# Job Security, Stability and Production Efficiency*

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**Abstract**

We study a 2-sided labor market with a set of heterogeneous firms and workers in an environment where jobs are secured by regulation. Without job security Kelso and Crawford have shown that stable outcomes and efficiency prevail when all workers are (weak) gross substitutes to each firm, in the sense that increases in other workers' salaries can never cause a firm to withdraw an offer from a worker whose salary has not risen. It turns out that by introducing job security, stability and efficiency may still prevail, and even for a significantly broader class of production functions.

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# 1 Introduction

Since the work of Kelso and Crawford (1982) the 2-sided many-to-one matching model has emerged as the prominent tool to analyze labor markets whenever firms and workers are heterogeneous. The notion of stability, initially due to Gale and Shapley (1962), is the standard solution concept for matching models in general and for labor markets in particular. A stable outcome is an allocation of workers to firms (of which one firm is the outside option of unemployment) and a salary vector for the workers such that no combination of a single firm and a set of workers can improve their position while disregarding the others (there is no 'blocking coalition'). Underlying the logic of this solution concept is the notion of a free, unregulated, competitive market, where any coalition can withdraw from the market if the market does not provide them with a desired outcome.

A fundamental question about stability, as with any game theoretic (or economic) solution concept is its existence. An elegant, yet in-existent solution concept falls short of being satisfactory. In their original paper, Kelso and Crawford prove existence, as well as efficiency, under the assumption that firms' preferences over sets of workers exhibit "gross-substitutability" (on which we elaborate in the sequel). Much of the follow-up literature followed in their footsteps and assumes gross-substitutes production function. In fact, Gul and Stacchetti (1999) have shown that existence of a stable outcome may not be guaranteed beyond gross substitutes production functions and the theory then becomes mute for such markets. To remedy this, we consider the following research question: Can one weaken the requirements underlying the notion of stability, in some natural way, to obtain existence for a larger class of markets?

In reality many labor markets are regulated and in particular much of the regulation provides various degrees of job security to workers.[1] Job security regulation, within the context of a matching model, should be be seen as a hurdle on the formation of blocking coalitions. Thus, under such regulation, one should expect stability to hold for a larger class of production functions. This is exactly the line of thought we pursue.

The theoretical literature on matching seems to be mute about the possibility and implications of job security. Thus, partly to remedy the existence problem of stable outcomes and partly motivated by observations about real labor markets, the present paper proposes a new solution concept, JS-stability. We do so by revising the notion of stability so it accounts for a regulated labor market. In particular we would like to model a regulated market where firms cannot unilaterally fire employees (or where such costs of firing are prohibitively high). In such labor markets for a

[1] In most European countries many employees have indefinite contracts which make it very difficult and very costly for an employer to terminate a contract. In the UK, for example, the tenure necessary to qualify for such protection was lowered in 1999 from 24 to 12 months (Marinescu, 2009). In Germany, the 1951 Dismisal Protection Act which is still largely valid today acknowledges that workers have the right to keep their jobs, and, for example, fixed term contracts are allowed only for a period of up to 18 months (Emmenegger and Marx, 2011). High job security exists in many non-European countries as well. In India, as another example, the Industrial Disputes Act of 1976 requires that written permission to retrench workers be obtained, normally from the relevant state government (Fallon and Lucas, 1991).

firm to be part of a blocking coalition it must account for its current employees and ensure their utility is not compromised. More simply, such a firm must retain its workers at their current salary level. Technically, such regulation implies fewer blocking coalitions. Consequently, the requirements underlying the implied notion of stability, which we refer to as JS-stability (where JS stands for Job Security), becomes easier to satisfy.

It is no surprise, therefore, that we can guarantee the existence of JS-stable outcomes in some markets where no stable outcomes exist. As previously discussed, a key assumption for most results on labor markets is that of gross-substitutability. In the Kelso and Crawford model that we adopt, such gross-substitutability is a necessary and sufficient condition for a variety of results (see, among others, Kelso and Crawford (1982), Gul and Stacchetti (1999) and Ausubel (2006)). Our treatment, on the other hand, goes substantially beyond the scope of gross-substitutability and allows for a broader class of preferences. In fact, existence and optimality of a JS-stable outcome is guaranteed for the class of 'Almost Fractionally Sub-additive' valuations (AFS), which we formally define in the sequel. Furthermore, it is shown that this class is a maximal class for which such existence and optimality of a JS-stable allocation hold.

The 'gross substitutes' assumption, and in fact any assumption on substitutability, obviously rules out the treatment of markets where complementarities exist among the workers. Such complementarity is vital for the analysis of many particular markets. Two leading examples are those of the matching between players and sports clubs — as clubs are focused on generating a team spirit and building synergy among their players — and of the matching between universities and academics.[2] The class of AFS production functions, which is central to our analysis, allows for some limited form of complementarity among workers (see Example 1) and so we can undoubtedly argue that our work goes beyond substitutability.[3] In addition, it is straightforward to verify that the class of AFS production functions is a superset of the class of gross substitutes. In fact, it has been shown by Lehmann et al. (2006) to be substantially larger than the class of gross substitutes in some natural measure theoretic terms.[4]

## 1.1 Our contribution

Our contribution is conceptual as well as technical.

- Our conceptual contribution is two-fold:

[2]As an example for this complementarity we note that this research might not have been conducted had Lavi and Smorodinsky or Fu and Kleinberg not been members of the same department.

[3]In the neoclassical matching literature, starting with Becker (1973), the assumption of complementarities (otherwise known as 'positive assortative matching') is cardinal for many of the results. Note that the notion of complementarities assumed in that literature is related to the synergy between a firm and a (single) worker. In contrast, we refer to complementarities among workers with respect to a firm's production function.

[4]Technically, for a natural measure over the set of all production functions, the set of all functions exhibiting gross-substitutes has measure zero while the class $AFS$ has positive measure.

1. We introduce a new solution concept for the many-to-one matching model — JS-stability. This solution concept, tailored to analyze labor markets with employment protection, is inspired by the prevalence of regulation in many countries (in the EU in particular) which puts significant restrictions on firms' ability to fire employees. The on-going public debate of such regulation has not been part of the matching literature so far and JS-stability provides a (preliminary) means of such analysis.
2. We introduce a new class of production functions, dubbed 'almost fractionally sub-additive' (AFS) functions. Recall that a production function is called submodular if it exhibits decreasing productivity. It is well-known that the class of submodular production functions strictly contains the class of gross-substitutes production functions (and in fact significantly expands it). Our class AFS strictly contains and significantly expands the class of submodular production functions.

- For these new concepts we prove the following theorems:

1. We provide analogs of the welfare theorems to markets with job security. On the one hand, if firms' production functions are almost fractionally sub-additive (AFS) then any efficient outcome is sustained as a JS-stable outcome. On the other hand, although there may be inefficient JS-stable outcomes, we provide a tight bound on the efficiency loss such an outcome entails. In fact, in cardinal terms, summing over all players' utilities (as expressed with a numeraire good), the social welfare of any JS-stable outcome is at least 50% of the most efficient outcome.
2. We show that the family of AFS production functions is maximal with respect to obtaining our welfare theorems.
3. We provide a natural decentralized mechanism which yields a JS-stable outcome in equilibrium.

## 1.2 Related Literature

We briefly discuss three strands of related literature:

- **Beyond gross-substitutes:** The existence of stable outcomes under weaker notions of substitutability has received recent attention in the literature on matching with contracts, a model of labor markets and stability that originates with Hatfield and Milgrom (2005) and generalizes Kelso and Crawford (1982). Within this model Hatfield and Kojima (2010) define two notions, 'bilateral substitutes' and 'unilateral substitutes', that extend the original substitutes condition and still ensure existence of stable outcomes. Sönmez and Switzer (2013) and Sönmez (2013) demonstrate the applicability of these extended classes in the context of the 'cadet-branch matching problem'. However, these new classes shed no light on the original Kelso and Crawford (1982) model. When folding back the new classes of production function into the Kelso and Crawford (1982) model one obtains that the classes of 'bilateral substitutes', 'unilateral substitutes' and 'gross substitutes' are one and the same.[5] This is no surprise given the maximality theorem of Gul and Stacchetti (1999) which argues that one cannot go beyond the class of 'gross substitutes' without considering weaker notions of stability as we do.

- **Dynamic matching markets:** A sequence of papers starting with Compte and Jehiel (2008), and more recently Pereyra (2013), study a dynamic two-sided matching model of labor markets with existing workers who are guaranteed to be matched with at least as good partners as their current ones. Although these papers share a similar motivation to ours, these papers have very different models: They considered matchings with non-transferable utilities (i.e., non-negotiable salaries), and their markets have specific sets of workers (i.e., existing workers) who have secured jobs. Additional papers in this strand are Kurino (2011) in the context of on-campus housing for college students (where freshmen apply to move in and graduating seniors leave) and Ünver (2010) in the context of kidney exchange. These papers focus on the unit demand case (one-to-one matching) and utilities are non-transferable.

- **Job Security:** The lion's share of the theoretical literature on job security and employment protection legislation makes use of partial and general equilibrium in dynamic models. A common thread of all these models is that the work force is assumed *homogeneous* (e.g., Gavin, 1986, Lazear, 1990, Acemoglu and Shimer, 2000, Bertola, 2004), which is in sharp contrast with our heterogeneity assumption. Typically, a firm's productivity depends on the size of the workforce but not on the exact composition of workers it employs. Whereas our model is static and with complete information these models are dynamic and information stochastically unravels with time (e.g., workers' productivity and firms' technology). Whereas our work is more concerned with existence and efficiency of stable outcomes with regulation, their focus is on the impact of regulation on unemployment rates. Interestingly, the findings of this literature, both theoretically and empirically, are inconclusive (see the survey by Bertola (1999)). Although the current paper does not discuss unemployment rates we argue for the relevance of the new notion of stability to such an analysis. In particular, comparison of unemployment rates in stable vs. JS-stable outcomes may shed light on this important topic.

### 1.3 Other matching markets

The notion of JS-stability is primarily motivated by regulatory intervention designed to increase job security in labor markets. However, it may also have relevance in the study of immigration and

[5] The way to embed the latter model in the former is by restricting attention to contracts of the form of a triplet $(m, n, s)$, interpreted as a contract where worker $m$ is employed by firm $n$ for the salary $s$.

community formation. In this context, matching takes place between countries on the one hand and citizens on the other hand. Thus, firms are replaced by countries and workers by citizens. In such a matching market there is also clear asymmetry in the flexibility to divorce. Typically, once citizenship is granted to someone it is (almost) impossible to revoke. On the other hand, although there exists a barrier for citizens to immigrate and replace their current citizenship with a different one such a barrier is clearly lower (which can be evidenced empirically). Thus, a variant of JS-stability to such a NTU setting may correctly represent the feasible community structure in a model of immigration.[6] In fact, there may be additional many-to-one matching markets where divorce costs on both sides of the market are highly asymmetric and so the notion of JS-stability becomes an adequate tool for their analysis.

### 1.4 Paper structure

Section 2 introduces the model and details the new solution concept as well as the class of production functions we study. One particular aspect of our model is that each worker requires a minimal wage in order to work for any given firm. These minimal wages not only differ across workers but may differ across firms for any given worker and so the model is quite asymmetric. Section 3 focuses on a special case of our model where such asymmetry does not exist and the minimal wage is set to zero for all workers and all firms. It furthermore shows that this reduction is technically meaningless and results which are true for the special case are true for the more general model with asymmetry. Section 4 provides the main results, and Section 5 discusses future research.

## 2 Model

A labor market is composed of a set of $N$ firms and $M$ workers such that each firm hires as many workers as it wishes, but each worker is allowed to work only at one firm. Each firm pays its workers a salary and the utility of each worker depends on which firm he works for and the salary he receives. The firms' objective function is their profit and each firm's profit is the difference between the value of its production (in salary units) and the salaries it pays out. Note, in particular, there are no externalities among workers nor among firms.

The formal model we use is due to Kelso and Crawford (1982) — A labor market is a tuple $(N, M, v, b)$ where $N$ is a finite set of firms and $M$ is a finite set of workers with quasi-linear utility function (in the sequel we abuse notation and use $N$ and $M$ to denote the cardinality of these sets as well). $v = \{v^n\}_{n \in N}$, where $v^n : 2^M \to \Re_+$ is firm $n$'s monotonically increasing production function, as measured in the same units as salaries.[7] We calibrate $v^n(\emptyset) = 0$. $b = \{b^n_m\}_{m \in M, n \in N}$, where $-b^n_m$ is the valuation, in salary terms, of worker $m$ for working at firm $n$ without being paid.

[6] We thank Yoram Weiss for pointing out this connection between JS-stability and community formation.
[7] $v^n$ is monotonically increasing if $C \subset D \implies v^n(C) \leq v^n(D)$.

In fact we typically think of $b_m^n$ as the minimal salary requested by worker $m$ for working at firm $n$ and hence the negation sign. Thus, the quasi-linear utility for this worker is $u_m(n,s) = s - b_m^n$ when her salary is $s$.[8] Hereinafter firm 0 will denote unemployed workers and we calibrate $b_m^0 = 0$ for all $m$. We refer to worker $m$ as *"salary-driven"* if $b_m^n = 0$ for all $n$.

As productivity is measured in salary units, the profit of firm $n$ from employing a set of workers $C$ when workers' salaries are $\{s_m\}_{m \in M}$ is $\Pi^n(C; s) = v^n(C) - \sum_{m \in C} s_m$. We often abbreviate the tuple $(N, M, v, b)$ to $(v, b)$ as the sets of workers and firms are implicitly encoded in $(v, b)$.

For any two disjoint sets of employees, $C$ and $D$, we denote by $v(D|C) = v(D \cup C) - v(C)$ the marginal productivity of $D$ given $C$ and we also abuse notation and write $m$ to denote the singleton set $\{m\}$ as well (hence $v(m)$ will denote the productivity of a single worker, $m$).

Our results require the following assumption that relates each worker's minimal salaries to his marginal productivity. This "marginal productivity assumption" (MP), originally made by Kelso and Crawford (1982), states that the marginal productivity of any firm from any employee is at least the employee's minimal desired salary. Formally,

$$\forall n,\ C \subset M,\ m \in M \setminus C,\quad v^n(m|C) \geq b_m^n \geq 0. \tag{MP}$$

Notice that MP trivially holds for any "salary-driven" worker, i.e., a worker with all minimal salaries being equal to zero. More generally, we view this as a behavioral assumption on the way workers set minimal salaries. In particular, Kelso and Crawford (1982) justify this assumption by writing *"This is a natural restriction, since if a worker's marginal product, net of the salary required to compensate him or her for the disutility of work at a given firm, were negative, the firm could agree to let the worker do nothing for a salary of zero."* (section 2, page 1486).

An *assignment* of workers is a partition $A = \{A^0, A^1, \ldots, A^N\}$ of the set of workers, where $A^n$ denotes all workers employed by firm $n$, with $A^0$ interpreted as the set of unemployed workers. An *allocation* is a pair $(A, s)$ where $A$ is an assignment of workers and $s \in \Re_+^M$ is a vector of salaries. Such an allocation implies that any employee $m \in A^n$ works for firm $n$ at a salary $s_m$, whenever $n > 0$ and $m \in A^0$ implies that $m$ is unemployed and receives no salary.

**Definition 1.** An allocation $(A, s)$ is *individually rational (IR)* if (1) $v^n(A^n) - \sum_{m \in A^n} s_m \geq 0\ \forall n \in N$; and (2) $s_m \geq b_m^n$ for all $n \in N$ and $m \in A^n$.

The first part of this definition requires that each firm has a non-negative net profit and the second part requires that each employed worker is paid her minimal required salary.

[8]The model and results in Kelso and Crawford (1982) make use of an abstract utility function for workers, not necessarily of a quasi-linear form. In particular the units of such functions are abstract utilities in contrast with our quasi-linear functions whose units are in salary terms. Thus, as opposed to the Kelso-Crawford model, we can discuss a cardinal measure of social welfare and consequently measure efficiency levels, which is central to our results. However, we do so without ignoring non-salary related components of the work-package as these are embedded in the minimal salary component ($b$ in our model) which is dependent on the specific worker and the specific firm.

## 2.1 Stability and Job Security

The central solution concept we adopt is that of stability. However our notion of stability is a central innovation of our work and is weaker than the standard stability notions in two-sided markets. The stability notion we introduce is inspired by markets where job security is guaranteed by regulatory means. In particular, we consider the following simple yet somewhat extreme assertion - once a worker is employed by a firm for a certain salary only the worker can decide to quit whereas the firm cannot lower the salary nor can it fire the worker. Thus, the stability notion we introduce is an adaptation of the standard notion of stability to such regulatory restrictions. Formally:

**Definition 2.** A coalition $\{n, C\}$ is a **blocking** coalition for an allocation $(A, s)$ if and only if $C \subset M \setminus A^n$ and there exists a vector of salaries, $\hat{s} \in \Re_+^C$, such that:

- $u_m(n, \hat{s}_m) \geq u_m(k, s_m) \quad \forall k \in N, m \in A^k \cap C$ (workers in $C$ are better-off),
- $v^n(C|A^n) \geq \sum_{m \in C} \hat{s}_m$ (firm $n$ is better-off),

with at least one of the inequalities being strict.

The following alternative definition of a blocking coalition is more intuitive although less convenient for the proofs that follow:

**Definition 3.** A coalition $\{n, \hat{C}\}$ is a **JS-blocking** coalition for an allocation $(A, s)$ if and only if $A^n \subset \hat{C}$ and there exists a vector of salaries, $\hat{s} \in \Re_+^{\hat{C}}$, such that:

- $u_m(n, \hat{s}_m) \geq u_m(k, s_m) \quad \forall k \in N, m \in A^k \cap \hat{C}$,
- $v^n(\hat{C}) - \sum_{m \in \hat{C}} \hat{s}_m \geq v^n(A^n) - \sum_{m \in A^n} s_m$,

with at least one of the inequalities being strict.

The classical notion of a blocking coalition, as in Kelso and Crawford (1982), is similar to the above definition except it does not require that $A^n \subset \hat{C}$. The connection between these two definitions is as follows:

**Lemma 1.** *Let $C \subset M \setminus A^n$. The coalition $\{n, C\}$ is a blocking coalition for the allocation $(A, s)$ if and only if $\{n, A^n \cup C\}$ is a JS-blocking coalition for the allocation $(A, s)$.*

*Proof.* Note that, without loss of generality, we may assume that $\hat{s}_m = s_m$ for any $m \in A^n$ in the definition of JS-blocking. With this at hand the proof is immediate. □

**Definition 4.** An allocation $(A, s)$ is called *Job Security stable (JS-stable)* if it is IR and there exist no blocking coalitions.[9] [10]

In words, the requirement for JS-stability, beyond IR, is that there exists no firm and no set of agents currently not working for this firm such that the firm can offer better working terms for these agents (first set of inequalities) while maintaining its current set of workers and increasing its profits (second inequality). This is a weaker notion than the core allocation defined by Kelso and Crawford (1982). While Kelso and Crawford require that an allocation be immune to a deviation by a coalition of workers and a firm where such workers may (partly) replace the firm's current working force, our notion ignores this possibility as it is banned by regulation.

JS-stability models an extreme version of regulation related to job security. Thus, the inefficiency induced under JS-stability may be seen as a lower bound on the efficiency implications of some more realistic regulation. Indeed, as we demonstrate in this work, in spite of our modeling choice, efficiency partly prevails. This suggests that weaker forms of regulation designed for job security do not necessarily contradict efficiency.

### 2.2 Efficiency

The *efficiency level* of an assignment $A$ is $P(A) = \sum_n v^n(A^n) - \sum_{m \in A^n} b^n_m$ (recall that $v^n(\cdot)$ and $b^n_m$ are all measured in salary units). An assignment is *efficient* if it maximizes the efficiency, over all possible assignments.

### 2.3 AFS production functions

The structure that we assume on the production technology significantly expands SM (let alone GS). To define this structure we recall the following definition from cooperative game theory: For any $C \subseteq M$, a vector of non-negative weights $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ is a *"fractional cover"* of $C$ if for any $m \in C$, $\sum_{\{D \subseteq C: m \in D\}} \lambda_D = 1$.[11] An example of a fractional cover of the set $\{a, b, c\}$ is $\lambda_D = \frac{1}{2}$ for any subset with two workers and $\lambda_D = 0$ otherwise.

**Definition 5.** A firm's production function $v$ is *Fractionally Sub-additive on* $C \subseteq M$ if for any fractional cover $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ of $C$, $v(C) \leq \sum_{D \subseteq C, D \neq \emptyset} \lambda_D v(D)$.

We can offer the following intuition for this notion: assume a firm can either make use of the set $C$ of workers during a single period or it can break $C$ into subsets of workers (possibly overlapping)

[9] Arguably, one could consider a stronger notion of stability where blocking coalitions may involve more than a single firm. To demonstrate this consider the JS-stable allocation of example 4. Note that by switching between workers $b$ and $c$ we improve the firms' utility without jeopardizing any of the workers. This suggests that for an alternative notion of a blocking coalition, involving more than a single firm, the allocation in the example might not be stable. We re-visit this issue in our concluding remarks where we introduce the JS-Core.

[10] Requiring individual rationality as part of the definition of JS-stability implies that a firm that is not profitable, and thus in danger of bankruptcy, need not comply with job security regulation.

[11] The term used in cooperative game theory is "balanced collection of weights", see Osborne and Rubinstein (1994).

and deploy the subsets sequentially, each for a fraction of a period, such that any employee works a full period of time. The production function is fractionally sub-additive on $C$ if the latter option is always at least as productive as the former. In the example of a fractional cover preceding Definition 5, the firm will (weakly) prefer having the three workers work in three shifts of pairs, each for half of a time period, over employing all three workers simultaneously for a single time period.

**Definition 6.** A firm's production function $v$ is *Fractionally Sub-additive*, denoted $v \in FS$, if for any $C \subseteq M$, $v$ is Fractionally Sub-additive on $C$.[12]

This class of production functions was recently studied in the context of combinatorial auctions, starting with Nisan (2000), Feige (2009), and Dobzinski et al. (2010).

The Bondareva-Shapley theorem gives a useful characterization of FS in terms of "supporting salary vectors":

**Definition 7.** A vector of salaries, $s$, is called a *supporting salary vector* for the production function $v$ and a subset of workers $C \subset M$ if (1) $\sum_{m \in C} s_m = v(C)$; and (2) For any $D \subset C$, $\sum_{m \in D} s_m \leq v(D)$.[13]

**Theorem 1 (Bondareva-Shapley Theorem).** *$v$ is* Fractionally Sub-additive on $C \subseteq M$ *if and only if there exists a non-negative supporting vector of salaries for $v$ on $C$.*[14]

*FS* continues to enforce substitutability in the sense that, for any $v \in FS$ and any two sets $S, T$, $v(S \cup T) \leq v(S) + v(T \setminus S) \leq v(S) + v(T)$ (as the weights $\lambda_S = 1$ and $\lambda_{T \setminus S} = 1$ are a fractional cover of $S \cup T$). Therefore, we extend this class slightly further, to allow some (limited) form of complementarity. The way we expand the class *FS* is by relaxing some of the restrictions on the grand coalition:

**Definition 8.** A firm's production function $v$ is *Almost Fractionally Sub-additive*, denoted $v \in AFS$, if:

1. For any $C \subset M$ (excluding $C = M$) $v$ is Fractionally Sub-additive on $C$; and
2. $v(M) \leq \frac{\sum_{m \in M} v(M \setminus m)}{|M|-1}$.

One thing to note is that *AFS* allows for a certain type of complementarities: a single worker and the set of all other workers may be complements. This is demonstrated in the following example:

[12] Originally, a similar notion was introduced by Bondareva (1963) and Shapley (1967) in the context of value functions for cooperative games. However, Bondareva and Shapley actually take interest in the reversed inequalities and refer to such value functions as *balanced*.

[13] Originally, Bondareva (1963) and Shapley (1967) consider the case where these inequalities are reversed and refer to a collection of such vectors as the *core* of a cooperative game

[14] Bondareva and Shapley actually prove that the core is not empty if and only if the value function is balanced, which is equivalent to the stated theorem.

**Example 1.** Assume there are 3 workers, denoted $a, b, c$ and let the production function $u$ be defined by: $u(a) = u(b) = u(c) = 3$, $u(\{a, b\}) = u(\{a, c\}) = 6$, $u(\{b, c\}) = 4$, $u(\{a, b, c\}) = 8$. We leave it to the reader to verify that $u \in AFS$ (but not in $FS$). Note that the worker $a$ and the pair $\{b, c\}$ are complements.

The complementarity displayed in Example 1 is possible as we do not require the fractional sub-additivity to hold on the full set of workers but only on strict subsets.

A technical observation about functions in $AFS$ which we use in the sequel is:

**Lemma 2.** $v \in AFS \implies \sum_{m\in M} v(m|M \setminus m) \leq v(M) \leq \sum_{m\in M} v(m)$.

*Proof.* To prove the left inequality note that:

$$\sum_{m\in M} v(m|M\setminus m) = \sum_{m\in M} v(M) - v(M\setminus m) = |M|\cdot v(M) - \sum_{m\in M} v(M\setminus m) \leq |M|\cdot v(M) - (|M|-1)\cdot v(M),$$

where the last inequality follows from the definition of $AFS$. Thus, $\sum_{m\in M} v(m|M \setminus m) \leq v(M)$.

To prove the right inequality we proceed by induction on $|M|$. The claim trivially holds for $|M| = 1$. For $|M| > 1$, notice that for any $m \in M$ we have by the inductive assumption $v(M \setminus m) \leq \sum_{k\in M\setminus m} v(k)$ (since the restriction of $v$ to the set of workers $M \setminus m$ is also a production function in AFS, and the inductive assumption holds for this production function). Thus,

$$v(M) \leq \frac{\sum_{m\in M} v(M \setminus m)}{|M| - 1} \leq \frac{\sum_{m\in M} \sum_{k\in M\setminus m} v(k)}{|M| - 1} = \sum_{k\in M} v(k),$$

as claimed. □

## 2.4 The hierarchy of sets of production functions

At this point it is useful recall additional more restricted classes of production functions that have been previously studied in the literature and put the classes GS and AFS in context. A more detailed discussion, along with formal definitions and proofs can be found in many previous papers, see for example Lehmann et al. (2006).

Kelso and Crawford (1982) introduced the class of Gross Substitutes production functions, denoted here as $GS$. A production function is in $GS$ if, whenever it is optimal for the firm to hire a worker $m$ at a given vector of salaries, it remains optimal to hire $m$ whenever the salaries of workers other than $m$ are weakly increased while fixing the salary of $m$. It has long been recognized that the notion of substitutes that this class captures is very restricted. To illustrate this point, consider the following example:

**Example 2.** [Lehmann et al. (2006)] Assume there are 3 workers denoted $a, b, c$ and let the production function $v$ be defined by: $v(a) = v(b) = 2$, $v(c) = 4$, and every other subset of workers $S$

has $v(S) = 4$. It seems natural to refer to these workers as substitutes (they are certainly not complements, and they exhibit decreasing marginal productivity), but this production function is not in GS: Consider the salary vector $s_a = 0, s_b = 1, s_c = 2$. At these salaries, the firm will maximize utility by hiring $\{a, b\}$. However, if we increase the salary of $a$ to $s'_a = 2$, the firm will maximize utility by hiring $c$ alone.

A production function $v$ is called submodular if it exhibits decreasing marginal productivity. More formally, for every two sets of workers $S \subset T$ and for any worker $x \notin T$, $v(x|T) \leq v(x|S)$. We denote by *SM* the set of submodular production functions. Lehmann et al. (2006) show that $GS \subset SM$.

The production function in the above example is submodular but not in GS, hence the inclusion is strict. In fact, Lehmann et al. (2006) argue that the difference between these two classes is quite significant, not a mere anecdote, in the following sense. A production function can be represented by a vector in $(2^M - 1)$-dimensional Euclidean space that specifies the value of the production function on every non-empty set. Under this natural representation, Lehmann et al. (2006) prove that the set *GS* has Lebesgue measure zero while *SM* has positive measure.

Lehmann et al. (2006) show that $SM \subset FS$ and that the inclusion is strict, as they demonstrate:

**Example 3.** [Lehmann et al. (2006)] Consider the following symmetric production function on three workers: any set of one or two workers produce 2, while the set of all three workers produce 3. This is clearly not in SM, and it can be easily verified that it belongs to FS.

Consequently we can conclude that $SM \subset AFS$ and so *SM* substantially expands *GS* in the above measure-theoretic sense.

# 3 Salary Driven Workers

Before we state our results we consider a variation of a labor market which we refer to as labor markets with *salary driven* workers. We use the notion of 'salary-driven' to emphasize that workers do not care about any aspect of their job, except for their salary. This manifests itself by setting the minimal wage requested by all workers to zero, across all firms ($b^n_m = 0 \ \forall m \in M, n \in N$).

For labor markets with salary driven workers the notions of individual rationality and JS-stability become simpler:

**Lemma 3.** *Let $b^n_m = 0 \ \forall m \in M, n \in N$. An allocation $(A, s)$ is* individually rational (IR) *if and only if $v^n(A^n) - \sum_{m \in A^n} s_m \geq 0 \ \forall n \in N$.*

The proof is straightforward and therefore omitted.

**Lemma 4.** *Let $(A, s)$ be an allocation in a salary driven market $(v, 0)$. Then $(n, C)$ is a* blocking coalition *if and only if $v^n(C|A^n) > \sum_{m \in C} s_m$.*

*Proof.* A sufficient condition: Clearly if $v^n(C|A^n) > \sum_{m\in C} s_m$ then by setting $\hat{s} = s$ the coalition $(n, C)$ is a blocking coalition.

A necessary condition: Assume now that $(n, C)$ is a blocking coalition and so there exits some vector of salaries $\hat{s}$ such that $v^n(C|A^n) \geq \sum_{m\in C} \hat{s}_m$ and $\hat{s}_m \geq s_m \quad \forall m \in C$, with one of the inequalities being strict. Assume the strict inequality is $v^n(C|A^n) > \sum_{m\in C} \hat{s}_m$ then clearly $v^n(C|A^n) > \sum_{m\in C} s_m$ as well and we are done. If, on the other hand, $v^n(C|A^n) = \sum_{m\in C} \hat{s}_m$ but for some $\hat{m} \in C$, $\hat{s}_{\hat{m}} > s_{\hat{m}}$ then once again we have $v^n(C|A^n) > \sum_{m\in C} s_m$ and we are done. □

As before an allocation $(A, s)$ is called *Job Security stable (JS-stable)* if it is IR and there exist no blocking coalitions.[15]

### 3.1 Labor markets, without loss of generality, have salary driven workers

In this section we construct tools that will enable us to prove results about labor markets by restricting attention to labor markets that have salary driven workers. Let $(v, b)$ be a labor market. We denote by $(v - b, 0)$ a labor market with salary driven workers and production functions $(v - b)^n(B) = v^n(B) - \sum_{m\in B} b^n_m$. The marginal productivity (MP) assumption implies that $(v - b)^n$ remains monotone. Similarly, if $(A, s)$ is some allocation then $s - b$ is the following vector of salaries: If $m \in A^k$ then $(s - b)_m = s_m - b^k_m$.

**Lemma 5.** *Let $(v, b)$ be a labor market. $v^n \in AFS$ if and only if $(v - b)^n \in AFS$.*

*Proof.* We prove the first direction and assume $v^n \in AFS$. To prove that $(v - b)^n \in AFS$ we need to show two things:

1. For any $C \subset M, C \neq M$, $v - b$ is fractionally Sub-additive on $C$. Indeed, let $\{\lambda_D\}_{D\subseteq C, D\neq\emptyset}$ be a fractional cover of $C$ and so $\sum_{m\in C} b_m = \sum_{D\subseteq C, D\neq\emptyset} \lambda_D \sum_{m\in D} b_m$. Consequently:

$$(v-b)(C) = v(C) - \sum_{m\in C} b_m \leq \sum_{D\subseteq C, D\neq\emptyset} \lambda_D v(D) - \sum_{D\subseteq C, D\neq\emptyset} \lambda_D \sum_{m\in D} b_m = \sum_{D\subseteq C, D\neq\emptyset} \lambda_D (v-b)(D).$$

2. $(v - b)(M) \leq \frac{\sum_{m\in M}(v-b)(M\setminus m)}{|M|-1}$. Indeed

$$(v - b)(M) = v(m) - \sum_m b_m \leq \frac{\sum_{m\in M} v(M \setminus m)}{|M| - 1} - \frac{\sum_m \sum_{k\neq m} b_k}{|M| - 1} = \frac{\sum_{m\in M}(v - b)(M \setminus m)}{|M| - 1}.$$

The opposite direction of the proof is similar and hence omitted. □

[15] The model of salary driven workers may be viewed as a combinatorial auction model with a seller who owns goods (the "workers") and buyers (the "firms") who have valuations for subsets of goods and compete for these goods. The interpretation of IR and efficiency in this model is straightforward, however the notion of a 'blocking coalition' and hence the notion of JS-stability forms a technical relaxation of the notion of a core for which the authors have compelling explanation.

Let $P_{(v,b)}(A)$ be the efficiency level of the assignment $A$ for the labor market $(v,b)$.

**Lemma 6.** $P_{(v,b)}(A) = P_{(v-b,0)}(A)$.

*Proof.* This is quite straightforward:

$$P_{(v,b)}(A) = \sum_n (v^n(A^n) - \sum_{m\in A^n} b^n_m) = \sum_n (v-b)^n(A^n) = P_{(v-b,0)}(A).$$

□

**Lemma 7.** *A is an efficient assignment for* $(v-b,0)$ *if and only if it is an efficient assignment for* $(v,b)$.

*Proof.* This follows directly from Lemma 6. □

**Lemma 8.** *The allocation* $(A,s)$ *is an (IR) allocation for the labor market* $(v,b)$ *if and only if* $(A,s-b)$ *is an (IR) allocation for* $(v-b,0)$.

*Proof.* Let $(A,s)$ be an (IR) allocation for $(v,b)$. Then, for each firm $n$, $v^n(A^n) \geq \sum_{m\in A^n} s_m$ which can be rewritten as $(v-b)^n(A^n) \geq \sum_{m\in A^n}(s_m - b^n_m) = \sum_{m\in A^n}(s-b)_m$. In addition, for each worker, $m$, $s_m \geq b^n_m$, where $m \in A^n$. Equivalently , $(s-b)_m \geq 0$ which means that $(A,s-b)$ is (IR) in the labor market $(v-b,0)$.

The proof of the opposite direction is similar and hence omitted. □

**Lemma 9.** *The coalition* $(n,C)$ *is a blocking coalition for the allocation* $(A,s)$ *in the labor market* $(v,b)$ *if and only if it is a blocking coalition for the allocation* $(A,s-b)$ *in the labor market* $(v-b,0)$.

*Proof.* Assume that $(n,C)$ is a blocking coalition for the allocation $(A,s)$ in the labor market $(v,b)$. Then there exists some vector of salaries $\{\hat{s}_m\}_{m\in C}$ such that:

- $\hat{s}_m - b^n_m \geq s_m - b^k_m$ for all $k$ and for all $m \in C \cap A^k$,
- $v^n(C|A^n) \geq \sum_{m\in C}\hat{s}_m$, implying $(v-b)^n(C|A^n) \geq \sum_{m\in C}\hat{s}_m - b^n_m$

with at least one of the inequalities being strict.

Let us set $\bar{s}_m = \hat{s}_m - b^n_m,\ \ \forall m \in C$. The above system of inequalities is equivalent to:

- $\bar{s}_m \geq s_m - b^k_m = (s-b)_m\ \ \forall k$ and $m \in A^k \cap C$,
- $(v-b)^n(C|A^n) \geq \sum_{m\in C}\bar{s}_m$,

with at least one of the inequalities being strict, implying the desired conclusion.

The proof of the opposite direction is similar and hence omitted. □

**Lemma 10.** *The allocation* $(A,s)$ *is a JS-stable allocation for* $(v,b)$ *if and only if the allocation* $(A,s-b)$ *is a JS-stable allocation for* $(v-b,0)$.

*Proof.* This is a direct consequence of Lemmas 8 and 9. □

## 4 Results

We now return to general labor markets and state our four main results. The first two connect JS-stability with efficiency and can be viewed as analogs for the First and Second Welfare Theorems. In particular, we show that whenever production functions are in $AFS$, efficient outcomes are JS-stable. In our third result we show that one cannot extend these results beyond the class of $AFS$ production functions. Our final result makes a connection between the notion of JS-stability and, inspired by cooperative game theory, a Nash equilibrium of a natural auction-like non-cooperative game played among the firms. All these four results parallel central results in the literature on stability in labor markets. We informally summarize our results and their parallels towards the end of this section.

### 4.1 A $\frac{1}{2}$-First Welfare Theorem

As one can expect, JS-stability does not guarantee efficiency. On the other hand the inefficiency of any JS-stable outcome is bounded:

**Theorem 2.** *If $(A, s)$ is a JS-stable allocation and $\bar{A}$ is an efficient assignment, then $P(A) \geq \frac{1}{2}P(\bar{A})$.*

Note that this result is not restricted to the class of AFS production functions; it holds for arbitrary non-AFS monotone production technologies that satisfy the MP assumption.

*Proof.* We first prove our result for labor markets with salary driven workers, denoted $(v, 0)$. Indeed, for every firm $n$ we have $v^n(\bar{A}^n \setminus A^n | A^n) \leq \sum_{m \in \bar{A}^n \setminus A^n} s_m$. Thus, we have

$$v^n(\bar{A}^n) \leq v^n(\bar{A}^n \cup A^n) \leq \sum_{m \in \bar{A}^n \setminus A^n} s_m + v^n(A^n)$$

Therefore

$$\sum_{i=1}^{n} v^n(\bar{A}^n) \leq \sum_{i=1}^{n} \left( \sum_{m \in \bar{A}^n \setminus A^n} s_m + v^n(A^n) \right) \leq \sum_{i=1}^{n} \left( \sum_{m \in \bar{A}^n} s_m + v^n(A^n) \right) \leq$$

$$\leq \sum_{m \in M} s_m + \sum_{i=1}^{n} v^n(A^n) = \sum_{i=1}^{n} \sum_{m \in A^n} s_m + \sum_{i=1}^{n} v^n(A^n) \leq 2 \sum_{i=1}^{n} v^n(A^n),$$

where the last inequality follows from (IR) of the assignment $A = (A^n)_{n \in N}$. This proves the claim for labor markets with salary driven workers.

Now let $(A, s)$ be a JS-stable allocation for an arbitrary labor market $(v, b)$ and let $\bar{A}$ be an efficient assignment for $(v, b)$. Therefore, $(A, s - b)$ is a JS-stable allocation for $(v - b, 0)$ (Lemma 10) and $\bar{A}$ is efficient for $(v - b, 0)$ (Lemma 7). Now:

$$P_{(v,b)}(A) = P_{(v-b,0)}(A) \geq \frac{1}{2} P_{(v-b,0)}(\bar{A}) = \frac{1}{2} P_{(v,b)}(\bar{A}),$$

where the left and right equalities follow from Lemma 6 and the inequality follows from the first part of the proof. □

This bound on the efficiency loss is tight as suggested by the following example:

**Example 4.** Consider a labor market with two salary-driven workers $a, b$ and two firms with unit-demand production functions, $v^1$, $v^2$, defined as follows. $v^1(a) = v^2(b) = 2$, $v^1(b) = v^2(a) = 1$.[16] The following allocation is JS-stable: firm 1 is matched to worker $b$, firm 2 is matched to worker $a$, and both salaries are 1. This allocation has welfare of 2, while the efficient allocation has welfare of 4.

The following example demonstrates that in the absence of the (MP) assumption the inefficiency is potentially unbounded:

**Example 5.** Consider a market with two workers and one unit demand firm, who values each worker for 2. Each worker has a minimal salary of 1. Notice that MP is violated in this setting, since the marginal production of each worker, given the other worker, is zero, while his minimal salary is 1. Assigning both workers to the firm, with a salary of 1 to each worker, is a JS-stable outcome. This outcome has zero welfare, while the optimal welfare is 1.

### 4.2 A Second Welfare Theorem

**Theorem 3.** *Let $(v, b)$ be a labor market. If $v^n \in AFS$ for all $n \in N$ then for any efficient assignment $A$ there is a salary vector $s$, such that $(A, s)$ is a JS-stable allocation.*

Note, in particular, that the existence of a JS-stable outcome is guaranteed under the conditions of Theorem 3. The proof considers two cases. First, if an efficient allocation does not allocate all workers to a single firm, we can invoke the Bondareva-Shapley theorem and find supporting vectors of salaries for each firm. It turns out that supporting vectors of salaries make the efficient allocation JS-stable. Second, if an efficient allocation allocates all workers to the same firm, it turns out that setting the salary of each worker to be her marginal productivity results in a JS-stable outcome.

*Proof.* We begin by proving our result for an arbitrary salary driven labor market $(v, 0)$, with production functions in $AFS$.

**Case 1: No efficient assignment assigns all workers to a single firm**: Therefore if $A = (A^1, \cdots, A^n)$ is some efficient assignment then $A^k \neq M$ for any firm $k$. Thus, we can apply Theorem

[16] A unit-demand production function satisfies $v(B) = \max\{v(x) : x \in B\}$ for any $B \subseteq M$. In words, a coalition can only produce as much as its top producing member. These production functions are in GS, hence also in AFS.

1 (which is our version of the Bondareva Shapley theorem) and conclude that for each $k \in N$, there exists a supporting vector of salaries, $\{s^k_m\}_{m \in A^k}$, for $(v^k, A^k)$. For any $m \in M$ let $n(m)$ denote the firm for which $m \in A^{n(m)}$ and set $s_m = s^{n(m)}_m$. We show that the allocation $(A, s)$ is JS-stable. IR follows immediately from the definition of a supporting vector of salaries. To finish our proof we must show that an arbitrary coalition, $(n, B)$, where $B \subset M \setminus A^n$, cannot be a blocking coalition. Denote $R^k = A^k \cap B$. As $A$ is efficient $v^n(A^n \cup B) + \sum_{k \neq n} v^k(A^k \setminus R^k) \leq \sum_{k \in N} v^k(A^k)$. Therefore $v^n(A^n) + v^n(B|A^n) \leq \sum_{k \in N} v^k(A^k) - \sum_{k \neq n} v^k(A^k \setminus R^k) = v^n(A^n) + \sum_{k \neq n} v^k(R^k|A^k \setminus R^k)$. As $\{s^k_m\}_{m \in A^k}$ is a vector of supporting salaries for $(v^k, A^k)$ we have $v^n(B|A^n) \leq \sum_{k \neq n} v^k(R^k|A^k \setminus R^k) \leq \sum_{k \neq n} \sum_{m \in R^k} s^k_m = \sum_{m \in B} s_m$, implying that $(n, B)$ is not a blocking coalition.

**Case 2: There is an efficient assignment that assigns all workers to firm** $n$: Efficiency implies that for any $k \neq n$ and any $m \in M$, $v^k(m) + v^n(M \setminus m) \leq v^n(M) = v^n(m|M \setminus m) + v^n(M \setminus m)$, therefore $v^k(m) \leq v^n(m|M \setminus m)$.

Now set $s_m = v^n(m|M \setminus m)$ for every $m \in M$. We show that this yields a JS-stable allocation:

- **IR**: By Lemma 2 $v^n(M) \geq \sum_{m \in M} v^n(m|M \setminus m) = \sum_{m \in M} s_m$, and IR follows from Lemma 3.

- **No blocking coalition**: For every firm $k \neq n$ and for every subset $B \subseteq M$, we apply Lemma 2:
$$v^k(B) \leq \sum_{m \in B} v^k(m) \leq \sum_{m \in B} v^n(m|M \setminus m) = \sum_{m \in B} s_m,$$
Thus no blocking coalition follows from Lemma 4.

So far we have proven our claim for a salary driven labor market. The proof for an arbitrary labor market follows from Lemmas 5, 7 and 10. □

Note that the proof of Theorem 3 is constructive and so it is suggestive of an algorithm that, given an efficient allocation, computes the salaries that support it as a JS-stable outcome.

## 4.3 The maximality of the set of production technologies $AFS$

We now turn to show that the set of production functions, $AFS$, is maximal with respect to the property that any efficient assignment can also be supported as a JS-stable allocation. In other words, if one of the firms has a production function that is not in $AFS$ it could be the case that some efficient assignment is not supported by a JS-stable allocation. In fact, we will show that it could be that none of the efficient assignments are supported by a JS-stable allocation. Formally:

**Theorem 4.** *If $\bar{v} \notin AFS$ then there exists a labor market $(v, 0)$, where $v^1 = \bar{v}$ and for all $n > 1$ $v^n \in AFS$ and if $A$ is an efficient assignment then for no vector of salaries $s$ is $(A, s)$ a JS-stable allocation of the market $(v, 0)$.*

Theorem 4 argues that $AFS$ is a maximal domain of production functions such that the second welfare theorem holds. In particular if $v \in AFS$ then a JS-stable outcome is guaranteed to exist. However, Theorem 4 does not argue maximality of $AFS$ with respect to the existence of JS-stable outcomes (that are not necessarily efficient). We take a slight detour before returning to prove Theorem 4 and show that such general existence cannot be guaranteed outside the class of Symmetrically Fractionally Sub-additive (SFS) functions, defined as follows:

**Definition 9.** A valuation $v$ is called *symmetrically fractionally sub-additive* if for any $B \subseteq M$ with $|B| \geq 2$, $v(B) \leq \frac{1}{|B|-1} \sum_{x \in B} v(B \setminus x)$. Let $SFS$ denote the set of all symmetric fractionally sub-additive valuations.

Compared with AFS, SFS removes the requirement $v(B) \in FS$ for all strict subsets $B$ of $M$. Instead, it requires only the *symmetric* FS inequalities to hold for $v(B)$, namely, the inequality that corresponds to the fractional cover $\{B - x\}_{x \in B}$ with equal weights $\frac{1}{|B|-1}$. Since we only remove requirements, SFS strictly contains AFS.

As intuition why production functions that violate SFS need not admit any JS-stable outcome, consider the following example with three workers.

**Example 6.** Suppose three salary-driven workers $M = \{a, b, c\}$, and a production function $u \notin SFS$ since $u(abc) > \frac{1}{2}[u(ab) + u(bc) + u(ac)]$ (recall that SFS requires a reversed inequality). Rearranging, $u(abc) < u(a|bc) + u(b|ac) + u(c|ab)$. Given this productiom function, we construct a second production function, such that the resulting market will not admit any JS-stable outcome. Fix a small $\epsilon > 0$ and define $v(x) = u(x|M - x) - \epsilon$ for any $x \in M$ (so $u(abc) < v(a) + v(b) + v(c)$). One can verify that it is not possible to allocate the three workers to the two firms in a JS-stable way. In particular it is instructive to check why the following two assignments are not JS-stable:

- If the first firm (with production function $u$) receives all three workers, any corresponding JS-stable salaries must satisfy $s_a + s_b + s_c \leq u(abc) < v(a) + v(b) + v(c)$. Thus, there must exist a worker $x$ with $s_x < v(x)$, violating JS-stability.

- If the second firm (with production function $v$) receives exactly one worker, $x$, and the first firm receives the two other workers, any corresponding JS-stable salaries must satisfy $s_x \leq v(x) < u(x|M - x)$, again violating JS-stability.

Proposition 1 below extends this example to show that JS-stability cannot be guaranteed beyond SFS:

**Proposition 1.** *If $u \notin SFS$ then there exist $k \leq n - 1$ unit-demand production functions $v_1, ..., v_k$ such that the salary driven labor market with $k + 1$ workers, $((v_1, ..., v_k, u), 0))$, does not admit any JS-stable allocation.*

We use the following technical lemma for the proof of Proposition 1.

**Lemma 11.**

1. $u \in SFS \;\Leftrightarrow\; \forall B \subseteq M$, $\sum_{m \in B} u(m|B \setminus m) \leq u(B)$.

2. $u \in SFS \;\Rightarrow\; \forall B \subseteq M$, $u(B) \leq \sum_{m \in B} u(m)$.

This result is similar to Lemma 2 and in fact the proof of Lemma 2 applies verbatim to this lemma as well and so the proof is omitted.

*Proof.* (of Proposition 1) Since $u \notin SFS$, Lemma 11 implies that there exists $B \subseteq M$ such that $\sum_{x \in B} u(x|B \setminus x) > u(B)$. We construct the following tuple of unit-demand valuations. For every worker $x \in M \setminus B$ we have two unit-demand valuations $v_x^{(1)} = v_x^{(2)}$ such that $v_x^{(i)}(x) = u(M) + 1$ and $v_x^{(i)}(y) = 0$ for any worker $y \neq x$. Additionally define a unit-demand valuation $v_B$ as follows. Choose a small enough $\epsilon > 0$ such that (i) $\sum_{x \in B}(u(x|B \setminus x) - \epsilon) > u(B)$, and (ii) $\forall x \in B$ such that $u(x|B \setminus x) > 0$, $\epsilon < u(x|B \setminus x)$. Then define

$$v_B(x) = \begin{cases} \max(0, u(x|B \setminus x) - \epsilon) & x \in B \\ 0 & x \notin B \end{cases}$$

We show that there does not exist a JS-stable allocation for this labor market. Note that in every possible JS-stable allocation in this labor market, every worker $x \in M \setminus B$ must be allocated to either the firm with valuation $v_x^{(1)}$ or $v_x^{(2)}$ and its salary must be $v_x^{(1)}(x)$. As a result, note that if in some JS-stable allocation a firm with valuation $v_x^{(i)}$ is being allocated some worker $y \in B$, $y$'s salary must be zero. Suppose by contradiction that there exists a JS-stable allocation with salaries $p$ and in which the firm with valuation $u$ is allocated a set of workers $T_u$, the firm with valuation $v_B$ is allocated a set of workers $T_v$, and $T_u \cup T_v \subseteq B$.

If $T_v = \emptyset$ or $v_B(T_v) = 0$ (in which case $p(T_v)$ is 0), then we have $p(B) = \sum_{x \in T_u} p_x \leq u(T_u) \leq u(B) < \sum_{x \in B}(u(x|B \setminus x) - \epsilon)$. Thus, there exists a worker $x \in B \setminus T_v$ with $p_x < u(x|B \setminus x) - \epsilon \leq v_B(x)$. Since in this case the firm with valuation $v_B$ will desire such a worker $x$, this cannot be a JS-stable allocation.

Otherwise, $v_B(T_v) > 0$. Let $x^* = \arg\max_{x \in T_v} v_B(x)$, then $v_B(x^*) = u(x^*|B \setminus x^*) - \epsilon$. Since $p(B \setminus T_u) = p(T_v) \leq v_B(x^*)$ we have,

$$\begin{aligned} u(B \setminus T_u|T_u) - p(B \setminus T_u) &> u(B \setminus T_u|T_u) - u(x^*|B \setminus x^*) \\ &= (u(B) - u(T_u)) - (u(B) - u(B \setminus x^*)) \\ &= u(B \setminus x^*) - u(T_u) \geq 0, \end{aligned}$$

where the last inequality follows since $T_u \subseteq B \setminus x^*$. Once again this contradicts the assumption that the allocation is JS-stable. $\square$

| | $\{a\}$ | $\{b\}$ | $\{c\}$ | $\{a,b\}$ | $\{a,c\}$ | $\{b,c\}$ | $\{a,b,c\}$ |
|---|---|---|---|---|---|---|---|
| $\tilde{u}$ | 5 | 3 | 3 | 6 | 6 | 6 | 9 |
| $\bar{v}$ | 3 | 3 | 3 | 6 | 6 | 4 | 8 |

Figure 1: An example with four workers $M = \{a, b, c, d\}$ and two firms with production functions $\tilde{u}, \bar{v}$. The table describes all values for nonempty subsets of $\{a, b, c\}$. We additionally have (1) $\tilde{u}(d) = 1.1$ and $\forall S \subseteq M \setminus \{d\}, S \neq \emptyset$, $\tilde{u}(d|S) = 0$, (2) $\forall S \subseteq M$, $\bar{v}(d|S) = 0$. Allocating all workers to $\tilde{u}$ with salaries $s_a = s_b = s_c = 3$ and $s_d = 0$ is JS-stable.

Apparently one can easily verify that for three workers the two classes, AFS and SFS, are the same. Hence, existence of JS-stable outcomes cannot be guaranteed beyond AFS while Theorem 4 guarantees existence with the class AFS.

Unfortunately, with four or more workers SFS strictly contains AFS as demonstrated by the production function $\bar{v}$ in Figure 1 ($\bar{v}(abc) = 8 > 3 + 4 = \bar{v}(a) + \bar{v}(bc)$).

Consider another firm with a production function $\tilde{u}$ as in Figure 1. For this pair of firms the unique efficient allocation is to assign $\{d\}$ to the firm with $\tilde{u}$ and $\{a, b, c\}$ to the firm with $\bar{v}$. To see that this allocation cannot be made JS-stable, note that JS-stable salaries $s$ for this allocation must satisfy $s_a \geq 3.9$ (since $\tilde{u}(a|d) = 3.9$) and $s_b + s_c \geq 4.9$ (since $\tilde{u}(bc|d) = 4.9$). However we also need $s_a + s_b + s_c \leq \bar{v}(abc) = 8$, a contradiction.

The proof of Theorem 4 builds on this example, and generalizes it to any arbitrary production function $\bar{v}$ in SFS minus AFS. We next discuss some interim observations needed in order to prove Theorem 4.

For any valuation $v$ and a positive number $r$ let $v + r$ be the valuation defined as follows: $(v + r)(D) = v(D) + r$, $\forall D \subseteq M$.

**Lemma 12.** *For any monotone valuation $v$, there exists some positive number $R$ such that for any $r \geq R$, $v + r \in FS$*

*Proof.* If $v(M) = 0$, then $v$ is already in $FS$. Otherwise, let $R$ be $(|M| - 1)v(M)$, and we show for any $r > R$, $v + r \in FS$, by constructing supporting salary vectors for every $S \subseteq M$ (Definition 7). For any $S \subseteq M$, consider the vector $s \in \mathbb{R}^S$, where $s_x = \frac{r+v(S)}{|S|}$, for each $x \in S$. It is straightforward to see that $\sum_{x \in S} s_x = r + v(S) = (v + r)(S)$. Then for any proper subset $T \subsetneq S$,

$$(v+r)(T) \geq \sum_{x \in T} \frac{r}{|T|} = \sum_{x \in T} r \cdot \frac{|S|}{|T|} \cdot \frac{1}{|S|} \geq \sum_{x \in T} r \left(1 + \frac{1}{|M|-1}\right) \cdot \frac{1}{|S|} \geq \sum_{x \in T} \frac{r + v(S)}{|S|} = \sum_{x \in T} s_x.$$

In the second inequality we used the fact $\frac{|S|}{|T|} \geq \frac{|M|}{|M|-1}$, and in the last inequality we used the fact $r > R \geq (|M| - 1)v(S)$. This shows that indeed $s$ is a supporting salary vector, and therefore $v + r$ is in $FS$ by Theorem 1. □

For any $T \subset M$ let $v|_T(\cdot)$ denote the restriction of $v(\cdot)$ to $T$.

**Lemma 13.** *If $v \notin FS$ and for some proper subset $T \subset M$, $v|_T \in FS$ then $v(T) < v(M)$.*

*Proof.* For the sake of contradiction, suppose for $T \subsetneq M$, $v(T) = v(M)$. By the assumption that $v|_T$ is in $FS$, there exists a supporting salary vector $s$ on $T$. We extend $s$ by padding 0 for all elements in $M \setminus T$ and argue that we obtain a supporting salary vector for $M$, contradicting $v \notin FS$. To see this, observe that $\sum_{x\in M} s_x = \sum_{x\in T} s_x = v(T) = v(M)$, and for any $S \subsetneq M$, $\sum_{x\in S} s_x = \sum_{x\in S\cap T} s_x \leq v(S\cap T) \leq v(S)$. □

We are now ready to prove Theorem 4:

*Proof. (Theorem 4)*: If for some $B \subseteq M, \bar{v}(B) > \sum_{m\in B} \frac{\bar{v}(B\setminus m)}{|B|-1}$ then the conclusion follows from Proposition 1. Thus assume that for all $B \subset M$, $\bar{v}(B) \leq \sum_{m\in B} \frac{\bar{v}(B\setminus m)}{|B|-1}$. As $\bar{v} \notin AFS$ there exists some strict subset $T \subset M$ such that $\bar{v}|_T \notin FS$. In particular, let $T$ be a minimal such subset, namely any strict subset of $T$ is in $FS$. By Lemma 13 for any $T'$ that is a strict subset of $T$, $\bar{v}(T') < \bar{v}(T)$. In particular we may choose $\bar{\epsilon} > 0$ be such that for any $T'$ that is a strict subset of $T$, $\bar{v}(T') + \bar{\epsilon} < \bar{v}(T)$.

For any $\bar{\epsilon} > \epsilon > 0$ we define the valuation $u^\epsilon$ on $M$ as follows: $u^\epsilon(D) = r - \bar{v}(D^c)\ \forall D \neq T^c$ and $u^\epsilon(T^c) = r - \bar{v}(T) + \epsilon$, where $r = r(\epsilon)$ is large enough to guarantee that $u^\epsilon \in FS$ (recall Lemma 12).[17] Monotonicity of $u^\epsilon$ is straightforward from the construction and the choice of $\epsilon$.

Allocating $T$ to the firm with production function $\bar{v}$ and $T^c$ to the agent with production function $u^\epsilon$ is the unique optimal allocation. Note that it generates a social welfare of $r+\epsilon$ whereas any other allocation generates $r$.

Assume the theorem is false and that for any $\epsilon$ the unique optimal assignment of $(\bar{v}, u^\epsilon)$ can be supported by a JS-stable allocation $((T, T^c), s^\epsilon)$. By IR $\sum_{m\in T} s^\epsilon_m \leq \bar{v}(T)$, however by increasing the salary of some single worker in $T$ we can assume, without loss of generality, that $\sum_{m\in T} s^\epsilon_m = \bar{v}(T)$.

For any $D \subseteq T$, JS-stability implies

$$\sum_{m\in D} s^\epsilon_m \geq u^\epsilon(D|T^c) = u^\epsilon(D\cup T^c) - u^\epsilon(T^c) = \bar{v}(T) - \bar{v}(T\setminus D) - \epsilon = \sum_{m\in T} s^\epsilon_m - \bar{v}(T\setminus D) - \epsilon.$$

Therefore, for any $D \subseteq T$, $\sum_{m\in T\setminus D} s^\epsilon_m \leq \bar{v}(T\setminus D) + \epsilon$. This can be equivalently stated as follows:

$$\sum_{m\in D} s^\epsilon_m \leq \bar{v}(D) + \epsilon \quad \forall D \subseteq T.$$

Let $\bar{\epsilon} > \epsilon_n > 0$ be decreasing sequence with $\lim_n \epsilon_n = 0$ and let $s$ be an accumulation point of the set of salary vectors $\{s^{\epsilon_n}\}_{n=1}^\infty$. Then $\sum_{m\in T} s_m = \bar{v}(T)$ and $\sum_{m\in D} s_m \leq \bar{v}(D)\ \ \forall D \subset T$ which

[17] The set $D^c = M \setminus D$ denotes the complementary set of $D$ in $M$.

implies that $s$ is a supporting vector of salaries for $\bar{v}|_T$ on the set $T$, contradicting the assumption that $\bar{v}|_T \notin FA$. □

The results of this section leave open the question of the maximal set of production functions that guarantee the existence of (possibly inefficient) JS-stable outcomes. In particular, we do not know whether such an allocation necessarily exists in $SFS \setminus AFS$. We consider this to be a very interesting and technically challenging problem for future research.

### 4.4 JS-stability as an outcome of a decentralized mechanism

A labor market $(v, b)$ naturally induces the following complete information normal-form game played among the firms, the "Second-Price Item Bidding game" (SPIB):

1. Each firm proposes a vector of salaries $p^n = \{p^n_m\}_{m \in M}$, i.e., a salary for each worker. (Firm 0 proposes the vector $\mathbf{0}$.)

2. Each worker $m$ is then assigned a firm $n(m) \in \text{argmax}_{n \in N} p^n_m - b^n_m$. In words, each worker is assigned to one of the firms that offered the highest net salary. Ties can be resolved using any arbitrary tie-breaking rule.

3. The net salary that a worker $m$ receives is the second highest net salary offered to that worker. Formally, let $s^{(net)}_m$ be the second-highest net salary offered to $m$: $s^{(net)}_m = \max_{n \in N \setminus n(m)} p^n_m - b^n_m$. Then we set the total salary of $m$ to be $s_m = b^{n(m)}_m + s^{(net)}_m$.

Our last result shows connections between NE assignments in the SPIB game and JS-stable assignments. Note that the existence of a pure Nash equilibrium in the SPIB game is always guaranteed. In fact, in any profile of bids where one firm proposes an "infinite" salary to all workers, while all other firms propose a minimal salary (one that makes workers indifferent between working and staying unemployed) is such an equilibrium. In a complete information setting (i.e., firms know of each other's costs, only the workers and the auctioneer are unknowledgeable) such an outcome seems unreasonable – a firm that wins a set of workers by proposing salaries higher than the value of production of those workers cannot afford to pay the proposed salaries, and this "bluffing" could be easily exposed by other firms, by proposing reasonable salaries for these workers. In what follows we rule out such equilibria by restricting attention to *Nash equilibria with weak no overbidding*:

**Definition 10** (Weak no-overbidding)**.** Let $D^n(\vec{p})$ be the set of workers assigned to firm $n$ in the outcome of the SPIB game when the vector of proposed salaries is $\vec{p}$. We define "Nash equilibria with weak no-overbidding" to be those Nash equilibria of the SPIB game in which for any firm $n$, $v^n(D^n(\vec{p})) \geq \sum_{m \in D^n(\vec{p})} p^n_m$. In words, no firm violates IR if it pays the salaries it proposed to the set of workers assigned to it.

Christodoulou et al. (2008) and Bhawalkar and Roughgarden (2011) previously suggested a stronger notion of "no overbidding", where the above condition holds for any subset of workers $D \subseteq M$ (i.e., it should hold that $v^n(D) \geq \sum_{m \in D} p^n_m$ for all $D \subseteq M$).

**Theorem 5.** *There exists a pure Nash equilibrium with weak no-overbidding in the SPIB game induced by a labor market $(v, b)$ if and only if there exists a JS-stable outcome in this market. Furthermore, the assignments of workers to firms in the Nash equilibria outcomes are identical to the assignments in the JS-stable outcomes (though salaries need not be identical).*

In other words, this theorem ties *existence* of JS-stable outcomes to *existence* of pure NE with weak no-overbidding. It also asserts that workers' assignments in both types of outcomes are identical, while salaries may differ. In particular, in the construction used in our proof, the Nash equilibria salaries are generally lower than the JS-stable salaries (for the same assignments). This is a result of the one-shot second-price auction rule, that has a well-known tendency to decrease payments in a complete information setting. It seems plausible that a suitably defined dynamic best-response process will converge to those NE that have salaries comparable to the JS-stable salaries, but we leave this question open for further investigation.

The proof of the theorem relies on the following lemma.

**Lemma 14.** *A vector of bids $p$ is a NE of the market $(v, b)$ if and only if $p - b$ is a NE of the market $(v - b, 0)$. Furthermore, the assignments of workers to firms in these two Nash equilibria are identical.*

*Proof.* Let $D^n(p)$ denote the set of workers assigned to firm $n$ in the outcome of the SPIB game that corresponds to $(v, b)$. I.e., $D^n(p) \subseteq \text{argmax}_{m \in M}\{p^n_m - b^n_m\}$. Let $\tilde{D}^n(p - b)$ denote the set of workers assigned to firm $n$ in the outcome of the SPIB game that corresponds to $(v - b, 0)$. I.e., $\tilde{D}^n(p - b) \subseteq \text{argmax}_{m \in M}\{p^n_m - b^n_m\}$. Assuming that ties are broken in the same way in the two games, we have $\tilde{D}^n(p - b) = D^n(p)$. Similarly, the second-highest net salary offered to worker $m$ is identical in the two outcomes: $s^{(net)}_m = \max_{n \in N \setminus n(m)}\{p^n_m - b^n_m\} = \tilde{s}^{(net)}_m$. This implies that the resulting utilities of firm $n$ in the two outcomes of the two SPIB games are the same:

$$u^n(p) = v^n(D^n(p)) - \sum_{m \in D^n(p)} (b^n_m + s^{(net)}_m) = (v^n(D^n(p)) - \sum_{m \in D^n(p)} b^n_m) - \sum_{m \in D^n(p)} s^{(net)}_m = \tilde{u}^n(p - b).$$

Notice that this holds for any arbitrary vector of bids $p$ (not necessarily a NE bid vector). In particular, for any deviation $x^n$ of firm $n$, we have $u^n(p^{(-n)}, x^n) = \tilde{u}^n(p^{(-n)} - b^{(-n)}, x^n - b^n)$. Thus, $p$ has a profitable deviation in the game associated with $(v, b)$ if and only if $p - b$ has a profitable deviation in the game associated with $(v - b, 0)$. This immediately implies the claim. □

With this at hand we turn to prove Theorem 5.

*Proof.* (**Theorem 5**) Given lemma 14 and the results of section 3, it is sufficient to prove the Theorem for the case of salary driven workers. We begin by showing that if $\vec{p}$ is a NE with weak no-overbidding of the SPIB game induced by $(v, 0)$. Then, there exists a JS-stable outcome $(A, s)$, where the set of workers $A^n$ assigned to firm $n$ is exactly $D^n(\vec{p})$.

Set the salary of each worker $m \in M$ to be $s_m = p_m^{n(m)}$, i.e., the salary proposed by the firm that receives this worker. By weak no-overbidding, $v^n(D^n) \geq \sum_{m \in D^n} s_m$, hence this outcome is IR. The fact that $\vec{p}$ is a NE implies that, for any firm $n$ and $C \subset M \setminus A^n$, $v^n(C|A^n) \leq \sum_{m \in C} p_m^{n(m)} = \sum_{m \in C} s_m$. Thus, there does not exist a blocking coalition for $(A, s)$, and the claim follows.

Now we turn to show that if $((A^1, \cdots, A^N), (s_1, \ldots, s_M))$ is a JS-stable outcome for the market $(v, 0)$ then, there exists a NE with weak no-overbidding for the induced SPIB game, where each firm $n$ wins the set of workers $A^n$.

Consider the following strategy tuple, $\vec{p}$:

$$p_m^n = \begin{cases} s_m & m \in A^n \\ 0 & m \notin A^n \end{cases}$$

We claim that $\vec{p}$ is a NE with weak no-overbidding. Note that $\vec{p}$ induces an allocation where firm $n$ wins $A^n$, and pays zero. JS-stability implies $v^n(A^n) \geq \sum_{m \in A^n} s_m$, hence $p^n$ satisfies weak no-overbidding.

To verify that this is a Nash equilibrium, fix a firm $n$, and suppose towards a contradiction that there exists a strictly profitable deviation from $p^n$ for firm $n$. Suppose that the firm receives some set of workers $X$ in this deviation. Since $n$ pays a salary of zero for each worker $m \in A^n$ we can assume without loss of generality that $A^n \subset X$. To win any $m \in X \setminus A^n$, firm $n$ must submit a salary larger than $s_m$, therefore it will pay a salary of $s_m$ to $m$. Since $(n, X \setminus A^n)$ is not a blocking coalition, we have $v^n(X \setminus A^n | A^n) \leq \sum_{m \in X \setminus A^n} s_m$. Thus, $X$ does not strictly increase $n$'s utility, a contradiction and the claim follows.

We have now established Theorem 5 for salary driven workers. The Theorem for the general case follows from Lemmas 10 and 14. □

Recall the JS-stable assignment in example 5 and note that it cannot be obtained as a NE of the SPIB game. This emphasizes the necessity of assumption (MP) for the claim in theorem 5.

### 4.5 Summary of Results

The following table informally summarizes our main results while comparing these with the existing literature on unregulated labor markets:

| TYPE OF LABOR MARKET | UNREGULATED (existing literature) | REGULATED (our contribution) |
|---|---|---|
| Solution concept | Stable allocations | JS-stable allocations |
| Set of production function | GS | AFS and SFS |
| First welfare theorem | Stable allocations are efficient. | JS-stable allocations obtain half the maximal efficiency. |
| Second welfare theorem | Pareto efficient allocations are stable in GS. | Efficient allocations are JS-stable in AFS. |
| Maximality | Stable allocations are not guaranteed outside GS. | Efficient JS-stable allocations are not guaranteed outside AFS.<br><br>JS-stable allocations are not guaranteed outside SFS. |
| Non-cooperative foundations | Stable outcomes are Nash equilibria of the first price item bidding game (Bikhchandani, 1999). | JS-Stable assignments are Nash equilibria of the second price item bidding game (with no over-bidding). |

# 5 Discussion and Future Research

In this work we introduce JS-stability as a new solution concept for many-to-one matching markets. This concept is inspired by regulated labor markets where costs for firing employees are prohibitively high. Clearly any stable outcome, in the classical sense, is also JS-stable. However, there are JS-stable outcomes which are not stable. In fact, there is a large family of production functions which do not admit stable outcomes, yet JS-stable outcomes not only exist for them but in fact support all efficient outcomes. Unfortunately, JS-stability does not always guarantee efficiency. Surprisingly, it does guarantee a (multiplicative) upper bound of 50% on efficiency loss.

## 5.1 Regulated labor markets

Harnessing the many-to-one matching model for studying regulation in labor markets is novel, to the best of our knowledge. Thus, our work must be viewed as the first step of a research agenda that studies implications of regulatory intervention in labor markets. We highlight some natural follow-up questions which we leave for future research:

- **Unemployment rates:** Much of the work done by labor theorists around termination costs for workers focuses on the implications of such costs on the unemployment level. Two contradicting forces come into play. First, due to high termination costs employees will not be fired and hence unemployment should decrease. Second, at the hiring stage firms take the termination costs into account and so tend to hire less. The lion's share of the related work uses partial or general equilibrium analysis. In particular it assumes a homogeneous workforce. Our model, on the other hand, assumes heterogeneity of the workers and so may lead to conclusions that are different from those reached via the homogeneity assumption. In future work we shall compare employment levels in stable outcomes with those of JS-stable outcomes, when both exist (e.g., under gross-substitutes assumption). It can be demonstrated that, in general, such comparative statics can swing both ways. Consider the market in the following example, due to Fuhito Kojima, where JS-stable outcomes exhibit lower unemployment levels combined with a lower social welfare, when compared with a stable outcome of the same market:

**Example 7.** There are two firms, $A$ and $B$, and three workers $a, b, c$. Let $b_a^A = 1$ and $b_m^n = 0$ otherwise. Firm $A$ has a unit demand and $v^A(a) = 0, v^A(b) = 1, v^A(c) = 1.5$. For firm $B$, $v^B(a) = 4$, $v^B(b) = 6$, $v^B(c) = 2$ and $v^B(X) = 6$ otherwise.

Note that matching $b$ with $B$ and $c$ with $A$, both with zero salary, is a stable matching which leaves $a$ unemployed. On the other hand matching $b$ with $A$ at a salary of 1, $a$ and $c$ with $B$ at salaries 4 and 2 respectively, yields a JS-stable outcome with no unemployment (note that this matching is not efficient nor stable as $B$ and $b$ at a salary of 5 is a blocking coalition).

We leave it to the reader to find an example that demonstrates the opposite. Thus, one should refine the analysis and focus on specific domains of production functions where the outcome is conclusive. In addition, computing bounds on differences of unemployment under the two stability concepts is interesting. We leave these questions to future research.

- **The limitations of a static model:** Our model is a static one and hence is not adequate to capture a setting where information is revealed over time. Labor markets are, arguably, such markets. The quality of a worker may not be fully known to the firm when the worker is first hired and more information is revealed only after hiring takes place. That information may be critical to the firm's decision on whether or not to further employ the worker. Thus, the introduction of barriers to firing workers impacts the initial hiring decision. This consideration, which is critical for the analysis of job security, is not captured in our static model. Nevertheless, we argue that our model provides value for understanding such regulation. In particular, we think that the set of JS-stable outcomes provides an upper bound on the set of possible outcomes and an outcome that is not JS-stable will not prevail, even if firms take into account he regulation during the hiring process.

- **Severance payments:** As mentioned, JS-stability is inspired by prohibitive firing costs for employers. For example, a recent trend in labor theory is to study the implications of a requirement for severance payments when firms layoff employees (e.g., as suggested in Blanchard and Tirole (2008)). In fact, some countries, like Denmark, already implement such a policy (Andersen, 2012). It will be interesting to replace the notion of JS-stability with an alternative solution concept which models more moderate regulation than tenure within the framework of many-to-one matching models. One thing to note is that the conclusions of such a model may also depend on the recipient of such severance payments. Does the employee or the state enjoy them (in which case they could be modeled as deadweight costs)? The research agenda may well go beyond severance payments and study other regulatory means designed for job protection and job security such as insurance institutions.

- **Risk-averse workers**: Recall that some of our results refer to a cardinal notion of efficiency. For this notion to make sense we require that all utilities, for firms and for workers, are given in the same 'currency'. As a result our model assumes that firms' and workers' utilities are given in terms of money. Whereas for firms this is natural (as we identify utility with profits), for workers this is a limitation. Therefore, a study of JS-stability is called for when workers' utility functions go beyond additive-separable functions. This is particularly important if one would like to account for uncertainty without assuming workers are necessarily risk neutral.

## 5.2 The structure of the set of JS-stable outcomes

Apart from the natural appeal of stability as a solution concept in matching models it also exhibits a very elegant mathematical structure, as the set of stable outcomes forms a lattice under a natural order. A variety of observations then follows. These observations may have natural counterparts when considering the larger set of JS-stabile outcomes:

- A basic question we leave open refers to the maximal class of production functions which guarantees the existence of a JS-stable outcome (albeit not necessarily an efficient one). We do not know if there is a unique maximal such class of production functions, much less how to characterize the production functions in a maximal such class.

- Is there a similar lattice structure for the set of JS-stable outcomes? What kinds of production functions allow for such a lattice structure? We suspect that the answers will typically be negative but have not studied this in depth so far.

- A central corollary one can derive from the lattice structure of stable outcomes is the existence of 'best' and 'worst' stable outcomes for the firms as well as for the workers. However, such best and worst allocations may exist even without a lattice structure (e.g., see Hatfield and Kojima (2010)). Thus, the study of extreme allocations that are JS-stable may take place even prior to our full understanding of the existence of a lattice structure for JS-stable outcomes.

### 5.3 The JS-Core

The solution concept we focus on, JS-stability, is based on the inexistence of blocking coalitions composed of a single firm and some workers. However, a JS-stable allocation can conceivably allow for a situation where there exists a set of more than a single firm for which these firms can shuffle their current joint set of workers and possibly recruit additional workers to obtain an outcome that is better for all involved (the firms in the set, the current set of workers and the additional workers). Thus, the set of workers of such firms will happily agree with being laid-off, conditional on being recruited by some other firm in the set of firms. Such a possibility may imply that what we refer to as a JS-stable allocation may not necessarily be stable, even when Job-security provisions are instated.

This situation is demonstrated in the JS-stable outcome of Example 4 where two firms can switch workers, in particular workers $b$ and $c$, and consequently improve the situation for all. This observation begs a definition of a stronger notion of stability, the *JS-core*:

**Definition 11.** Given an allocation $(A, s)$, we say that a coalition composed of a set of firms $\hat{N} \subset N$, and a set of workers $C$ such that $\cup_{n \in \hat{N}} A^n \subset C$, is a *big blocking coalition* if there exists an allocation $(\hat{A}, \hat{s})$ such that $\cup_{n \in \hat{N}} \hat{A}^n = C$ and:

- $u_m(n, \hat{s}_m) \geq u_m(k, s_m) \quad \forall k \in N, n \in \hat{N}, m \in A^k \cap \hat{A}^n \cap C$ (workers in $C$ are better off in allocation $(\hat{A}, \hat{s})$),
- $v^n(\hat{A}^n) - \sum_{m \in \hat{A}^n} \hat{s}_m \geq v^n(A^n) - \sum_{m \in A^n} s_m$ for all $n \in \hat{N}$,

with at least one of the inequalities being strict.

**Definition 12.** An allocation $(A, s)$ is in the *Job Security Core (JS-Core)* if it is IR and there exist no big blocking coalitions.

Dropping the requirement in Definition 11 that $\cup_{n \in \hat{N}} A^n \subset C$ gives the definition of the *Core*. Some immediate observations are:

**Proposition 2.** *The following holds for the JS-Core:*

- *Any allocation in the JS-core is JS-stable.*
- *There are allocations that are JS-stable but are not in the JS-Core (see Example 4).*
- *Any allocation in the JS-core is Pareto optimal (otherwise the grand coalition is a blocking coalition).*
- *The Core is a subset of the JS-Core.*

Consider an allocation that is JS-stable yet not in the JS-core. For the big blocking coalition to succeed in pulling off a deviation all changes must occur simultaneously or alternatively there must exist strong guarantees in place for a job for those workers who voluntarily leave their current employer as well as strong guarantees for firms who commit to hiring new workers that some of their current workforce will voluntarily quit. Such a set of cross guarantees may be highly unreasonable when more than a single firm is involved. Thus, although the study of the JS-Core may shed additional light on the regulatory implications of tenure, the notion of JS-stability makes for a reasonable starting point for such analysis.

## 5.4 Centralized vs. decentralized job market

One can ask whether the notion of JS-stability is an adequate tool for the analysis of centralized, and typically highly specialized, markets such as the medical residency market, or should we consider it in the context of a decentralized market. We are more critical of the notion of JS-stability in the context of a centralized market. Recall that the only blocking coalitions considered are those consisting of a single firm and a collection of workers. In ordinary matching models, such a restriction is innocuous - if there is a blocking coalition involving several firms, then there is a smaller blocking coalition involving only a single firm. Here, as we have already observed, this is no longer the case. In particular, even if an outcome is JS-stable it may not be satisfactory from the central planner's point of view as some switching of workers between is a Pareto improvement (recall Example 4). A similar criticism might be applied towards JS-stability in the context of a decentralized setting. However, we argue such criticism is weaker in this context as it is unrealistic to expect the formation of blocking coalitions with more than a single firm for two reasons (a) often such firms compete and hence are not allowed to coordinate; and (b) such a blocking coalition hinges on a unrealistic long chain of trust among the players: if such a blocking coalition prevails the worker that volunteers to leave one firm bases this on some guarantee that he will be hired by some other firm, who simultaneously agrees to this only because some of its workers plan to quit and so on and so forth.

## 5.5 Matching with contracts

The simple many-to-one matching model that we use was extended by Hatfield and Milgrom (2005) to a model of 'matching with contracts'. In such a model a contract between a firm and an agent may specify various aspects related to employment, beyond the salary. It may specify working hours, shifts, insurance, job description, and many more. Therefore there may exist many possible contracts between a worker and a firm. Consequently, the firm's output (and, consequently, its profit) will not depend only on the set of employees it employs but also on the specific contracts signed between the employees and the firm. Hatfield and Milgrom extend the results from the classic matching model to the new matching-with-contracts paradigm under the gross-substitutes assumption. Echenique (2012) showed that under the gross substitutes assumption the two models are in fact equivalent and the seemingly multi-dimensional extension from wages to contracts boils down to a single-dimensional set. However, Hatfield and Kojima (2008; 2010) demonstrate the richness of the matching-with-contracts paradigm by extending some of the results beyond the familiar domain of gross substitutes. In particular they propose new notions of substitutability, called bilateral and unilateral substitutes. Sönmez and Switzer (2013) and Sönmez (2013) provide a realistic example of a market where these weaker assumptions hold and, as a result, offer new and more efficient allocation mechanisms for such markets. It turns out that when one folds the generalized contract model back into a matching model the two notions of bilateral and unilateral substitutability become equivalent to gross-substitutes and hence provide no further insight within the one-to-many matching model.

A major component of our analysis is the domain of production function which we analyze. In particular this domain, *AFS*, goes far beyond gross substitutes. Thus, the matching-with-contracts model is indeed a more general model and is not subject to the Echenique critique. Studying JS-stability in such a model is left to future research.

### 5.6 Auction theory

As noted frequently in the literature, matching models have natural connections with combinatorial auctions, where the buyers (the "firms") who have valuations for subsets of goods need to pay the sum of item prices (the "wages") in a bundle in order to keep the bundle. Our solution concept of JS-stability is then a relaxation of the Walrasian equilibrium in the auction context. A Walrasian equilibrium consists of allocations and item prices such that no bidder can improve her utility by adding items or dropping items or doing both. In comparison, equilibria corresponding to JS-stability would not allow bidders to drop items, although they are allowed to add items at current prices. If we refer to such equilibria as *equilibria conditional on buying* **at least** *a pre-specified subset of items* (abbreviated *conditional equilibria*), all of our results can be translated as properties of conditional equilibria. For example, the social welfare at a conditional equilibrium, whenever it exists, is always a 2-approximation to the optimal; the AFS valuation class is maximal with respect to the property that any welfare-optimal allocation can be supported as a conditional equilibrium. The SPIB game that we discussed in Section 4.4 corresponds to a natural *simultaneous second-price item auction*. In this auction, all bidders simultaneously submit their bids on all items, and then each wins the items on which she bids highest and pays the sum of second highest bids on those items. Previous work (Christodoulou et al., 2008; Bhawalkar and Roughgarden, 2011) showed that allocations at Nash equilibria with no overbidding in this auction give 2-approximation to the optimal social welfare when bidders' valuations are complement free. Our results immediately imply that for *all* valuations, as long as a Nash equilibrium under weak no-overbidding exists, it achieves a 2-approximation to the optimal social welfare. Our study also sheds light on the question of which valuation classes guarantee the existence of pure Nash equilibria in the simultaneous item auction.

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