Voting Power and Proportional Representation of Voters*

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Abstract

The paper provides a justification for the proportional representative (PR) election system for politically diversified societies. We employ the Shapley value concept to measure the political power of parties in a parliament. We prove that for the PR system if parties' sizes are uniformly distributed on the simplex, the expected ratio of a party size to its political power increases to 1, and the variance converges to 0, as the number n of parties increases. The rate of convergence is high and it is of the magnitude of $\frac{1}{n}$. Empirical evidence from the Netherlands elections supports our result.

Introduction

In many democracies parliaments are elected by proportional representative system (hereafter-PR). The PR system allocates seats in parliament to parties in proportion to their sup-

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porters. But does it represent the bargaining power of the parties? The answer in general is negative. As an example, let A, B and C be the only three parties represented in a parliament with 100 seats. Suppose that A, B and C have 45, 45 and 10 seats, respectively. A coalition of parties that have a simple majority (more than 50 seats) has the entire power. Any coalition of at least two parties has a majority and no party has a majority by itself. In this sense C has the same bargaining power as A or B even though C is much smaller in size. A similar argument applies to a large number of parties [1]. Suppose there are $n = 2m + 1$ parties in the parliament, m is an arbitrary integer. Assume there are $2m$ parties with $k > 1$ seats each, and one party, A, with 1 seat only. A majority consists of at least $mk + 1$ seats. Any coalition with $m + 1$ or more parties has a majority, and any coalition of less than $m + 1$ parties has no majority. Clearly, the smaller party A has the same bargaining power as any other party.

Many argue that "voting power" of parties should be closely related to their size. Nurmi [1981] advocates that the idea of proportional representation rests on the identity of distribution of parties' support and the distribution of parties' power. Nozick [1968, Note 4] refers to district systems and states that a system of proportional representation reflects legislators' power. The example above however demonstrates that the PR system does not satisfy this property, at least not for every distribution of parties' size. Yet, proportionality of a priori voting power to weight sounds a proper principle for a fair representative parliament. The purpose of this paper is to provide a justification for this principle. It is shown that on average, when parties' size is random, and the number of parties is large the ratio of voting power to weight is close to 1.

The literature offers several tools to measure voting power of a party. The most well-known tools are the Shapley value (Shapley [1953]) and the Banzhaf index (Banzhaf [1964, 1968]). Both measures are based on the probability of party to be a pivot. Namely, the voting power of a party is the probability that it turns a random coalition of parties from one with no majority into a winning coalition. While for the Banzhaf index all coalitions have the same probability to form, the Shapley value uses different probability

[1] The following example is from Lindner and Owen [2007].

distribution: coalitions of the same size are equally likely to be formed and all sizes have the same probability.

In this paper we show that irrespective of the quota required for majority, if parties' size is uniformly distributed (reflecting no prior information about their size), the expected ratio of a party size to its voting power, measured by the Shapley value, increases to 1, as the number of parties increases. This result fails to hold for the Banzhaf index. Furthermore, the rate of convergence is high and the error term is of the magnitude of $1/n$ where n is the number of parties. The variance of this ratio converges to 0, as the number of parties increases.

Even though the number of parties in most parliaments is relatively small our result may still be applicable. A relatively small number of parties in parliaments is often caused by a "threshold of participation" (see Rae et al. [1971]). A large number of parties often participate in the election, but in some cases only small number of them have seats in the parliament. For instance, the 2009 German federal election resulted with 6 parties in the parliament out of 29 competing parties. The 2006 Netherlands election resulted with 10 parties out of 23 competing parties. In addition, an electoral threshold induces some parties not to participate in elections as a distinct party.

Our result is also relevant to voting power of shareholders in a business company with relatively large number of shareholders. Since typically the number of shareholders is large our result asserts that profit sharing proportional to the number of shares reflects on average the voting power of shareholders.

The notion of voting power is well discussed in the literature. As mentioned above we focus here on the Shapley-Shubik index (Shapley and Shubik [1954]), which relies on the Shapley value for cooperative games (Shapley [1953]). This notion is uniquely derived by a set of four axioms and it assigns to every party in a given game a share in the total "cake". An axiomatization of the Shapley value for just voting games is given in Dubey [1975]. Young [1985] provides an alternative axiomatization of the Shapley value for the class of all $n-$person games in coalitional form which can also be used to characterize the

Shapley value on the class of voting games [2]. The Shapley value of a party is considered to measure its "real contribution" to the total cake, reflecting on its bargaining power in the cake division game. In the context of voting games, the Shapley-Shubik index measures voting power as an expected prize of a party ("the P-power", using terminology of Felsenthal and Machover [1998] [3]). Our main result can therefore be stated as follows: if parties' size are random and has uniform distribution the expected value of the ratio of the Shapley value of a party to its size increases to 1, when the number of parties increases. The parliamentary elections in the Netherlands provide an empirical evidence for our result. For each election in the Netherlands we calculated the average and the variance of the ratio of the Shapley value of a party to its size. The average is above 0.9, and the variance is impressively low.

Chang et al. [2006] confirmed our result through Monte-Carlo simulations for any majority quota, provided that it is not close to 1, as well as for the **normalized** Banzhaf index if the majority quota is 0.5 [4].

These simulations confirm the Penrose [1952] conjecture stating that asymptotically the ratio of voting power to size is the same accross parties. Penrose used informal language to describe the notion of voting power, one that was defined formally later in Banzhaf [1964]. The conjecture fails to hold for instance for the example above, but found to be correct for some special cases analytically (see Lindner and Machover [2004]) .

It is worth mentioning a very simple and related result by Shapley [1961]. Namely, for any number of players and any system of weights if the quota for a majority is random and has a uniform distribution then the expected Shapley value of every party coincides with its size.

[2] Young replaces the controversial additivity axiom by a more intuitive monotonicity axiom. For the axiomatization of the Banzhaf index and its relationship to the Shapley value see Lehrer [1988]. Another axiomatization of the Banzhaf index is by Dubey et al. [2005].

[3] Although Felsenthal and Machover [1998] expressed reservation regarding the Shapley-Shubik index, in Felsenthal and Machover [2005] they state that "for a priori P-power, the Shapley-Shubik index still seems to be the most reasonable candidate for measuring it ".

[4] The sum of Banzhaf indices of parties is normalized to 1.

Finally, in many parliaments around the world at least one party is relatively large. Nevertheless, this observation does not contradict the assumption that parties' size are uniformly distributed. For instance for $n = 10$, when parties' sizes are uniformly distributed on the simplex, the probaility that at least one party is larger than 0.2 is 0.92, and the probaility that at least one party is larger than 0.3 is 0.4 (see Holst [1980, Theorem 2.1].

The Model

Let $N = \{1, 2, \ldots, n\}$ be the set of parties. Suppose that $X_1, \ldots, X_n$ are n random variables that measure the size of the n parties. That is
$\sum_{i=1}^{n} X_i = 1$,$X_i \geq 0$ and $i = 1, \ldots, n$. Let

$$A_n = \{(x_1, \ldots, x_n) | \sum_{i=1}^{n} x_n = 1, x_i \geq 0, i = 1, \ldots, n\}$$

be the $n - 1$ dimensional simplex in $\mathbb{R}^n$. We assume that the realization $(x_1, \ldots, x_n)$ has a uniform distribution on A_n with respect to the volume of A_n. Let v_n be the volume of A_n and let $p_n = \frac{1}{v_n}$ be the (fixed) density function of $X = (X_1, \ldots, X_n)$ on A_n.

Let $\frac{1}{2} \leq q < 1$ be a quota and let V_n be the voting game on N defined for every realization $x \in A_n$ and all $S \subseteq N$ by

$$V_n(S, x) = \begin{cases} 1 & , \sum_{i \in S} x_i > q \\ 0 & ,\text{otherwise.} \end{cases}$$

We say that a subset S of N is a winning coalition if $\sum_{i \in S} x_i > q$ and it is a minimal winning coalition if it is a winning coalition and for all $i \in S$, $S \setminus \{i\}$ is not a winning coalition ($\sum_{j \in S \setminus \{i\}} x_j \leq q$).

Let $x \in A_n$ and let $\Theta_i(x)$ be the set of all coalitions S, $S \subseteq N \setminus \{i\}$, such that S is not a winning coalition and $S \cup \{i\}$ is a winning coalition. In this case we say that i is a pivot player to S. That is, $\Theta_i(x)$ is the set of all coalitions S, $S \subseteq N \setminus \{i\}$ such that i is pivot to S.

To derive the Shapley value of a player consider the $n!$ permutations of the players in N. For every $i \in N$ and every permutation $\Re$ let $P_i^{\Re}$ be the subset of players in N that precede i in the order $\Re$. For example, suppose that $N = \{1, 2, 3, 4\}$ and $\Re = \{2, 3, 1, 4\}$. Then $P_1^{\Re} = \{2, 3\}$.

The number of permutations of N where i is a pivot is

$$\phi(x, i) = \sum_{S \in \Theta_i(x)} |S|!(n - |S| - 1)!$$

Given the set N and the weights $x = (x_1, \ldots, x_n) \in A_n$, the Shapley value of V_n is

$$Sh_i(x) = \frac{\phi(x, i)}{n!}$$

That is, the Shapley value of a party i is the probability that i is a pivot in a random order where all orders are equally likely. An equivalent way to derive the Shapley value of $i \in N$ is through the following probability distribution over coalitions $S \subseteq N \setminus \{i\}$. All coalitions of the same size are equally likely to be formed and all sizes $0, 1, \ldots, n-1$ have the same probability, $\frac{1}{n}$. That is the probability of $S \subseteq N \setminus \{i\}$ is $\frac{1}{\binom{n-1}{|S|}n} = \frac{|S|(n-|S|-1)}{n!}$.

Given (N, x) the Banzhaf index, Bz, of $i \in N$ is:

$$Bz_i(x) = \frac{|\Theta_i(x)|}{2^{n-1}}$$

That is, every coalition $S \subseteq N \setminus \{i\}$ have the same probability to form irrespective of its size.

Thus the Shapley value and the Banzhaf index of a party i are both probability of i to be a pivot to a random coalition. The two measures differ in the probability distribution over coalitions.

Let Exp be the expected value operator and denote

$$Exp(Sh_i(X), n) = \int_{A_n} p_n Sh_i(X)\, dX$$

and

$$Exp(\frac{Sh_i(X)}{X_i}, n) = \int_{A_n} p_n \frac{Sh_i(X)}{X_i}\, dX.$$

Similarly,

$$Exp(\frac{Bz_i(X)}{X_i}, n) = \int_{A_n} p_n \frac{Bz_i(X)}{X_i}\, dX.$$

The following result is shown analytically.

Theorem For every $i \in N$, and $\frac{1}{2} \leq q < 1$

(1) $Exp(\frac{Sh_i(X)}{X_i}, n)$ is increasing in n and

$$\lim_{n \to \infty} Exp(\frac{Sh_i(X)}{X_i}, n) = 1$$

where $p_n = \frac{1}{Vol(A_n)}$ and Vol stands for volume.

(2) $Exp(\frac{Sh_i(X)}{X_i}, n) = 1 + O(\frac{1}{n})$

That is, the expected ratio between the Shapley value and the size of a party increases to 1 as n increases indefinitely. The rate of convergence is $\frac{1}{n}$ and it can be shown that the error term converges to zero exponentially. Figure 1 illustrates the rate of convergence for $q = 1/2$. In this case, $Exp(\frac{Sh_i(X)}{X_i}, n) \geq 0.9$ for $n \geq 10$.[5]

[5]Simulations in Chang et al. [2006] give a close result for the Shapley-Shubik index for any quota except quotas close to 1. Our analytical result holds for any quota lower than 1. The difference can be explained by the rate of convergence. For quota close to 1 the rate of convergence is relatively small, and the number of parties needed in this case is larger than is used in the simulations of Chang et al. [2006].

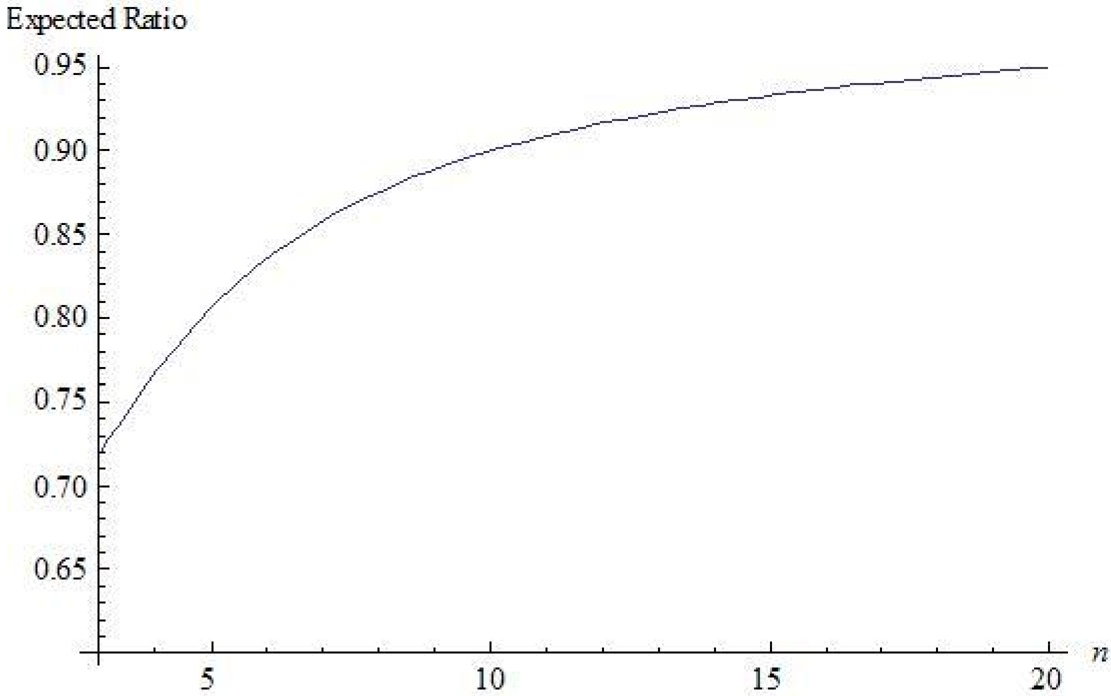

Figure 1: $Exp(\frac{Sh_i(X)}{X_i}, n)$ for $q = 0.5$

Examples

1. Suppose that $n = 2$ (two parties only). Then $X_i \sim U[0,1]$,$i = 1, 2$. Clearly

$$Sh_i(x) = \begin{cases} 1 & , x_i > q \\ 0 & , x_i \leq q \end{cases}$$

implying that

$$Exp(\frac{Sh_i(X)}{X_i}, 2) = \int_q^1 \frac{1}{x_i}\, dx_i = -\log q.$$

In particular for $q = \frac{1}{2}$ $Exp(\frac{Sh_i(X)}{X_i}, 2) = \log 2 < 1$

2. Suppose next that $n = 3$. The computation of $Exp(\frac{Sh_i(X)}{X_i}, 3)$ is more complicated. We show later on (see (6) below) that for $q = \frac{1}{2}$

$$Exp(\frac{Sh_i(X)}{X_i}, 3) = 2\log 2 - \frac{2}{3}$$

and $\log 2 < 2\log 2 - \frac{2}{3} < 1$ [6].

[6] We provide an explicit expression of $Exp(\frac{Sh_i(X)}{X_i}, n)$ for all q, $\frac{1}{2} \leq q < 1$ and all n.

To prove the Theorem we first state and prove the following proposition.

Let $c \in (0,1)$ and let $C_n(c)$ be the set of all elements in A_n such that $x_1 = c$.

$$C_n(c) = \{(c, x_2, \ldots, x_n) | \sum_{i=2}^{n} x_i = 1 - c, x_i \geq 0, i = 2, \ldots, n\}.$$

Proposition 1 *Suppose that the elements of $C_n(c)$ are uniformly distributed. Then for $n \geq 3$*

$$\int_{C_n(c)} p_n^{'} Sh_1(X)\, dX = \begin{cases} \frac{c(n-2)}{(1-c)n} & , 0 < c < 1-q \\ \frac{1}{n} + \frac{(n-2)(1-q)}{n(1-c)} & , 1-q \leq c \leq q \\ 1 & , q < c \leq 1, \end{cases}$$

where $p_n^{'}$ is the (fixed) density function of $X = (X_2, \ldots, X_n)$ on $C_n(c)$.

Note, that Proposition 1 is consistent with the well-known "oceanic games" result (Shapiro and Shapley [1978]), which states, that if there is a sequence of weighted majority games with one party of constant size $c, c < q < 1-c$ (a major party), and the size of any other (minor) party converges to zero, then the Shapley value of the major party converges to $\frac{c}{1-c}$. It was shown in Dubey and Shapley [1979] that the convergence of the Banzhaf index is different.

The proof of the Proposition relies on the following two well-known lemmas.

Lemma 1 *Let $Y_1, \ldots, Y_n$ be i.i.d. with exponential distribution. Then $(X_1, \ldots, X_n)$ and $(\frac{Y_1}{\sum_{j=1}^{n} Y_j}, \ldots, \frac{Y_n}{\sum_{j=1}^{n} Y_j})$ has the same distribution.*

For a proof see, for instance, Feller [1971].

Lemma 2 *Let $Y_1, \ldots, Y_n$ be i.i.d. random variables, each has an exponential distribution. For $1 \leq k \leq n$, let $\Sigma_k = \sum_{i=1}^{k} Y_i$. Then $\frac{\Sigma_k}{\Sigma_n}$ for $1 \leq k < n$ has the Beta distribution with parameters $(k, n-k)$.*

For a proof see Jambunathan [1954, Theorem 3].

Notice that the Beta distribution function is defined by

$$Prob(\frac{\Sigma_k}{\Sigma_n} < z) = \sum_{j=k}^{n-1} \binom{n-1}{j} z^j (1-z)^{n-1-j}$$

The next lemma is a consequence of the above two lemmas.

Lemma 3 *Suppose that* $X = (X_1, \ldots, X_m)$ *is uniformly distributed on* A_m*, where* $m \geq 2$*. Then*

$$\sum_{k=1}^{m-1} Prob(\sum_{i=1}^{k} X_i \leq z) = (m-1)z$$

Proof

From Lemmas 1 and 2, $\sum_{i=1}^{k} X_i$ has Beta distribution with parameters $(k, m-k)$. Hence,

$$\sum_{k=1}^{m-1} Prob(\sum_{i=1}^{k} X_i \leq z) = \sum_{k=1}^{m-1}\sum_{j=k}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} z^j (1-z)^{m-1-j}$$

By rearranging terms we have:

$$\begin{aligned}
\sum_{k=1}^{m-1} Prob(\sum_{i=1}^{k} X_i \leq z) &= \sum_{k=1}^{m-1} k \frac{(m-1)!}{k!(m-1-k)!} z^k (1-z)^{m-1-k} = \\
&= (m-1)z \sum_{k=1}^{m-1} \binom{m-2}{k-1} z^{k-1} (1-z)^{m-1-k} = \\
&= (m-1)z \sum_{k'=0}^{m-2} \binom{m-2}{k'} z^{k'} (1-z)^{m-2-k'} = \\
&= z(m-1)(z+1-z)^{m-2} = z(m-1) \, \square
\end{aligned}$$

Corollary 1 Suppose that X is uniformly distributed on $C_m(c)$, $m \geq 3$. Then

$$\sum_{k=2}^{m-1} Prob(\sum_{i=2}^{k} X_i \leq z) = (m-2)\frac{z}{1-c}$$

We are ready now to prove Proposition 1.

Proof of Proposition 1

For every permutation $\Re$ of N party 1 is pivot if $q - c < \sum_{i \in P_i^{\Re}} x_i \leq q$. Denote by $\Phi_{\Re} = \{(c, x_2, \ldots, x_n) \in C_n(c) | q - c < \sum_{i \in P_i^{\Re}} x_i \leq q\}$ the subset of $C_n(c)$, in which party 1 is pivot in $\Re$.

Let $\Re_k$ be the set of all orders $\Re$ of N such that there are exactly k parties that precede 1 in the order $\Re$. Note that $C_n(c)$ is a symmetric subset of $\mathbb{R}^n$ and so is $\Phi_{\Re}$ for every order $\Re$. Hence, if $\Re \in R_k$ and $\Re' \in R_k$

$$Prob(X \in \Phi_{\Re} | X \in C_n(c)) = Prob(X \in \Phi_{\Re'} | X \in C_n(c)) \equiv \Pi(c, k)$$

Since the orders of N are uniformly distributed, $Prob(\Re \in R_k) = \frac{1}{n}$. Thus

$$\int_{C_n(c)} p_n^{'} Sh_1(X)\, dX = \frac{1}{n}\sum_{k=0}^{n-1} \Pi(c,k) = \frac{1}{n}\sum_{k=1}^{n-1} \Pi(c,k) \tag{1}$$

Notice that

$$\Pi(c,k) = Prob(q - c < \sum_{i=2}^{k+1} X_i \leq q) \tag{2}$$

We distinguish two cases.

Case 1 $0 < c < 1 - q$

In this case $\Pi(c,0) = \Pi(c,n-1) = 0$ and by (2)

$$\frac{1}{n}\sum_{k=1}^{n-2} \Pi(c,k) = \frac{1}{n}\sum_{k=1}^{n-2} Prob(q-c < \sum_{i=2}^{k+1} X_i \leq q) = \frac{1}{n}[\sum_{k=1}^{n-2} Prob(\sum_{i=2}^{k+1} X_i \leq q) - \sum_{k=1}^{n-2} Prob(\sum_{i=2}^{k+1} X_i \leq q-c)]$$

By Corollary 1

$$\frac{1}{n}\sum_{k=1}^{n-2} \Pi(c,k) = \frac{n-2}{n(1-c)}[q-(q-c)] = \frac{n-2}{n}\frac{c}{1-c}$$

This together with (1) imply

$$\int_{C_n(c)} p_n^{'} Sh_1(X)\, dX = \frac{n-2}{n}\frac{c}{1-c}$$

as claimed.

Case 2 $1 - q \leq c \leq q$

In this case party 1 is a veto player meaning that every winning coalition must include 1. In this case $Prob(\sum_{i=2}^{k+1} X_i \leq q) = 1$ for every $k = 1, \ldots, n-1$, and in particular $\Pi(c, n-1) = 1$. Applying (2) we have

$$\frac{1}{n}\sum_{k=1}^{n-1} \Pi(c,k) = \frac{1}{n} + \frac{1}{n}\sum_{k=1}^{n-2} \Pi(c,k) = \frac{1}{n} + \frac{n-2}{n} - \frac{1}{n}\sum_{k=1}^{n-2} Prob(\sum_{i=2}^{k+1} X_i \leq q-c).$$

By Corollary 1

$$\frac{1}{n}\sum_{k=1}^{n-1} \Pi(c,k) = \frac{1}{n} + \frac{n-2}{n} - \frac{(n-2)(q-c)}{n(1-c)} = \frac{1}{n} + \frac{(n-2)(1-q)}{n(1-c)} \tag{3}$$

By (1) and (3)

$$\int_{C_n(c)} p_n^{'} Sh_1(X)\, dX = \frac{1}{n} + \frac{(n-2)(1-q)}{n(1-c)}$$

Note that if $c > q$ then 1 is dictator and is a pivot in every order $\Re$. In this case its Shapley value is 1. $\square$

We are now ready to prove the theorem.

Proof of the Theorem Without loss of generality we prove the theorem for $i = 1$. Let $f_{X_i}(x_i)$ be the density distribution function of X_i (derived from the fact that $X = (X_1, \ldots, X_n)$ has a uniform distribution on A_n.

Lemma 4 $f_{X_i}(x) = (n-1)(1-x)^{n-2}$

Proof By Lemmas 1 and 2

$$X_i \sim \frac{Y_i}{\sum_{j=1}^n Y_j} \sim \beta(1, n-1)$$

$$F_{X_i}(x) = Prob(X_i \leq x) = \sum_{j=1}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} =$$
$$= \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} - (1-x)^{n-1} = 1 - (1-x)^{n-1}$$

Consequently

$$f_{X_i}(x) = (n-1)(1-x)^{n-2}$$

as claimed. $\square$

Next define for every x_1,$0 \leq x_1 \leq 1$, the set $B_{n-1}(x_1) \subseteq \mathbb{R}^{n-1}$ by

$$B_{n-1}(x_1) = \{(x_1, \ldots, x_n) | \sum_{j=2}^n x_j = 1 - x_1\}$$

Also denote by $f_{X_{-1}}(x_2, \ldots, x_n | X_1 = x_1)$ the conditional density function of $(X_2, \ldots, X_n)$ on $B_{n-1}(x_1)$. Then

$$Exp(\frac{Sh_1(X)}{X_1}, n) = \int_{x \in A_n} p_n \frac{Sh_1(x)}{x_1} \, dx_1, \ldots, \, dx_n =$$
$$= \int_0^1 f_{X_1}(x_1) [\int_{B_{n-1}(x_1)} f_{X_{-1}}(x_2, \ldots, x_n | x_1) \frac{Sh_1(x)}{x_1} \, dx_2, \ldots, dx_n] \, dx_1 =$$

By Lemma 4

$$= \int_0^1 (n-1)(1-x_1)^{n-2} [\int_{B_{n-1}(x_1)} f_{X_{-1}}(x_2, \ldots, x_n | x_1) \frac{Sh_1(x)}{x_1} \, dx_2, \ldots, \, dx_n] \, dx_1 \qquad (4)$$

Note that $Sh_1(x) = 1$ whenever $x_1 > q$ (1 is a dictator in this case). Also, by Proposition 1 if $x_1 \le q$ then

$$\int_{B_{n-1}(x_1)} f_{X_{-1}}(x_2, \ldots, x_n | x_1) Sh_1(x)\, dx_2, \ldots,\, dx_n = \int_{C_n(x_1)} p_n^{'} Sh_1(x)\, dx = \tag{5}$$

$$= \begin{cases} \frac{n-2}{n}\frac{x_1}{1-x_1} & , x_1 \le 1-q \\ \frac{1}{n} + \frac{(n-2)}{n(1-x_1)}(1-q) & , 1-q < x_1 \le q \end{cases}$$

By (4) and (5)

$$\begin{aligned} Exp(\frac{Sh_1(X)}{X_1}, n) = & \int_0^{1-q} \frac{(n-1)(n-2)}{n}(1-x_1)^{n-3}\, dx_1 + \\ & + \int_{1-q}^{q} [\frac{(n-1)}{n}\frac{(1-x_1)^{n-2}}{x_1} + \\ & + \frac{(n-1)(n-2)}{n}\frac{(1-x_1)^{n-3}(1-q)}{x_1}]\, dx_1 + \\ & + \int_q^1 \frac{(n-1)(1-x_1)^{n-2}}{x_1}\, dx_1 \end{aligned} \tag{6}$$

But

$$\int_0^{1-q} \frac{(n-1)(n-2)}{n}(1-x_1)^{n-3}\, dx_1 = -\frac{n-1}{n}(1-x_1)^{n-2}\Big|_0^{1-q} = \frac{n-1}{n} - \frac{n-1}{n}q^{n-2} \tag{7}$$

and

$$\begin{aligned} 0 \le & \int_{1-q}^{q} [\frac{(n-1)}{n}\frac{(1-x_1)^{n-2}}{x_1} + \frac{(n-1)(n-2)}{n}\frac{(1-x_1)^{n-3}(1-q)}{x_1}]\, dx_1 \le \\ \le & \frac{n-1}{n(1-q)} \int_{1-q}^{q} [(1-x_1)^{n-2} + (n-2)(1-x_1)^{n-3}(1-q)]\, dx_1 = \\ = & \frac{n-1}{n(1-q)} [-\frac{(1-x_1)^{n-1}}{n-1} - (1-q)(1-x_1)^{n-2}]\Big|_{1-q}^{q} = \\ = & \frac{q^{n-1}}{n(1-q)} + \frac{(n-1)q^{n-2}}{n} - \frac{(1-q)^{n-2}}{n} - \frac{(1-q)^{n-2}(n-1)}{n} \end{aligned} \tag{8}$$

and

$$\begin{aligned} 0 \le & \int_q^1 \frac{(n-1)(1-x_1)^{n-2}}{x_1}\, dx_1 \le \frac{n-1}{q}\int_q^1 (1-x_1)^{n-2}\, dx_1 = \\ = & -\frac{1}{q}(1-x_1)^{n-1}\Big|_q^1 = \frac{(1-q)^{n-1}}{q} \end{aligned} \tag{9}$$

Consequently by (6),(7),(8) and (9)

$$\frac{n-1}{n}(1-q^{n-2}) \le Exp(\frac{Sh_1(X)}{X_1}, n) \le \frac{n-1}{n} + \frac{q^{n-1}}{n(1-q)} - (1-q)^{n-2} + \frac{(1-q)^{n-1}}{q}$$

Equivalently

$$-\frac{1}{n}-\frac{n-1}{n}q^{n-2}\leq Exp(\frac{Sh_1(X)}{X_1},n)-1\leq\frac{1}{n}+\frac{q^{n-1}}{n(1-q)}-(1-q)^{n-2}+\frac{(1-q)^{n-1}}{q}.$$

Since $0<q<1$ for every $l>0$

$$Exp(\frac{Sh_1(X)}{X_1},n)-1=O(\frac{1}{n}).$$

Using (6) it is straightforward to show that $Exp(\frac{Sh_1(X)}{X_1},n)$ increases in n, and the proof of the theorem is complete.$\Box$

Remarks

1. The assumption that A_n has uniform distribution is essential. As a trivial counter example, suppose that the distribution on A_n is such that $Prob(q<X_1\leq 1-\epsilon)=1$, $0<\epsilon<1-q$. In this case $Sh_1(X)=1$ and $X_1<1$ with probability 1. In addition, by Proposition 1, for $q=0.5$ if the size of Party 1 is c with probability 1, and the size of the other parties are distributed uniformly, then for n sufficiently large $Exp(\frac{Sh_1(X)}{X_1},n)$ converges to $\frac{1}{1-c}>1$.

2. The variance of the random ratio $\frac{Sh_i(X)}{X_i}$ can be calculated numerically using Monte-Carlo simulation and the approximation method of Owen [1975]. The simulation shows, that for $q=0.5$ the variance of $\frac{Sh_i(X)}{X_i}$ is small and converges to 0 when n increases (see Figure 2).

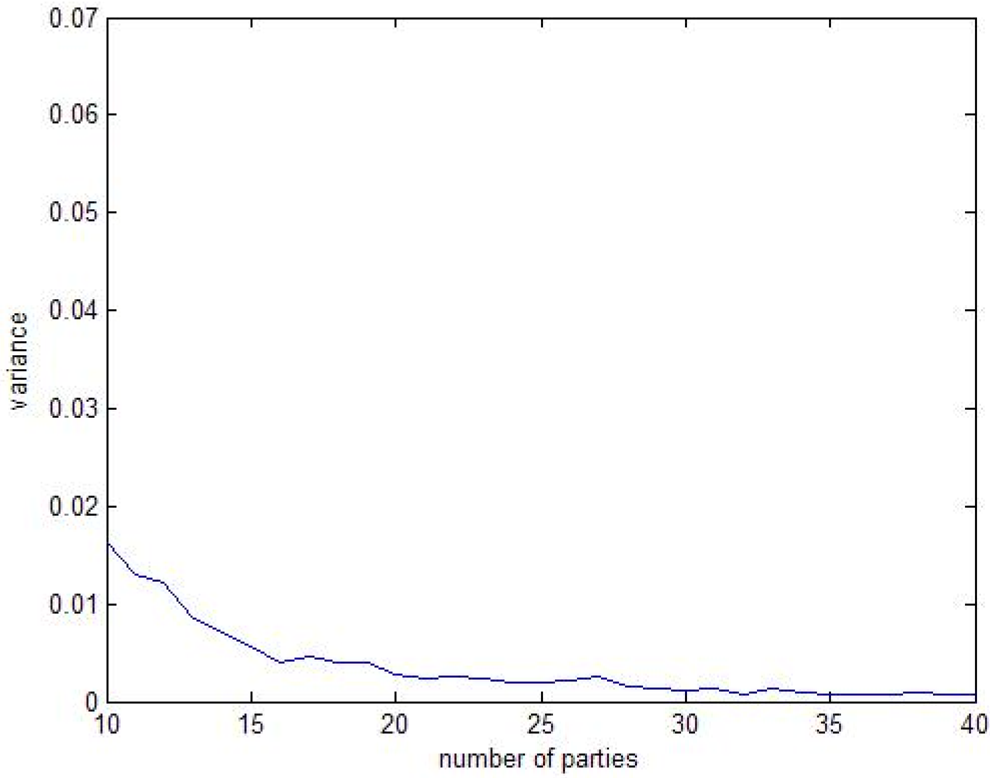

Figure 2: $Var(\frac{Sh_i(X)}{X_i}, n)$, $q = 0.5$

The next proposition follows from Neyman [1982] and states that if X is uniformly distributed on A_n, then the ratio $\frac{Sh_i(X)}{X_i}$ converges to 1 in probability, as n increases.

Proposition 2 Let $q \geq 0.5$. Suppose X is distributed on A_n with the uniform distribution. Then for any $\epsilon > 0$, there exists n' s.t. whenever $n > n'$ $Prob(|\frac{Sh_1(X)}{X_1} - 1| > \epsilon) < \epsilon$.

Note that Proposition 2 does not imply that $Exp(\frac{Sh_1(X)}{X_1})$ converges to 1, as $n \to \infty$, since the random variable $\frac{Sh_1(X)}{X_1}$ has no upper bound and for some realizations converges to infinity.

Proof [7]

First, we prove the following lemma.

Lemma 5 $\lim_{n\to\infty} Exp(\sum_{i=1}^{n} |Sh_i(X) - X_i|) = 0$, when X is uniformly distributed on A_n.

[7] This proof was contributed by Abraham Neyman.

Proof Since $\sum_{i=1}^{n} Sh_i(x) = \sum_{i=1}^{n} x_i = 1$,

$$\sum_{i=1}^{n} |Sh_i(x) - x_i| \leq 2 \tag{10}$$

for any realization $X = x \in A_n$.

Applying Neyman [1982, Main Theorem] we have that for every $\epsilon > 0$, there exists $\delta(\epsilon)$ s.t. for every n and for every $x \in A_n$

$$\max_{1 \leq i \leq n} x_i \leq \delta(\epsilon) \Rightarrow \sum_{i=1}^{n} |Sh_i(x) - x_i| < \frac{\epsilon}{3}. \tag{11}$$

Next we compute for every η, $0 < \eta < 1$, the probability that $X_i \leq \eta$. To that end denote

$$B_n^i(\eta) = \{x \in A_n | \eta \leq x_i \leq 1\}$$

$$A_n(1-\eta) = \{x \in \mathbb{R}_+^n | \sum_{i=1}^{n} x_i = 1 - \eta\}$$

It could be verified that

$$Vol(B_n^i(\eta)) = Vol(A_n(1-\eta)) = (1-\eta)^{n-1} Vol(A_n)$$

and

$$Prob(X_i > \eta) = \frac{Vol(B_n^i(\eta))}{Vol(A_n)} = (1-\eta)^{n-1}. \tag{12}$$

Hence for every i, $1 \leq i \leq n$,

$$Prob(X_i > \delta(\epsilon)) = (1 - \delta(\epsilon))^{n-1}.$$

Since $\delta(\epsilon)$ does not depend on n and $0 < \delta(\epsilon) < 1$, for n sufficiently large

$$Prob(X_i > \delta(\epsilon)) < \frac{\epsilon}{3n}.$$

This implies that

$$Prob(\exists i \in N \text{ s.t. } X_i > \delta(\epsilon)) \leq \frac{\epsilon}{3}. \tag{13}$$

Since X has uniform distribution over A_n

$$Exp(\sum_{i=1}^{n} |Sh_i(X) - X_i|) = \int_{\max_{1 \leq i \leq n} x_i \leq \delta(\epsilon)} \sum_{i=1}^{n} |Sh_i(x) - x_i| \, dx \cdot Prob(\max_{1 \leq i \leq n} X_i \leq \delta(\epsilon)) +$$
$$+ \int_{\exists i \in N \text{ s.t. } x_i > \delta(\epsilon)} \sum_{i=1}^{n} |Sh_i(x) - x_i| \, dx \cdot Prob(\exists i \in N \text{ s.t. } X_i > \delta(\epsilon)).$$

By (10), (11) and (13), for sufficiently large n

$$Exp(\sum_{i=1}^{n} |Sh_i(X) - X_i|) \leq \frac{\epsilon}{3} + \frac{2}{3}\epsilon = \epsilon,$$

as claimed. □

We proceed to prove the Proposition 2. Let $\epsilon_1 > 0$. By Lemma 5, for n sufficiently large

$$\sum_{i=1}^{n} Exp|Sh_i(X) - X_i| < \epsilon_1$$

Since the distribution of X on A_n is symmetric

$$nExp|Sh_1(X) - X_1| \leq \epsilon_1. \tag{14}$$

Clearly,

$$Prob(|\frac{Sh_1(X)}{X_1} - 1| > \epsilon) = Prob(|Sh_1(X) - X_1| > \epsilon X_1).$$

For any $1 > c > 0$,

$$\begin{aligned} Prob(|Sh_1(X) - X_1| > \epsilon X_1) &= Prob(|Sh_1(X) - X_1| > \epsilon X_1 \text{ and } X_1 > c) + \quad (15) \\ &+ Prob(|Sh_1(X) - X_1| > \epsilon X_1 \text{ and } X_1 \leq c) \leq \\ &\leq Prob(|Sh_1(X) - X_1| > \epsilon c \text{ and } X_1 > c) + Prob(X_1 \leq c) \end{aligned}$$

By the Markov inequality and (14),

$$Prob(|Sh_1(X) - X_1| > \epsilon c \text{ and } X_1 > c) \leq \frac{Exp(|Sh_1(X) - X|)}{\epsilon c} \leq \frac{\epsilon_1}{cn\epsilon}. \tag{16}$$

By (12)

$$Prob(X_1 \leq c) = 1 - (1 - c)^{n-1}. \tag{17}$$

From (15), (16) and (17) we have

$$Prob(|Sh_1(X) - X_1| > \epsilon X_1) \leq \frac{\epsilon_1}{cn\epsilon} + 1 - (1 - c)^{n-1}. \tag{18}$$

Let $c = \frac{\delta'}{n}$. From (18),

$$Prob(|Sh_1(X) - X_1| > \epsilon X_1) \leq \frac{\epsilon_1}{\epsilon\delta'} + 1 - (1 - \frac{\delta'}{n})^{n-1}$$

Since $(1-\frac{\delta'}{n})^{n-1} \to e^{-\delta'}$, as $n \to \infty$, there exists $n(\delta')$ s.t. if $n > n(\delta')$, then $(1-\frac{\delta'}{n})^{n-1} > e^{-\delta'} - \frac{\epsilon}{3}$. Hence,

$$Prob(|Sh_1(X) - X_1| > \epsilon X_1) \leq \frac{\epsilon_1}{\epsilon\delta'} + 1 - e^{-\delta'} + \frac{\epsilon}{3}.$$

Let δ' be sufficiently small s.t. $1 - e^{-\delta'} < \frac{\epsilon}{3}$. Let $n > n(\delta')$. Then

$$Prob(|Sh_1(X) - X_1| > \epsilon X_1) \leq \frac{\epsilon_1}{\epsilon\delta'} + \frac{2\epsilon}{3}.$$

Let ϵ_1 be sufficiently small, such that $\frac{\epsilon_1}{\epsilon\delta'} < \frac{\epsilon}{3}$. Then for n sufficiently large

$$Prob(|Sh_1(X) - X_1| > \epsilon X_1) < \epsilon \ \square$$

Empirical evidence

We analyzed all 26 elections of the Second Chamber ("Tweede Kamer") of the Netherlands' parliament since 1918 (the first time the PR system was introduced in the Netherlands). The data was taken from Mackie and Rose [1991], Van Der Eijk [1989],Lucardie and Voerman [1995],Irwin [1999],Lucardie [2003],Lucardie and Voerman [2004],Lucardie [2007] and Lucardie and Voerman [2011]. In these elections we only consider parties that entered the parliament [8]. For each party we calculated the ratio of its Shapley value to its size (the size is defined as the fraction of popular vote it received) [9].

Since parties in parliamnets change over time we could not use the average over elections of this ratio for every party. Instead we took for every election the average of this ratio over the parties for every election. For parliaments of 10 or more parties the average ratio is close to 1 and the variance is close to 0.

Figure 3 summarizes our findings.

[8] We also ignored parties classified as "others" in the data sources we used. In most cases those parties did not obtain sufficient votes to pass the electoral threshold.

[9] For parliaments of at least 10 parties we use for the Shapley value the approximation method of Owen [1975]

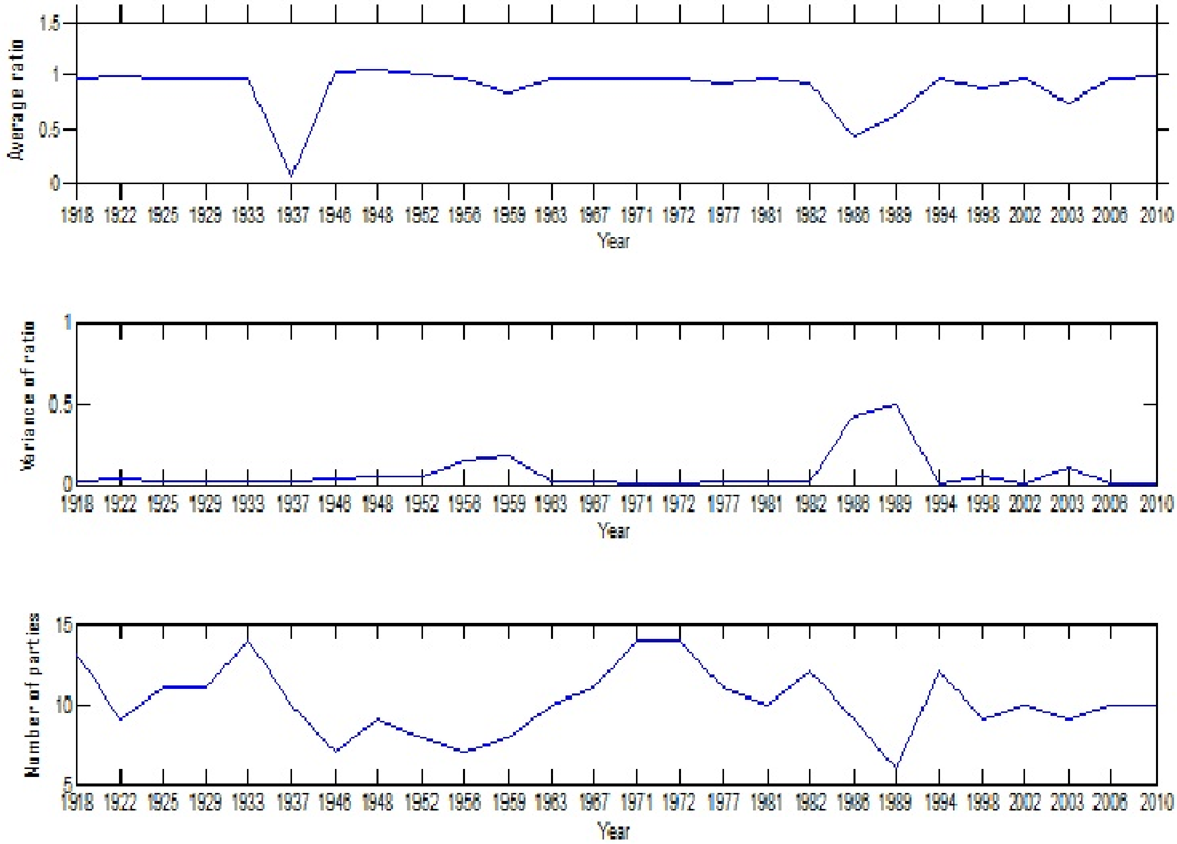

Figure 3: The data analysis for the elections in the Netherlands, 1918-2010

Remark 3 Similar calculations could be made for the Banzhaf index. It can be shown, that in most cases the expected value of the ratio of the party's Banzhaf index to its size is close to be a constant significantly larger than 1, and the variance is relatively high.

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