose further that we knew (from whatever reason) that there exists an admissible mechanism M and a BNE of M , e , which gives rise to the same outcome as D . Still, this does not mean that D 's outcome is “truly achievable,” because it may be the case that BNE is not the appropriate solution concept for the mechanism M . If M is a sequential mechanism, for example, then it may be the case that an appealing solution for the game it induces involves a refinement. In particular, e may be an “unreasonable” equilibrium of M .

Lastly, there is the issue of equilibrium existence. Suppose that we are interested in a mechanism M and in a certain refinement of BNE, for which there are no general existence results. We then need to study M in order to elicit the conditions under which it has the desired equilibrium, an analysis which may be of an independent interest.

These drawbacks lead me to take an alternative approach to the revelation approach, and consider the following games:

- (1) A 2-bidder first-price auction with independent and private values (IPV) is preceded by a single “take-it-or-leave-it” (TIOLI) stage: player 1 offers player 2 a bribe, which player 2 can either accept or reject; if he accepts it, he drops out of the auction, while if he rejects it, the pre-auction stage ends and the players go on to compete in the auction noncooperatively.
- (2) As in (1) above, except that when player 2 rejects an offer, he can counter it with an offer of his own, and now player 1 needs to decide whether to accept or reject it; the auction is reached if and only if both offers are rejected.
- (3) As in (2) above, but with a second-price auction.

These games are inspired by, and continue a line of research that was pioneered by, Esö and Schummer (2004, henceforth ES), who studied the second-price TIOLI game. Considering this class of simple bribing games, I address the following issues:

- The relationship between bribing and efficiency: Is it the case that bribing excludes Pareto-efficiency, or are there circumstances under which they can coexist in equilibrium?
- The relationship between bribing and revenue: How does bribing affects the seller's revenue?
- The significance of the auction format: How do the answers to the above-mentioned questions depend on whether the auction format is first- or second-price?

Finally, I consider auctions which are preceded by alternating-offers protocols of an arbitrary length and obtain a preliminary result for these games. A more comprehensive analysis is beyond the scope of this paper, as these games are virtually intractable.³

1.1 Summary of the results

In a bribing game, the amount a briber offers depends on his valuation (type); his behavior when he bribes is summarized by a *bribing function*, defined on his type-space. Considering the second-price TIOLI game, ES derived the unique *bribery-involving* (or *non-trivial*) equilibrium in which the bribing function is continuous. In this equilibrium, inefficiency results with a positive probability, because all the types of the briber above a certain threshold offer the same bribe, which is accepted by all the types of the respondent. The reason for this “nonseparation at the top” is that it is enough for a briber to signal that he is “sufficiently strong” in order to make sure that his bribe is accepted.

In contrast to ES, I show that with two rounds of bribing and a second-price auction, efficiency can be achieved. Specifically, I prove that there exists an efficient and bribery-involving equilibrium⁴ in monotonic strategies if and only if the expectation of the second mover's type is at least as large as one half of the maximal valuation. That is, with i 's

³Roughly speaking, the reason is that with more than two rounds, a player may reject a bribe in the hope of being offered a more generous bribe in the future; coupled with the ongoing signaling at the pre-auction stage, this feature makes such games very difficult to study.

⁴I call an equilibrium *efficient* if the ex-ante probability that it leads to a Pareto-efficient allocation is 1.

valuation denoted by θ_i , the condition is $\mathbb{E}(\theta_2) \geq \frac{1}{2}$.^{5,6} Bribing and efficiency can coexist in equilibrium (given this condition) because the second mover can reject an offer without triggering the noncooperative auction. When such equilibria exist, they all give rise to the same path of play, which is described as follows.

First, player 1 (the first mover) offers the difference between his valuation and his expected noncooperative payoff. More precisely, with the expected payoff in the noncooperative (dominant strategies) equilibrium of the second-price auction of type θ_i of player i denoted by $\pi_i^*(\theta_i)$, player 1's bribe is given by $b_1(\theta_1) = \theta_1 - \pi_1^*(\theta_1)$. Since b_1 is strictly increasing, player 1 reveals his type perfectly on the path. Seeing player 1's revealed type, player 2 employs an efficient acceptance rule; then, in case he rejects θ_1 's offer he responds with the counteroffer $\pi_1^*(\theta_1)$, which player 1 accepts. In such an equilibrium, player 1's *ex post* equilibrium payoff equals his *expected* noncooperative payoff. The reason for the full surplus extraction is player 2's positional advantage: he moves second, does not reveal any private information, and learns player 1's private information before making his move. This positional advantage translates to an expected payoff of $\pi_2^*(\theta_2) + C$, where $C = \int_0^1 [\theta_1 - \pi_1^*(\theta_1)] f_1(\theta_1) d\theta_1$ is the expected surplus extracted from player 1.

The aforementioned result is robust to the following change in the auction format: if the players are ex-ante symmetric with a type distribution whose expectation is at least $\frac{1}{2}$, then the aforementioned behavior can be sustained as the path of an *essentially perfect Bayesian equilibrium* (EPBE). This is a weakening of perfect Bayesian equilibrium (PBE), due to Blume and Heidhues (2006), which captures sequential rationality and facilitates analysis in games where the derivation of PBE is complicated. When the first-price auction is preceded by a single (TIOLI) round, only *trivial* monotonic EPBEs are possible, where the bribing function is identically zero. Whether the “no bribing” outcome can be sustained in an EPBE depends on the type distributions. I derive a sufficient condition for the existence of an EPBE and a sufficient condition for its nonexistence. Each of these conditions is satisfied by some distributions.

⁵Valuations are drawn from the unit interval. This is just a normalization.

⁶ $\mathbb{E}(\theta_2) \geq \frac{1}{2}$ is sufficient and necessary (for an equilibrium with all the above-mentioned properties) provided that player 1's type distribution, F_1 , satisfies $F_1''(0) > 0$. Otherwise, it is only a sufficient condition.

Finally, I offer a brief look into more general alternating bribe-offers games. Given an auction format M , and $T \in \mathbb{N} \cup \{\infty\}$, let Γ_T^M denote the extensive-form where M is preceded by T rounds of alternating offers. I show that for every M and T , the following is true: if Γ_T^M has a PBE where bribing occurs conditional on every realization of types, then Γ_T^{SPA} has a PBE with the same path of play (SPA stands for “second-price auction.”) In this sense, the second-price auction is the most “bribe-friendly” format.

1.2 Related literature

This paper contributes to a growing body of literature on collusion in one-shot auctions.⁷ Most of this literature takes the mechanism-design approach to collusion and studies direct revelation mechanisms which, in many cases, the cartel operates with the help of an incentiveless third party. Seminal contributions to this literature include Graham and Marshall (1987), Mailath and Zemsky (1991), Marshall and Marx (2007), and McAfee and McMillan (1992). These papers all take the standpoint of the cartel and seek to design mechanisms that are desirable for the bidders. The other side of the “mechanism-design coin” is to take the standpoint of the seller, and look for auction formats that are immune to collusion. Che and Kim (2009, henceforth CK) take this approach in a recent paper, where they derive a collusion-proof auction. It is important to note that the work in CK does not invalidate the contribution of the current paper. First, the current paper assumes that the seller is not strategic and that he employs a standard auction format (specifically, first- or second-price). CK considers a strategic seller who employs a nonstandard format. More importantly, the focus of the current paper is on pre-auction signaling among the bidders, an aspect which is absent from CK, since it restricts collusion to the following form: an incentiveless third party offers the cartel a collusive mechanism, which, if accepted unanimously, is implemented; if

⁷An related branch of research considers collusion in a repeated-game setting. Contributors to this literature include Aoyagi (2002, 2007), Athey and Bagwell (2001, 2008), Blume and Heidhues (2006), Hörner and Jamison (2007), Rachmilevitch (2009), and Skrzypacz and Hopenhayn (2004).

not, then the auction is played noncooperatively.^{8,9}

From the mechanism-design literature, the setting which is closest to the one considered here is that of the informed-principal problem, which originated in the pioneering work of Myerson (1983). For example, Maskin and Tirole (1990, henceforth MT) studied the following extensive-form game. The principal (first mover) offers a mechanism to the agent (second mover), who updates his belief about the principal's type once having seeing his offer. Then, the agent either accepts or rejects the mechanism, and the principal updates his belief about the agent's type upon seeing his response. If the mechanism is accepted, then the players play the game it specifies, while if it is rejected, the game ends, in which case they receive their reservation utilities.¹⁰ The MT game resembles my two-round game in that the first mover's proposal conveys (potentially, at least) information about his type, and the second mover's response conveys (again, potentially) information about his type.

Finally, two works that consider ES-like models (i.e., an auction which is preceded by a TIOLI stage) are Chen and Tauman (2006) and Kivetz and Tauman (2010). The former considers a second-price auction in an environment where, in addition to the cartel members, there is a random population from which the cartel members can hire shill bidders; the latter considers a first-price auction where the bidders' valuations are commonly known among the bidders.

1.3 Organization of the paper

The rest of the paper is organized as follows. The analysis of the second-price two-round game is in Section 2, and the analysis of the first-price games is in Section 3. Section 4 offers a preliminary result for auctions which are preceded by an alternating-offers protocol of an arbitrary length. Section 5 concludes, and proofs omitted from the text, as well as some

⁸Under some further conditions, CK also considers collusion where the proposal of the collusive mechanism is made not by a third party, but by one of the cartel members. However, they still retain the nonsequential structure of the pre-auction stage, which is signaling-free.

⁹Dequiedt (2007) and Pavlov (2008) also study collusion-proof auctions. In both of these papers, as in CK, there is no signaling among the bidders at the pre-auction stage.

¹⁰Similar extensive forms have also been utilized in more recent studies of the informed principal problem, e.g., Severinov (2008).

technicalities regarding equilibrium refinements, are provided in the appendices.

2 Second-price auction with two rounds of offers

2.1 The model

There are two players, player 1 and player 2, who are about to attend a second-price auction for a single indivisible good. Player i 's valuation for the good (his type) is an independent draw from F_i , a full-support distribution on $[0, 1]$ with a strictly positive and twice continuously differentiable density f_i .

The reserve price is zero. The tie-breaking rule employed by the auctioneer is not important; for simplicity, suppose that the auctioneer awards the good at random, with probability $\frac{1}{2}$ to each player, in case the bids are tied. The auctioneer, who owns the good initially, has no value for it (i.e., his valuation is zero). The players are expected-utility maximizers having quasi-linear utilities. The utility from nonparticipation (the outside option) is zero. Thus, given the bids (b_1, b_2) , the associated payoff for type θ_i of player i is $\mathbf{1}_{\{b_i > b_j\}}(\theta_i - b_j) + \mathbf{1}_{\{b_i = b_j\}}\frac{1}{2}(\theta_i - b_j)$. Throughout the paper, whenever player i and player j are mentioned in the same sentence, it is implicitly assumed that $j \neq i$.

Before the auction the players go through two rounds of alternating offers, where each player can try to bribe his rival (if he wishes) so that the latter will eliminate himself from the auction. Specifically, player 1 offers player 2 a nonnegative bribe, which player 2 can either accept or reject. Acceptance effectively ends the game: player 1 pays player 2 the offered amount, and in exchange player 2 eliminates himself from the auction.¹¹ If player 2 rejects player 1's offer he counters with a nonnegative bribe offer of his own. Now, player 1 can accept or reject this bribe, and in case of acceptance he eliminates himself from the auction. Not offering a bribe is a feasible action for either player, modeled as "offering zero." If both offers are rejected, then the pre-auction phase ends and the players turn to play the auction

¹¹It is assumed that the players can commit. The briber can commit to the bribe and the bribee commits to staying out of the auction.

noncooperatively, in which case it is *assumed* that they bid their valuations truthfully.¹²

A pure strategy for player 1 in this extensive form is a specification of (i) a bribing function and (ii) a family of acceptance rules—one for every possible continuation game. For player 2 a pure strategy is modeled as a family of functions $\{b_2(\cdot|x)\}_{x \in \mathbb{R}_+}$, where $b_2(\cdot|x)$ is the function prescribing behavior in the continuation game that follows the offer x . For each x , $b_2(\cdot|x): [0, 1] \rightarrow \mathbb{R}_+ \cup \{\text{“accept”}\}$, where $r \in \mathbb{R}_+$ is interpreted as “reject player 1’s offer and make the counteroffer r .” I restrict my attention to pure strategies.

I will employ the notation b_1 to denote player 1’s bribing function in the first round, and σ to denote a generic strategy profile.

Definition 1. *A strategy profile is **monotonic** if:*

- *For player 1: The bribing function b_1 is weakly increasing and absolutely continuous.*
- *For player 2: Each function $b_2(\cdot|x)$ is weakly increasing, where the action “accept” is identified with the number -1 , and it is absolutely continuous on $\{\theta_2 \in [0, 1] | b_2(\theta_2|x) \neq \text{“accept”}\}$.*

Recall that a perfect Bayesian equilibrium (PBE) is a strategy-belief pair, (σ, μ) , such that σ_i prescribes a best-response for i against σ_j in each of i ’s information sets, and beliefs are derived from Bayes’ rule whenever possible.¹³ I will sometimes abuse language a little, and call a PBE *monotonic* if its strategy profile is monotonic. The idea behind (or justification for) a monotonic PBE, is that players who have a higher valuation for the good also have a higher willingness to pay for their rival’s abstention. Monotonic PBEs are PBEs in which the behavior of a briber can be interpreted as expressing this willingness to pay.

Note that Definition 1 imposes monotonicity on b_1 but does not refer to player 1’s acceptance rules. It is straightforward that in any PBE profile $\sigma = (\sigma_1, \sigma_2)$, the acceptance rules that σ_1 prescribes for player 1 are monotonic in the sense that whenever type θ_1 accepts

¹²This is a reasonable assumption, because bidding truthfully is a dominant strategy for a player independently of his information.

¹³I do not introduce formal notation for beliefs (as a collection of probability distributions), since it will not be needed in the sequel.

a certain bribe $b > 0$, so does every $\theta'_1 < \theta_1$.¹⁴ The acceptance rules of player 2 are also monotonic in this sense.

Remark 1: Monotonic strategies, by definition, rule out discontinuous bribing functions. This may seem a little restrictive, because some signaling games with a continuum of types have equilibria in discontinuous strategies which are otherwise well-behaved. For example, any step-function is ruled out by Definition 1, but such functions are encountered in several standard adverse-selection settings, such as the *bilateral trade problem*.¹⁵ The result for a second-price auction with two rounds of offers (see Theorem 1 below) continues to hold if “absolutely continuous” is replaced by “absolutely continuous or constant on some interval.” The latter requirement is, of course, satisfied by step-functions. Since I do not know whether the results for the first-price bribing games can be obtained under this weaker condition, I employ the stronger definition of a “monotonic strategy,” in order to provide an exposition which is as unified as possible for the two formats.

Definition 2. A PBE is **efficient** if the ex-ante probability that it leads to a Pareto-efficient allocation is 1. Otherwise, it is **inefficient**.

Remark 2: Our game is a signaling game with a continuum of types. In such games equilibria are typically insensitive to the behavior of a single (bordering) type. For example, suppose that there is an equilibrium in which all types $\theta_2 < \frac{1}{2}$ accept a certain bribe b^* , and all types $\theta_2 > \frac{1}{2}$ reject that bribe and counter it with the same offer. Then the threshold type $\theta_2 = \frac{1}{2}$ is indifferent between these two actions; either action can therefore be supported in equilibrium. Throughout the paper, any statement of the form “ σ is the unique PBE profile such that...” means that it is unique up to the behavior of such bordering types. Similarly, I will sometimes abuse language a little, as follows: statements of the form “type θ_j accepts a bribe b iff $b > x$ ” should be interpreted as saying only that bribes $b > x$ are accepted and bribes $b < x$ are rejected.

¹⁴If type θ'_1 weakly prefers competing in the auction to the sure payoff $b > 0$, then he expects to win with a strictly positive probability, which implies that $\theta_1 > \theta'_1$ strictly prefers the noncooperative auction to b .

¹⁵See Leininger, Linhart, and Radner (1989).

2.2 Characterization of efficient equilibria

Some further restrictions on strategies and beliefs are needed for Theorem 1. Specifically, I will focus on (monotonic) PBEs with the following property:

- (A) Whenever player 1 sees an unexpected offer $x \leq 1$ from player 2, any belief he may form assigns probability 1 to the event $\{\theta_2 \geq x\}$.

(A) is found in equilibria that survive *iterated deletion of weakly dominated strategies*. Unfortunately, however, this refinement presents substantial difficulties. Therefore, throughout this section I will implicitly assume (A) in the definition of a PBE. In Appendix B, I explain the problems raised by iterative deletion of weakly dominated strategies, and I show that a weaker version of it—*one round of deletion of weakly dominated strategies*—overcomes these problems and implies (A).

Remark 3: I do not impose a condition, analogous to (A), which would restrict player 2's beliefs in case he sees an unexpected offer from player 1. The reason is that player 1 may offer a bribe which exceeds his valuation, in the hope that it will be rejected and trigger a generous counteroffer. In fact, this is what happens in a *completely pooling equilibrium*, where all the types of player 1 offer the same bribe. Such equilibria exist for a large class of priors.

Recall that $\pi_i^*(\theta_i)$ denotes θ_i 's payoff in the dominant-strategy equilibrium of the noncooperative auction. It is readily verified (through integration by parts) that $\pi_i^*(\theta_i) = \int_0^{\theta_i} F_j(t)dt$. Let $\tilde{\pi}_1(\theta_1, x)$ denote the expected payoff of type θ_1 from competing in the auction against player 2 whose type is distributed on $[x, 1]$ according to $F_2|_{\{\theta_2 \geq x\}}$.^{16,17} For stating Theorem 1, we first need to define the following strategy profile, σ^* :

- Player 1 offers bribes according to $b_1(\theta_1) = \theta_1 - \pi_1^*(\theta_1)$. When player 2 sees an offer of the form $x - \pi_1^*(x)$ for some $x \in [0, 1]$, he rejects it if and only if $\theta_2 > x$, in which

¹⁶Given a cdf F and an event A , $F|_A$ denotes the associated conditional distribution.

¹⁷Specifically, $\tilde{\pi}_1(\theta_1, x) = 0$ for $\theta_1 \leq x$ and $\tilde{\pi}_1(\theta_1, x) = \int_x^{\theta_1} (\theta_1 - t) \frac{f_2(t)}{1-F_2(x)} dt$ for $\theta_1 > x$. Note that $\tilde{\pi}_1(\theta_1, 0) = \pi_1^*(\theta_1)$.

case he counters with $\pi_1^*(x)$; offers greater than $1 - \pi_1^*(1)$ are also accepted by player 2. Player 1's acceptance policy of counterbribes b is as follows:

(1) If player 1's initial offer was of the form $x - \pi_1^*(x)$ for some $x \in [0, 1]$ and player 2 counters with b , then (i) if $b = \pi_1^*(x)$ then 1 accepts b if and only if b is weakly greater than $\tilde{\pi}_1(\theta_1, x)$, and (ii) if $b \neq \pi_1^*(x)$ 1 accepts b if and only if $\theta_1 < 2b$,¹⁸

(2) If player 1's initial offer was strictly greater than $1 - \pi_1^*(1)$, then b is accepted if and only if $b > 0$.

2.2.1 The main result for the two-round second-price game

Theorem 1. (1) *There exists a system of beliefs μ^* such that (σ^*, μ^*) is a PBE if and only if $\mathbb{E}(\theta_2) \geq \frac{1}{2}$.*

(2) *If (σ, μ) is an efficient monotonic PBE and the density of F_1 , f_1 , satisfies $f_1'(0) > 0$, then (i) $\mathbb{E}(\theta_2) \geq \frac{1}{2}$, and (ii) σ induces the same path of play as σ^* .*

Remark 4: Suppose that we restrict attention to priors (F_1, F_2) such that $f_1'(0) > 0$. In this case, the condition $\mathbb{E}(\theta_2) \geq \frac{1}{2}$ is sufficient and necessary for the existence of an efficient and monotonic equilibrium. Moreover, all these equilibria are necessarily bribery-involving. Note: (i). Even without the condition $f_1'(0) > 0$, the inequality $\mathbb{E}(\theta_2) \geq \frac{1}{2}$ is sufficient for the existence of an equilibrium with the three properties—efficiency, monotonicity, and non-triviality. However, it is an open question whether $\mathbb{E}(\theta_2) \geq \frac{1}{2}$ is necessary for the existence of an equilibrium with the three properties (i.e., it is an open question whether there exists such an equilibrium in a case where $f_1'(0) \leq 0$ and $\mathbb{E}(\theta_2) < \frac{1}{2}$); (ii) Without the condition $f_1'(0) > 0$, the fact that an equilibrium is efficient and monotonic does not imply that it is bribery-involving (see the Example below).

¹⁸Alternatively, we could prescribe the following off-path reaction to player 1: the rule that instructs to accept b if and only if $\theta_1 < 2b$ is in place only for b 's which are less generous than the expected counteroffer (i.e., b 's that correspond to $y < x$) while those which are more generous than the expected counteroffer are also accepted.

In the equilibria described in Theorem 1, player 1 reveals his type by offering his surplus (the difference between what he gets in a competition-free world and what he gets under competition) and player 2 employs an efficiency acceptance rule, which is possible because player 1's surplus is a strictly increasing function of his type. When player 2 rejects θ_1 's offer, he reacts to it by making the counteroffer $\pi_1^*(\theta_1)$, which player 1 accepts.¹⁹ As a result, player 1's ex post equilibrium payoff equals his expected noncooperative payoff, $\pi_1^*(\theta_1)$. The reason that $\mathbb{E}(\theta_2) \geq \frac{1}{2}$ is necessary for the path of σ^* to be supported in equilibrium is that it is equivalent to the following condition, which is necessary for this purpose:

$$\pi_1^*(\theta_1) \leq \frac{\theta_1}{2} \quad \forall \theta_1 \in [0, 1]$$

To see this necessity, assume by contradiction that $\pi_1^*(\theta_1) > \frac{\theta_1}{2}$ for some θ_1 , suppose that this θ_1 is the type that was truthfully revealed through the first offer, and suppose further that $\theta_2 > \theta_1$. In this case player 2 has profitable deviation from the equilibrium action: to offer $\pi_1^*(\theta_1) - \epsilon$ instead of $\pi_1^*(\theta_1)$, for some small $\epsilon > 0$. This offer is necessarily accepted by player 1, because he assigns probability 1 to $\{\theta_2 \geq \pi_1^*(\theta_1) - \epsilon\}$ and will therefore accept the bribe if $\pi_1^*(\theta_1) - \epsilon \geq \theta_1 - (\pi_1^*(\theta_1) - \epsilon)$, which is clearly satisfied for a sufficiently small $\epsilon > 0$.

Lemma 1. $\mathbb{E}(\theta_2) \geq \frac{1}{2}$ if and only if $\pi_1^*(\theta_1) \leq \frac{\theta_1}{2} \quad \forall \theta_1 \in [0, 1]$.

Proof. Suppose that $\pi_1^*(\theta_1) \leq \frac{\theta_1}{2} \quad \forall \theta_1 \in [0, 1]$. Taking $\theta_1 = 1$ gives $1 - \mathbb{E}(\theta_2) \leq \frac{1}{2}$, or $\mathbb{E}(\theta_2) \geq \frac{1}{2}$. Conversely, suppose that $\mathbb{E}(\theta_2) \geq \frac{1}{2}$. We need to prove that $\gamma(\theta_1) \geq 0$, where $\gamma(\theta_1) \equiv \frac{\theta_1}{2} - \pi_1^*(\theta_1)$. Note that $\gamma'(\theta_1) = \frac{1}{2} - F_2(\theta_1)$, hence $\gamma''(\theta_1) = -f_2(\theta_1) < 0$, so γ is concave. Also, $\gamma(0) = 0$ and $\gamma(1) = \frac{1}{2} - (1 - \mathbb{E}(\theta_2)) = \mathbb{E}(\theta_2) - \frac{1}{2} \geq 0$, where the inequality is by assumption. Therefore, $\gamma(\theta_1) \geq \theta_1\gamma(1) + (1 - \theta_1)\gamma(0) = \theta_1\gamma(1) \geq 0$. \square

¹⁹The reader may suspect that once player 1 reveals his type and player 2 rejects his offer—which, in an efficient equilibrium, is a signal that player 2 is stronger—it must be the case that player 2 counters with an arbitrarily small bribe, because player 1 would agree to any offer from a stronger opponent. This cannot be a part of an equilibrium, however, because then low types of player 2 would have an incentive to mimic higher types. In addition, note that an unexpected bribe indicates that player 2 is not following the equilibrium and therefore player 1's beliefs are unrestricted in such an information set. In particular, player 1 need not infer that 2 is of a higher type, and therefore need not be willing to accept any bribe.

The technical condition $f_1'(0) > 0$ rules out equilibria in which the event “no bribing” occurs with a strictly positive probability. Without it, “no bribing” can be a part of an equilibrium.

Example of a “no bribing” equilibrium: Suppose that F_i is uniform for each $i = 1, 2$. Consider the strategy where player 1 offers zero independently of his type, player 2 accepts a bribe b if and only if $b > \theta_2$, player 1 accepts a counterbribe b if and only if $2b > \theta_1$, and the rejections of player 2 are followed by the counterbribe zero. Call this profile σ^N (the superscript N stands for “no bribing”). It is not hard to verify that this is indeed a PBE, when coupled with appropriate beliefs. The following proposition makes this statement formal.

Proposition 1. *If F_i is uniform for each i , then there exists a system of beliefs μ such that (σ^N, μ) is a PBE.*

To see the rule of the condition $f_1'(0) > 0$, consider the above-mentioned profile σ^N . Suppose that player 1 adhered to his strategy and offered nothing, and it is now player 2’s turn to move. If he deviates and offers some small $\epsilon \in (0, \frac{\theta_2}{2})$, then all types $\theta_1 \leq 2\epsilon$ would accept it, because player 1 forms beliefs that assign probability 1 to $\{\theta_2 \geq \epsilon\}$. Suppose that player 1 is “as optimistic as possible,” and he assigns probability 1 to $\{\theta_2 = \epsilon\}$ so only $\theta_1 \leq 2\epsilon$ accept the bribe. Let $E = \{\theta_1 > 2\epsilon\}$. Conditional on E player 2’s payoff from the deviation is the same as his payoff from the equilibrium strategy, because the auction is played noncooperatively in either case. Conditional on the complement of E , the payoffs are the same if F_1 is uniform, as player 2 wins with certainty and pays ϵ in expectation. However, if $f_1'(0) > 0$, then in a neighborhood of 0 strong types of player 1 are relatively more likely than weak types, so by excluding them player 2 gains relatively “a lot.” Thus, he strictly prefers to deviate to some small $\epsilon > 0$, and the equilibrium unravels.

2.2.2 Comparison with the inefficiency of the ES equilibrium

As was shown by ES, efficiency (with ex-ante probability 1) is impossible in equilibrium when there is only one round of offers before a second-price auction, because perfect signaling of the briber’s type is not incentive-compatible for him. For example, when type $\theta_1 = 1$ mimics

a type $1 - \epsilon$, the lowered bribe will surely be accepted, because player 2 cannot hope to obtain more than ϵ in the auction. This shows that there must be pooling among all the types of player 1 above a certain threshold.

This may no longer be true with two rounds of bribing, however, because the second round makes it possible for player 2 to reject an offer without triggering the noncooperative auction. Thus, it is not enough for player 1 to signal that his type is “sufficiently high” in order to secure acceptance. One can think of the following message being implied by player 2 when he rejects a relatively high offer: “I know that you are of a very high type, but my type is even higher, so you better accept my counteroffer and not compete against me in the auction.”

Furthermore, the game with two rounds of bribing looks fundamentally different from that of ES also for the *low types* of player 1. In the ES model, player 1 will never offer an amount which exceeds his valuation, but he may very well do so in the two-round game, hoping that his bribe will be rejected and trigger a generous counteroffer. Indeed, this consideration gives rise to equilibria in which there is complete pooling in the first round, where all the types of player 1 offer the same strictly positive offer. Low types follow the equilibrium, despite the fact that they may end up with a negative ex post payoff, because there is a positive probability that player 2 is of an extremely high type, in which case he rejects the common bribe and counters it with a positive offer of his own.

2.2.3 Payoffs

In the equilibria described in Theorem 1, player 1’s ex post payoff equals his expected noncooperative payoff, $\pi_1^*(\theta_1)$. Player 2’s expected payoff can be computed directly from the strategy σ^* : $\int_0^{\theta_2} [\theta_2 - \pi_1^*(t)]f_1(t)dt + \int_{\theta_2}^1 [t - \pi_1^*(t)]f_1(t)dt \equiv \Pi_2(\theta_2)$. Note that $\Pi_2(\theta_2) = \pi_2^*(\theta_2) + C$, where $C = \int_0^1 [t - \pi_1^*(t)]f_1(t)dt$ is the expected surplus extracted from player 1.

3 First-price auctions

In this section I consider the case where the auction format is first-price. The basic model is the same as described in Section 2, but the assumption of truthful bidding is (obviously)

dropped. Hence, in the first-price games, a player’s strategy also prescribes bidding behavior in the auction after every possible history. I further restrict attention to bribing functions that have at most finitely many discontinuity points.²⁰ Finally, I assume that a player does not bid more than his valuation in the auction.

The first-price games give rise to auctions in which exactly one player’s type is commonly known. For example, consider, in the two-round game, a PBE whose strategy profile prescribes a strictly increasing bribing function (on some interval) for player 1, and play unfolded as follows: after player 1 revealed his type player 2 rejected his offer, made a counteroffer which did not reveal θ_2 , and this counter was rejected by player 1. Then, play reaches an auction with the aforementioned asymmetric information, as only θ_1 is common knowledge. In any BNE of this auction, all the types of the informed player submit the “minimally winning” bid if they find winning worthwhile. In terms of modeling, this is made possible by allowing the informed player to submit, in addition to the regular bids, bids of the form r^+ , for every $r \in \mathbb{R}$. A bid r^+ wins with certainty against any $r' \leq r$ and loses against any $r' > r$. When a player wins with a bid r^+ , he pays r .²¹

3.1 The first-price TIOLI game

In the first-price TIOLI game, a PBE is called *monotonic* if the bribing function is continuous and weakly increasing. Unlike the original ES game, this game does not have a monotonic bribery-involving equilibrium.

To show this, I first establish that any such candidate equilibrium must exhibit several properties; in particular, the bribing function takes a particular form: it must look like—qualitatively speaking—the bribing function in the unique continuous equilibrium of ES. The bribing function in this equilibrium is a continuous function which is strictly increasing on some interval $I = [0, \theta_1^*)$ and constant on $[\theta_1^*, 1]$, for some $\theta_1^* \in (0, 1)$. Call such a bribing

²⁰This assumption is needed only in Theorem 4.

²¹In a repeated first-price auction model, Blume and Heidhues (2006) consider the possibility of submitting a special bid, 0^+ , which is identical to 0 except that it wins for sure if the highest competing bid is 0. Rachmilevitch (2009) follows Blume and Heidhues and also allows for the special bid 0^+ in a repeated auction model. The set $\{r^+ | r \in \mathbb{R}\}$ is a natural generalization of 0^+ .

function an *ES function*.

Lemma 2. *In every bribery-involving monotonic equilibrium of the first-price TIOLI game, the bribing function b is an ES function. Moreover, (1) if b is locally invertible at ϵ , then player 2 accepts $b(\epsilon)$ if and only if $\theta_2 \leq A(b(\epsilon)) \equiv b(\epsilon) + \epsilon$, and (2) if b is locally invertible at ϵ and $b(\epsilon)$ is rejected, then player 2 bids ϵ^+ in the auction.*

With the lemma, the following theorem can be stated and proved.

Theorem 2. *In the first-price TIOLI game there does not exist a bribery-involving monotonic equilibrium.*

Proof. Assume by contradiction that such an equilibrium exists. Let b denote 1's bribing function in one such equilibrium. By Lemma 2, it is an ES function. Consider player 1 of type $\epsilon > 0$ where b is locally invertible. I will show that there exist an $\epsilon' \in (0, \epsilon)$ and an $\epsilon'' \in (\epsilon', \epsilon)$ such that the (non-detectable) deviation $b(\epsilon) \mapsto b(\epsilon')$, followed by the bid ϵ'' in the auction, is profitable for player 1 independently of θ_2 . This will be done by considering the following cases.²²

Case 1: $\theta_2 \leq A(b(\epsilon')) < A(b(\epsilon))$: Here the deviation is profitable because player 1 pays a lower bribe.

Case 2: $\theta_2 > A(b(\epsilon))$: Under the deviation, player 1's payoff is $\epsilon - \epsilon'' > 0$, while under the original strategy it is 0.

Case 3: $\theta_2 \in (A(b(\epsilon')), A(b(\epsilon)))$: Here, the deviation is profitable provided that $\epsilon - \epsilon'' > \epsilon - b(\epsilon)$, which is clearly satisfied for a sufficiently small ϵ'' . \square

Theorem 2 says that every (monotonic) equilibrium must be trivial, but does not say anything regarding whether such an equilibrium exists. Deriving a PBE in a first-price bribing game is complicated, because it requires the prescription of mutual best-responses in all off-path auctions, which are asymmetric first-price auctions, and in which Bayes' rule does

²²In the three cases, Lemma 2 is invoked in obvious ways; for brevity, I omit detailed explanations.

not apply. Deriving such best-responses is not trivial by itself, and it becomes even more complicated when adding the extra condition that the expected payoffs from such off-path auctions be sufficiently low, so that no player will want to trigger them by deviating from the equilibrium's path.

This problem could potentially be mitigated if not all the off-path auctions needed to be addressed. Note that only player 1 can trigger these off-path auctions, and player 2 has no influence whatsoever on whether they are triggered; hence, ignoring them is legitimate if player 1 finds it suboptimal to trigger them in the first place.

This is precisely the idea behind *essentially perfect Bayesian equilibrium* (EPBE), a solution concept due to Blume and Heidhues (2006). Roughly speaking, this concept requires beliefs to be derived from Bayes' rule whenever possible, and requires optimality of strategies only after *relevant histories*; behavior after *irrelevant histories* is not required to be optimal. More formally, an EPBE is a triple $(\sigma, \mu, \{\mathcal{H}_i\})$, consisting of a strategy profile, a system of beliefs, and a collection of histories for every player—the *relevant* ones—which contains all the histories on the path of play and satisfies the following additional conditions: (a) once a player leaves the set of his relevant (potentially private) histories, \mathcal{H}_i , he never reenters, and (b) σ induces a best-response for every player after every relevant history, against any strategy that coincides with it on the set of relevant histories.

The following theorem identifies conditions under which the first-price TIOLI game has an EPBE.

Theorem 3. *Suppose that $F_1 = F_2 \equiv F$, where F satisfies $2F(t) + tf(t) \geq 1$ for all $t \in (0, 1]$, $f = F'$. Then, the first-price TIOLI game has an EPBE. In this EPBE, there is no bribing.*

Proof. Let F be a distribution that satisfies the above-mentioned condition. Declare all the histories where player 1 offered a strictly positive bribe as irrelevant for him. Consider the following strategy: player 1 is instructed to offer zero independently of his type, and is instructed to bid truthfully in any auction that follows a rejection of a strictly positive bribe. After the rejection of zero, he is instructed to bid as in the one-shot (unique) symmetric BNE of the auction. Player 2 is instructed to reject zero and bid as in the symmetric BNE following his rejection, and is instructed the following response to positive bribes: any $b > 0$

is accepted if and only if $\theta_2 \leq b$, and in case of rejection player 2 bids α^+ in the auction, where $\alpha = \max\{0, \theta_2 - b\}$. I will prove that the above description amounts to an EPBE (when appropriate beliefs are specified).

Optimality for player 2: Given player 1's truthful bidding in the aforementioned off-path auctions, player 2's behavior is optimal given the belief that assigns probability 1 to $\{\theta_1 = \alpha\}$.

Optimality for player 1: We only need to verify ex-ante optimality (because only relevant histories are considered). Consider then a deviation to a bribe $b > 0$. Clearly we can assume that $b \leq 1$. The payoff corresponding to such a deviation is:

$$F(b)(\theta_1 - b) + \int_b^1 \max\{0, \theta_1 - t + b\} f(t) dt = F(b)(\theta_1 - b) + \int_b^{\min\{\theta_1 + b, 1\}} (\theta_1 - t + b) f(t) dt.$$

Suppose first that $\min\{\theta_1 + b, 1\} = 1$. In this case player 1's expected payoff is $F(b)(\theta_1 - b) + \int_b^1 (\theta_1 - t + b) f(t) dt$. The derivative of this expression (wrt b) is $-F(b) - f(b)b < 0$. Hence, if such a deviation b is profitable, then there exists a profitable deviation b such that $\theta_1 + b \leq 1$. Suppose, then, that b is such a deviation. Player 1's expected payoff is:

$$F(b)(\theta_1 - b) + \int_b^{\theta_1 + b} (\theta_1 - t + b) f(t) dt = -2bF(b) + \int_0^{\theta_1 + b} (\theta_1 + b) f(t) dt - \int_b^{\theta_1 + b} t f(t) dt.$$

Thus, in order to show that the deviation is not profitable, it is enough to show that for every $\theta_1, b \in [0, 1]$, the following holds:²³

$$-2bF(b) + \int_0^{\theta_1 + b} (\theta_1 + b) f(t) dt - \int_b^{\theta_1 + b} t f(t) dt \leq \int_0^{\theta_1} F(t) dt$$

, or:

$$F(\theta_1 + b)(\theta_1 + b) \leq \int_0^{\theta_1} F(t) dt + 2bF(b) + \int_b^{\theta_1 + b} t f(t) dt. \quad (1)$$

When $b = 0$, equation (1) is satisfied as equality (this follows from integration by parts). Therefore, it is enough to show that the derivative of the RHS exceeds that of the LHS; that is, that $F(\theta_1 + b) \leq 2F(b) + bf(b)$, which holds by assumption. \square

²³This is where the assumption of ex-ante symmetric bidders is invoked—it guarantees that the non-cooperative payoff is the same as the one of the second-price auction.

Lemma 3. *There exist distribution functions $F: [0, 1] \rightarrow [0, 1]$ such that $2F(t) + tf(t) \geq 1$ for all $t \in (0, 1]$, where $f = F'$.*

Corollary 1. *There exists a common type distribution such that the (symmetric) first-price TIOLI game has an EPBE.*

Proof. Combine Theorem 3 and Lemma 3. □

Theorem 3 is not robust to changes in the type distributions. It is enough that the density is strictly increasing near zero and that the expected type is at least one half in order for the equilibrium from Theorem 3 to unravel. This is stated and proved formally below; though the result is stated for PBE, it also holds for EPBE.²⁴

Theorem 4. *Suppose that $F_1 = F_2 \equiv F$, where $f = F'$ satisfies $f'(0) > 0$ and $\mathbb{E}(\theta) \geq \frac{1}{2}$. Then, there does not exist a trivial equilibrium in the first-price TIOLI game.*

Proof. Make the aforementioned assumption on the common prior and assume by contradiction that a trivial equilibrium exists. In particular, it follows that there exists a $\delta > 0$ such that player 1's bribing function is identically zero on $[0, \delta]$.²⁵ Let s_2 denote player 2's strategy in this equilibrium. Consider a deviation by $\theta_1 \in [0, \delta]$ to a positive bribe $b > 0$ followed by the bid x , in case that b is rejected; denote this deviation by $D = (b, x)$. The bribe b is accepted by player 2 if and only if $\theta_2 \leq q$, for some threshold $q = q(b) \in [b, 1]$. Since the auction format is first-price, a type $\theta_2 > q$ who rejects b does not bid more than $\theta_2 - b$ in the auction. Consider then the alternative strategy for player 2, \bar{s}_2 , which coincides with s_2 , except that it prescribes every $\theta_2 > q$ the bidding function $B(\theta_2) \equiv \theta_2 - b$. Clearly,

²⁴Likewise, Theorem 2 also applies to EPBEs.

²⁵This follows from the assumption that the bribing function can have—by definition—only finitely many discontinuity points. To see this, let c be the minimum discontinuity point and let b denote the bribing function. Suppose first that $c = 0$. It is easy to see that in equilibrium $b(0) = 0$ (otherwise, type $\theta_1 = 0$ is guaranteed a negative payoff, in contradiction to equilibrium). Hence, if such a $\delta > 0$ does not exist then there are types arbitrarily close to 0 who offer a strictly positive bribe which is bounded from below by some $\beta > 0$; this is not incentive-compatible for $\theta_1 \in (0, \beta)$. Therefore, $c > 0$. Any $\delta \in (0, c)$ satisfies the aforementioned statement. To see why, note that if b is strictly positive at some $\theta_1 < c$, then it is strictly positive on some interval, which implies that the ex-ante probability of bribing is positive, in contradiction to the assumption that the equilibrium is trivial (given a $\beta > 0$, all $\theta_2 < \beta$ accept the bribe β).

θ_1 's expected payoff from the deviation D , computed against s_2 , is at least as large as the one he would have obtained from this deviation if player 2's strategy were \bar{s}_2 . Therefore, any D whose expected payoff, when computed against \bar{s}_2 , is strictly greater than the equilibrium payoff $\pi_1^*(\theta_1)$, is a profitable deviation. I will prove the existence of a profitable deviation $D = (b, x)$ in the set of deviations that satisfy $x \geq b$ and $b \leq \frac{\theta_1}{2}$. Let \mathcal{D} denote the set of these deviations. When 2 employs \bar{s}_2 the expected payoff corresponding to the deviation $D = (b, x)$ is:

$$F(q)(\theta_1 - b) + (1 - F(q)) \times \{\text{Prob}[x \geq B(\theta_2)|\theta_2 \geq q](\theta_1 - x)\}. \quad (2)$$

First, I argue that if $q = 1$ then this deviation is profitable. To see this, assume by contradiction that it is not. Then, since in this case the payoff corresponding to the deviation is $\theta_1 - b$, it follows that:

$$\theta_1 - b \leq \pi_1^*(\theta_1) < \frac{\theta_1}{2} \Rightarrow \frac{\theta_1}{2} < b,$$

in contradiction to $D \in \mathcal{D}$.²⁶ Consider then $q < 1$. In this case, since $\text{Prob}[x \geq B(\theta_2)|\theta_2 \geq q] = \text{Prob}[\theta_2 \leq x + b|\theta_2 \geq q] = \frac{F(b+x) - F(q)}{1 - F(q)}$, the expected payoff (2) can be written as:

$$F(q)(\theta_1 - b) + (F(b+x) - F(q))(\theta_1 - x) = F(q)(x - b) + F(b+x)(\theta_1 - x). \quad (3)$$

Since $x \geq b$ and $q \geq b$, the RHS of (3) is weakly greater than:

$$F(b)(x - b) + F(b+x)(\theta_1 - x). \quad (4)$$

Therefore, in order to establish the existence of a profitable deviation in \mathcal{D} , it is sufficient to find a $\theta_1 \in [0, \delta]$ for which there exist non-negative $b \leq \frac{\theta_1}{2}$ and $x \geq b$ such that (4) is greater than the equilibrium payoff $\pi_1^*(\theta_1) = \int_0^{\theta_1} F(t)dt$. I argue that for a sufficiently small θ_1 the desired inequality holds for $b = x = \frac{\theta_1}{2}$. That is, I argue that the for a sufficiently small θ_1 the following strict inequality holds:

$$L(\theta_1) \equiv F(\theta_1)\frac{\theta_1}{2} > \int_0^{\theta_1} F(t)dt \equiv R(\theta_1).$$

²⁶The inequality $\pi_1^*(\theta_1) < \frac{\theta_1}{2}$ follows from Lemma 1 (it is easy to see that the inequality which is proved in Lemma 1 is strict on $(0, 1)$, and there is no loss of generality in assuming $\theta_1 \in (0, 1)$.)

Note that $L(0) = R(0) = 0$. Hence, it suffices to show that there exists an $\epsilon > 0$, such that $L' \geq R'$ on $[0, \epsilon]$, with a strict inequality on $(0, \epsilon] \equiv I$. That is, we need to prove that the following holds on I : $f(\theta_1)\frac{\theta_1}{2} + \frac{F(\theta_1)}{2} > F(\theta_1)$, or:

$$\nu(\theta_1) \equiv f(\theta_1)\theta_1 > F(\theta_1).$$

Since $\nu(0) \geq 0 = F(0)$, it is enough to prove that $\nu' > F' = f$ on I . Namely, that $f'(\theta_1)\theta_1 > 0$ on I . This follows from $f'(0) > 0$. \square

3.2 The first-price game with two rounds of offers

Up to this point I have considered two separate modifications of the ES game. First, I kept the auction format fixed and changed the bribing protocol, and second, I kept the protocol fixed and changed the format. The first modification made it possible for efficiency and bribing to coexist in equilibrium. The major implication of the second modification was equilibrium nonexistence. It would be interesting to see what force would prevail if we made both modifications simultaneously, and considered two rounds of offers and a first-price format. Will the first-price format annihilate equilibrium, or will the additional pre-auction round give rise to an equilibrium in which bribes and efficiency coexist?

Since the two-round first-price game is fairly complicated (there are many off-path asymmetric auctions), I focus on one particular task: to support the path of σ^* in an equilibrium. As in the first-price TIOLI game I will employ EPBE instead of PBE, and I will consider ex-ante symmetric bidders with type distribution F that satisfies $\mathbb{E}(\theta) \geq \frac{1}{2}$.

In order to complete the path of σ^* to an equilibrium, the following requirements need to be addressed:

- (I) Bidding in the auction that follows a rejection of a counterbribe $\pi_1^*(\theta_1)$ needs to be optimal for both players.
- (II) An efficient acceptance rule coupled with the counter $\pi_1^*(x)$ (in case that $b_1(x)$ is rejected) needs to be optimal for player 2. In particular, $\pi_1^*(\theta_1)$ needs to be the minimal bribe that type θ_1 is willing to accept after his offer is rejected.

- (III) Player 1 should have the right incentives to offer $b_1(\theta_1) = \theta_1 - \pi_1^*(\theta_1)$ when he makes the first move in the game.

As we will see shortly, both (I) and (III) are easy to accommodate. Requirement (II), however, presents substantial difficulties. To see this, consider the case where the first offer was $b_1(x)$ and player 2 deviated and countered with $\pi_1^*(x) - \epsilon$, for some $\epsilon > 0$. To support σ^* 's path, we need to equip player 1 with a response which is always optimal (i.e., which is optimal independent of whether $\theta_1 = x$), and which prescribes rejection, for any $\epsilon > 0$, when $x = \theta_1$. Whereas deriving such a response was not hard in the case of a second-price auction, it is substantially more difficult in the first-price case.

This problem could be solved if prescribing rejection is always (i.e., independent of whether $\theta_1 = x$) optimal. A simple way to achieve this is to equip player 1 with a bid that wins for sure, that is the minimally winning bid, and that gives him the desired payoff. With this bid denoted by r , it must satisfy $\theta_1 - r \geq \pi_1^*(x)$. Therefore, prescription of this behavior is only possible following own-deviations that mimic a type x which is not “too far away” from θ_1 ; in particular, this construction is not feasible after histories in which player 1 made a non-detectable “large” deviation.

Interestingly, if $\mathbb{E}(\theta) \geq \frac{1}{2}$ then these histories can be ignored if we assume that player 1 never deviates in a way that may lead to a negative ex post payoff. To see this formally, recall that $\mathbb{E}(\theta) \geq \frac{1}{2}$ is equivalent to $\frac{\theta_1}{2} \geq \pi_1^*(\theta_1)$ for all θ_1 . Now consider type θ_1 who mimics another type x . By assumption, this deviation cannot lead to a negative ex post payoff, so in particular the payoff conditional on player 2's acceptance is nonnegative, that is,

$$\theta_1 - x + \pi_1^*(x) \geq 0 \Rightarrow \theta_1 - x + \frac{x}{2} \geq 0$$

where the implication follows from $\frac{x}{2} \geq \pi_1^*(x)$ for all $x \in [0, 1]$. Hence, we obtain $\theta_1 \geq \frac{x}{2}$. Now, assume by contradiction that $\pi_1^*(x) > \theta_1$. Combining these inequalities yields $\pi_1^*(x) > \frac{x}{2}$, a contradiction.

Declaring as irrelevant for player 1 those private histories of his where he makes a non-detectable “large” deviation (a deviation that can lead to a negative ex post payoff), one can complete the prescription of optimal behavior after any other history, so as to support

σ^* 's path as the path of an EPBE. I now turn to the specification of this behavior. In doing so, I will allow further sophistication regarding tie breaking: I will assume that both players can submit bids of the forms, r , r^+ , and r^{++} , where r^{++} relates to r^+ in the same way that r^+ relates to r .

Requirement (I) is easy to accommodate, because such auctions only happen off the path, in which case the beliefs of neither player are restricted by Bayes' rule. In particular, the following behavior can be supported as optimal: Let player 1 bid θ_1^+ (i.e., the + version of truthful bidding) and let player 2 bid 1 (i.e., the maximal valuation) independently of his type. Given player 2's bid, it is clear that player 1's bid is optimal; given player 1's truthful bidding, player 2's bid is supported as optimal by the belief that assigns probability 1 to the event $\{\theta_1 = 1\}$. With this bidding in the off-path auctions, we can address requirement (III). Note that player 1's expected payoff when he is of type θ_1 and he mimics type x when making his offer is:

$$U(x, \theta_1) \equiv F(x)(\theta_1 - x + \pi_1^*(x)) + (1 - F(x))\pi_1^*(x) = F(x)(\theta_1 - x) + \pi_1^*(x).$$

This follows from the fact that conditional on rejection of his offer, it is a best-response for player 1 to accept the counter $\pi_1^*(x)$, because if he rejects it, he triggers an auction where player 2 submits the bid 1. It is straightforward to verify that θ_1 is the unique maximizer of $U(\cdot, \theta_1)$ for all $\theta_1 \in [0, 1]$. Therefore, (III) is also taken care of.

Finally, we need to show that the efficient decision rule together with the counter $\pi_1^*(x)$ which follows the rejection of $b_1(x)$ is optimal for player 2. In order to do that, I first establish that $\pi_1^*(x)$ is the minimally acceptable bribe by player 1 if $b_1(x)$ is rejected.

Suppose that the first offer, $b_1(x)$, was rejected by player 2. In order to make sure that a counter $\pi_1^*(x) - \epsilon$ is rejected by player 1, it is enough to equip him with a bid r in the post-rejection auction such that r wins with certainty, it is the minimally winning bid, and $\theta_1 - r > \pi_1^*(x) - \epsilon$. Note that this is possible, since $\theta_1 \geq \pi_1^*(x)$ (I consider only relevant histories). Let player 1 then reject $\pi_1^*(\theta_1) - \epsilon$ and bid α^+ in the auction, where $\alpha = \theta_1 - \pi_1^*(x)$. Call this bidding function B_1 . Now we need to find a bidding function B_2 for player 2, such that for each i there exist beliefs that support B_i as a best-response against B_j . Given the

first offer $b_1(x)$, let B_2 be defined as follows:²⁷

$$B_2(\theta_2, x) = \begin{cases} x - \pi_1^*(x) & \text{if } \theta_2 \leq x - \pi_1^*(x) \\ (x - \pi_1^*(x))^{++} & \text{if } \theta_2 > x - \pi_1^*(x). \end{cases}$$

It is clear that B_2 is optimal against B_1 , given the belief that assigns probability 1 to the event $\{\theta_1 = x\}$. In turn, B_1 is supported as optimal against B_2 by any belief that assigns probability 1 to the event $\{\theta_2 \leq x - \pi_1^*(x)\}$. Therefore, there exist mutual best-responses in the off-path auctions that follow the rejection of the off-path counters $\pi_1^*(x) - \epsilon$, and these best-responses support the rejection of the deviation $\pi_1^*(x) - \epsilon$ as a best-response for player 1. Therefore, $\pi_1^*(x)$ can indeed be supported as the minimally acceptable counterbribe in the continuation game that follows the rejection of the first offer $b_1(x)$.

Therefore, player 2 effectively has three options when he faces the first offer $b_1(x)$: he can accept it, reject it and counter with $\pi_1^*(x)$ which will be accepted, or reject it and trigger the noncooperative auction. The latter is done by countering with any lower-than-expected counterbribe, and the auction it triggers is such that player 2 expects player 1 to bid α^+ , where $\alpha = x - \pi_1^*(x)$. Thus, the best that player 2 can obtain in such an auction is $\theta_2 - (x - \pi_1^*(x))$. Note that he can obtain weakly more than that by countering with $\pi_1^*(x)$, since $\theta_2 - (x - \pi_1^*(x)) \leq \theta_2 - \pi_1^*(x)$ follows from $\pi_1^*(x) \leq \frac{x}{2}$. Therefore, the option of initiating the noncooperative auction can be ignored. Finally, the optimality of the behavior prescribed by σ^* follows from the indifference condition $x - \pi_1^*(x) = \theta_2 - \pi_1^*(x)$ which holds exactly at $\theta_2 = x$. We therefore have the following result:

Theorem 5. *Suppose $F_1 = F_2 \equiv F$, where the distribution F satisfies $\mathbb{E}(\theta) \geq \frac{1}{2}$. Then, there exists an EPBE in the first-price two-round game, whose path coincides with the path of σ^* .*

²⁷This function B_2 is independent of $\epsilon > 0$. That is, player 2 is prescribed the same bidding function in any of the auctions that follow the rejection of the off-path counters $\pi_1^*(x) - \epsilon$.

4 Alternating-offers protocols of arbitrary length

Define a *2-bidder auction* to be a pair of functions $M = (q, \zeta)$, $q: \mathbb{R}_+^2 \rightarrow [0, 1]^2$ and $\zeta: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$, such that $\zeta_i(b_1, b_2) = 0$ if $b_i = 0$ and $q_1(b_1, b_2) + q_2(b_1, b_2) \leq 1$ for all $(b_1, b_2) \in \mathbb{R}_+^2$. Given the bids (b_1, b_2) , $q_i(b_1, b_2)$ is i 's winning probability and $\zeta_i(b_1, b_2)$ is his expected payment to the seller. Let \mathcal{M} be the set of all these auctions.

Given $M \in \mathcal{M}$ and $T \in \mathbb{N} \cup \{\infty\}$, let Γ_T^M be the extensive-form where M is preceded by T rounds of alternating (bribe) offers. A (public) *history* in such a game is an element of \mathbb{R}_+^T , generically denoted by h . A PBE of such a game is called *completely collusive* if bribing occurs conditional on every realization of types.

Theorem 6. *For every $M \in \mathcal{M}$, $T \in \mathbb{N} \cup \{\infty\}$, and a completely collusive PBE of Γ_T^M , (σ, μ) , Γ_T^{SPA} has a PBE with the same path of play as (σ, μ) .*

Proof. Let $M \in \mathcal{M}$ and $T \in \mathbb{N} \cup \{\infty\}$, and let (σ, μ) be a completely collusive PBE of Γ_T^M . Note that in this equilibrium every auction is an off-path auction; let h denote a generic history which leads to an off-path auction, and let $\phi_i(h)$ denote i 's expected payoff in this off-path auction. Let σ' be the following profile of Γ_T^{SPA} : it is identical to σ except that it instructs truthful bidding in any auction. Let μ' be any system of beliefs that satisfies Bayes' rule on σ' 's path, and such that at each off-path auction which is preceded by h , μ_i assigns probability 1 to the event $\{\theta_j = 1\}$ if $\phi_i(h) = 0$, and assigns probability 1 to the event $\{\theta_j = \theta_i - \phi_i(h)\}$ otherwise. It is straightforward that (σ', μ') is a PBE of Γ_T^M which has the same path as (σ, μ) . \square

5 Conclusion

I have examined three simple collusion-games, in the spirit of Esö and Schummer (2004). I started by considering a second-price auction which is preceded by two rounds of bribing. Given the technical condition $f_1'(0) > 0$, this game has a monotonic and efficient equilibrium if and only if the expected type of the second mover is at least as large as one half of the maximal valuation. In this equilibrium, bribing occurs conditional on every realization of types and the second mover enjoys the entire collusive surplus. This equilibrium is unique

(up to off-path behavior). Without the aforementioned technical condition, it is still an equilibrium (provided that the expected type of the second mover is at least one half of the maximal valuation), but it is an open question as to whether it is unique. Its path can also be supported in an EPBE when the auction format is first-price, provided that the players are ex-ante symmetric with a type distribution whose expectation is at least one half of the maximal valuation. Finally, in a first-price TIOLI bribing game, every monotonic EBPE is necessarily trivial; it was shown that such an EPBE exists for some type distributions but not for others.

Regarding efficiency, it was shown that when a first- or second-price auction is preceded by alternating bribe-offers, a minimum of two rounds is necessary for efficient collusion. This minimum actually suffices (under a large class of circumstances) in the case of a second-price auction. Regarding the seller's problem, the first-price format is superior to the second-price format. First, if collusion assumes a simple TIOLI form, then there always exists a bribery-involving equilibrium under the second-price format (the ES equilibrium), but never under the first-price format. Second, any completely collusive equilibrium of any alternating-offers bribing game (and in particular, a first-price bribing game) can be replicated under the second-price format. In this sense, the second-price auction is the most "bribe-friendly" auction.

The asymmetry which is assumed in the games I studied has a simple real-life interpretation. The second mover is an incumbent (or a professional bidder) and the first mover is an entrant (or an occasional bidder). The newcomer needs to move first if he wishes to reach a secret agreement with the veteran professional, and the latter's surplus is interpreted as an experience rent.

Finally, it should be noted that the analysis of the second-price bribing game applies to an N -player environment, $N > 2$, where collusion occurs between two players and the behavior of all the others is fixed at their dominant strategies. This is done simply by re-defining valuations to be the valuations of competing in the auction against the other $N - 2$ players.²⁸

²⁸In principle, the same is true for the first-price case: the theorems for the first-price format continue to hold in a multiplayer environment, where the $N - 2$ non-cartel-members are unaware of collusion. Here,

5.1 Ideas for future research

The bribing games studied here are bargaining games with incomplete information, where uncertainty (also) exists about the disagreement payoffs (those of the noncooperative auction).²⁹ The existing literature on bargaining hardly ever addresses such uncertainty,³⁰ whose treatment is largely confined to the cooperative (or axiomatic) framework.³¹ Chun and Thomson (1990a, 1990b), Livne (1988), Peters and van Damme (1991), and Smorodinsky (2005), consider uncertain disagreement points in Nash’s (1950) bargaining model, but this uncertainty differs from the one I have considered. Instead of being uncertain about each other’s type, the players face *public* uncertainty, conditional on the realization of which the disagreements payoffs are deterministic. Studying private-information-based uncertainty is of both practical importance and theoretical interest, because in many bargaining situations the disagreement alternative is to “compete,” or to “fight,” in which case the players have payoff-relevant private information.

A second avenue for future research concerns the pre-game negotiation process. Essentially any competition (e.g., Bertrand, Cournot, auctions, and more) can be embedded in a richer setting, where prior to the competition the players exchange payoff-relevant messages (as opposed to “cheap talk”). Since, at least in principle, there is no reason to restrict the number of pre-play rounds, one is led to consider game theoretic models with a rather special timing frame, where time has order type $\omega + 1$. In such a setting, tasks which are routine in

however, there is no obvious assumption regarding the behavior of these $N - 2$ players. One alternative is to assume that they submit bids as in an N -player noncooperative equilibrium. This assumption, however, is problematic, because it implies that the noncartel bidders are (generically) not playing best-responses and are unaware of that.

²⁹Each player is also uncertain about how much it is worth for him to exclude his rival from the auction.

³⁰Skrzypacz and Fuchs (2010) study bargaining where the outside option is uncertain, but as opposed to the setting considered here it depends on (external) market conditions, not on the private information of the players.

³¹The noncooperative literature mostly studies uncertainty regarding gains from trade in bilateral trade problems (see, e.g., Ausubel, Cramton, and Deneckere (2002)). Another branch of research—not directly connected to the current paper, but which nevertheless belongs to the “bargaining under uncertainty” literature—considers uncertainty about time preferences. It includes Cramton (1984), Rubinstein (1985), and Chatterjee and Samuelson (1987), among others.

standard models, such as defining strategies, are not straightforward.³²

6 Appendix A: Proof of Theorem 1

The following lemmas will be needed for the proof of Theorem 1. For formulating them, the following notation is needed. Let:

$$Z(x, \theta_1) \equiv F_2(x)(\theta_1 - x + \pi_1^*(x)) + (1 - F_2(x))\max\{\pi_1^*(x), \tilde{\pi}_1(\theta_1, x)\}$$

where $\tilde{\pi}_1(\theta_1, x)$ denotes θ_1 's expected payoff in the non-cooperative auction against player 2 whose type is distributed on $[x, 1]$ according to the conditional distribution $F_2|_{\{\theta_2 \geq x\}}$.

Condition 1. (C1) *The distribution function F_2 , defined on $[0, 1]$, satisfies condition C1 if $x = 1$ is a maximizer of $Z(\cdot, 1)$.*

Note that condition C1 is simply the incentive compatibility constraint of the maximal type of player 1 in the first information set in the two-round second-price game. As the following lemma shows, if this constraint is satisfied for the maximal type (i.e. if C1 holds), then it is satisfied for all types.

Lemma 4. *If C1 is satisfied, then $x = \theta_1$ is a maximizer of $Z(\cdot, \theta_1)$ for all $\theta_1 \in [0, 1]$.*

Proof. Suppose that C1 is satisfied. Assume by contradiction that $x = \theta_1$ does not maximize $Z(\cdot, \theta_1)$ over $[0, 1]$, for some $\theta_1 \in [0, 1]$. Clearly we can assume that $\theta_1 > 0$. Since the objective is continuous and $[0, 1]$ is compact, a maximizer $x^* \in [0, 1]$ exists. First, I argue that $x^* < \theta_1$. To see this, assume by contradiction that $x^* \geq \theta_1$. Then, x^* is a maximizer of $Z(\cdot, \theta_1)$ over $[\theta_1, 1]$. On this sub-domain, the objective is:

$$Z(x, \theta_1) = F_2(x)(\theta_1 - x + \pi_1^*(x)) + (1 - F_2(x))\pi_1^*(x) = F_2(x)(\theta_1 - x) + \pi_1^*(x). \quad (5)$$

³²Aumann and Hart (2003) consider a model with “infinity plus one” periods, where a strategic-form two-person game is preceded by infinitely many rounds of cheap talk.

The FOC is $f_2(x)(\theta_1 - x)$, which is negative on $(\theta_1, 1]$, hence on this range the maximum is at $x^* = \theta_1$, a contradiction. Hence, as was argued, $x^* < \theta_1$.

It is easy to see that there is a unique $m = m(\theta_1) \in (0, \theta_1)$ such that the objective satisfies (5) on $[m, 1]$. Since on that sub-domain θ_1 is the unique maximizer of $Z(\cdot, \theta_1)$, we only need to consider $[0, m]$, where the objective takes the form:

$$Z(x, \theta_1) = F_2(x)(\theta_1 - x + \pi_1^*(x)) + \int_x^{\theta_1} (\theta_1 - t)f_2(t)dt$$

Since x^* gives the objective a higher value than θ_1 does, it follows that:

$$Z(x^*, \theta_1) = F_2(x^*)(\theta_1 - x^* + \pi_1^*(x^*)) + \int_{x^*}^{\theta_1} (\theta_1 - t)f_2(t)dt > \int_0^{\theta_1} (\theta_1 - t)f_2(t)dt = Z(\theta_1, \theta_1)$$

therefore:

$$F_2(x^*)(\theta_1 - x^* + \pi_1^*(x^*)) > \int_0^{x^*} (\theta_1 - t)f_2(t)dt$$

Adding $F_2(x^*)(1 - \theta_1) = \int_0^{x^*} (1 - \theta_1)f_2(t)dt$ to both sides gives:

$$F_2(x^*)(1 - x^* + \pi_1^*(x^*)) > \int_0^{x^*} (1 - t)f_2(t)dt$$

therefore:

$$Z(x^*, 1) = F_2(x^*)(1 - x^* + \pi_1^*(x^*)) + \int_{x^*}^1 (1 - t)f_2(t)dt > \int_0^1 (1 - t)f_2(t)dt = Z(1, 1)$$

, in contradiction to *C1*.

□

The following lemma guarantees that the incentive constraint of the maximal type (i.e. *C1*) is indeed satisfied.

Lemma 5. *Every distribution F_2 satisfies condition C1.*

Proof. Assume by contradiction that there exists an $x < 1$ such that $Z(x, 1) > Z(1, 1) = \pi_1^*(1)$. By the previous lemma, it must be that this x is sufficiently small, so that $\tilde{\pi}_1(1, x) \geq \pi_1^*(x)$, and consequently the value of the objective satisfies:

$$Z(x, 1) = F_2(x)(1 - x + \pi_1^*(x)) + \int_x^1 (1 - t)f_2(t)dt > \int_0^1 (1 - t)f_2(t)dt$$

or,

$$F_2(x)(1 - x + \pi_1^*(x)) > \int_0^x (1 - t)f_2(t)dt$$

Since $\int_0^x (1 - t)f_2(t)dt = \int_0^x [x - t + (1 - x)]f_2(t)dt = \pi_1^*(x) + \int_0^x (1 - x)f_2(t)dt = \pi_1^*(x) + F_2(x)(1 - x)$, we obtain:

$$F_2(x)(1 - x + \pi_1^*(x)) > \pi_1^*(x) + F_2(x)(1 - x)$$

and therefore:

$$F_2(x)\pi_1^*(x) > \pi_1^*(x)$$

, a contradiction. □

With the lemmas at hand, we can turn to the proof of the theorem.

Proof. (I). Sufficiency—If $\mathbb{E}(\theta_2) \geq \frac{1}{2}$, then there exist beliefs μ such that (σ^, μ) is an efficient monotonic PBE:* By Lemma 1, $\pi_1^*(\theta_1) \leq \frac{\theta_1}{2}$ for all $\theta_1 \in [0, 1]$. This fact will be useful shortly.

Let μ be a system of beliefs such that (i) when 2 sees an offer of the form $x - \pi_1^*(x)$ for some $x \in [0, 1]$, he assigns probability 1 to the event $\{\theta_1 = x\}$, while when he sees an offer strictly greater than $1 - \pi_1^*(1)$ he assigns probability 1 to the event $\{\theta_1 = 1\}$, and (ii) in the information set following the rejection of the first offer b_1 where 2 offered the counter-bribe b_2 , 1's beliefs are the following: (ii.a) if $b_1 = x - \pi_1^*(x)$ for some $x \in [0, 1]$ then 1 believes that θ_2 is distributed according to $F_2|_{\{\theta_2 \geq x\}}$ in case that $b_2 = \pi_1^*(x)$, and he believes that $\{\theta_2 = \min\{b_2, 1\}\}$ otherwise, (ii.b) if $b_1 > 1 - \pi_1^*(1)$ then 1 believes that $\{\theta_2 = 1\}$.

It is easy to see that σ^* is monotonic and efficient (i.e., leads to an efficient allocation of

the good with probability 1); hence, all that we need to show is that (σ^*, μ) is a PBE. That is, that σ_i^* prescribes a best-response for i against σ_j^* , in each of i 's information sets (given his belief).

Start with player 1 of an arbitrary type, $\theta_1 \in [0, 1]$. First, I argue that he does not have a strictly profitable deviation in the first node in the game tree. To see this, note that if there exists such a deviation to a different bribe, then there exists such a deviation where he mimics a different type (because all offers above $1 - \pi_1^*(1)$ are accepted by player 2). But this contradicts the fact that $x = \theta_1$ is a maximizer of $Z(\cdot, \theta_1)$ (which is guaranteed by Lemma 3 and Lemma 4).

Next, consider the information set where 1 responds to 2's counter-offer. Suppose first that 1's initial offer was of the form $x - \pi_1^*(x)$ for some $x \in [0, 1]$ and 2 countered with $b = \pi_1^*(y)$ for some y . If $y = x$, then clearly following σ_1^* —by definition—is a best-response. If $y \neq x$, then since 1's belief assigns probability 1 to the event $\{\theta_2 = b\}$, it follows that accepting the bribe iff $b \geq \theta_1 - b$ —as instructed by σ_1^* —is a best-response. Suppose, on the other hand, that 1's initial offer was strictly greater than $1 - \pi_1^*(1)$ and was rejected by 2; then, since 1's belief assigns probability 1 to the event $\{\theta_2 = 1\}$, it follows that accepting b is a best-response.

Consider now player 2 of an arbitrary type $\theta_2 \in [0, 1]$. In his first information set in the game he responds to 1's offer. If this offer takes the form $x - \pi_1^*(x)$ for some $x \in [0, 1]$, he concludes that $\theta_1 = x$. First, I argue that any counter-offer $b < \pi_1^*(x)$ is rejected. To see this, assume by contradiction that there exists a counter-offer $b < \pi_1^*(x)$ which is accepted by 1. This offer is accepted iff $\theta_1 \leq 2b$, or $\frac{x}{2} \leq b$. Combining these inequalities we obtain $\frac{x}{2} < \pi_1^*(x)$, a contradiction.

Hence, player 2 has effectively three options: (a) to accept the bribe $x - \pi_1^*(x)$, (b) to reject and counter with $\pi_1^*(x)$, which is accepted by player 1,³³ (c) to compete in the non-cooperative auction against type $\theta_1 = x$.

Suppose first that $\theta_2 < x$. Here, we need to prove that acceptance is a best-response. If he rejects then the best he can obtain is $\max\{\theta_2 - \pi_1^*(x), 0\} < x - \pi_1^*(x)$, and therefore adhering to σ_2^* is a best-response.

³³The counter-bribe $\pi_1^*(x)$ is accepted by player 1 of type $\theta_1 = x$ because $\pi_1^*(x) \geq \tilde{\pi}_1(x, x) = 0$.

Suppose, on the other hand, that $\theta_2 \geq x$. Here, we need to prove that countering with $\pi_1^*(x)$ is a best-response. First, note that doing so is weakly better than accepting 1's offer, and is strictly better if $\theta_2 > x$. This follows from $\theta_2 - \pi_1^*(x) \geq x - \pi_1^*(x)$. Secondly, this is also better than competing the auction, because $\theta_2 - \pi_1^*(x) \geq \theta_2 - x$.

Next, consider the case where 2 sees an offer strictly greater than $1 - \pi_1^*(1)$. Since he believes that $\theta_1 = 1$, acceptance of this offer is a best-response. We conclude σ_2^* prescribes a best-response for player 2. Since the analogous conclusion has already been established for player 1, we conclude that σ_i^* prescribes i a best-response against σ_j^* in each of i 's information sets (given the beliefs). That is, (σ^*, μ) is a PBE.

(II). Necessity—Suppose that $f_1'(0) > 0$. Then, if there exists an efficient monotonic PBE, then it has the same path as σ^* , and $\mathbb{E}(\theta_2) \geq \frac{1}{2}$.

Let b_1 denote 1's bribing function.

Claim 1: b_1 is strictly increasing

Proof of Claim 1: Assume by contradiction that there exists some non-degenerate interval I on which b_1 is constant, taking the value b^* .

Case 1: There exists such an I with $b^* > 0$.

Let $\underline{\theta}_1 \equiv \inf I$ and consider the following cases.

Case 1.1: $\underline{\theta}_1 > 0$. Let $\Theta_1 \subset I$ be the set of those $\theta_1 \in I$ given which the (conditional) probability of efficiency is 1. Since the ex-ante probability of efficiency is 1, this is a full-measure subset.

Let $\epsilon \in (0, \min\{\frac{b^*}{2}, |I|\})$.³⁴ By efficiency, all $\theta_2 \in (\underline{\theta}_1, \underline{\theta}_1 + \epsilon) \equiv J$ reject b^* .³⁵ Let $\beta: J \rightarrow \mathbb{R}_+$ denotes 2's counter-bribing function on this domain.

³⁴ $|I|$ denotes the length of the interval I .

³⁵Otherwise, there would exist a $\theta_1 \in J \cap \Theta_1$ conditional on which the probability of efficiency is strictly less than 1.

Case 1.1.1: β is constant on its domain. Let b' denote its level. Note that necessarily $b' > 0$. To see this, assume by contradiction that $b' = 0$ and consider a type $\theta_2 \in J$. When he rejects b^* and counters with $b' = 0$, the offer b' is rejected by player 1 with probability 1,³⁶ hence, this behavior gives θ_2 a payoff which is bounded from above by $\epsilon < b^*$, and therefore this cannot be a part of an equilibrium. Thus, $b' > 0$.

Since types $\theta_1 \in \Theta_1$ arbitrarily close to $\underline{\theta}_1$ reject b' ,³⁷ it follows that there exists a $\theta_2 < \underline{\theta}_1$ who rejects b^* and counters with b' .³⁸ But note that this b' is rejected with probability 1, and hence rejecting b^* and countering with b' is not a best-response for such a $\theta_2 < \underline{\theta}_1$.

Case 1.1.2: There exists a $\theta_2 \in J$ such that $\beta^* \equiv \beta(\theta_2) > \inf\{\beta(x)|x \in J\}$. Let $\underline{\theta}_2$ be the infimum type of 2 who offers β^* . Let $\delta \in (0, \beta^*)$ be sufficiently small such that $K \subset J$, where $K \equiv (\underline{\theta}_2, \underline{\theta}_2 + \delta)$. Note that $K \subset I$. The event $E \equiv \{(\theta_1, \theta_2) \in K \times K\}$ has an ex-ante positive probability. Thus, it is sufficient to prove that conditional on E the probability of inefficiency is strictly positive.

Conditional on E the following path of play is realized: 1 offers b^* , which is rejected and countered by some $\tilde{b} \geq \beta^*$, which is accepted by 1.³⁹ Since conditional on E the probability that $\{\theta_2 < \theta_1\}$ is strictly positive, it follows that the probability of inefficiency conditional on E is strictly positive.

Case 1.2: $\underline{\theta}_1 = 0$. Here, b_1 is constant at the level $b^* > 0$ on an interval of the form $[0, x)$ for some $x > 0$. All $\theta_2 < b^*$ accept this bribe, hence the ex-ante probability of inefficiency is positive, a contradiction.⁴⁰

Case 2: $b^* = 0$ for every interval I on which b_1 is constant.

In this case $b_1 \equiv 0$ on an interval of the form $[0, x)$ for some $x > 0$. Wlog, suppose that x is the supremum number for which this is true. Since all $\theta_2 > 0$ reject the offer 0, there

³⁶In equilibrium player 1 will accept the counter offer $b' = 0$ only if he is certain that $\theta_2 \geq \theta_1$. Thus, only type $\theta_1 = \underline{\theta}_1$ can accept it in equilibrium.

³⁷If such a $\theta_1 \in \Theta_1$ accepts it then the conditional probability of efficiency is less than 1.

³⁸Otherwise, a θ_1 sufficiently close to $\underline{\theta}_1$ would be better off accepting b' , as his expected payoff in the non-cooperative auction is strictly smaller than b' .

³⁹The counter bribe \tilde{b} is accepted by 1 because in the non-cooperative auction 1's expected payoff is bounded from above by $\delta < \beta^* \leq \tilde{b}$.

⁴⁰More precisely, the probability of inefficiency is bounded from below by $\int_0^{b^*} [\int_0^{\theta_2} \frac{f_1(t)}{F_1(\theta_2)} dt] f_2(\theta_2) d\theta_2$.

is a probability $F_1(x) > 0$ that play will go to the second round of the bribing phase, where the continuation game is a TIOLI game, played with the following beliefs: the distribution of θ_2 is given by F_2 and the distribution of θ_1 is given by $G \equiv F_1|_{\{\theta_1 < x\}}$. That is,

$$G(t) = \begin{cases} 1 & \text{if } t \geq x \\ \frac{F_1(t)}{F_1(x)} & \text{if } t < x \end{cases}$$

Let $g \equiv G'$ denote its density. Let $b_2 \equiv b_2(\cdot|0)$ denote 2's bribing function in this continuation game.

I argue that $b_2 \equiv 0$ on $[0, x)$. To see this, assume by contradiction that $b_2(\theta_2) > 0$ for some $\theta_2 < x$. If there exists a $\theta'_2 \in (\theta_2, x)$ such that $b_2(\theta'_2) = b_2(\theta_2) \equiv b^*$, then b_2 is constant at the level $b^* > 0$ on some interval J . Wlog, suppose that J is the maximal such interval (i.e. the union of all such intervals). All types θ_1 sufficiently close to $\inf J$ accept the bribe b^* , and therefore the probability of inefficiency is positive. If, on the other hand, there does not exist such a θ'_2 , then b_2 is strictly increasing on (θ_2, x) , and again we conclude that the probability of inefficiency is strictly positive (given each revealed type $\theta'_2 \in (\theta_2, x)$, all types above and close to him accept his bribe, because bidding in the auction is truthful). Thus, $b_2 \equiv 0$ on $[0, x)$.

Let $0 < \theta_2 < x$ and let $\epsilon \in (0, \frac{\theta_2}{2})$. When player 1 is contemplating acceptance vs. rejection of ϵ (in the continuation game which follows the rejection of 0), he compares the two associated payoffs. The payoff from acceptance is obvious, ϵ . The payoff from rejection is the expected payoff in the non-cooperative auction, which is bounded from above by $\max\{\theta_1 - \epsilon, 0\}$. To see why this is an upper bound, note that there are exactly two possibilities: (i) ϵ constitutes a detectable deviation (i.e. there does not exist a $\theta_2 \geq x$ such that $b_2(\theta_2) = \epsilon$), or (ii) there exists some $\theta_2 \geq x$ such that $b_2(\theta_2) = \epsilon$. In case (i) the fact that $\max\{\theta_1 - \epsilon, 0\}$ is an upper bound on the payoff 1 expects follows from the fact that 2 will bid his valuation truthfully in the auction and 1 believes that $\theta_2 \geq \epsilon$. As for case (ii), note that if θ_2 offers ϵ in equilibrium then $\theta_2 \geq \epsilon$; otherwise, he would be better off offering zero.

Therefore, a sufficient condition for acceptance of ϵ is $\theta_1 - \epsilon < \epsilon$, or $\theta_1 < 2\epsilon$. Therefore,

θ_2 's expected payoff from the deviation ϵ is bounded from below by:

$$G(2\epsilon)(\theta_2 - \epsilon) + (1 - G(2\epsilon)) \int_{2\epsilon}^x M(\epsilon, t) \frac{g(t)}{1 - G(2\epsilon)} dt \quad (6)$$

where $M(\epsilon, t)$ is defined as the minimum utility player 1 of type $\theta_1 = t$ can impose on player 2 of type θ_2 following the offer ϵ , given that the players bid truthfully in the auction. Specifically, $M(\epsilon, t)$ is given by:

$$M(\epsilon, t) = \begin{cases} \min\{\theta_2 - \epsilon, \theta_2 - t\} & \text{if } t \leq \theta_2 \\ 0 & \text{if } t > \theta_2 \end{cases}$$

Since $M(\epsilon, t) \geq 0$ and $x > \theta_2$, the expression in (6) is bounded from below by:

$$G(2\epsilon)(\theta_2 - \epsilon) + (1 - G(2\epsilon)) \int_{2\epsilon}^{\theta_2} M(\epsilon, t) \frac{g(t)}{1 - G(2\epsilon)} dt = G(2\epsilon)(\theta_2 - \epsilon) + \int_{2\epsilon}^{\theta_2} (\theta_2 - t)g(t)dt \quad (7)$$

The equality in (7) follows from the fact that in the range $2\epsilon \leq t \leq \theta_2$ we have $M(\epsilon, t) = \min\{\theta_2 - t, \theta_2 - \epsilon\} = \theta_2 - t$, where the latter equality follows from $t \geq 2\epsilon > \epsilon$. Therefore, θ_2 's expected payoff if he deviates to ϵ is bounded from below by the RHS of (7). Setting (for convenience) $z \equiv 2\epsilon$, his payoff is bounded from below by:

$$\psi(z) \equiv G(z)(\theta_2 - \frac{z}{2}) + \int_z^{\theta_2} (\theta_2 - t)g(t)dt$$

Note that $\psi(0)$ equals θ_2 's equilibrium payoff.⁴¹ Therefore, it must be that ψ is locally non-increasing in a neighborhood of 0. I will now prove that this is not the case. Indeed, the first two derivatives of ψ are zero and the third derivative is positive (all evaluated at 0).

$$\psi'(z) = g(z)\frac{z}{2} - \frac{G(z)}{2} \Rightarrow \psi'(0) = 0$$

$$\psi''(z) = \frac{1}{2}g'(z)z \Rightarrow \psi''(0) = 0$$

$$\psi'''(z) = \frac{1}{2}(g''(z)z + g'(z)) \Rightarrow \psi'''(0) = \frac{g'(0)}{2} = \frac{f_1'(0)}{2F(x)} > 0$$

⁴¹In this equilibrium player 2 of type θ_2 offers 0 (in this continuation game), which is rejected. This leads to a non-cooperative play of the auction against player 1 whose type is distributed according to G , giving the expected payoff $\int_0^{\theta_2} (\theta_2 - t)g(t)dt = \psi(0)$.

Thus, Claim 1 is proved— b_1 is strictly increasing on $[0, 1]$. In this case, by assumption, it is absolutely continuous.

Let $\Theta_1 \subset [0, 1]$ be the set of types of 1, given which the (conditional) probability of efficiency is 1. By assumption the ex-ante probability of efficiency is 1 and therefore Θ_1 is a full-measure subset of $[0, 1]$.⁴²

Fix $\theta_1 \in \Theta_1$. Wlog, suppose that θ_1 is a point where b_1 is differentiable. Type θ_1 reveals himself through his bribe offer (like any other type). Note that if there exists an opponent type $\theta_2 > \theta_1$ who accepts $b_1(\theta_1)$, then so do all $\theta'_2 \in (\theta_1, \theta_2)$, because strategies are monotonic. This contradicts the assumption that conditional on θ_1 the probability of efficiency is 1. Hence, all $\theta_2 > \theta_1$ reject $b_1(\theta_1)$.

Consider now a respondent type $\theta_2 < \theta_1$. The payoff of such a θ_2 in the non-cooperative auction is zero (because of truthful bidding) and therefore if he rejects $b_1(\theta_1)$ it follows that he bribes player 1 in the next round and his bribe is accepted with certainty.⁴³ Also, by monotonic strategies, if $\theta_2 < \theta_1$ rejects $b_1(\theta_1)$ then so does every $\theta'_2 \in (\theta_2, \theta_1)$; again, this leads to a contradiction to the assumption that the probability of efficiency given θ_1 is 1. Therefore, all $\theta_2 < \theta_1$ accept $b_1(\theta_1)$.

Next, I argue that all the types who reject θ_1 's offer (i.e. all $\theta_2 > \theta_1$) react to it with the same counter offer. To see this, consider first $\theta_2 = \theta_1 + \epsilon$ where $\epsilon \in (0, b_1(\theta_1))$ and let β denote his counter-offer.⁴⁴ Clearly $\beta < \theta_1$, because if $\beta \geq \theta_1$ then 2's payoff will be at most $\epsilon < b_1(\theta_1)$, independently of 1's response. Secondly, it must be that $\beta > 0$; to see this, assume by contradiction that $\beta = 0$. If it is accepted by player 1, then $\theta_2 < \theta_1$ would have an incentive to offer $\beta = 0$, in contradiction to efficiency.⁴⁵ If $\beta = 0$ is rejected then θ_2 , by offering it, triggers the non-cooperative auction, in which his payoff is bounded by $\epsilon < b_1(\theta_1)$. Hence, $\beta > 0$. Thirdly, I argue that β must be accepted by 1, because in this equilibrium rejection of $b_1(\theta_1)$ signals that $\theta_2 > \theta_1$, and therefore 1's unique best-response is

⁴²With respect to the measure F_1 .

⁴³By assumption, strategies are pure; in particular, player 1 employs a deterministic acceptance rule.

⁴⁴There is no loss of generality in assuming that $b_1(\theta_1) > 0$ because we are in the case where b_1 is strictly increasing.

⁴⁵Otherwise, $b(\theta_1) \geq \theta_1$, which clearly cannot be a part of an equilibrium.

to accept β .⁴⁶ Consider now $\theta'_2 \in (\theta_1, \theta_2)$ who counters with β' . By monotonic strategies, $\beta' \leq \beta$. By the argument that was just applied to β , it follows that θ_1 accepts β' ; then, since θ_2 's counter-offer is a best-response, it follows that $\beta = \beta' < \theta_1$.⁴⁷ Finally, the strict inequality implies that all $\theta'_2 > \theta_2$ also counter $b_1(\theta_1)$ with β . First of all, countering with β is better than competing in the auction; secondly, if they countered with another offer, say $\alpha < \beta$, then type θ_2 (as well as θ'_2) would have a profitable deviation: $\beta \mapsto \alpha$.⁴⁸ Therefore, as was argued, all $\theta_2 > \theta_1$ counter with the same offer, β .

Next, I argue that type $\theta_2 = \theta_1$ is exactly indifferent between accepting $b_1(\theta_1)$ and rejecting it. To see this, note first that all types $\theta_2 = \theta_1 - \epsilon$ prefer acceptance to rejection, hence $b_1(\theta_1) \geq \theta_1 - \epsilon - \beta$ for all $\epsilon > 0$. Taking $\epsilon \rightarrow 0$ gives $b_1(\theta_1) \geq \theta_1 - \beta$. Similarly, types $\theta_2 = \theta_1 + \epsilon$ prefer rejection to acceptance, which implies $b_1(\theta_1) \leq \theta_1 - \beta$. Thus, we obtain the indifference condition:

$$b_1(\theta_1) = \theta_1 - \beta \Rightarrow \beta = \beta(\theta_1) \equiv \theta_1 - b_1(\theta_1) \quad (8)$$

Next, I argue that there exists an open neighborhood of θ_1 , N , such that when type θ_1 of player 1 mimics a type $x \in N \cap \Theta_1$, $x \neq \theta_1$, and $b_1(x)$ is rejected and countered with $\beta(x)$, θ_1 's best response is to accept this counter-offer $\beta(x)$. This is clearly the case for all $x > \theta_1$, because in this case rejection signals that $\theta_2 \geq x > \theta_1$ and bidding in the auction is truthful. Consider then $x < \theta_1$. Assume by contradiction that there does not exist such an N . That is, for every $\epsilon > 0$ there exists an $x \in \Theta_1 \cap [0, \theta_1)$ which is ϵ -close to θ_1 , and such that when θ_1 mimics x and $b_1(x)$ is rejected, it is a best response to reject $\beta(x)$. Then for each such x

⁴⁶Remark: the reader may suspect that because of this signaling it must be the case that $\beta = 0$. However, this is not true; i.e., does not follow from PBE. To see this, consider an equilibrium in which 2's rejection is followed by the offer $\beta > 0$. One cannot deduce (without any further assumptions) that undercutting β by some $\epsilon > 0$ is a profitable deviation for 2. The reason is that such a detectable deviation tells player 1 that σ_2 is not followed, and hence he need not conclude that $\theta_2 > \theta_1$; consequently, one cannot rule out the possibility that $\beta - \epsilon$ would be rejected.

⁴⁷Here, the assumption of pure strategies (in particular, deterministic acceptance rules) is important: if player 1 were allowed to randomize between acceptance and rejection, then θ_2 could have had multiple best-responses.

⁴⁸Because such an α , if it existed, would be accepted with certainty by player 1.

the following holds:⁴⁹

$$\beta(x) \leq \text{Prob}(\theta_2 \leq \theta_1 | \theta_2 \geq x) \quad (9)$$

Taking $\epsilon \rightarrow 0$, (9) yields $\beta(\theta_1) \leq 0$.⁵⁰ Since $\beta(\theta_1)$ is a bribe offer, it is non-negative, hence $\beta_1(\theta_1) = 0$, or $b_1(\theta_1) = \theta_1$. That is, player 1 of type θ_1 offers his true valuation. I argue that this behavior cannot be a part of an equilibrium. To see this, note that this behavior gives player 1 the payoff 0, because rejection of the offer means that $\theta_2 \geq \theta_1$, in which case player 2 reacts with the counter offer $\beta(\theta_1) = 0$, and 1's payoff would be 0 whether he accepts this counter offer or rejects it and goes to compete in the auction; on the other hand, trivially, 1's payoff is 0 if the offer $b_1(\theta_1) = \theta_1$ is accepted. Consider now a deviation to another offer, $b' \in (0, \theta_1)$. Since all $\theta_2 \in (0, b')$ accept this offer, it gives a strictly positive expected payoff. Hence, as was argued, $b_1(\theta_1) = \theta_1$ cannot be a part of an equilibrium.

Therefore, such a neighborhood N exists; that is, if θ_1 mimics a nearby type $x \in \Theta_1$ and the offer $b_1(x)$ is rejected, then it is indeed his best-response to accept the counter-offer $\beta(x)$.

Consider then θ_1 who mimics a nearby type $x \neq \theta_1$, $x \in \Theta_1$; according to what was just established, the expected payoff associated with this deviation is:

$$\Pi(x|\theta_1) \equiv F_2(x)(\theta_1 - b_1(x)) + (1 - F_2(x))\beta(x) \quad (10)$$

Note that $\Pi(x|\theta_1)$ is differentiable (wrt x) at any point where b_1 is differentiable; the FOC is:

$$\{f_2(x)(\theta_1 - b_1(x)) - F_2(x)b_1'(x) - f_2(x)\beta(x) + (1 - F_2(x))\beta'(x)\}|_{x=\theta_1} = 0$$

This gives $b_1'(\theta_1) = 1 - F_2(\theta_1)$. From (8) it follows that $\beta'(\theta_1) = F_2(\theta_1)$ at each differentiability point θ_1 . Since β is absolutely continuous (it is a difference of two absolutely continuous functions), $\beta(\theta_1) = c + \int_0^{\theta_1} F_2(t)dt$ for some constant c . Moreover, $c = 0$. To see this, suppose first that $c < 0$. In this case $\beta(\theta_1) < 0$ for all sufficiently small θ_1 's, which contradicts the fact that a bribe offer must be non-negative. Suppose on the other hand that $c > 0$. Then from (8)

⁴⁹Because valuations belong to the unit interval, the expected payoff in the non-cooperative auction is bounded from above by the winning probability.

⁵⁰From (8) we know that β is continuous, because it is the difference of two absolutely continuous functions, and hence is absolutely continuous. Thus, $\beta(x) \rightarrow \beta(\theta_1)$ as $x \rightarrow \theta_1$.

it follows that $\theta_1 - b_1(\theta_1) \geq c > 0$ for arbitrarily small θ_1 's, in contradiction to $b_1(\theta_1) \geq 0$ for all θ_1 . Therefore, $\beta(\theta_1) = \int_0^{\theta_1} F_2(t)dt$. From (10) we have that $\Pi(\theta_1|\theta_1) = \beta(\theta_1)$. Therefore,

$$\Pi(\theta_1|\theta_1) = \int_0^{\theta_1} F_2(t)dt = F_2(\theta_1)\theta_1 - \int_0^{\theta_1} t f_2(t)dt = \pi_1^*(\theta_1)$$

where the second equality follows from integration by parts and π_1^* denotes 1's expected payoff in the non-cooperative equilibrium.

Finally, I argue that $\pi_1^*(\theta_1) = \beta(\theta_1) \leq \frac{\theta_1}{2}$ for every $\theta_1 \in \Theta_1$. To see this, assume by contradiction that there exists some θ_1 for which $\beta(\theta_1) > \frac{\theta_1}{2}$. Consider the case where player 1 of type θ_1 reveals himself through the offer $b_1(\theta_1)$ and it is now 2's turn to respond, and $\theta_2 > \theta_1$. I argue that 2 has a profitable deviation: to counter with $\frac{\theta_1}{2} + \epsilon$, where $\epsilon > 0$ is sufficiently small, so that $\frac{\theta_1}{2} + \epsilon < \beta(\theta_1)$. To prove that this deviation is profitable, it is sufficient to prove that it will be accepted by player 1 with certainty. Note that this deviation by player 2 is a detectable deviation: player 1 expects to see the offer $\beta(\theta_1)$, but instead he sees a different offer. Thus, he needs to revise his beliefs, and by assumption he assigns probability 1 to the event $\{\theta_2 \geq \frac{\theta_1}{2} + \epsilon\}$. Thus, if he rejects 2's offer he expects his payoff in the auction to be no greater than $\theta_1 - (\frac{\theta_1}{2} + \epsilon) = \frac{\theta_1}{2} - \epsilon < \frac{\theta_1}{2} + \epsilon$. Thus, 2's offer is accepted and hence constitutes a profitable deviation. That $\beta(\theta_1) \leq \frac{\theta_1}{2}$ for all $\theta_1 \in [0, 1]$ follows from the fact that Θ_1 is dense in $[0, 1]$.⁵¹ □

7 Appendix B: Deletion of weakly dominated strategies

In the case of the second-price two-rounds game I restrict attention to those PBE in which player 1, upon seeing a sign of a sure deviation by 2, adopts a belief according to which player 2 did not offer a bribe which exceeds his valuation (condition (A)). This condition is satisfied

⁵¹Efficient and monotonic PBE may differ in the prescriptions they assign the players off the path. For example, consider the equilibrium which is identical to the one described in this proof, except that when player 2 sees an offer strictly greater than $1 - \pi_1^*(1)$ but smaller than his own type, he rejects it because he believes that $\theta_1 = 0$. Another example obtains if we change 1's acceptance rule when he faces an offer b which was offered by 2 after the rejection of an initial offer greater than $1 - \pi_1^*(1)$.

by any PBE that survives *iterated deletion of weakly dominated strategies*. Unfortunately, however, this refinement is problematic because of the following reasons. First of all, when deleting weakly dominated strategies “order matters”. A natural remedy to this difficulty is to assume that in every round of deletion all the weakly dominated strategies are deleted.⁵² Even if we adopt the maximal deletion approach, one serious problem still remains: it is not known how to describe the entire process, and hence it is not known whether there exists strategy profile that survive it. I therefore consider the following weaker version of this refinement—I will consider a single round of deletion. That is, keeping the strategy set of i to be the entire set of his strategies in the game, I delete all the weakly dominant strategies for j ; let Σ_i^1 be the set of i 's strategies that survive this process.

Lemma 6. *Every $\sigma \in \Sigma_1^1 \times \Sigma_2^1$ satisfies (A). Moreover, $\sigma^* \in \Sigma_1^1 \times \Sigma_2^1$.*

Therefore, one can take the formal solution concept in Theorem 1 to be *PBE that survives one round of deletion of weakly dominated strategies*, instead of assuming (A) explicitly.

Proof. First, I will prove that any strategy that instructs player 2 to counter some offer with a bribe which is greater valuation is deleted. Consider a strategy σ_2 that instructs 2 to counter b with some $x > \theta_2$ for some b and θ_2 . Let b^* denote one particular first offer. Let σ'_2 be identical to σ_2 except that it changes the aforementioned counters to 0. Clearly σ'_2 does at least as good as σ_2 against any strategy 1 may play. Consider the strategy for player 1 where all θ_1 offer b^* and accept any counter-bribe if b^* is rejected. Clearly, σ'_2 does strictly better than σ_2 against this strategy (for some θ_2 's).

Now, I will prove that $\sigma_i^* \in \Sigma_i^1$ for each i . Start with $i = 1$. Suppose that σ_1^* is weakly dominated by some σ_1 . First, I argue that σ_1 prescribes the same bribing function as σ_1^* . To see this, let $\theta_1 \in [0, 1]$. Let x denote that bribe prescribed by σ_1^* ($x = \theta_1 - \pi_1^*(\theta_1)$) and let y be the bid prescribed by the alternative strategy. The case $y > x$ is impossible, because the alternative strategy must do at least as good as σ_1^* against any strategy of 2; in particular, against the strategy that instructs 2 to accept any bribe. Thus, $y \leq x$. Consider then the strategy where 2 accepts a bribe iff it is greater than his valuation and responds to any other

⁵²This is the common approach to deletion of weakly dominated strategies. See, for example, Fudenberg and Tirole (1991, p. 461).

rejected bribe with the offer 0. In this case, since the alternative strategy gives an expected payoff which is at least as large as $\pi_1^*(\theta_1)$, it follows that $y = 0$. Finally, consider the strategy where 2, independently of his type, responds to x with a the counter θ_1 , and counter every other offer with 0. It follows that if one strategy of player 1 weakly dominates another, then they must have the same bribing function.

It follows that both σ_1 and σ_1^* prescribe the same bribing function. Suppose now that player 1 follows this bribing function, offers $\theta_1 - \pi_1^*(\theta_1)$, and his offer is rejected and countered with b . If $b = \pi_1^*(\theta_1)$ then σ_1 must prescribe acceptance, because player 2 may be playing σ_2^* . If $b \neq \pi_1^*(\theta_1)$ then σ_1 must assign acceptance iff $2b > \theta_1$, because it may be the case that player 2 plays a strategy that instructs him to offer his valuation following a rejection. Therefore, there does not exist a strategy that weakly dominates σ_1^* .

Consider now $i = 2$. Bribes greater than $1 - \pi_1^*(1)$ must be accepted because it may be the case that player 1 plays the following strategy: all $\theta_1 < 1$ offer 0 and $\theta_1 = 1$ offers the aforementioned bribe, and he rejects all counter-bribes. Since player 1 maybe playing σ_1^* , it follows from the proof of Theorem 1 that if σ_2 does at least as good against σ_1^* as σ_2^* , then the responses it prescribes to offers in $[0, 1 - \pi_1^*(1)]$ coincide with the ones prescribed by σ_2^* . \square

8 Appendix C: Additional proofs

Proof of Lemma 2: Let b denote 1's bribing function in a non-trivial equilibrium in monotonic strategies. The proof is in seven steps.

Step 1: If θ_1 reveals himself through $b(\theta_1) \equiv \beta > 0$, then the set of its acceptors is an interval.

Proof of Step 1: If all the types of player 2 accept β , then clearly this set is an interval— $[0, 1]$. Suppose then that rejection happens on the path. In this case, the corresponding on-path auction is such that θ_1 is common knowledge. Let B denote his bid in the BNE of this auction. Let θ_2 be an arbitrary type who accepts β in equilibrium,⁵³ and consider $\theta'_2 < \theta_2$. Assume by contradiction that θ'_2 rejects β in equilibrium. This rejection implies

⁵³Such a type exists, because $\beta > 0$ (any $\theta_2 < \beta$ accepts it).

$\theta'_2 - B \geq \beta$. Therefore $\theta_2 - B > \beta$, in contradiction to the fact that θ_2 accepts β in equilibrium. ■

Therefore, if $\beta > 0$ is a revealing offer then the set of its acceptors is $[0, A]$, where $A = A(\beta)$. Note that $A(\beta) \geq \beta$.⁵⁴

Step 2: If θ_1 reveals himself through $b(\theta_1) > 0$ and his offer is rejected, and if such a rejection happens in equilibrium, then he bids his true valuation in the post-rejection auction.

Proof of Step 2: Let β be a revealing offer which is rejected in equilibrium (by some types of player 2). Since the set of its acceptors is $[0, A(\beta)]$, it follows that in the post-rejection auction player 2's types is distributed on $[A(\beta), 1]$ according to $F_2|_{\{\theta_2 \geq A(\beta)\}}$. Let B denote player 1's bid in this auction. Note that every $\theta_2 \in [A(\beta), 1]$ submits the minimally winning bid in the auction, B^+ (because every such θ_2 rejected a strictly positive bribe). Since player 1 does not have a profitable deviation, any ϵ -increase of B is non-profitable. Hence, $B = \theta_1$. ■

Step 3: If $b(\theta_1) > 0$ is a revealing offer and $\theta_1 < 1$, then there exist at least one type θ_2 who rejects $b(\theta_1)$ in equilibrium.

Proof of Step 3: Assume by contradiction that such a $b(\theta_1)$ is accepted by all $\theta_2 \in [0, 1]$. Since b is continuous, $\theta_1 < 1$, and b is locally increasing at θ_1 , there exists a $\theta'_1 > \theta_1$ who makes a revealing offer that satisfies $b(\theta'_1) > b(\theta_1)$. Since this is a part of an equilibrium, it follows that there exists a θ_2 who rejects $b(\theta'_1)$ in equilibrium (otherwise, there is no reason to offer $b(\theta'_1)$). As we saw in Step 2, player 1 bids his valuation truthfully in the auction. Therefore, the incentive constraints of the aforementioned θ_2 imply $b(\theta_1) \geq \theta_2 - \theta_1 > \theta_2 - \theta'_1 \geq b(\theta'_1)$, in contradiction to the monotonicity of b . ■

Therefore, it follows from Step 2 and Step 3, that when a revealing offer $b(\theta_1) > 0$ is rejected

⁵⁴To see this, assume by contradiction that there exists such a β for which $A(\beta) < \beta$. Consider a type $\theta_2 \in (A(\beta), \beta)$. Such a type is supposed to reject β because it is strictly greater than $A(\beta)$, in contradiction to the fact that it is not incentive compatible to reject a bribe which exceeds one's valuation.

and $\theta_1 < 1$, player 1 bids his valuation truthfully in the post-rejection auction. Therefore, every θ_2 for which $\theta_2 - \theta_1 < b(\theta_1)$ accepts the bribe. That is, every $\theta_2 < \theta_1 + b(\theta_1)$ accepts the bribe. Therefore, for every revealing $\beta = b(\theta_1)$ we have $\theta_1 + \beta \leq A(\beta)$. Moreover, I argue that the latter inequality is satisfied as equality. Otherwise, if $\theta_1 + \beta < A(\beta)$ then types $\theta_2 \in (\theta_1 + \beta, A(\beta))$ accept the bribe even though it is strictly better for them to compete in the auction. Therefore, $A(\beta) = \theta_1 + \beta$ for each revealing $\beta > 0$.

Step 4: $b(0) = 0$.

Proof of Step 4: If $b(0) > 0$, then since all types $\theta_2 < b(0)$ accept this bribe, it gives $\theta_1 = 0$ a strictly negative payoff. Therefore, it cannot be a part of an equilibrium. ■

Step 5: If θ_1 reveals himself, so does every $\theta'_1 < \theta_1$.

Proof of Step 5: Assume by contradiction that there are $\underline{\theta}_1 < \bar{\theta}_1 < 1$ such that $b(\underline{\theta}_1) = b(\bar{\theta}_1) \equiv \beta$ and b is strictly increasing to the right of $\bar{\theta}_1$. Wlog, suppose that these θ_1 's are the endpoints of the interval where b is constant at the level β .

Case 1: $\underline{\theta}_1 > 0$. Let $\epsilon > 0$ such that $x = \underline{\theta}_1 - \epsilon > 0$ and $y = \bar{\theta}_1 + \epsilon < 1$ be two points where b is invertible. For types $z \in \{x, y\}$ the equilibrium payoff is $F_2(A(b(z)))[z - b(z)]$. Note that necessarily $A(b(x)) < 1$, or else $b(y) > b(x)$ cannot be a part of an equilibrium. Note also that $b(x) \approx b(y)$ whereas $F_2(A(b(y)))$ is significantly larger than $F_2(A(b(x)))$. Therefore, type x has a profitable deviation (to $b(y)$), a contradiction.

Case 2: $\underline{\theta}_1 = 0$. In this case, it follows from Step 4 that b is constant at the level $\beta = 0$ on some interval of the form $[0, m]$. By assumption the equilibrium is bribery-involving, hence $m < 1$. Consider $\theta_1 = m + \delta$ for an arbitrarily small $\delta > 0$. In equilibrium this θ_1 reveals himself through the a small offer $b(m + \delta) > 0$, since the bribing function is continuous. Rejection clearly happens on the path, because for a sufficiently small δ type $\theta_2 = 1$ prefers rejection and non-cooperative play of the auction to acceptance of $b(m + \delta) \approx 0$. Therefore, by Step 2, following a rejection of this offer type $\theta_1 = m + \delta$ submits his valuation in the auction. Therefore, all types $\theta_2 < m + \delta$ accept the bribe.

Consider then a type $\theta_1 \in (0, m)$. By offering $b(m + \delta)$ he can guarantee to himself a payoff which is bounded from below by $F_2(m + \delta)[\theta_1 - b(m + \delta)]$. Since $b(m + \delta) \rightarrow 0$ as

$\delta \rightarrow 0$, every $\theta_1 \in (0, m)$ can guarantee to himself an expected payoff of at least $F_2(m)\theta_1$ in this equilibrium. Since these types prefer to initiate the non-cooperative auction by offering 0, it follows that the expected payoff in the auction for every such type is at least $F_2(m)\theta_1$. Fix a $\theta_2 \in (0, m)$. His bid in the post-rejection-of-0-auction, call it $B = B(\theta_2)$, satisfies $0 < B \leq \theta_2$.⁵⁵ Fix $\theta_1 \in (0, B)$. Clearly, his bid in this auction is strictly smaller than B . Since player 2 is employing a monotonic bribing function in the auction,⁵⁶ it follows that the winning probability of θ_1 is bounded from above by $F_2(\theta_2) < F_2(m)$ and therefore his expected payoff in the auction is strictly smaller than $F_2(m)\theta_1$, a contradiction. ■

Step 6: There exists a type θ_1 who reveals himself.

Proof of Step 6: Otherwise, b is constant. By Step 4 it is identically 0, in contradiction to the assumption that the equilibrium is bribery-involving. ■

Step 7: There exists a θ^* such that b is constant on $[\theta^*, 1]$.

Proof of Step 7: As we saw earlier, the following holds for each type θ_1 who reveals himself through a strictly positive bribe: $A(b(\theta_1)) = \theta_1 + b(\theta_1)$. Consider the function $g(\theta_1) = \theta_1 + b(\theta_1)$. It is strictly increasing, continuous, and satisfies $g(0) = 0$ and $g(1) > 1$ (because the equilibrium is bribery-involving). Therefore, there exists a unique $\theta_1^* < 1$ such that $\theta_1^* + b(\theta_1^*) = 1$. Clearly, no type of player 1 will offer more than $b(\theta_1^*)$ in equilibrium. ■

⁵⁵ $B \leq \theta_2$ by assumption. To see that it is strictly positive, assume by contradiction that $B = 0$. Therefore, there is a positive probability that player 1 bids 0 in the auction, because if this probability is 0 then θ_2 's expected payoff from the bid $B = 0$ is 0, whereas the bid $\frac{\theta_2}{2}$ gives a strictly positive expected payoff. Look at a $\theta_1 > 0$ who bids 0; by the same arguments, there must be a positive probability that player 2 bids 0. Thus, there is a positive measure $\mu_i > 0$ of types of player i who bid 0 in this auction. Since neither θ_1 nor θ_2 profit from deviating to a positive small $\epsilon > 0$, it follows that *each one* of them beats the a full-measure subset of the set of the rival's 0-bidding types, which is impossible in equilibrium.

⁵⁶To see this, let $\theta'_2 > \theta_2$, and denote by B' and B their respective bids. Similarly, let P' and P be the respective winning probabilities with these bids. Assume by contradiction that $B' < B$. Therefore, $P' \leq P$. The incentive constraint for high type implies $P'[\theta'_2 - B'] \geq P[\theta'_2 - B]$. Similarly, the constraint for the low type is $P[\theta_2 - B] \geq P'[\theta_2 - B']$. Adding them up and canceling terms gives $P' = P$. Plugging this result into the low type's constraint gives $B' \geq B$, a contradiction.

Therefore, b is an ES function. □

Proof of Lemma 3: Let $F(x) \equiv -2x^2 \log x + x^2$. Note that $F(1) = 1$ and $f(x) = F'(x) = -4x \log x > 0$ for all $x \in (0, 1]$. Therefore, in order to prove that F is a distribution function, it is enough to prove that $F(x) \rightarrow 0$ as $x \rightarrow 0$. That is, to prove that $-x^2 \log x \rightarrow 0$ as $x \rightarrow 0$. This follows from $0 \leq -x^2 \log x \leq x$ for all $x \in (0, 1]$.

Next, I argue that $2F(x) + xf(x) \geq 1$ for all $x \in (0, 1]$. That is, the the following holds on $(0, 1]$:

$$2x^2[1 - 4\log x] \geq 1$$

Or, setting $t = x^2$, $1 - 2\log t \geq \frac{1}{2t}$ (note that t and x run on the same range— $(0, 1]$). It is readily verified that if $1 - 2\log t < \frac{1}{2t}$ for some $t \in (0, 1)$ then $1 - 2\log t < \frac{1}{2t}$ for all t in a neighborhood of 0. Hence, it suffices to prove that for every $n \in \mathbb{N}$ the following holds:

$$\frac{n}{2} \leq 1 - 2\log\left(\frac{1}{n}\right) \tag{11}$$

The proof is by induction; (11) is clearly satisfied for $n = 1$; hence I will prove that its satisfaction for n implies its satisfaction for $n + 1$. That is, it is enough to show:

$$\frac{n+1}{n} \left[1 - 2\log\left(\frac{1}{n}\right)\right] \leq 1 - 2\log\left(\frac{1}{n+1}\right)$$

or,

$$\frac{1}{n+1} \leq 2\left[\log\left(\frac{1}{n}\right) - \log\left(\frac{1}{n+1}\right)^{\frac{n}{n+1}}\right] = -2\log\left[n\left(\frac{1}{n+1}\right)^{\frac{n}{n+1}}\right]$$

Again, I proceed by Induction; namely, it is enough to show that:

$$-\frac{n+1}{n+2} \log\left[n\left(\frac{1}{n+1}\right)^{\frac{n}{n+1}}\right] \leq -\log\left[(n+1)\left(\frac{1}{n+2}\right)^{\frac{n+1}{n+2}}\right]$$

Rearranging we obtain $(n+1)\left(\frac{1}{n+2}\right)^{\frac{n+1}{n+2}} \leq n^{\frac{n+1}{n+2}}\left(\frac{1}{n+1}\right)^{\frac{n}{n+2}}$, or $\left[\frac{1}{n(n+2)}\right]^{\frac{n+1}{n+2}} \leq \left(\frac{1}{n+1}\right)^{\frac{-2}{n+2}} = (n+1)^{\frac{2}{n+2}}$, which is obviously true. □

Proof of Proposition 1: Consider player 1. The expected payoff corresponding to a deviation $x \leq \theta_1$ is $x(\theta_1 - x) + (1-x)\left(\frac{\theta_1 - x}{1-x}\right)(\theta_1 - \frac{\theta_1 + x}{2}) = \frac{\theta_1}{2} - \frac{x^2}{2}$, which is clearly maximized

at $x = 0$. It is straightforward that a deviation to $x > \theta_1$ cannot be profitable. Next, consider player 2 who needs to respond to a bribe $b \leq \theta_2$. Assume by contradiction that he reacts to it with a deviation to a positive counter-offer, $x \in (0, 1]$. Let player 1's beliefs be such that he assigns probability 1 to $\{\theta_2 = x\}$, hence accepting x if and only if $\theta_1 \leq 2x$. It is then clear that only $x \in (0, \frac{1}{2}]$ can be candidates for profitable deviation of player 2, each such $x < \frac{1}{2}$ giving an expected payoff of $2x(\theta_2 - x) + (1 - 2x)(\frac{\theta_2 - 2x}{1 - 2x})(\theta_2 - \frac{\theta_2 + 2x}{2}) = \frac{\theta_2^2}{2}$, so clearly adhering to the equilibrium strategy is optimal. Therefore, σ^N can be supported in equilibrium. \square

9 References

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