Central Bank Balance Sheet Policies Without Rational Expectations*

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Abstract

We study the effects of central bank balance sheet policies—namely, quantitative easing and foreign exchange interventions—in a model where people form expectations through the level-k thinking process, which is consistent with experimental evidence on the behavior of people in strategic environments. We emphasize two main theoretical results. First, under a broad set of conditions, central bank interventions are effective under level-k thinking, while they are neutral in the rational expectations equilibrium. Second, forecast errors about future endogenous variables are predictable by balance sheet interventions. We confirm these predictions using data on mortgage purchases by US government sponsored enterprises.

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1 Introduction

The balance sheets of central banks are among the most important and widely-used stabilization policy tools.[1] A recent example of their use is the policy of quantitative easing (QE), which is a central bank's purchase of long-term public bonds and private risky assets financed with central bank liabilities. Several central banks in developed countries have recently used quantitative easing to stimulate their economies when the conventional policy tool, the nominal interest rate, reached its effective zero lower bound. Yet balance sheet policies are not strictly confined to liquidity traps: foreign exchange (FX) interventions are another type of balance sheet policies, which have arguably been used more often across countries and over time. An FX intervention is a central bank's purchase of foreign sovereign bonds denominated in foreign currency, which is usually financed by selling holdings of domestic sovereign bonds. Advanced economies had to routinely rely on FX interventions during periods of fixed exchange rate arrangements (e.g., during the Gold Standard, the Bretton Woods, and the European Exchange Rate Mechanism). Moreover, during the recent financial crisis, some economies have again resorted to such interventions to tame speculative capital flows (e.g., in Switzerland and Israel), and to stimulate domestic production (e.g., in the Czech Republic). Emerging economies have also been using FX interventions to limit exchange rate fluctuations and to accumulate buffers against sudden stops.

Despite their popularity, central bank balance sheet policies are not yet well understood. First, from an empirical perspective, identifying a causal effect of these policies is challenging, as they are usually implemented in response to economic events, thereby creating an endogeneity problem. There is nonetheless evidence, which we discuss below, in favor of the effects of QE and FX interventions on asset markets and the real economy. There is still uncertainty, however, about the magnitude of these effects and, more importantly, about the mechanisms through which these policies operate. Second, from a theoretical perspective, a wide class of standard macroeconomic models predicts that balance sheet policies are completely irrelevant. This *irrelevance result* is essentially the celebrated Modigliani-Miller proposition, a cornerstone in the theory of corporate finance, applied to central-bank interventions, as noted by Wallace (1981). Curdia and Woodford (2010, Section 1) further observe that the irrelevance result continues to hold even when markets are incomplete. The intuition of the irrelevance result is a combination of two effects. First, when a central bank purchases, for example, private risky assets and issues safe liabilities (as in the case of QE), investors understand that gains or losses incurred on the central bank's portfolio will be directly transferred to the fiscal authority and, through

[1] Bernanke (2012); Draghi (2015); Yellen (2016) discuss the importance of the balance-sheet policies in the US and the Eurozone during the recent financial crisis.

taxes, they will indirectly return to investors. As a result, investors will reduce their individual demand for risky assets to hedge against this new tax risk. Second, the demand for an asset depends also on the future resale price of the asset. Thus, each investor must forecast how all the other investors will respond to the central bank's policy in order to predict future demands for risky assets and, hence, their prices. In a world where the irrelevance result holds, investors not only anticipate the tax risk, they also believe that all the other investors will anticipate the tax risk, that these other investors believe that all the other investors will anticipate the tax risk, and so on. Formally, the entire hierarchy of "higher-order beliefs" must be formed correctly. In this case, investors expect that the overall future demand for risky assets will absorb the central bank's intervention and, as a consequence, that future asset prices will not respond to the new policy. Taken together, the two effects imply that current prices will be unchanged, making central bank interventions irrelevant.

Higher-order beliefs are formed correctly when agents hold rational expectations. While the assumption of rational expectations represents a cornerstone in macroeconomics, it is nonetheless a strong one. Laboratory experiments have repeatedly demonstrated that standard equilibrium analysis, which is based on rational expectations, fails to predict people's behavior, especially when players are confronted with novel strategic situations. Instead, the evidence suggests that a process known as level-k thinking is a better description of how players form beliefs about their opponents and, hence, make decisions. This process assumes that agents interrupt the formation of higher-order beliefs at some finite level k, either due to the complexity of the economic environment or because agents believe that other agents are less sophisticated.[2]

Level-k thinking can be particularly relevant in macroeconomic settings, especially when people are confronted with new policies—such as QE—which share two distinctive features. First, they are widely advertised by policymakers and media, thus, they are likely to receive considerable attention by the public.[3] Second, the novelty of these policies implies that data on their effects is likely to be scarce, making it very costly for households and firms to predict their consequences as well as the response of the other agents. Agents must then form beliefs about endogenous variables with little guidance by past experience and by policymakers, thus, they are unlikely to hold rational expectations. In these cases, level-k thinking provides a plausible alternative as it does not require the knowledge of past policy effects and, in addition, acknowledges that the process of

[2] Crawford et al. (2013) provide a recent review of level-k thinking in game theory.

[3] This is in contrast to the models in which people are inattentive to policy changes, such as Gabaix (2016), or people observe government policies with noise, such as Angeletos and Lian (2018). The unprecedented public attention brought to the Fed's balance sheet policies by the recent financial crisis has likely made these policies common knowledge.

higher-order belief formation may be disrupted.

In this paper, we introduce the level-k thinking process into a dynamic stochastic equilibrium model along the lines of De Long et al. (1990), which allows us to derive all our results in a closed-form. We show that the assumption that people forecast future endogenous variables through the level-k thinking process invalidates the irrelevance result and provides a new channel for central bank balance sheet policies. We then verify empirically some of our theoretical predictions.

We model the level-k thinking process of belief formation as in Garcıa-Schmidt and Woodford (forthcoming) and Farhi and Werning (2017). All agents are assumed to be perfectly aware of current balance sheet policies as well as of their own income and asset positions. However, each agent is characterized by a "level of thinking," which determines her expectations about the effects of balance sheet policies on future endogenous variables, such as taxes and asset prices. More specifically, expectations are constructed according to an iterative procedure. First, "level-1 thinking" posits that, after observing the policy change, agents do not update their expectations. As a result, "level-1 thinkers" make consumption and portfolio decisions under their old expectations; in particular, they do not hedge against the future tax risk, as required for the irrelevance result to hold. Next, level-2 thinkers are assumed to believe that the economy is populated only by level-1 thinkers. Thus, upon observing a policy change, they expect future variables to coincide with the equilibrium outcomes of an economy populated only by level-1 thinkers. Notice that, unlike level-1 thinkers, these more sophisticated agents revise their expectations following a policy intervention. However, they may still hold non-rational expectations. Proceeding recursively, we can define the expectations and, hence, the behavior of level-k thinkers, for any finite k. Having characterized the expectations of every agent, we follow Garcıa-Schmidt and Woodford (forthcoming) and compute the equilibrium of an economy populated by agents with different levels of thinking.[4] The resulting notion of equilibrium is known as *reflective* equilibrium.

Our first main result shows that, when agents are level-k thinkers, balance sheet policies have an impact on asset prices, while they are neutral when expectations are rational. Intuitively, since agents do not hold rational expectations about future endogenous variables, they underestimate the tax risk emanating from policy interventions and incorrectly forecast the behavior of future assets prices. As a result, they demand lower risk premia, which boosts asset prices and makes balance sheet policies effective. Inter-

[4]Garcıa-Schmidt and Woodford (forthcoming) assume that the average expectation is updated in a continuous fashion following a first-order differential equation. This assumption is equivalent to assuming that beliefs are formed in a discrete way as we have just discussed and that the economy is populated by all types of thinkers. Moreover, Angeletos and Lian (2017) point out that a notion of reflective equilibrium "smooths out" some of the unappealing properties of level-k equilibria (i.e., a reflective equilibrium with a degenerate distribution over k).

estingly, even when all the agents correctly understand the tax risk, which is the case in our model when agents are level-2 thinkers or higher, balance sheet policies may still be effective. This is because even more sophisticated agents fail to form all the higher-order beliefs correctly: while they predict the tax risk correctly, they believe that the other agents will have incorrect beliefs, or that these other agents believe that the other agents will have incorrect beliefs, and so on.

We derive a number of properties of balance sheet policies in the reflective equilibrium. First, as the average level of sophistication in the economy increases, the reflective equilibrium converges to the rational expectations equilibrium. However, we show that this convergence may be non-monotonic when the policy intervention is expected to persist over time. Specifically, an increase in the sophistication of agents has two opposing effects. On the one hand, as the economy becomes more sophisticated, more agents foresee the fiscal consequences of balance sheet policies, bringing the policy closer to full neutrality. On the other hand, more sophisticated agents become endogenously more forward-looking, thus making persistent policies more effective. Second, we entertain the possibility that a fraction of agents holds rational expectations and show that their presence does not make the effects of balance sheet policies disappear; on the contrary, they can amplify the effectiveness of these policies. Third, we consider the effects of "learning" in our model by letting the sophistication of agents increase over time. As a result, the economy converges to the rational expectations equilibrium in the long run. One important consequence of this learning process is that balance sheet policies become less effective over time.

We also consider an extended version of our economy and show that the purchases of long-term *nominal* public bonds paid by issuing nominal reserves (or selling short-term bonds) and the purchases of foreign bonds paid by selling domestic bonds (i.e., sterilized FX intervention) are also effective in the reflective equilibrium, while they are completely neutral in the rational expectations equilibrium. The intuition behind this result is an application of our first main result: long-term nominal bonds and foreign bonds may be safe in the currency of their denomination, but they are risky in real terms due to inflation and foreign exchange risk. As a result, central bank purchases of these assets change the tax risk faced by households. If the households do not fully internalize the extent of this risk, they cannot undo these interventions.

Our second main result characterizes the behavior of forecast errors of asset prices after policy interventions. We show that individual and cross-sectional-average forecast errors are related to policy interventions. Importantly, this result can help differentiate the mechanism proposed in this paper from other theories of balance sheet policies in the literature. First, predictable forecast errors are absent in the standard models that assume limited market participation but retain the assumption of rational expectations. Second, in models with incomplete information in which agents form expectations rationally, predictable forecast errors would arise only if agents had imperfect information about policy interventions. If, instead, agents had access to all the relevant information regarding the policy, forecast errors would no longer be predictable. Instead, in our model, agents are fully aware of the policy intervention, yet they make mistakes due to their inability to form rational expectations.

Finally, we confirm that the empirical evidence is consistent with the predictability of cross-sectional-average forecast errors by balance sheet policies. We focus on the mortgage market in the US and use purchases of mortgages by government-sponsored enterprises (GSEs) such as Fannie Mae and Freddie Mac as a proxy for quantitative easing. In particular, we follow Fieldhouse, Mertens and Ravn (2018), who identify "exogenous and unexpected" changes in mortgage purchases by the GSEs using a narrative approach in the spirit of Romer and Romer (2010). As predicted by our model, we first verify that these exogenous changes in mortgage purchases affect the conventional mortgage rate. We then use the Blue Chip Financial Forecasts survey data to show that exogenous purchases by the GSEs also predict conventional mortgage rate forecast errors. Finally, using these empirical estimates together with our stylized model, we calculate that the number of level-1 thinkers (those who do not change their expectations after policy interventions) in the data is 86 percent.

Related literature. Our paper is related to several strands of literature. First, we contribute to the literature that incorporates deviations from rational expectations into macroeconomic models. One useful way to think of such deviations, as highlighted in, for example, Woodford (2013), is to divide them into the so-called "inductive" and "eductive" approaches of belief formation.[5] An example of the inductive approach is statistical learning, which estimates econometric models with past data and then uses them to make predictions about the future.[6] The eductive approach, instead, assumes that agents understand the model and use it to form expectations through a process of reflection, which is potentially independent of past experience. The formation of beliefs via level-k thinking, which we examine in this paper, is an example of the eductive approach.

The level-k thinking belief-formation process has been widely used in behavioral game theory to rationalize the behavior of subjects in various laboratory and field experiments playing full-information games (Nagel, 1995; Stahl and Wilson, 1995; Bosch-Domenech

[5]The term "eductive" comes from Binmore (1987).

[6]Econometric learning has been widely used in macroeconomics (see Evans and Honkapohja, 2012; De Grauwe, 2012 for reviews) and finance (e.g., Barberis et al., 2015 and references therein).

et al., 2002).[7] Interestingly, these papers show that deviations from Nash-equilibrium behavior in many simple games are most stark on the first round of play, when agents face novel strategic environments, with no prior experience. In these games, subjects usually exhibit levels of thinking no higher than 3.[8]

Most closely related to our paper are Garcıa-Schmidt and Woodford (forthcoming) and Farhi and Werning (2017), who use level-k to study the policy of forward guidance.[9] Crucially, these papers abstract from aggregate risk. Our mechanism, instead, hinges entirely on this type of risk: under level-k thinking, agents incorrectly believe that balance sheet policies insulate them from aggregate risk or, perhaps, that other agents believe that this is so, hence, they demand lower risk premia.

Another related paper is Gabaix (2016) that augments an otherwise standard New Keynesian model with subjective "discounting" of expectations about future variables, which helps resolve a number of puzzles in the New Keynesian literature. In contrast, in our environment, agents perfectly understand current and future policy changes, however, they do not revise their expectations about endogenous variables. In fact, one can view the level-k process of belief formation as a way to generate the subjective discounting of the future in Gabaix (2016).

We also contribute to the theoretical literature that studies the effectiveness of balance sheet policies. An important starting point is the irrelevance result in Wallace (1981). Backus and Kehoe (1989) show the irrelevance of *sterilized* foreign-exchange intervention in an international setting. To deviate from the irrelevance result, the literature has proposed various frictions. In particular, incomplete information and market segmentation. The former friction generates the so-called "signaling" channel and the latter generates the "portfolio balance" channel.

According to the signaling channel, changes in the composition of a central bank's balance sheet do not have a direct effect on the economy. Instead, they serve as a signal of the central bank's objectives and economic fundamentals, about which agents have incomplete information. Mussa (1981), Bhattacharya and Weller (1997), Popper and Mont-

[7] Camerer et al. (2004) propose a related model of "cognitive hierarchy" in which level-k thinkers assume that the other players are not only level-$(k-1)$, like in this paper, but also level-$(k-2)$ and so on. This alternative assumption retains most of the tractability of level-k thinking, but outperforms it in some applications.

[8] In several papers, laboratory experiments mimic macroeconomic situations. For example, Kneeland (2016) studies coordinated attack games, such as currency attacks, in a laboratory setting and concludes that a model with the level-k belief formation process better fits the responses of subjects to public information than a model with dispersed information rational expectations. Giamattei (2015) runs an experiment in which price setters respond to central bank's attempt to reduce inflation. He shows that subjects' price choices are better approximated by a model with level-k thinking rather than with rational expectations.

[9] The level-k thinking process of belief formation was first used in macroeconomics by Evans and Ramey (1992, 1998) to analyze conventional monetary policy. Qiu (2018) quantifies the effects of conventional monetary policies in a calibrated New Keynesian model, extended with level-k thinking.

gomery (2001), Vitale (1999, 2003) extend this idea to FX interventions.[10] The portfolio-balance channel posits that changes in the supplies of different assets affect asset prices due to the segmentation of assets markets. Segmentation can occur because of fixed costs of entry or because of limited market participation by yet-unborn people in models with overlapping generations. Kouri (1976) and, more recently, Gabaix and Maggiori (2015), Fanelli and Straub (2016), Amador et al. (2017), and Cavallino (2017) apply this idea to FX interventions, while Vayanos and Vila (2009), Curdia and Woodford (2011), Chen et al. (2012), Hamilton and Wu (2012), and Silva (2016) apply this idea to quantitative easing; finally, Krishnamurthy and Vissing-Jorgensen (2011) summarize the recent literature on quantitative easing.[11] In the current paper, we propose a "bounded rationality" channel of balance sheet policies, derive the implications that can distinguish it from other prominent channels, and provide empirical support for it.

Our empirical exercise is related to the literature that measures the effects of balance sheet policies on macroeconomic and financial variables. Using high-frequency financial data, Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), and Hancock and Passmore (2011) found that large-scale purchases of mortgage-backed securities (MBS) by the United States Federal Reserve have affected mortgage market yields; those effects have then spread to other assets markets. Di Maggio et al. (2016) and Chakraborty et al. (2016) found evidence of the effects of the Federal Reserve's MBS purchases on mortgage lending. Fieldhouse et al. (2017) show that purchases of MBS by government-sponsored enterprises in the US, which resemble a policy of QE, affected not only mortgage rates and lending, but also residential investment.[12] Instead, we show that identified

[10]Some papers have also considered the case in which the central bank cannot commit to a desired monetary policy and uses the balance sheet policy as a costly signal about its future intentions (see Jeanne and Svensson (2007) and Bhattarai et al. (2015) for a discussion of FX and QE interventions, respectively).

[11]Reis (2017) proposes that quantitative easing is a powerful stabilization tool in times of fiscal crisis. Sterk and Tenreyro (2013) show that standard open-market operations have sizable effects on the real economy in the presence of durable goods and asset markets segmentation. Goncharov et al. (2017) propose a political economy explanation of the effects of balance sheet policies by noting that central bankers are averse to negative profits on central bank balance sheet under greater political pressure.

[12]Chodorow-Reich (2014) estimates that surprise announcements about the Federal Reserve's quantitative easing policies during the years of 2008-09 had negative effects on the credit default swap (CDS) spreads of life insurance companies and banks. At the same time, Stroebel and Taylor (2012) find no assets markets effects of the Federal Reserve's MBS purchases in its first round of quantitative easing.

Dominguez and Frankel (1990, 1993) estimate significant effects of sterilized FX interventions. Sarno and Taylor (2001) discuss earlier literature that often found no significant effects. One potential reason that can explain the absence of evidence of the effects of FX interventions is "leaning against the wind" by governments. This type of policy introduces reverse causality that biases estimates. To deal with this bias, Kearns and Rigobon (2005) study a "natural experiment" in which Japan and Australia "exogenously" changed their FX policies, resulting in statistically and economically significant changes in their exchange rates. Another reason why it might be hard to detect significant effects of FX interventions is that some studies focus on advanced countries, where FX interventions are often a small fraction of the overall size of the bond market (Menkhoff, 2013). To address this concern, Dominguez et al. (2013) and Kohlscheen and Andrade changes in balance sheet policies affect the *forecast errors* of asset prices by financial market experts. Our empirical exercise is related to Coibion and Gorodnichenko (2012), who also study the responses of forecast errors to various macroeconomic shocks. However, they do not relate forecast errors of asset prices to central bank balance sheet shocks.

The rest of the paper is organized as follows. Section 2 presents the real model and studies purchases of private risky assets by the central bank. Section 3 presents two extensions of the real model that introduce a nominal friction in the form of money demand. The first extension studies purchases of nominal long-term bonds financed by the issuance of short-term nominal reserves. The second one adds an international dimension to the model and explores foreign exchange interventions. Section 4 presents and tests the implications of the model. Section 5 concludes.

2 Risky Assets Purchases in a Real Closed-Economy Model

We now present a closed, endowment economy that we will refer to as the "simple model." The structure of the model is close to the model in De Long et al. (1990), with the main difference that in our simple model agents are neither noise traders nor they hold rational expectations. Instead, they form their expectations according to the level-k thinking process. We use this model to investigate the effects of purchases of private risky assets by the government.[13] The purchases of mortgage-backed securities by the Federal Reserve during the Great Recession (also referred to as QE1) are an example of such policies.

2.1 Assets, Agents, and Expectations

Time is discrete, infinite, and indexed by $t = 0, 1, 2, \ldots$. There are two assets in the economy. First, there is a one-period riskless asset, available in perfectly elastic supply, that pays off a real dividend $r > 0$. Because this asset last only for one period, r represents also the net return on this asset. Second, there is a risky asset, available in fixed supply $\overline{X}$, that pays off a risky dividend r^x_{t+1} (in units of the consumption good) in the following period

(2014) use data from the Czech Republic and Brazil, respectively, and employ a high-frequency identification strategy. They find a significant effect of FX interventions on nominal exchange rates. Chamon et al. (2017) also find significant effects by applying a synthetic control approach to Brazilian data. Blanchard et al. (2014) present evidence of significant effects of FX interventions from a cross-section of developing countries.

[13] One can decompose the effects of the balance sheet policies on the real economy into two parts. First, the policy might affect asset prices. Second, after observing changes in asset prices, agents adjust their behavior, which in turn changes real quantities. For example, after observing an increase in house prices, companies can start building more houses. In this paper, we only consider the first part. We leave the investigation of the second part (i.e., the endogenous response of real variables) to future research.

and trades at price q_t in period t. The dividend on the risky asset satisfies $r_t^x = r^x + \epsilon_t^x$, where r^x is constant and ϵ_t^x is assumed to be independent over time and normally distributed, with zero mean and standard deviation σ_x. We denote the gross return on the riskless asset as $R \equiv 1 + r$.

Households. There are overlapping generations (OLG) of households. Each household lives for two periods. In the first period, the household receives a real endowment w, which she can use to buy the two assets described above. In the second period, she gets a return on her portfolio, pays taxes, and consumes. In each period, there is an equal mass of size one of "young" and "old" households. To make their consumption and portfolio choices, households need to form expectations about future variables. We describe expectations in detail below; for now, we use a tilde on top of the expectation operator to emphasize the fact that households may use a probability distribution over future variables that differs from their true distribution.

Specifically, given beliefs and prices, in period t households choose consumption c_{t+1}, investment in the safe asset s_{t+1}, and investment in the risky asset x_{t+1}, so as to maximize

$$-\frac{1}{\gamma}\widetilde{\mathbb{E}}_t e^{-\gamma c_{t+1}},$$

subject to the current-period budget constraint

$$s_{t+1} + q_t x_{t+1} \leq w, \tag{1}$$

and the future-period budget constraint

$$c_{t+1} + T_{t+1} \leq R s_{t+1} + \left(r_{t+1}^x + q_{t+1}\right) x_{t+1}. \tag{2}$$

It is worth commenting on our choice of the OLG framework. It is well known that, in an OLG environment, asset purchases by the government can affect the economy even under rational expectations. In such environments, in fact, future generations are excluded from participating in asset markets that operate before they are born. The OLG model is therefore an example of a limited participation model, where the Wallace irrelevance result may not apply. In this sense, the level-k thinking process through which agents form expectations—the focus of this paper—is not necessary to make asset purchases effective.

There are two main reasons that lead us to choose this particular environment. First, maximization of exponential preferences with Gaussian shocks is equivalent to maximization of mean-variance preferences. This property turns out to be extremely convenient in those settings, such as ours, where agents face uninsurable risk. In fact, the model presented in this section has been a workhorse model in the finance literature starting from the seminal contribution of De Long et al. (1990). Second, with the appropriate choice of policy, we can guarantee that the Wallace irrelevance result holds even in our OLG environment. We can thus achieve tractability without departing from the key benchmark of irrelevance of asset purchases in the rational expectations equilibrium (REE). In fact, Wallace (1981) also uses a two-period OLG model to derive his irrelevance result.

To ensure that asset purchases are irrelevant in the REE, it is enough to assume that the government transfers the profits from its portfolio choice at time t to the old households at time $t+1$. Under this assumption, the young households, who trade assets to save for the future, are also those who will bear the tax risk when they will be old. The existence of a policy that makes asset purchases irrelevant is not an artifact of our environment: similar policies can be shown to exist in virtually all limited-participation models, provided that the government has access to a rich enough set of policy tools. Once we guarantee that the irrelevance result holds when expectations are rational, we can attribute any effects of asset purchases to the departure from such expectations.

Expectations. We allow household expectations to deviate from rational expectations. More precisely, consistent with the idea that households understand policy announcements but may be unable (or may think that other agents are unable) to solve for the equilibrium of the economy, we make the following assumptions. First, for future *exogenous* variables—that is, $\{r^x_{t+1}\}$—we assume that expectations coincide with the *true* distribution of such variables.[14] Second, for future *endogenous* variables, we assume that expectations are described by a sequence of one-period-ahead conditional distributions. More precisely, letting Z_{t+1} denote the vector of endogenous variables at some future time $t+1$, we assume that, conditional on information at t, households expect Z_{t+1} to be distributed according to some cumulative distribution function $\widetilde{\phi}_t$.[15] Given these one-period-ahead conditional distributions, it is immediate to derive n-period-ahead distributions for any n. In the model of this section, $Z_{t+1} = (q_{t+1}, T_{t+1}, B_{t+1})$, where B_{t+1} denotes the issuance of safe debt by the government, which we discuss below. For convenience, we denote the sequence of one-period-ahead beliefs starting from period t as $\widetilde{\Phi}_t \equiv \{\widetilde{\phi}_s\}_{s\geq t}$.

[14]In a recent work, Bordalo et al. (2016) study the consequences of incorporating "diagnostic expectations", where agents over-weigh the future likelihood of events that occurred in the recent past, on the volatility and predictability of credit spreads. In contrast, we assume that agents' expectations about future shocks coincide with their true distribution. It would be interesting to combine "diagnostic expectations" with level-k thinking. We leave this exercise for future research.

[15]We use a "tilde" to stress that the distribution $\widetilde{\phi}_t$ can potentially differ from the distribution ϕ_t implied by equilibrium.

Government. In the simple model, we do not distinguish between monetary and fiscal authorities. Instead, we consider a consolidated government that conducts fiscal policy and implements purchases of private risky assets. Government purchases of private risky assets can then capture not only the purchases of mortgage-backed securities by the Federal Reserve during the Great Recession, but also the purchases of private assets by the US Treasury conducted in October 2008, known as the Troubled Asset Relief Program (TARP). We will use capital letters to denote government policies. The government controls real per capita taxes $\{T_{t+1}\}$, real purchases of private risky assets $\{X_{t+1}\}$, and the *real* amount of public bonds $\{B_{t+1}\}$. We let $\Pi_t \equiv \{T_{t+1}, X_{t+1}, B_{t+1}\}$.

Without loss of generality, we focus on time 0 and assume that the government announces the entire path of risky asset purchases $\{X_{t+1}\}$. We refer to these purchases as "quantitative easing." The consolidated budget constraint of the government is

$$q_t X_{t+1} + RB_t = (r_t^x + q_t)\, X_t + B_{t+1} + T_t. \tag{3}$$

The left-hand side represents government's outlays, consisting of purchases of risky assets, $q_t X_{t+1}$, and repayment of bonds, RB_t. The right-hand side is government income. Although not necessary for our analysis, it is convenient to require that the government finances its purchases of risky assets by issuing debt. Formally,

$$B_{t+1} = q_t X_{t+1}. \tag{4}$$

After combining equations (3) and (4), we have that the taxes levied on the old generation at time $t+1$ equal

$$T_{t+1} = -\left(r_{t+1}^x + q_{t+1} - Rq_t\right) X_{t+1}. \tag{5}$$

The interpretation of (5) is straightforward. The government finances asset purchases by issuing safe assets. In the next period, the government raises taxes to pay for the interest on its borrowing and to cover any loss on its portfolio of risky assets. Note that, if the government makes positive profits, taxes can become negative, hence, old households receive a transfer.

2.2 Beliefs and Equilibrium Concepts

When studying deviations from rational expectations, it is useful to start with a more general notion of equilibrium known as temporary equilibrium (Hicks, 1939; Lindahl, 1939; Grandmont, 1977; Woodford, 2013). A temporary equilibrium generalizes the standard REE insofar as it does not impose restrictions on beliefs about endogenous variables, which are then free to deviate from their equilibrium counterparts. More specifically, a temporary equilibrium takes as given household beliefs about future endogenous variables and requires only that *(i)* households optimize given these beliefs and that *(ii)* markets clear in every period.

Definition (Temporary Equilibrium)**.** Conditional on beliefs $\{\widetilde{\Phi}_t\}$, a temporary equilibrium is a collection of household choices $\{c_t, x_{t+1}, s_{t+1}\}$, government policies $\{\Pi_t\}$, and prices $\{q_t\}$ such that

1. Given beliefs and prices, households optimize for all t;
2. Risky-asset and bond markets clear for all t

$$x_{t+1} + X_{t+1} = \overline{X}, \tag{6}$$
$$s_{t+1} = B_{t+1}; \tag{7}$$

3. The government constraints (4) and (5) are satisfied for all t.

Suppose that, at time t, agents' expectations are described by the sequence of conditional distributions $\widetilde{\Phi}_t$. Given these beliefs, a temporary equilibrium is a collection of (potentially stochastic) endogenous variables, which satisfy household optimality, market clearing, and government constraints for every t. Equilibrium variables can be described with a sequence of one-period ahead conditional distributions, which we denote with $\Phi_t \equiv \{\phi_s\}_{s \geq t}$. As a result, we can represent a temporary equilibrium as a mapping from beliefs to equilibrium distributions, which we formally write as

$$\Phi_t = \Psi(\widetilde{\Phi}_t, \{X_{t+1}\}). \tag{8}$$

In general, the sequence of future distributions Φ_t may differ from the original sequence of household beliefs $\widetilde{\Phi}_t$, except when agents hold rational expectations.[16]

Definition (Rational Expectations Equilibrium)**.** A REE is a temporary equilibrium that satisfies

$$\widetilde{\Phi}_t = \Phi_t, \text{ for all } t.$$

Note that REE beliefs are a fixed point of (8).[17]

[16] The discrepancy between beliefs and equilibrium outcomes can in principle open the door to learning. The notion of temporary equilibrium, however, does not allow households to update their beliefs when observing equilibrium variables, such as prices. We make this assumption to emphasize the implications of non-rational expectations, which is the novel channel of this paper.

[17] It is standard in macroeconomics to define rational expectations equilibrium by requiring that a per-

Level-k process of belief formation. The definition of temporary equilibrium is silent about the origin of beliefs. We now consider a specific process of belief formation, known as level-k thinking, where k denotes the level of sophistication of an agent. By assumption, agents know the correct model of the economy and understand policy announcements. Instead, the process introduced in this section concerns agents' beliefs about future *endogenous* variables.

We start by assuming that, before the policy intervention, the economy is in its REE, that is, all agents hold rational expectations about future variables. We begin with level-1 agents, the lowest level of sophistication. We assume that, after the policy intervention in period $t = 0$, these agents do not change their beliefs about future endogenous variables. Formally, the beliefs of level-1 agents are $\widetilde{\Phi}_t^1 = \Phi_t^{SQ}$, for all periods $t \geq 0$, where the additional superscript denotes "level-1" beliefs and Φ_t^{SQ} denotes household beliefs in the REE *before* the policy intervention, which we refer to as the "status quo."[18] Having specified beliefs of level-1 agents, we can use the mapping (8) to obtain the distributions generated in the temporary equilibrium of an economy populated by level-1 thinkers. We then move to level-2 agents and assume that their beliefs coincide with the temporary equilibrium distributions just obtained. Proceeding recursively, we can define the beliefs of level-k agents for any $k \geq 1$.

Formally, level-k agents' beliefs are defined as follows. Given level-k thinkers' beliefs $\widetilde{\Phi}_t^{k-1}$, $k \geq 2$, we use (8) to obtain the distributions of endogenous variables in the temporary equilibrium, for all t. We then assume that these distributions coincide with the beliefs of level-k thinkers. The entire process of belief formation is thus described by the following recursion:

$$\widetilde{\Phi}_t^{k+1} = \Psi(\widetilde{\Phi}_t^k, \{X_{t+1}\}), \tag{9}$$

for all $k \geq 1$ and $t \geq 0$.

Reflective Equilibrium. Having defined the beliefs of level-k thinkers for any k, we introduce the notion of equilibrium that we will use to investigate government interventions. We follow Garcıa-Schmidt and Woodford (forthcoming) and consider an economy populated by households who are heterogeneous in their levels of sophistication k. In particular, the population is divided into different groups depending on their beliefs. Each

ceived law of motion equals an actual law of motion of variables (Stokey, 1989; Ljungqvist and Sargent, 2012). In our notation, the sequence $\widetilde{\Phi}_t$ represents the perceived law of motion and Φ_t is the actual law of motion. We do not summarize these laws of motion with functions or conditional distributions (as it is usually done in macroeconomics) but rather with a *sequence* of conditional distributions because government policy $\{X_{t+1}\}$ may take a non-recursive form.

[18] Note that the beliefs of level-1 agents do not incorporate the effects of policy interventions not only at the start of the policy in period $t = 0$, but also at later dates $t > 0$. This will not be the case if agents can update their beliefs over time, for example, through learning. We discuss this possibility in Section 2.6.

group contains households with the same level of sophistication k and has mass given by the probability density function $f(k) \geq 0$, with $\sum_{k=1}^{\infty} f(k) = 1$. One advantage of this approach is that the economy is not indexed by a discrete level of sophistication. Instead, by changing the mean of $f(k)$, we can vary the average level of sophistication in the economy and perform comparative statics in a continuous way.

Definition (Reflective Equilibrium)**.** A reflective equilibrium is a collection of beliefs $\{\widetilde{\Phi}_t^k\}_k$, household choices $\{c_t^k, x_{t+1}^k, s_{t+1}^k\}$, government policies $\{\Pi_t\}$, and prices $\{q_t\}$ such that

1. Given beliefs and prices, households optimize for all t;

2. Risky-asset and bond markets clear for all t

$$\sum_{k=1}^{\infty} f(k) x_{t+1}^k + X_{t+1} = \overline{X}, \tag{10}$$

$$\sum_{k=1}^{\infty} f(k) s_{t+1}^k = B_{t+1}; \tag{11}$$

3. The government constraints (4) and (5) are satisfied for all t;

4. Beliefs are generated through the mapping (9), starting from $\widetilde{\Phi}_t^1 = \Phi_t^{SQ}$, for all t.

2.3 Equilibrium Effects of Risky Assets Purchases

We now solve the household problem and then derive the temporary equilibrium for a general sequence of balance sheet policies. Let $\mathcal{R}_{t+1} = r_{t+1}^x + q_{t+1} - Rq_t$ be the realized one-period excess return on one unit of risky asset. Let $\widetilde{\Sigma}_t$ denote the variance of $\mathcal{R}_{t+1}$ under distribution $\widetilde{\phi}_t$. The other moments are denoted analogously by adding a tilde on top.

So far, we have treated the sequence of distributions $\widetilde{\Phi}_t$ as arbitrary. In what follows, it will be useful to impose more structure on beliefs. Specifically, in this section we assume that every element of $\widetilde{\Phi}_t$, for example, $\widetilde{\phi}_s$, $s \geq t$, is such that any endogenous variable Z_{s+1} can be represented as a linear function of the contemporaneous shock:

$$Z_{s+1} = \alpha_s + \beta_s \epsilon_{s+1}^x, \tag{12}$$

where α_s and β_s are deterministic (vector) functions of time. We use a subscript i, $i \in \{q, T, B\}$, to denote any element of α_s and β_s. Here, α_s represents the expected value of Z_{s+1}, while β_s captures the expected sensitivity of Z_{s+1} to the aggregate shock.

While the assumption of linearity might seem restrictive at this stage, it turns out that it will be automatically satisfied in the reflective equilibrium. This result follows from the fact that the mapping (9) preserves linearity and that the initial condition is given by REE variables, which take the form of (12).

To solve the model, we combine the two budget constraints of the household into a single intertemporal budget constraint, plug it into the objective function, and take the first-order conditions with respect to x_{t+1}. Since preferences are exponential, beliefs are linear, and shocks are normally distributed, the household problem becomes a standard mean-variance portfolio optimization problem, which has a simple closed-form solution.

Lemma 1. *When beliefs satisfy (12), the household asset demand is*

$$x_{t+1} = \frac{\widetilde{\mathbb{E}}_t(\mathcal{R}_{t+1})}{\gamma \widetilde{\Sigma}_t} + \frac{\widetilde{cov}_t\,(\mathcal{R}_{t+1}, T_{t+1})}{\widetilde{\Sigma}_t}, \tag{13}$$

where the volatility of risky asset return is $\widetilde{\Sigma}_t = \sigma_x^2(1+\beta_{q,t})^2$ *and the covariance between the asset return and taxes is* $\widetilde{cov}_t(\mathcal{R}_{t+1}, T_{t+1}) = \sigma_x^2(1+\beta_{q,t})\beta_{T,t}$.

This result, which we prove in Appendix A.1, is well known in the finance literature. The first term on the right-hand side of (13) shows that the demand for risky assets is proportional to their excess return and inversely proportional to the coefficient of absolute risk aversion γ times the volatility of excess returns. The second term captures a hedging motive coming from the fact that the return on the risky asset may correlate with taxes. This correlation may be non-zero when, as it will be the case here, the government conducts balance sheet policies and some households realize that assets returns will affect taxes. If, for example, households expect future taxes to be negatively affected by the return on risky assets, then these assets are a bad hedge against future tax risk. As a result, demand (13) will be lower.

Equation (13) suggests that, by affecting beliefs of future taxes, balance sheet policies can potentially influence investors demand and, by market clearing, equilibrium asset prices. Equilibrium future asset prices, in turn, feed back into asset demand. Below, we show that this feedback between beliefs of future prices, asset demand, and equilibrium prices leads to the irrelevance of balance sheet policies when expectations are rational.

A convenient property of (13) is that it does not depend on the optimal choice of consumption nor on investment in the riskless asset. As a result, we can impose the market-clearing condition in the risky-asset market, i.e., equation (6), and solve for the endogenous price without reference to the other markets.[19] Formally, using (12), we can express

[19]Note, however, that the particular functional forms are not responsible for the qualitative results that balance sheet policies are neutral in the REE and effective in the reflective equilibrium. However, as we

the risky-asset price at time t as

$$q_t = \frac{1}{R}\left[r^x + \alpha_{q,t} - \gamma\sigma_x^2\left(1+\beta_{q,t}\right)^2\left(\overline{X} - X_{t+1} - \frac{\beta_{T,t}}{1+\beta_{q,t}}\right)\right]. \tag{14}$$

Equation (14) illustrates that the risky-asset price in the temporary equilibrium is, in general, a function of both government holdings of risky assets and of household beliefs about future prices and taxes.

Finally, equilibrium taxes and issuance of safe bonds by the government are obtained by substituting (14) into the financing constraint (4) and into the government budget constraint (5). Taken together, equations (4), (5), and (14) define the mapping in (8).

2.4 Neutrality under Rational Expectations

We next solve for the response of the economy to balance sheet policies in the REE, in which expectations are linear in fundamental shocks as in (12). By definition, in a REE subjective beliefs must be equal to equilibrium distributions. We can then use the equilibrium tax process (5) to conclude that beliefs about one-period-ahead taxes must satisfy $\beta_{T,t} = -X_{t+1}^G$. In addition, since the contemporaneous realization of the shock does not appear in equation (14), the equilibrium asset price must satisfy $\beta_{q,t} = 0$, hence, $q_{t+1} = \widetilde{\mathbb{E}}_t q_{t+1} = \alpha_{q,t}$. The fact that the asset price is independent of the aggregate shock is not surprising, since neither demand nor supply of risky assets are stochastic. With these two observations, we can rewrite equation (14) in a familiar way:

$$q_t = \frac{r^x + q_{t+1} - \gamma\sigma_x^2\overline{X}}{R},$$

This equation is a standard asset pricing equation showing that the current price equals the discounted sum of expected dividends and future resale price minus the risk premium. After imposing a no-bubble condition, we can solve the above equation forward to obtain:

$$q_t = \frac{r^x - \gamma\sigma_x^2\overline{X}}{R-1} \equiv q^{REE}. \tag{15}$$

The following proposition summarizes the key property of q^{REE}.

Proposition 1. *In the REE, the price of the risky asset does not depend on balance sheet policies.*

Proposition 1 states that, when agents correctly anticipate future taxes, government intervention is irrelevant. This is the celebrated result that, in an economy where the Ri-

mentioned earlier, working with an exponential utility function and normal distributions allows us to express all the results explicitly.

cardian equivalence holds, asset purchases by the government—or, equivalently, by the central bank—are irrelevant. The reason is that, when expectations are rational, households correctly anticipate that future taxes will be risky because of government purchases and, thus, adjust their demand for risky assets. In the end, equilibrium prices are unaffected.

2.5 Non-neutrality under Level-k Thinking

We now depart from rational expectations and assume that, following an announcement of intervention in period $t = 0$, households form expectations following the level-k process described in Section 2.2. As explained in that section, we assume that, before the announcement, the economy is in the REE. Thus, absent a policy change, households would correctly forecast the behavior of future prices and taxes. Formally, status-quo beliefs $\widetilde{\Phi}_t^{SQ}$ are a fixed point of (8) when $X_{t+1} = B_{t+1} = 0$, for all t.

We begin with level-1 beliefs and use (8) to obtain level-k beliefs recursively. It is instructive to consider the recursion for the risky-asset price, which is obtained from equation (14) after noting that *(i)* $\beta_{q,t}^k = 0$, for all $k \geq 1$ and $t \geq 0$, because shocks do not enter the risky-asset market-clearing condition; *(ii)* $\beta_{T,t}^1 = 0$, for all $t \geq 0$, because level-1 agents do not update their beliefs; and *(iii)* $\beta_{T,t}^k = -X_{t+1}$, for $k > 1$ and $t \geq 0$, because more sophisticated thinkers take into account the government budget constraint (5). As a result, we obtain

$$q_t^k = \begin{cases} \dfrac{r^x + q^{REE} - \gamma\sigma_x^2 \left(\overline{X} - X_{t+1}\right)}{R}, & k = 1, \\ \dfrac{r^x + q_{t+1}^{k-1} - \gamma\sigma_x^2 \overline{X}}{R}, & k > 1, \end{cases} \tag{16}$$

where q_t^k denotes the temporary equilibrium price when all agents hold level-k beliefs. By assumption, this price coincides with level-$(k+1)$ agents' beliefs.

To gain some intuition about equation (16), remember that agents need to form expectations about next period's asset price and taxes. The first line of equation (16) reflects the fact that, following a policy of asset purchases, level-1 agents do not revise their expectations about future asset prices and taxes. Instead, they think that these variables will coincide with the status quo before the policy intervention, where $q_t = q^{REE}$ and $T_t = 0$ (taxes follow from equation (5)). Thus, level-1 thinkers behave as if the X_{t+1} units of risky assets purchased by the government have disappeared from the economy. In turn, since they expect the economy and, hence, their future consumption to be less risky, they require a lower risk premium and, as a result, the asset price increases. Moving to the second line of equation (16), we see that level-k thinkers, for $k > 1$, expect next period's asset price to coincide with the asset price computed in the previous iteration, q_{t+1}^{k-1}. On the tax side, these agents correctly anticipate the risk contained in future taxes, hence, they fully hedge against such risk. As a result, the quantity X_{t+1} disappears from the pricing equation.

The iterative process implied by (16) is depicted in Figure 1. The horizontal axis represents time and the vertical axis plots the level k. A bold dot corresponding to time t and level k represents the temporary equilibrium price q_t^k. The diagram shows visually that if one wants to compute, for example, the asset price at time 0 in a temporary equilibrium with level-5 agents, one has to "iterate diagonally" and compute the asset price at time 1 in a temporary equilibrium with level-4 agents, q_1^4, the asset price at time-2 in a temporary equilibrium with level-3 agents, q_2^3, and so on.

Importantly, these iterations always stop at the point where the temporary equilibrium with level-1 agents is reached. This is because, beyond that point, agents will no longer revise their expectations following a policy announcement. In fact, Figure 1 suggests that the economy displays a strong form of *endogenous discounting*. To see this, suppose that the economy is populated by level-k agents, with $k \leq 5$, and that, at time 0, the government announces that it will purchase risky assets in period 6. From (16), level-k agents do not react to events happening more than k periods ahead. Therefore, the announcement of the government will have no effect on asset prices at $t = 0$. Formally, if we iterate equation (16), we have that the temporary equilibrium price at time t,

$$q_t^k = q^{REE} + \frac{\gamma \sigma_x^2 X_{t+k}}{R^k}, \tag{17}$$

depends only on government purchases k-periods ahead, discounted at rate R.

Having characterized beliefs for any level of sophistication, we turn to the reflective equilibrium, which assumes that heterogeneous agents coexist in the economy. Using Lemma 1, the market-clearing condition in the reflective equilibrium can be written explicitly in period t as

$$X_{t+1} + \sum_{k=1}^{\infty} f(k) \left(\frac{r^x + \alpha_{q,t}^k - q_t R}{\gamma \sigma_x^2} + \beta_{T,t}^k \right) = \overline{X}. \tag{18}$$

Using the beliefs of level-k households obtained above, we can solve (18) for the price of the risky asset, which is given in the following proposition.

Proposition 2. *Given a sequence of balance sheet policies $\{X_{t+1}\}$, the asset price in the reflective equilibrium satisfies*

$$q_t = q^{REE} + \gamma \sigma_x^2 \sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{R^k}. \tag{19}$$

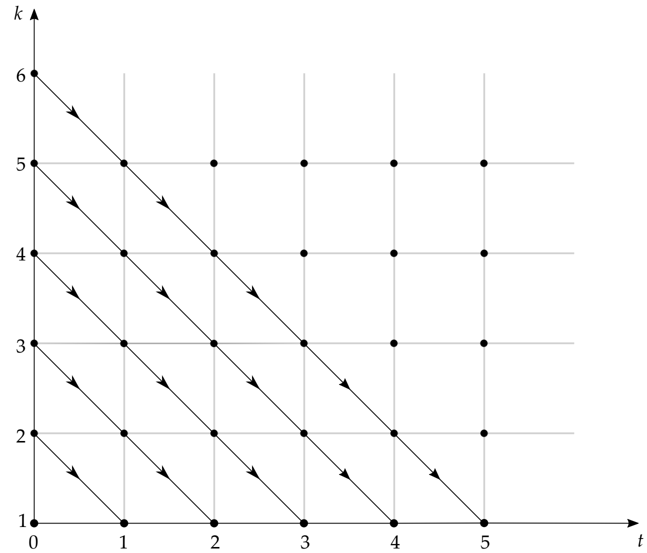

Figure 1: The price q_t^k of the risky asset in the temporary equilibrium where agents hold level-k beliefs. The horizontal axis plots time and the vertical axis plots the level of sophistication k. Every bold dot represents q_t^k. The arrows point to the direction of "diagonal iterations" required to compute the price q_t^k.

Proposition 2 shows that, in the reflective equilibrium, balance sheet policies are an effective tool for controlling the price of risky assets. Moreover, since the risk-free real return is fixed at R, any change in the risky asset price can be immediately interpreted as a change in the risk-premium required by investors. This is not surprising: the reason for why prices change is that some households fail to understand how asset purchases translate into future tax risk.

An important consequence of equation (19) is that the average level of sophistication of the agents in the economy—defined as the mean of $f(k)$—has two counteracting effects on the strength of balance sheet policies. First, a lower average level of sophistication implies that fewer agents internalize the future fiscal consequences of balance sheet policies. This effect tends to make balance sheet policies stronger. Second, following the discussion of Figure 1, a lower average level of sophistication implies a higher endogenous discounting, thus, more agents disregard the effects of balance sheet policies in the distant future. This effect tends to make balance sheet policies weaker.

To illustrate these two effects formally, we compute the risky-asset price following the announcement, at time 0, of a one-time purchase of risky assets in the following period, i.e., $X_2 > 0$ and $X_t = 0$, $t \geq 1$, $t \neq 2$. For the sake of specificity, we also assume that $f(k)$ is the pdf of an exponential distribution (i.e., $f(k) = (1-\lambda)\lambda^{k-1}$, $\lambda \in [0,1)$). With this distribution, the average level of sophistication in the economy $\bar{k}$ is $1/(1-\lambda)$. From equation (19), the risky asset price at time 0 equals

$$q_0 - q^{REE} = \frac{\gamma \sigma_x^2}{R^2} \left(\frac{1}{\bar{k}} - \frac{1}{\bar{k}^2} \right) X_2.$$

The nonlinear effect of $\bar{k}$ on the price is clear. If $\bar{k} = 1$, then the policy is completely ineffective because the discounting effect is so strong ($\bar{k} = 1$ if and only if all agents are level-1) that households do not react even to policies implemented in the very near future. As $\bar{k}$ increases above 1, the endogenous discounting effect becomes weaker and the policy gains power, provided that $\bar{k}$ is below 2. The policy strength peaks at $\bar{k} = 2$ and then declines again to zero as $\bar{k}$ approaches infinity, that is, when the equilibrium approaches the REE.

2.6 Discussion of Alternative Assumptions

We now discuss how the results we have obtained so far change when we make two additional assumptions, namely, *(i)* a fraction of agents hold rational expectations and *(ii)* agents use a simple learning mechanism.

Presence of Rational Expectations Agents. In the simple model, we assumed that all agents are level-k thinkers. A natural question is whether the presence of households who hold correct expectations about future endogenous variables can restore the neutrality result of balance sheet policies. The answer to this question is negative. Intuitively, if the presence of such agents was enough to guarantee the neutrality of balance sheet policies then, following any such policy, the price of risky assets would remain at its REE value. In this case, as we have seen in the proof of the REE benchmark, the demand for risky assets by rational-expectations agents would drop exactly by the amount of the government intervention. At the same time, however, the demand for risky assets by level-k thinkers would drop by less than the government intervention, because some of these agents fail to hedge against the tax risk. As a result, the market-clearing condition would not be satisfied. We formalize this logic in Appendix A.4.1.

Interestingly, the presence of rational-expectations agents can amplify the effects of balance sheet policies on the risky-asset price. Intuitively, rational-expectations agents are fully forward looking relative to any level-k thinker with finite k. As a result, when a balance sheet policy persists over time, rational-expectations agents take into account the entire future path of the policy. Example 2 in Appendix A.4.1 illustrates this point formally.

Equilibrium Unraveling and Long-Run Neutrality. Laboratory and field experiments provide some evidence that people "learn" to play the Nash equilibrium after several repetitions of simple one-shot games (Nagel, 1995). In the simple model, we assumed that households always initiate their eductive belief-formation process from the status quo that corresponds to the REE without policy actions. As a result, if the policy is stationary over time, the model produces stationary outcomes and, in particular, there is no convergence of the reflective equilibrium towards the REE over time.

In Appendix A.4.2, we consider a simple process of "learning." Specifically, we assume that, after observing the policy, the level of sophistication of the agents increases over time. We show that this simple process leads to a form of "equilibrium unraveling," whereby the economy converges to the REE and balance sheet policies become ineffective. Interestingly, as the power of balance sheet policies fades away, it is no longer enough for the government to keep its policy constant to exert a constant effect on the risky-asset price. Instead, asset purchases need to grow over time to counteract the unraveling effect. This logic suggests that, when a policy of asset purchases is repeated over time, subsequent rounds may turn out to be less effective.

Although we focused only on these two extensions, the simple model is tractable enough to accommodate a number of other extensions. In particular, it is easy to allow for the interaction between level-k thinking and other frictions that are invoked to generate non-neutrality of the balance-sheet policies (e.g., incomplete information and segmented markets). Similarly, we could introduce alternative learning protocols (e.g., adaptive learning by level-1 thinkers) and an endogenous supply of risky asset. We leave the analysis of these extensions to future research.

3 Purchases of Domestic and Foreign Nominal Public Bonds

The previous section presented our simple model in which the government intervened by acquiring private real risky assets. Balance sheet policies, however, are not confined to purchases of private assets. For example, in November 2010, the Federal Reserve announced the purchase of $600 billion worth of US Treasury securities (this operation was dubbed as "QE2"), and in September 2011, communicated a plan to purchase long-term public bonds by selling short-term bonds (this intervention was dubbed as "Operation Twist," after a similar policy action implemented in 1961).[20] Moreover, there are many historical cases of central banks around the world purchasing foreign public bonds. In

[20] See the November 3, 2010 FOMC statement for the details on QE2, the September 21, 2011 FOMC statement for further information about the 2011 Operation Twist, and Alon et al. (2011) for the discussion of the 1961 Operation Twist.

some cases, such purchases are financed by selling domestic public bonds (i.e., the so-called "sterilized foreign exchange interventions").

When default risk is negligible, public bonds (usually fixed-income securities) are riskless in *nominal* terms. However, in the presence of inflation or exchange rate risk, public bonds are not riskless in *real* terms. We can then use the insights developed in the previous section to analyze domestic and foreign public bond purchases. We pursue this as follows. First, in Section 3.1, we augment our simple model with long-term public bonds and inflation risk to study the purchases of long-term public bonds. Second, in Section 3.2, we extend our simple model to an international setting and study foreign exchange (FX) interventions. We present these two extensions separately to keep our notation simple, albeit at the cost of lengthening the exposition.

3.1 Purchases of Domestic Long-Term Public Bonds

To study the effects of long-term public bonds purchases, we extend the simple model in two ways. First, we add a nominal friction in the form of a utility service from money balances, which is necessary to generate the demand for money. Second, we introduce nominal long-term bonds. There are thus four assets in the economy: *(i)* a riskless real asset, which pays a gross return $R \equiv 1 + r > 1$ and is available in perfectly elastic supply; *(ii)* money, which is issued by the monetary authority; *(iii)* a one-period nominal bond, which pays a continuously compounded nominal interest rate i_t and is issued by the fiscal authority; and *(iv)* a nominal long-term bond, which pays one unit of currency every period, trades at price q_t in real terms, and is issued by the fiscal authority. We assume that each long-term bond is a perpetuity that matures with probability $\delta \in [0, 1]$ in every period, independently of the other bonds. The expected time to maturity of a long-term bond is thus equal to $1/\delta$ in every period. Finally, for the sake of simplicity, we do not consider private risky assets in this extension. None of the results are affected by this simplification.

The only source of aggregate risk in the economy is given by shocks to the money supply. In particular, we assume that the money supply follows a moving average stochastic process of the form $\log M_{t+1} = \log \overline{M} + \epsilon_t^m - \epsilon_{t-1}^m (1 + v)/v$, where $v > 0$ and $\overline{M} > 0$. The disturbances ϵ_t^m are assumed to be independent over time and normally distributed with zero mean and standard deviation σ_m. Note that ϵ_t^m affects both the current and the following period's money supply.[21]

[21] The specific form of the money supply—i.e., the presence of the lagged shock ϵ_{t-1}^m and the parameter v that also appears in the household preferences—allows us to streamline the analysis. Under these assumptions, in fact, there is no inflation risk between two consecutive periods, making one-period nominal bonds riskless and long-term bonds risky in real terms. It is straightforward to solve the model under alterna-

Households. Each household maximizes

$$-\frac{1}{\gamma}\widetilde{\mathbb{E}}_t \exp\left[-\gamma\left(c_{t+1}-\frac{m_{t+1}\left[\log\left(m_{t+1}/\overline{m}\right)-1\right]}{v}\right)\right] \tag{20}$$

by choosing real assets s_{t+1}, short-term nominal bonds b_{t+1} (expressed in units of period-t consumption), long-term nominal bonds d_{t+1} (expressed in units of period-t consumption), real money balances m_{t+1}, and consumption c_{t+1}, subject to the current period budget constraint

$$P_t s_{t+1} + P_t b_{t+1} + P_t q_t d_{t+1} + P_t m_{t+1} + P_t T_t^y \leq P_t w, \tag{21}$$

and the next period budget constraint

$$P_{t+1}\left(c_{t+1}+T_{t+1}^o\right) \leq P_{t+1} R s_{t+1} + e^{i_t} P_t b_{t+1} + \left[1+(1-\delta)\,P_{t+1}q_{t+1}\right] d_{t+1} + P_t m_{t+1}. \tag{22}$$

Here, T_t^y and T_t^o are real taxes paid by the young and old generations, respectively, P_t is the nominal price level, and $\overline{m}$ is a positive constant. Unlike in the simple model, preferences are assumed to depend on real money balances m_{t+1}, which is a standard way of introducing the demand for money in macroeconomic models. The particular functional form assumed here simplifies the analysis by making money demand independent of the consumption choice. Note that utility is increasing in m_{t+1}, for $m_{t+1} \leq \overline{m}$, and decreasing in m_{t+1}, for $m_{t+1} > \overline{m}$. We thus restrict our analysis to the case with $m_{t+1} \leq \overline{m}$.

As in the simple model, households' expectations are captured by the sequence of one-period ahead distributions $\widetilde{\Phi}_t \equiv \{\widetilde{\phi}_s\}_{s\geq t}$, where $\widetilde{\phi}_s$ is a distribution, conditional on information available at time s, of the vector of endogenous variables Z_{s+1}. In the extended model, $Z_t \equiv (p_t, i_t, q_t, T_t^y, T_t^o, B_{t+1}^{CB})$, where $p_t \equiv \log P_t$ and where B_{t+1}^{CB} is part of government policies, which we define below. As in the simple model, level-k thinkers will form expectations in a recursive way, starting from a status-quo distribution $\widetilde{\Phi}_t^{SQ}$ that corresponds to the linear REE before the intervention. We do not write this recursion explicitly here, but we formally construct level-k beliefs $\widetilde{\Phi}_t^k$ in the proof of Proposition 3.

Government. We separately specify the behavior of the fiscal authority (e.g., the Treasury or the Finance Ministry) and the monetary authority (e.g., the central bank).

The fiscal authority controls real per capita taxes $\{T_t^y, T_t^o\}$, the *real* amount of one-period public nominal bonds $\{B_{t+1}\}$, and the *real* amount of nominal long-term bonds. Without any loss, we can assume that the real outstanding amount of long-term bonds is held constant at D, implying that the fiscal authority simply replaces maturing bonds

tive processes for money supply that lead to a one-period-ahead inflation risk. However, this clutters the exposition without adding any important economic insights.

with newly issued long-term bonds. In addition, the fiscal authority receives transfers $\{Tr_t\}$ from the monetary authority in every period. The fiscal authority budget constraint is thus

$$e^{i_{t-1}}P_{t-1}B_t + D = P_t(T_t^y + T_t^o) + P_tB_{t+1} + P_tq_t\delta D + Tr_t.$$

The left-hand side represents payments on short-term and long-term bonds, respectively. The right-hand side sums up all sources of revenue: taxes, issuance of one-period bonds, replacement of matured long-term bonds, and transfers from the monetary authority. Finally, note that we are implicitly assuming that the original quantity D of long-term bonds was issued at some date before period t.

The monetary authority controls the nominal money supply $\{M_{t+1}\}$, the *real* amount of one-period interest-paying reserves $\{B_{t+1}^{CB}\}$, and purchases of long-term public bonds $\{D_{t+1}^{CB}\}$. Since reserves and short-term public bonds will be perfect substitutes in equilibrium, they will pay the same interest rate i_t.[22] The budget constraint of the monetary authority is

$$M_t + e^{i_{t-1}}P_{t-1}B_t^{CB} + q_tD_{t+1}^{CB} + P_tTr_t = M_{t+1} + B_{t+1}^{CB} + [1 + (1-\delta)\,P_tq_t]D_t^{CB}.$$

The left-hand side represents outlays consisting of repayment to money holders, payments on reserves, purchases of long-term bonds, and transfers to the fiscal authority. The right-hand side represents central bank revenues consisting of issuance of money, issuance of reserves, and income from coupons and sales of long-term bonds.

To save on notation and without loss of generality, we assume that central bank bond holdings before the intervention are zero. Moreover, again without loss of generality, we consider only balance sheet policies consisting of purchases of long-term bonds entirely financed by issuance of reserves. We again refer to such policies as "quantitative easing." Formally, we require

$$B_{t+1}^{CB} = q_tD_{t+1}^{CB}. \tag{23}$$

We finally let $\Pi_t = \{T_{t+1}^y, T_{t+1}^o, M_{t+1}, B_{t+1}, D, B_{t+1}^{CB}, D_{t+1}^{CB}\}$ summarize the policy tools of the (consolidated) public sector.

As in the simple model, to isolate the effects of the deviation from rational expectations, we ensure that balance sheet policies are irrelevant in the REE. To achieve this in our OLG setting, we assume that government interventions do not shift risk across generations. In the simple model, there were no outstanding bonds before the intervention, thus, it was enough to use taxes on the old generations. A different scenario arises when there

[22] We assume that only cash M_t, which we refer to as "money," provides utility benefits to households. This is an extreme form of the assumption that cash is more liquid than central bank reserves and short-term government bonds.

is a positive supply of outstanding bonds, as in the model of this section. We therefore assume that, in every period, the fiscal authority purchases all the outstanding bonds and money from the old generation and finances these purchases, together with the interest accrued on the bonds, by taxing the old generation. At the same time, the fiscal authority issues short-term bonds, sells long-term bonds and money, and transfers the proceeds to the young generation. Formally, we impose the following fiscal rules:

$$P_t T_t^y = - P_t q_t (D - D_{t+1}^{CB}) - P_t (B_{t+1} + B_{t+1}^{CB}) - M_{t+1}, \tag{24}$$

$$P_t T_t^o = P_{t-1} e^{i_{t-1}} (B_t + B_t^{CB}) + [1 + (1-\delta)\, P_t q_t](D - D_t^{CB}) + M_t. \tag{25}$$

It is straightforward to adapt the definition of reflective equilibrium to the environment of this section, which we formally do in the proof of Proposition 3 in Appendix A.2.

To streamline the analysis, we assume that the demand and supply of money are negligibly small. Specifically, we first solve for the equilibrium, and then we let $\overline{m}$ and $\overline{M}$ approach zero, such that the ratio $\overline{m}/\overline{M}$ approaches one. This "cashless limit" is a standard assumption employed in, for example, the New Keynesian literature to eliminate the real effects of nominal money supply above and beyond its effects on inflation and the nominal interest rate. We can thus abstract from money holdings when computing equilibria.

The following proposition summarizes the effects of quantitative easing on prices in the reflective equilibrium of the economy.

Proposition 3. *Consider a sequence of quantitative easing policies* $\{D_{t+1}^{CB}\}$. *In the cashless limit of the REE, the short-term interest rate,* $i_t = r - (\epsilon_t^m - \epsilon_{t-1}^m)/v$, *the price level,* $p_t = vr - \epsilon_{t-1}^m/v$, *and the long-term bond price,* $q_t = (1 - vr)\,/\,(R - 1 + \delta) + \epsilon_t^m\,/\,(vR) \equiv q_t^{REE}$, *are all independent of balance sheet policies.*

In the cashless limit of the reflective equilibrium, the short-term nominal interest rate and the price level coincide with their REE counterparts, while the long-term bond price satisfies

$$q_t = q_t^{REE} + \frac{(1-\delta)^2}{R^2 v^2} \sigma_m^2 \sum_{k=1}^{\infty} f(k) \left(\frac{1-\delta}{R}\right)^{k-1} D_{t+k}^{CB}.$$

In particular, q_t *is increasing in the amount of long-term bonds purchased by the central bank.*

We relegate the proof of this proposition to Appendix A.2 as it repeats the steps discussed in Section 2 and does not add new economic insights. We only discuss the intuition here.

The first notable conclusion of Proposition 3 is that, as in the simple model, balance sheet policies are irrelevant in the REE benchmark. The endogenous variables, however, do depend on money supply shocks in the REE. To gain intuition, recall that the contemporaneous shock ϵ_t^m has a positive impact on money supply in the current period and a negative impact on money supply in the following period. Thus, a positive realization of ϵ_t^m has two effects on the economy. On the one hand, there is a direct effect through market clearing in the money market: an increase in the money supply is accompanied by a lower interest rate, which raises the demand for real money balances, and by a higher price level, which lowers the supply of real money balances. On the other hand, a positive realization of ϵ_t^m brings news of a decline in next period's money supply. This news is accompanied by a further decline in the current interest rate, because expected inflation drops, and by an increase in the price level, because a lower interest rate leads to an even higher demand for real money balances. Although not necessary for the main message of the paper, our choice of the money supply process has the convenient property that the direct and the news effects balance out exactly, thus, the price level is unaffected by contemporaneous money supply shocks. The nominal interest rate, instead, declines. Finally, the real price of the long-term bond is positively affected by ϵ_t^m because this shock leads to a drop in the price level in the following period, which is good news for assets that make fixed payments in nominal terms.

Another notable feature of Proposition 3 is that, in the reflective equilibrium, the price of long-term bonds depends on policy interventions. Importantly, higher-order thinkers (i.e., $k > 1$) correctly anticipate that central bank purchases of long-term bonds will be reflected in future taxes, hence, they adjust their demand for these bonds. However, level-1 thinkers fail to understand this logic and do not change their demand schedule. As a result, to guarantee market clearing, the price of the long-term bond has to increase. Note that the strength of the intervention is positively related to $1 - \delta$ (i.e., the probability of "survival" of a long-term bond), therefore, interventions that happen further into the future have progressively smaller effects on the current price.[23]

Perhaps surprisingly, when it comes to the short-term interest rate, balance sheet policies are ineffective even in the reflective equilibrium. The reason is that, as we have discussed above, the money supply process we have chosen implies that the price level and, hence, the inflation rate are fully predictable one-period ahead. The absence of (one-period-ahead) inflation risk together with the fact that risk-free real rates are fixed imply that asset purchases have no impact on the nominal interest rate.[24]

Finally, it is instructive to relate our analysis to Vayanos and Vila (2009). In this influen-

[23] In the extreme case of $\delta = 1$, long-term bonds become short-term bonds and, since we have assumed that the central bank finances its purchases by issuing essentially short-term bonds, there is no change in the households' holdings of bonds and asset purchases are irrelevant even in the reflective equilibrium.

[24] For a different process of money supply that creates one-period ahead inflation risk, the short-term interest rate would depend on the interventions.

tial paper, the authors propose a model with segmented markets where risk-averse arbitrageurs do not fully arbitrage away shocks to the supply of bonds of different maturities because arbitrage is risky. As a result, a shock to the supply of bonds of a certain maturity impacts the whole yield curve. Interestingly, the model and the conclusions of Vayanos and Vila (2009) can be reinterpreted in the context of our setting where the central bank purchases are akin to shocks to the supply of bonds and our risk-averse households play the role of the risk-averse arbitrageurs. This interpretation of their model is particularly instructive because it highlights a crucial assumption that those who invoke the model of Vayanos and Vila (2009) as a proof of the effectiveness of balance sheet policies have often underweighted. Specifically, while in the model of Vayanos and Vila (2009) there is no connection between taxes and shocks to the supply of bonds, things are different when such shocks originate from balance sheet policies. In fact, when the connection between taxes and purchases of bonds is taken into account, as we emphasize in this paper, and agents form expectations rationally, then asset purchases are irrelevant even if agents are risk averse. However, as long as some agents fail to predict future taxes, then balance sheet policies gain power. In fact, we could think of the model without taxes of Vayanos and Vila (2009) as resembling our model where all households are level-1 thinkers and, thus, do not understand that asset purchases will impact future taxes.

3.2 Foreign Exchange Interventions

We now discuss the effects of international balance sheet policies, such as sterelized FX interventions. To do this, we extend the model presented in the previous section to an international setting by using elements of the models in Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006). Since this is a straightforward extension, we briefly sketch the model and present the main results, relegating a detailed presentation to Appendix A.3.

There are two countries, home and foreign, that are endowed with the same good that is freely traded across borders. Households in both countries can trade domestic money, one-period nominal bonds issued by both countries, and riskless real assets. Without loss of generality, we abstract from private risky assets and long-term nominal public bonds. There are two sources of risk in the world economy. Home- and foreign-country money supplies are, respectively, $\log M_{t+1} = \log \overline{M} + \epsilon_t^m$ and $\log M_{t+1}^* = \log \overline{M}^* + \epsilon_t^{m*}$, where the disturbances ϵ_t^m and ϵ_t^{m*} are assumed to be independent from each other, independent over time, and normally distributed, with zero mean and standard deviations σ_m, and σ_m^*. In this environment, we study the effects of purchases of foreign-country public bonds by the home-country central bank. The following proposition summarizes our main results.

Proposition 4. *Given a sequence of home-government purchases of foreign nominal public bonds*

$\{B_{F,t+1}\}$, in the cashless limit of the reflective equilibrium, the home-country nominal interest rate and price level are

$$\begin{pmatrix} i_t \\ p_t \end{pmatrix} = \begin{pmatrix} i_t^{REE} \\ p_t^{REE} \end{pmatrix} + \begin{pmatrix} 1/v \\ 1 \end{pmatrix} \frac{\gamma\sigma_m^2}{(1+v)^2} \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{F,t+k},$$

where $i_t^{REE} \equiv r - \epsilon_t^m / (1+v)$ and $p_t^{REE} \equiv vr + \epsilon_t^m / (1+v)$ are the nominal interest rate and the price level in the REE. The expressions for the foreign country are analogous. The exchange rate satisfies

$$e_t = e_t^{REE} + \frac{\gamma[\sigma_m^2 + (\sigma_m^*)^2]}{(1+v)^2} \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{F,t+k},$$

where $e_t^{REE} \equiv (\epsilon_t^m - \epsilon_t^{m*}) / (1+v)$ is the nominal exchange rate in the REE. In particular, i_t, p_t, and e_t are increasing in the amount of foreign bonds purchased by the central bank.

The formal proof is in Appendix A.3.1. Once again, in the benchmark with rational expectations, the interest rate, the price level, and the exchange rate take a particularly simple form, which is independent of balance sheet policies. First, the interest rate is given by the risk-free real rate minus the shock to the money supply. Intuitively, to stimulate agents to hold more money, the opportunity cost of holding money, that is, the nominal interest rate i_t, must go down. Second, in this economy, the nominal interest rate i_t equals the constant real interest rate r plus expected inflation $\mathbb{E}_t p_{t+1} - p_t$.[25] As a result, the expected inflation must decline following a shock to the money supply. Since the economy is stationary and the future expected price level is constant, the drop in expected inflation is achieved through an increase in the current price level p_t. Third, the law of one price requires that nominal exchange rate depreciates (i.e., e_t goes up, after a positive shock to the home-country money supply).

More importantly, Proposition 4 shows that, in the reflective equilibrium, balance sheet policies affect asset prices. In particular, the nominal interest rate and the price level—and, therefore, the exchange rate—are now functions of the entire path of bond purchases. In the reflective equilibrium, some households fail to anticipate that future taxes will now be risky in real terms since bonds promise a risk-free *nominal* payment. In particular, since the FX intervention is sterilized, the tax risk in the home country will be a combination of the shocks to the money supplies in the two countries. As before, if households do not hold rational expectations and, hence, fail to hedge the tax risk, asset

[25]Note that inflation risk reduces the demand for nominal government bonds, but at the same time, this risk makes future real taxes positively correlated with real bond returns. The last effect increases the demand for nominal bonds. In the REE, the two effects cancel each other out and the nominal interest rate equals the real rate plus expected inflation.

prices will be affected.

4 Testing the Model Predictions

When households do not hold rational expectations, as we assumed throughout the paper, they make systematic mistakes. In fact, we can use the simple model (or any of its extensions) to derive closed-form expressions for the errors that agents make following central bank interventions. Importantly, these predictions can help us differentiate the mechanism in this paper from other mechanisms that maintain the assumption of *rational expectations*, but assume either market segmentation (i.e., the portfolio balance channel) or asymmetric information between the government and private agents (i.e., the signaling channel).

We begin by deriving the households' forecast errors in the reflective equilibrium. We focus on the price of the risky asset; the forecast errors for the other prices can be obtained analogously. Remember that a level-k thinker assumes that the world is populated only by level-$(k-1)$ thinkers; thus, she expects the price of the risky asset at some future period s to be q_s^{k-1}. We denote her forecast error with $u_{t,s}^k \equiv q_s - \widetilde{\mathbb{E}}_t^k q_s = q_s - q_s^{k-1}$. Proposition 2 gives us an expression for q_s and the price q_s^{k-1} can be obtained by iterating equation (16) $k-1$ times forward. We thus have the following expression for the *average* forecast error:

$$\overline{u}_{t,s} \equiv \sum_{k=1}^{\infty} f(k) u_{t,s}^k = \frac{\gamma \sigma_x^2}{\overline{k}} \cdot \frac{\mu^{t-s}}{\overline{k}\,(R-\mu)+\mu} X_{t+1}, \tag{26}$$

where, for the second equality, we have assumed that f is exponential and that asset purchases start from X_{t+1} and decay exponentially at rate μ. Importantly, the size of the average forecast error depends on the size of the intervention X_{t+1}. This relation between the average forecast error and the size of balance sheet policies is a peculiar prediction of our mechanism. It can help us differentiate, for example, our model from those with rational expectations and symmetric information among private agents, in which there are no systematic forecast errors.

Heterogeneous information models. There is, however, an alternative class of models in which agents form expectations rationally but possess heterogeneous information. Some of these models can potentially generate non-neutrality of balance sheet policies together with predictable forecast errors (both individually and on average across agents). For example, if some agents do not have accurate information about government intervention due to noisy information (Lucas, 1972; Woodford, 2001; Angeletos and La'O, 2010), sticky information (Mankiw and Reis, 2002; Reis, 2006a,b), or rational inattention (Sims, 2003; Maćkowiak and Wiederholt, 2009), they will make predictable forecast errors from the view point of an econometrician who is perfectly aware of the policy implementation.

The two types of models are, however, not observationally equivalent. A crucial difference between them is that, in models with heterogeneous information, individual agents hold rational expectations *conditional* on their information set. More formally, models with heterogeneous information predict that an agent's forecast error is orthogonal to any variable contained in the agent's information set. For example, if agents are aware of the policy of asset purchases, which is publicly announced, then this policy should not predict future forecast errors. On the contrary, level-k agents use their information incorrectly and, hence, make forecast errors that are predictable even with their information. To distinguish the two types of models empirically, therefore, one would need a proxy for individual information about government interventions. Unfortunately, the econometrician rarely observes individual information sets.[26] As a result, the empirical exercise in this section has to be understood as a joint test of rational expectations and full information about the government interventions described below.[27] However, one can argue that, when it comes to forecasting financial variables by financial institutions, which is the case in our empirical exercise below, these forecasters are likely to pay a great deal of attention to government interventions. Therefore, if incomplete information was the main friction, it would be unlikely that government interventions could predict forecast errors.

4.1 Predictability of Forecast Errors in the Data

In this section, we use a specific instance of balance sheet policy to test the central prediction of our model. Specifically, we focus on average forecast errors of mortgage rates and show that they respond significantly to purchases of mortgages that resemble the balance sheet policies discussed in this paper.[28] In particular, we project forecast errors about conventional mortgage rates at different horizons on "exogenous and unexpected" purchases

[26] Bordalo et al. (2018) provide evidence on the predictability of individual forecast errors by variables that are included in individual information sets, namely, individual forecast updates. This points to non-rational expectations. Since we seek to estimate the forecast-error effect of GSEs' assets purchases, which is an aggregate time series, if we were to repeat the empirical strategy in Bordalo et al. (2018), we would necessarily lose the cross-sectional dimension of our forecast data.

[27] Sheffrin (1996) discusses popular tests of the null hypothesis of the full information rational expectations. Pesaran and Weale (2006) provide a comprehensive survey of the empirical findings. Coibion et al. (forthcoming) provide a recent survey of macroeconomic studies using survey-based data on expecations, with a particular focus on inflation expectations, and Gennaioli and Shleifer (2018) is a recent survey of the finance literature, focusing on the global financial crisis.

[28] We leave the analysis of the other types of balance sheet policies (i.e., long-term public bonds purchases as well as sterilized FX interventions) for future research.

of mortgages by quasi-government agencies, also known as government-sponsored enterprises (GSEs), such as Fannie Mae and Freddie Mac.

We follow Fieldhouse, Mertens and Ravn (2018) (henceforth referred to as "FMR"), who argue that the purchases of mortgages by GSEs resemble the purchases of private risky assets by the Federal Reserve in the recent financial crisis. The authors provide a comprehensive description of the institutional details of the operations of the GSEs. Here, we briefly describe some of the details that are relevant for understanding our empirical results.

The GSEs have been routinely buying mortgages from mortgage issuers since their incorporation in the 1960s. They finance their purchases with debt securities that command a "liquidity and safety" premium similar to the one of Treasury securities. Although most of these purchases are motivated by the cyclical developments in the mortgage market (e.g., stimulating housing starts in recessions), some purchases are related to non-cyclical regulatory events (e.g., those invoked by a desire to increase homeownership among lower-income households or by concerns regarding structural budget deficits). FMR use narrative records to identify the motivation behind any considerable change in the GSEs' mortgage purchasing behavior and construct a list of major regulatory events that are not related to cyclical considerations.[29] We call these events "exogenous."

To quantify the impact of these exogenous events, FMR use various sources to obtain an estimate of the projected impact, denoted by m_t, of the agencies' capacity to purchase mortgages during the first year following the moment when a policy is announced publicly. Therefore, m_t can be thought of as news about future purchases by the GSEs following the exogenous events.[30] We take m_t directly from FMR.

Empirical strategy. To estimate the effect of the asset purchases by the GSEs, we follow FMR and use the Jordà (2005) local projections method, implemented by two-stage least squares (2SLS). Specifically, in the first stage, we project the cumulative commitments $\sum_{j=0}^{h} p_{t+j}$ to purchase mortgages by the GSEs over $h+1$ months, expressed in constant dollars, on the non-cyclical narrative instrument m_t, also expressed in constant dollars, and a host of controls:

$$\frac{\sum_{j=0}^{h} p_{t+j}}{X_t} = \alpha_h^{(1)} + \gamma_h^{(1)} \frac{m_t}{X_t} + \varphi_h^{(1)}(L) Z_{t-1} + u_{t+h}^{(1)}. \tag{27}$$

[29]The interested reader may consult FMR, which contains a detailed discussion of the construction of these narrative events.

[30]A similar approach is used in the literature on the effects of fiscal policies. See, for example, Ramey and Zubairy (2018) who used news about military spending as an instrument for government spending.

We express the left-hand side variable as well as m_t on the right-hand side as ratios of X_t , a deterministic trend in real personal income obtained by fitting a third-degree polynomial of time to the log of personal income, deflated by the core personal consumption expenditures (PCE) price index. In equation (27), we also control for lagged values of the left-hand side variable, lagged growth rates of the core PCE price index, a nominal house price index, total mortgage debt, the log level of real mortgage originations, housing starts, and lags of several interest rate variables: the 3-month T-bill rate, the 10-year Treasury rate, the conventional mortgage interest rate, and the BAA-AAA corporate bond spread. The superscript (1) denotes first-stage regression coefficients and errors.

In the second stage, we estimate

$$y_{t+h} = \alpha_h^{(2)} + \gamma_h^{(2)} \left(\frac{12}{8} \times \frac{\sum_{j=0}^{7} p_{t+j}}{\widetilde{X}_t} \right) + \varphi_h^{(2)} \left(L \right) Z_{t-1} + u_{t+h}^{(2)}, \tag{28}$$

where y_{t+h} is any variable of interest in month $t+h$—such as the realized mortgage rate, or the mortgage rate forecast error—and $\widetilde{X}_t$ is a long-run trend in annualized mortgage originations. Since, in the first stage, we estimate the reaction of the GSEs' cumulative commitments at various horizons, we pick a specific horizon of eight months to use as an indicator of policy actions. The reason for this choice is that the F-statistics of the first stage is maximized at this horizon. By doing this, we again follow FMR. We estimate $\gamma_h^{(2)}$ by 2SLS, i.e., we replace the term multiplying $\gamma_h^{(2)}$ in (28) with its predicted value in the first stage (27). The regressions on both stages include twelve lags of the dependent variables.

Data. We use data from October 1982 to December 2006. The choice of the starting date is dictated by the availability of the forecast data. The choice of the end date avoids using data from the Great Recession when the GSEs faced a particularly turbulent experience, which culminated in their conservatorship by the government in September 2008. All data sources, except for data on forecasts, are identical to those used in FMR. We list them in Appendix B.

To measure mortgage rate forecasts, we use a survey of expectations by major financial institutions collected in the Blue Chip Financial Forecast (BCFF) database.[31] The BCFF contains monthly surveys of around forty financial institutions that forecast major financial indicators, including mortgage rates at horizons up to six quarters. The surveys are usually conducted in the last few days of a month and released on the first date of the following month. We focus on the median forecast across forecasters at any point in time.

[31] The Blue Chip Financial Forecasts dataset is proprietary. It can either be purchased directly from the official website or obtained through the institutions subscribed to this dataset.

Importantly, the Blue Chip survey asks participants to forecast the average value of a variable over the current and future calendar quarters. As a result, there is no fixed forecast horizon at a monthly frequency. For example, a January forecast of the mortgage rate r_t in the second quarter of a particular year is a three-month-ahead forecast, while a February forecast of the same variable is a two-month-ahead forecast. We thus employ the following definition of forecast errors about next-quarter average mortgage rate:

$$\widetilde{u}_{t,t+``1:3"} \equiv \frac{r_{t+3-mod(t+2,3)} + r_{t+4-mod(t+2,3)} + r_{t+5-mod(t+2,3)}}{3} - f_t^{``1:3"},$$

where $f_t^{``1:3"}$ is the median forecast of next-quarter mortgage rate at time t in the BCFF database.[32] The notation "1:3" emphasizes the fact that the horizon of this forecast varies from one to three months and $mod(t+2,3)$ is the remainder of the division of $t+2$ by 3. Similarly, we define forecast errors of mortgage rates in the subsequent quarters as

$$\widetilde{u}_{t,t+``(3n-2):3n"} \equiv \frac{\sum_{i=0}^{2} r_{t+3n+i-mod(t+2,3)}}{3} - f_t^{``(3n-2):3n"}, \quad (29)$$

where $n = 1, 2, 3, 4$.

Null hypothesis. As long as forecasters working for financial institutions are aware of significant purchases by the GSEs and hold rational expectations, the forecast errors $\widetilde{u}_{t,t+``(3n-2):3n"}$ should not be predictable by such purchases. To test this, we can simply use $\widetilde{u}_{t+1,t+1+``(3n-2):3n"}$ in place of y_{t+h} in equation (28) and verify whether the coefficient $\gamma_1^{(2)}$ is statistically different from zero. Note that, by regressing the forecast errors based on the information available to forecasters at the beginning of month $t+1$ on the GSEs' purchases in month t, we avoid the possibility that these interventions were not implemented before forecasters were asked to predict future prices.

Results. We begin by estimating the effects of the GSEs' exogenous mortgage purchases on mortgage yields in our sample from October 1982 to December 2006. In doing this, we confirm that the main conclusion in FMR does not change much when we use our restricted data sample. The left panel of Figure 2 shows the impulse response function of the conventional mortgage rate—i.e., the coefficients $\gamma_h^{(2)}$ in the second stage equation (28) when the dependent variable is r_{t+h}—following an exogenous increase in the GSEs' purchases by 1 percent of trend originations. Our results are only slightly different from

[32] Note that $t = 1$ corresponds to January 1982, $t = 2$ to February 1982, and so on.

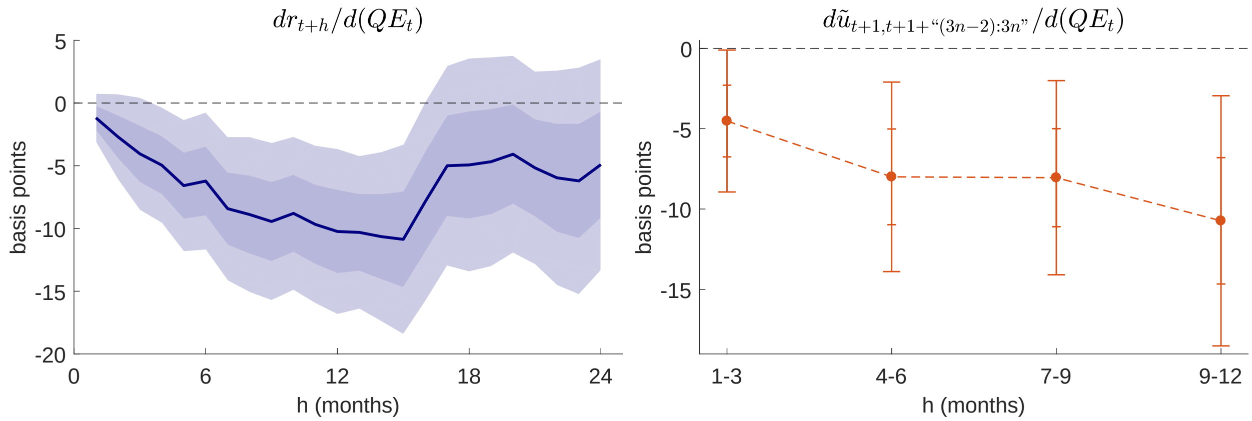

Figure 2: The left panel presents the conventional mortgage-rate impulse response function to an exogenous change in the GSEs' purchases of mortgages. The notation QE_t refers to the term multiplying $\gamma_h^{(2)}$ on the right-hand side of equation (28). The right panel shows the response of the conventional mortgage-rate forecast errors at various forecasting horizons to an exogenous change in the GSEs' purchases of mortgages. The labels on the horizontal axis in the right panel represent the varying forecast horizon. For example, the first label "1-3 months" indicates that the forecast horizon varies from one to three months, when forecasting next calendar quarter mortgage rate at a monthly frequency. In both panels, the confidence intervals are one and two Newey and West (1987) standard deviation error bands.

those in FMR.[33]

Next, we turn to the estimation of the response of mortgage-rate forecast errors to purchases by GSEs. The right panel of Figure 2 presents the estimates of coefficients $\gamma_1^{(2)}$ in equation (28) when the dependent variable is $\widetilde{u}_{t+1,t+1+\text{``}(3n-2):3n\text{''}}$, $n = 1, 2, 3, 4$, along with one- and two-standard-error confidence intervals. Consistently with the predictions of our model, forecast errors react negatively and significantly to the GSEs' mortgage purchases, which suggests that forecasters tend to under-react to news about such interventions. Moreover, under the additional assumption that forecasters working for financial institutions are aware of significant purchases by the GSEs, imperfect information models would fail to predict the under-reaction in the forecast errors.

We repeat our analysis for the "nowcast" error. We define the "nowcast" error using equation (29) where n is set to zero and $f_t^{\text{''}-2:0\text{''}}$ denotes the "nowcast"—the current-calendar-quarter average forecast of mortgage rates. It is clear that, when the nowcast is released in the beginning of the first month of a quarter, it is effectively a forecast of the mortgage rate during the whole quarter ahead. On the other hand, the nowcast released in the last month of the quarter is likely to depend on the data that has become available during the first part of the quarter that is being nowcasted. As a result, our measure of

[33]One notable difference between our results and those reported in FMR is the value of the first-stage F-statistics. While the authors estimate the F-statistics to be higher than ten in their longer sample, the value of F-statistics is just slightly above five in our smaller sample. However, quantitatively, the results reported in Figure 2 are close to those presented in Figure VII of FMR, suggesting that the weak instrument bias is small.

the nowcast error $\widetilde{u}_{t+1,t+1+\text{"}-2:0\text{"}}$ is an average between a true nowcast and a forecast at a short horizon. Hence, we expect our nowcast error to still be predictable, but perhaps to a smaller degree than the forecast errors at more distant horizons. Consistent with this logic, we find that the point estimate of $\gamma_1^{(2)}$ is -0.8 basis points with the standard deviation of 1.3 basis points.

Finally, we can use our theoretical model and the empirical results obtained so far to compute the average level of sophistication of agents in the data.[34] Specifically, we first assume that all of the reaction of forecast errors can be attributed to the bounded rationality channel of this paper and then we use equations (19) and (26) to compute the average level of sophistication $\bar{k}$ by taking the ratio of the impulse responses:

$$\bar{k} = \frac{\partial q_s / \partial X_{t+1}}{\partial \bar{u}_{t,s} / \partial X_{t+1}}. \tag{30}$$

The key property of (30) is that it is independent of the specific process of balance sheet policies $\{X_{t+1}\}$, thus, we can use it to estimate $\bar{k}$ independently of the exact details of the asset purchase programs in our data (the formula, however, does rely on the assumption that the distribution $f(k)$ is exponential). In addition, the same formula holds true if we replace the price of the risky asset in the numerator with the one-period return on the risky asset. Therefore, if we identify the conventional mortgage rate in our data with the one-period return on the risky assets in our theoretical model, then we can use equation (30) to get an estimate of $\bar{k}$. Note that, because we use the BCFF forecasts, we define the forecast errors as in equation (29). As a result, the numerator in (30) has to be consistent with this definition of forecast errors. Specifically, the numerator of the formula must be a moving average of the realized values, rather than simply the realized value in period s. Formally, in the case of the one-period return, the numerator must be $\partial(\sum_{i=0}^{2} r_{t+3n+i-mod(t+2,3)}/3)/\partial X_{t+1}$.

Guided by our theory, we compute $\bar{k}$ by taking the impulse response of the moving average of the realized mortgage rates and dividing it by the responses of the forecast errors at the same horizon from the right panel of Figure 2. The estimates of $\bar{k}$ for each horizon are presented in Appendix Table A1 and the average across the four horizons is 1.17. This number implies that 86 percent of the agents in the sample consists of level-1 thinkers, who do not change their forecasts after the policy intervention, while only 14 percent achieves higher levels of thinking. This is a pretty low estimate of $\bar{k}$, especially considering that our sample is made up of major financial institutions. At the same time,

[34] We implicitly assume that the BCFF survey participants have the same average level of sophistication as the agents who price mortgages. We leave the alternative assumption of different levels of sophistication to future research.

when interpreting the numbers in this exercise, one may want to keep in mind that we assumed that all the agents are perfectly aware of the policy interventions. If, on the contrary, some agents do not react to new policies simply because they are not aware of them, then these agents would be incorrectly classified as being of the lowest level of thinking.

5 Conclusion

In this paper, we have shown that the irrelevance result, whereby balance sheet policies—such as, purchases of private risky assets, long-term public bonds, and sterilized FX interventions by the public sector—are irrelevant in a standard model with rational expectations, is invalidated in a model where agents form expectations according to the level-k thinking process. We test the main implications of our channel by showing that forecast errors of asset prices (i.e., mortgage rates) are predictable by the balance sheet policies. Specifically, we show that previously identified exogenous and unexpected purchases of mortgages by quasi-government agencies predict forecast errors of mortgage rates at different horizons.

There are many important directions ahead, which we leave for future research. First, our analysis suggests that the "bounded-rationality channel" may be an important determinant of the impact of balance sheet policies on the economy. The next step will be to quantify the size of our channel relative to the other channels proposed in the literature (mainly the portfolio balance channel and the signaling channel). Second, to present our mechanism in the cleanest way, we have made assumptions that delivered results in closed forms. In particular, we have chosen to work with an endowment economy. The next step is to allow output to respond to balance sheet policies so as to evaluate the efficacy of such policies at stabilizing the economy. Finally, we have studied the response of the economy to an exogenous path of asset purchases. A crucial step is to understand when the central bank finds it optimal to use balance sheet policies as a stabilization tool.

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Appendix

A Proofs

A.1 Proof of Lemma 1

If shocks are normally distributed and forecasts are linear in shocks, then $-\widetilde{\mathbb{E}}_t e^{-\gamma c_{t+1}} = -e^{-\gamma(\widetilde{\mathbb{E}}_t c_{t+1} - \frac{\gamma}{2}\widetilde{\mathbb{V}}_t c_{t+1})}$. As a result, the household optimization problem can be rewritten as

$$\begin{aligned}
\max_{c_{t+1}, s_{t+1}, x_{t+1}} \; & \mathcal{U}\left(c_{t+1}\right) \\
\text{s.t.:} \; & s_{t+1} = w - q_t x_{t+1}, \\
& c_{t+1} = R s_{t+1} + r^x_{t+1} x_{t+1} + q_{t+1} x_{t+1} - T_{t+1}.
\end{aligned}$$

where $\mathcal{U}\left(c_{t+1}\right) \equiv \widetilde{\mathbb{E}}_t c_{t+1} - \frac{\gamma}{2}\widetilde{\mathbb{V}}_t c_{t+1}$. Substituting out s_{t+1}, we can express consumption as

$$c_{t+1} = R\left(w - q_t x_{t+1}\right) + r^x_{t+1} x_{t+1} + q_{t+1} x_{t+1} - T_{t+1}.$$

Using (12), expectations and variance of consumption are, respectively,

$$\begin{aligned}
\widetilde{\mathbb{E}}_t c_{t+1} &= R\left(w - q_t x_{t+1}\right) + r^x x_{t+1} + \alpha_{q,t} x_{t+1} - \alpha_{T,t}, \\
\widetilde{\mathbb{V}}_t c_{t+1} &= \left[\left(1 + \beta_{q,t}\right) x_{t+1} - \beta_{T,t}\right]^2 \sigma_x^2.
\end{aligned}$$

The optimization problem can then be rewritten as

$$\max_{x_{t+1}} \quad R\left(w - q_t x_{t+1}\right) + r^x x_{t+1} + \alpha_{q,t} x_{t+1} - \alpha_{T,t} - \frac{\gamma}{2}\left[\left(1 + \beta_{q,t}\right) x_{t+1} - \beta_{T,t}\right]^2 \sigma_x^2.$$

Optimal investment choice requires

$$\frac{\partial \mathcal{U}(c_{t+1})}{\partial x_{t+1}} = -q_t R + r^x + \alpha_{q,t} - \gamma\left(1 + \beta_{q,t}\right)\left[\left(1 + \beta_{q,t}\right) x_{t+1} - \beta_{T,t}\right]\sigma_x^2 = 0,$$

which can be rearranged as

$$x_{t+1} = \frac{r^x + \alpha_{q,t} - q_t R}{\gamma \sigma_x^2 \left(1 + \beta_{q,t}\right)^2} + \frac{\beta_{T,t}}{1 + \beta_{q,t}}.$$

A.2 Proof of Proposition 3

We start by presenting the formal definition of reflective equilibrium for this economy.

Definition (Reflective Equilibrium)**.** A reflective equilibrium is a collection of beliefs $\{\widetilde{\Phi}^k_t\}_k$, household choices $\{c^k_t, b^k_{t+1}, d^k_{t+1}, m^k_{t+1}, s^k_{t+1}\}$, government policies $\{\Pi_t\}$, and prices $\{i_t, p_t, q_t\}$ such that

1. Given beliefs and prices, households choose consumption and portfolio optimally for all t;
2. Bonds and money markets clear for all t;
3. The constraints (23), (24), and (25) are satisfied for all t;

4. Beliefs are generated recursively, starting from $\widetilde{\Phi}_t^1 = \Phi_t^{SQ}$, for all t.

We now prove this proposition in five steps.

Step 1: household optimization. Once again, we solve for household optimal decisions under the conjecture that the distributions $\{\widetilde{\phi}_t\}$ are such that, conditional on information at time t, the vector of endogenous variables Z_{t+1} is linear in the underlying shocks of the economy. This conjecture will be verified in all the equilibria we consider below. Formally,

$$Z_{t+1} = \alpha_t + \beta_t \epsilon_{t+1}^m + \vartheta_t \epsilon_t^m, \tag{A.1}$$

where α_t, β_t, and ϑ_t are deterministic functions of time. In particular, β_t captures the sensitivity with respect to the contemporaneous innovation to money supply ϵ_{t+1}^m, while ϑ_t captures the sensitivity with respect to the previous period's innovation.

We can combine the budget constraints (21) and (22) to get

$$c_{t+1} + T_{t+1}^o = R\left(w - b_{t+1} - q_t d_{t+1} - m_{t+1} - T_t^y\right) + e^{i_t}\frac{P_t}{P_{t+1}} b_{t+1} + \frac{1}{P_{t+1}} d_{t+1} + (1-\delta)\, q_{t+1} d_{t+1} + \frac{P_t}{P_{t+1}} m_{t+1}.$$

Taking a first-order Taylor approximation around $(i_t, \pi_{t+1}, p_{t+1}) = (r, 0, vr)$ and assuming that r is close to zero, we have

$$\begin{aligned} c_{t+1} + T_{t+1}^o &= R\left(w - T_t^y\right) + \left(e^{i_t - \pi_{t+1}} - R\right) b_{t+1} + \left(\frac{1}{P_{t+1}} + (1-\delta)\, q_{t+1} - R q_t\right) d_{t+1} + \left(e^{-\pi_{t+1}} - e^r\right) m_{t+1} \\ &\approx R\left(w - T_t^y\right) + (b_{t+1}, d_{t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}, \end{aligned}$$

where we denoted the vector of real returns on bonds and money as $\mathcal{R}_{t+1} = (\mathcal{R}_{b,t+1}, \mathcal{R}_{d,t+1}, \mathcal{R}_{m,t+1})' \equiv (i_t - p_{t+1} + p_t - r, 1 - p_{t+1} + (1-\delta)\, q_{t+1} - R q_t, -(p_{t+1} - p_t) - r)'$. Our strategy of log-linearizing the budget constraint and treating it as exact follows Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006). Alternatively, we could avoid the linearization by assuming that households hold mean-variance instead of exponential preferences.

We can thus concisely write the household problem as

$$\max_{(b_{t+1}, d_{t+1}, m_{t+1})} (b_{t+1}, d_{t+1}, m_{t+1})\, \widetilde{\mathbb{E}}_t \mathcal{R}_{t+1} - \frac{\gamma}{2} \widetilde{\mathbb{V}}_t \left[(b_{t+1}, d_{t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1} - T_{t+1}^0\right] - \frac{m_{t+1}\left[\log\left(m_{t+1}/\overline{m}\right) - 1\right]}{v}.$$

Note that the variable of consumption can be expanded as follows

$$\begin{aligned} \widetilde{\mathbb{V}}_t \left[(b_{t+1}, d_{t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1} - T_{t+1}^o\right] =& \widetilde{\mathbb{V}}_t T_{t+1}^o + (b_{t+1}, d_{t+1}, m_{t+1})\, \widetilde{\Sigma}_t \,(b_{t+1}, d_{t+1}, m_{t+1})' \\ &+ 2\,(b_{t+1}, d_{t+1}, m_{t+1}) \cdot \widetilde{cov}_t\left(-T_{t+1}^o, \mathcal{R}_{t+1}\right), \end{aligned}$$

where the variance-covariance matrix $\widetilde{\Sigma}_t \equiv \widetilde{\mathbb{V}}_t(\mathcal{R}_{t+1})$ is such that $(\widetilde{\Sigma}_t)_{1,1} = (\widetilde{\Sigma}_t)_{3,3} = (\widetilde{\Sigma}_t)_{1,3} = (\widetilde{\Sigma}_t)_{3,1} = (\beta_{p,t})^2 \sigma_m^2$, $(\widetilde{\Sigma}_t)_{2,2} = (\beta_{p,t} - (1-\delta)\, \beta_{q,t})^2 \sigma_m^2$, $(\widetilde{\Sigma}_t)_{1,2} = (\widetilde{\Sigma}_t)_{2,1} = (\widetilde{\Sigma}_t)_{3,2} = (\widetilde{\Sigma}_t)_{2,3} = \beta_{p,t}(\beta_{p,t} - (1-\delta)\, \beta_{q,t})\sigma_m^2$. We use $(A)_{m,n}$ to denote the (m,n)'th element of matrix A.

Optimal portfolio choice implies

$$\widetilde{\Sigma}_t \cdot \begin{pmatrix} b_{t+1} \\ d_{t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{\gamma}\widetilde{\mathbb{E}}_t \mathcal{R}_{t+1} + \frac{1}{\gamma}\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{v}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + \widetilde{cov}_t\left(T^o_{t+1}, \mathcal{R}_{t+1}\right). \tag{A.2}$$

Note that the matrix $\widetilde{\Sigma}_t$ is not invertible because the return on money, the return on short-term bonds, and the one-period ahead return on long-term bonds have the same risk profile. By taking the difference between the first and second lines of equation (A.2), we obtain the following money demand function:

$$m_{t+1} = \overline{m}e^{-vi_t}.$$

Step 2: temporary equilibrium. First, the market-clearing conditions in assets markets in period t are

$$\begin{aligned} B_{t+1} + B^{CB}_{t+1} &= b_{t+1}, \\ D - D^{CB}_{t+1} &= d_{t+1}, \\ \frac{\overline{M}}{P_t}e^{\epsilon^m_t - \epsilon^m_{t-1}\frac{1+v}{v}} &= \overline{m}e^{-vi_t}. \end{aligned}$$

In the cashless limit as $\overline{M}/\overline{m} \to 1$, the latter condition implies

$$p_t = vi_t + \epsilon^m_t - \epsilon^m_{t-1}\frac{1+v}{v}. \tag{A.3}$$

Second, we combine optimality conditions and market clearing:

$$\widetilde{\Sigma}_t \begin{pmatrix} B_{t+1} + B^{CB}_{t+1} \\ D - D^{CB}_{t+1} \\ \frac{\overline{M}}{P_t}\left(1 + \epsilon^m_t - \epsilon^m_{t-1}\frac{1+v}{v}\right) \end{pmatrix} = \frac{1}{\gamma}\widetilde{\mathbb{E}}_t \begin{pmatrix} i_t - p_{t+1} + p_t - r \\ 1 - p_{t+1} + (1-\delta)\, q_{t+1} - Rq_t \\ -\left(p_{t+1} - p_t\right) - r - \frac{1}{v}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + \widetilde{cov}_t\left(T^o_{t+1}, \mathcal{R}_{t+1}\right)$$

or, in the cashless limit,

$$(\widetilde{\Sigma}_t)_{1:2,1:2} \begin{pmatrix} B_{t+1} + B^{CB}_{t+1} \\ D - D^{CB}_{t+1} \end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix} i_t - \widetilde{\mathbb{E}}_t p_{t+1} + p_t - r \\ 1 - \widetilde{\mathbb{E}}_t p_{t+1} + (1-\delta)\, \widetilde{\mathbb{E}}_t q_{t+1} - Rq_t \end{pmatrix} + \widetilde{cov}_t\left(T^o_{t+1}, (\mathcal{R}_{t+1})_{1:2}\right), \tag{A.4}$$

where $(\widetilde{\Sigma}_t)_{1:2,1:2}$ is the upper-left sub-matrix of $\widetilde{\Sigma}_t$, and $(\mathcal{R}_{t+1})_{1:2}$ is the vector with the first two elements of the vector $\mathcal{R}_{t+1}$.

The second line of (A.4) can be rewritten as

$$\begin{aligned} &(\widetilde{\Sigma}_t)_{2,1}\left(B_{t+1} + B^{CB}_{t+1}\right) + (\widetilde{\Sigma}_t)_{2,2}\left(D - D^{CB}_{t+1}\right) \\ =&\frac{1}{\gamma}\left(1 - \alpha_{p,t} - \vartheta_{p,t}\epsilon^m_t + (1-\delta)\,\alpha_{q,t} + (1-\delta)\,\vartheta_{q,t}\epsilon^m_t - Rq_t\right) + \widetilde{cov}_t\left(T^o_{t+1}, \mathcal{R}_{d,t+1}\right). \end{aligned}$$

This equation can be solved for q_t as a function of ϵ^m_t, ϵ^m_{t-1}, and balance sheet policies:

$$q_t = \frac{1 - \alpha_{p,t} + (1-\delta)\,\alpha_{q,t} + \left(-\vartheta_{p,t} + (1-\delta)\,\vartheta_{q,t}\right)\epsilon^m_t}{R} - \frac{\gamma}{R}\widetilde{cov}_t(c_{t+1}, \mathcal{R}_{d,t+1}), \tag{A.5}$$

where the covariance term is a function of the elasticities to the monetary shock:

$$\widetilde{cov}_t(c_{t+1}, \mathcal{R}_{d,t+1}) = [(1-\delta)\,\beta_{q,t} - \beta_{p,t}]\sigma_m^2 \cdot \left[((1-\delta)\,\beta_{q,t} - \beta_{p,t})(D - D_{t+1}^{CB}) - \beta_{p,t}(B_{t+1} + B_{t+1}^{CB}) - \beta_{T,o,t}\right]. \tag{A.6}$$

Similarly, the nominal interest rate on short-term bonds is obtained from the first line of (A.4):

$$i_t = r + \alpha_{p,t} + \vartheta_{p,t}\epsilon_t^m - p_t + RP_t, \tag{A.7}$$

where, for convenience, we let

$$RP_t \equiv \gamma\sigma_m^2\beta_{p,t}\left[\beta_{p,t}(B_{t+1} + B_{t+1}^{CB}) + (\beta_{p,t} - (1-\delta)\,\beta_{q,t})\,(D - D_{t+1}^{CB}) + \beta_{T,o,t}\right].$$

The price level p_t is obtained by combining (A.3) and (A.7)

$$\begin{aligned} p_t &= vi_t + \epsilon_t^m - \frac{1+v}{v}\epsilon_{t-1}^m \\ &= \frac{v}{1+v}(r + \alpha_{p,t}) + \frac{1 + v\vartheta_{p,t}}{1+v}\epsilon_t^m - \frac{1}{v}\epsilon_{t-1}^m + \frac{v}{1+v}RP_t. \end{aligned} \tag{A.8}$$

Third, we consider the budget constraint of the government. In the cashless limit, after taking a first-order Taylor expansion, taxes on the old generation are

$$T_t^o = B_t(1 + i_{t-1} - \pi_t) + (D - D_t^{CB})(1 + (1-\delta)\,q_t - p_t) + q_{t-1}D_t^{CB}(1 + i_{t-1} - \pi_t). \tag{A.9}$$

Similarly, in the cashless limit, taxes on the young generation are

$$\begin{aligned} T_t^y &= (D_{t+1}^{CB} - D)q_t - B_{t+1}^{CB} - B_{t+1} \\ &= -Dq_t - B_{t+1}. \end{aligned} \tag{A.10}$$

Note that, to derive (A.9) and (A.10), we have used the fact that issuance of reserves satisfies (23).

Step 3: rational expectations equilibrium. In the REE, the equilibrium distribution of endogenous variables must be equal to the agents' beliefs about these variables. Specifically, for the price level, we need to make sure that the coefficients $\alpha_{p,t}^{REE}$, $\beta_{p,t}^{REE}$, and $\vartheta_{p,t}^{REE}$ satisfy

$$\begin{aligned} p_{t+1} &= \frac{v}{1+v}(r + \alpha_{p,t+1}^{REE}) + \frac{1 + v\vartheta_{p,t+1}^{REE}}{1+v}\epsilon_{t+1}^m - \frac{1}{v}\epsilon_t^m + \frac{v}{1+v}RP_t \\ &\overset{REE}{=} \alpha_{p,t}^{REE} + \beta_{p,t}^{REE}\epsilon_{t+1}^m + \vartheta_{p,t}^{REE}\epsilon_t^m, \end{aligned}$$

for all realizations of the shocks. The latter is satisfied if and only if

$$\begin{aligned} \vartheta_{p,t}^{REE} &= -\frac{1}{v}, \\ \beta_{p,t}^{REE} &= \frac{1 + v\vartheta_{p,t+1}^{REE}}{1+v} = 0. \end{aligned}$$

Similarly, for the price of the perpetuity,

$$\vartheta_{q,t}^{REE} = 0,$$
$$\beta_{q,t}^{REE} = \frac{-\vartheta_{p,t+1}^{REE} + (1-\delta)\,\vartheta_{q,t+1}^{REE}}{R} = \frac{1}{Rv}.$$

In addition, the sensitivities of taxes on the old generation are

$$\vartheta_{T,o,t}^{REE} = \frac{D - D_{t+1}^{CB}}{v} + \frac{1}{vR} D_{t+1}^{CB} \left[1 + r + \alpha_{p,t}^{REE} - \frac{v}{1+v}(r + \alpha_{p,t+1}^{REE})\right],$$
$$\beta_{T,o,t}^{REE} = \frac{1-\delta}{Rv}(D - D_{t+1}^{CB}),$$

while the sensitivities of taxes on the young generation are

$$\vartheta_{T,y,t}^{REE} = 0,$$
$$\beta_{T,y,t}^{REE} = -D\beta_{q,t}^{REE} = -\frac{D}{Rv}.$$

Using (23) to derive B_{t+2}^{CB}, we obtain

$$\vartheta_{B,t}^{REE} = -\vartheta_{q,t}^{REE} D_{t+2}^{CB} = 0,$$
$$\beta_{B,t}^{REE} = -D_{t+2}^{CB}\beta_{q,t}^{REE} = -\frac{D_{t+2}^{CB}}{Rv}.$$

Finally, for the interest rate we get

$$\vartheta_{i,t}^{REE} = \frac{1}{v},$$
$$\beta_{i,t}^{REE} = -\frac{1}{v}.$$

To simplify notation, we omit time subscripts from those coefficients that are constant over time. We thus have

$$\left(\vartheta_p^{REE}, \vartheta_i^{REE}, \vartheta_q^{REE}, \vartheta_{T,y}^{REE}, \vartheta_{T,o}^{REE}, \vartheta_B^{REE}\right) = \left(-\frac{1}{v}, \frac{1}{v}, 0, 0, \frac{D}{v}, 0\right),$$
$$\left(\beta_p^{REE}, \beta_i^{REE}, \beta_q^{REE}, \beta_{T,y}^{REE}, \beta_{T,o,t}^{REE}, \beta_{B,t}^{REE}\right) = \left(0, -\frac{1}{v}, \frac{1}{Rv}, -\frac{D}{Rv}, \frac{1-\delta}{Rv}\left(D - D_{t+1}^{CB}\right), -\frac{D_{t+2}^{CB}}{Rv}\right).$$

In the REE, the top left entry of the variance-covariance matrix of $\mathcal{R}_{t+1}$, which we denote with Σ^{REE}, is given by

$$(\Sigma^{REE})_{1,1} = \frac{(1-\delta)^2}{R^2 v^2}\sigma_m^2.$$

Finally, for the non-random part of the endogenous variables, we have

$$\begin{aligned}
\alpha_{p,t}^{REE} =& \frac{v}{1+v}\left(\alpha_{p,t+1}^{REE}+r\right), \\
\alpha_{i,t}^{REE} =& r+\alpha_{p,t}^{REE}-\frac{v}{1+v}(r+\alpha_{p,t+1}^{REE}), \\
\alpha_{q,t}^{REE} =& \frac{1-\alpha_{p,t+1}^{REE}+(1-\delta)\,\alpha_{q,t+1}^{REE}}{R}-\frac{\gamma}{R}cov_{t+1}^{REE}(c_{t+2},\mathcal{R}_{d,t+2}), \\
\alpha_{T,y,t}^{REE} =& -D\alpha_{q,t}^{REE}-B_{t+2}, \\
\alpha_{B,t}^{REE} =& -\alpha_{q,t}^{REE}D_{t+2}^{CB},
\end{aligned}$$

$$\begin{aligned}
\alpha_{T,o,t}^{REE} =& \left(D-D_{t+1}^{CB}\right)+B_{t+1}\left(1+r+\alpha_{p,t}^{REE}\right)-\left(D-D_{t+1}^{CB}+B_{t+1}\right)\frac{v}{1+v}(r+\alpha_{p,t+1}^{REE}) \\
&+(1-\delta)\left(D-D_{t+1}^{CB}\right)\left[\frac{1-\alpha_{p,t+1}^{REE}+(1-\delta)\,\alpha_{q,t+1}^{REE}}{R}-\frac{\gamma}{R}cov_{t+1}^{REE}(c_{t+2},\mathcal{R}_{d,t+2})\right] \\
&+\left[\frac{1-\alpha_{p,t}^{REE}+(1-\delta)\,\alpha_{q,t}^{REE}}{R}-\frac{\gamma}{R}cov_{t}^{REE}(c_{t+1},\mathcal{R}_{d,t+1})\right]\cdot D_{t+1}^{CB}\left[1+r+\alpha_{p,t}^{REE}-\frac{v}{1+v}(r+\alpha_{p,t+1}^{REE})\right],
\end{aligned}$$

where we use $cov_t^{REE}(\cdot,\cdot)$ to denote the covariance operator under rational expectations. It is immediate to verify that

$$cov_t^{REE}(c_{t+1},\mathcal{R}_{d,t+1})=0,$$

for all t. We can then solve the system of equations above as follows:

$$\begin{aligned}
\alpha_{p}^{REE} =& vr, \\
\alpha_{i}^{REE} =& r, \\
\alpha_{q}^{REE} =& \frac{1-vr}{R-1+\delta}. \\
\alpha_{T,y,t}^{REE} =& -D\frac{1-vr}{R-1+\delta}-B_{t+2}, \\
\alpha_{T,o,t}^{REE} =& \left(D-D_{t+1}^{CB}\right)(1-vr)+RB_{t+1} \\
&+\left[D_{t+1}^{CB}+\frac{(1-\delta)\left(D-D_{t+1}^{CB}\right)}{R}\right]\frac{R\,(1-vr)}{R-1+\delta}, \\
\alpha_{B,t}^{REE} =& -\frac{1-vr}{R-1+\delta}D_{t+2}^{CB}.
\end{aligned}$$

Therefore, in the REE,

$$\begin{aligned}
i_t &= r-\frac{1}{v}\left(\epsilon_t-\epsilon_{t-1}\right), \\
p_t &= vr-\frac{1}{v}\epsilon_{t-1}, \\
q_t &= \frac{1-vr}{R-1+\delta}+\frac{1}{vR}\epsilon_t.
\end{aligned}$$

Step 4: level-k beliefs. Equations (23), (A.5), (A.7)-(A.10) define a mapping $\Psi(\cdot)$ from beliefs about policy actions and endogenous variables into distributions over the endogenous variables $Z_t = (p_t, i_t, q_t, T_t^y, T_t^o, B_{t+1}^{CB})$.

By iterating $\Psi(\cdot)$, starting from $\widetilde{\Phi}_t^{SQ}$, we can compute the beliefs of level-k agents, for any $k \geq 1$. It is easy to see that some β's and ϑ's coincide with their REE counterparts: $\beta_p^k = \beta_p^{REE}$, $\beta_i^k = \beta_i^{REE}$, $\beta_q^k = \beta_q^{REE}$, $\beta_{T,y}^k = \beta_{T,y}^{REE}$, $\vartheta_p^k = \vartheta_p^{REE}$, $\vartheta_i^k = \vartheta_i^{REE}$, $\vartheta_q^k = \vartheta_q^{REE}$, $\vartheta_{T,y}^k = \vartheta_{T,y}^{REE}$, $\vartheta_{T,o}^k = \vartheta_{T,o}^{REE}$, $\vartheta_B^k = \vartheta_B^{REE}$, for all $k \geq 1$. In particular, the latter imply that $(\widetilde{\Sigma}_t^k)_{1,1} = (\Sigma^{REE})_{1,1}$, for all $k \geq 1$.

We are left to derive $\beta_{T,o,t}^k$, $\beta_{B,t}^k$, and the α^k's, for all $k \geq 1$. For $k = 1$, since we assume that the expectations of level-1 households coincide with the distributions of REE variables before the intervention by the central bank, the sensitivity of taxes to the current shock coincides with its REE counterpart *in the absence* of asset purchases:

$$\begin{aligned} \beta_{T,o,t}^1 &= \frac{1-\delta}{Rv} D, \\ \beta_{B,t}^1 &= 0. \end{aligned}$$

Proceeding recursively for $k > 1$, we have

$$\begin{aligned} \beta_{T,o,t}^k &= \frac{1-\delta}{Rv} \left(D - D_{t+1}^{CB} \right), \\ \beta_{B,t}^k &= -\frac{1}{Rv} D_{t+2}^{CB}. \end{aligned}$$

From equation (A.8) and $\beta_q^k = \beta_q^{REE} = 0$, the constant part of the price level p_t remains

$$\alpha_{p,t}^k = vr,$$

for any $k \geq 1$. For the constant part of i_t, we get $\alpha_{i,t}^k = r$. For the constant part of q_t, from equation (A.5), we have

$$\alpha_{q,t}^k = \begin{cases} \frac{1-vr}{R-1+\delta}, & k = 1, \\ \frac{1-vr+(1-\delta)\alpha_{q,t+1}^{k-1}}{R} - \frac{\gamma}{R} \widetilde{cov}_{t+1}^{k-1} \left(c_{t+2}, \mathcal{R}_{d,t+2} \right), & k > 1, \end{cases}$$

where the expression for $k = 1$ follows from the fact that, following the policy announcement, level-1 thinkers do not change their expectations about the future. We use equation (A.6) to express the covariance term as follows:

$$\widetilde{cov}_t^k \left(c_{t+1}, \mathcal{R}_{d,t+1} \right) = \begin{cases} -\frac{(1-\delta)^2}{R^2 v^2} D_{t+1}^{CB} \sigma_m^2, & k = 1, \\ 0, & k > 1. \end{cases}$$

Thus, $\alpha_{q,t}^k$ can be rewritten as

$$\alpha_{q,t}^k = \begin{cases} \frac{1-vr}{R-1+\delta}, & k = 1, \\ \frac{1-vr}{R-1+\delta} + \frac{\gamma}{R} \cdot \frac{(1-\delta)^2}{R^2 v^2} D_{t+2}^{CB} \sigma_m^2, & k = 2, \\ \frac{1-vr+(1-\delta)\alpha_{q,t+1}^{k-1}}{R}, & k \geq 3. \end{cases}$$

We can rewrite the piece corresponding to $k \geq 3$ as follows:

$$\begin{aligned}\alpha_{q,t}^{k} - \frac{1-vr}{R-1+\delta} &= \frac{1}{R}\left((1-\delta)\,\alpha_{q,t+1}^{k-1} + 1 - vr - R\frac{1-vr}{R-1+\delta}\right) \\ &= \frac{1-\delta}{R}\left(\alpha_{q,t+1}^{k-1} - \frac{1-vr}{R-1+\delta}\right)\end{aligned}$$

and apply the "diagonal iterations" to obtain

$$\begin{aligned}\alpha_{q,t}^{k} - \frac{1-vr}{R-1+\delta} &= \left(\frac{1-\delta}{R}\right)^{k-2}\left(\alpha_{q,t+k-2}^{2} - \frac{1-vr}{R-1+\delta}\right) \\ &= \left(\frac{1-\delta}{R}\right)^{k-2}\frac{\gamma}{R}\cdot\frac{(1-\delta)^2}{R^2 v^2}D_{t+k}^{CB}\sigma_m^2,\end{aligned}$$

for $k \geq 3$. As a result, $\alpha_{q,t}^{k}$ satisfies

$$\alpha_{q,t}^{k} - \frac{1-vr}{R-1+\delta} = \begin{cases} 0, & k=1, \\ \frac{\gamma}{R}\cdot\left(\frac{1-\delta}{R}\right)^{k}\cdot\frac{1}{v^2}D_{t+k}^{CB}\sigma_m^2, & k \geq 2. \end{cases} \tag{A.11}$$

Step 5: reflective equilibrium. The market-clearing condition for long-term bonds is

$$\sum_{k=1}^{\infty} f(k)(\Sigma^{REE})_{1,1}(D - D_{t+1}^{CB}) = \frac{1}{\gamma}\sum_{k=1}^{\infty} f(k)(1 - \widetilde{\mathbb{E}}_t^k p_{t+1} + (1-\delta)\,\widetilde{\mathbb{E}}_t^k q_{t+1} - Rq_t) + \sum_{k=1}^{\infty} f(k)\widetilde{cov}_t^k(T_{t+1}^o, \mathcal{R}_{d,t+1}),$$

which, after substituting out $\widetilde{cov}_t^k(T_{t+1}^o, \mathcal{R}_{d,t+1}) = \beta_{T,o,t}^k\left((1-\delta)\,\beta_q^k - \beta_p^k\right)\sigma_m^2$, $\widetilde{\mathbb{E}}_t^k p_{t+1} = \alpha_{p,t}^k + \vartheta_p^k \epsilon_t^m$, and $\widetilde{\mathbb{E}}_t^k q_{t+1} = \alpha_{q,t}^k + \vartheta_q^k \epsilon_t^m$, can be rewritten as

$$\begin{aligned} q_t &= \frac{1-vr}{R-1+\delta} + \frac{1}{Rv}\epsilon_t^m + \frac{\gamma}{R}\cdot\frac{(1-\delta)^2}{R^2v^2}\sigma_m^2\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1} D_{t+k}^{CB} \\ &= q_t^{REE} + \frac{\gamma}{R}\cdot\frac{(1-\delta)^2}{R^2v^2}\sigma_m^2\sum_{k=1}^{\infty} f(k)\left(\frac{1-\delta}{R}\right)^{k-1} D_{t+k}^{CB}. \end{aligned}$$

A.3 A Model with Foreign Exchange Interventions

In this section, we present the detailed discussion of the extension that investigates the effects of international balance sheet policies, such as sterilized FX interventions. Relative to the simple model in Section 2, this extension features both a nominal friction in the form of the demand for money and an international dimension, which borrows elements of the open-economy models of Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006).[35]

There are two countries: home and foreign. Foreign-country variables will bear an asterisk. Both countries produce the same good, which is traded freely across borders. As a result, the law of one price necessarily holds in equilibrium and we have $P_t = E_t P_t^*$, where P_t and P_t^* are the nominal price levels in

[35] These two papers feature deviations from full-information rational expectations equilibria. Jeanne and Rose (2002) introduce "noise traders," while Bacchetta and Van Wincoop (2006) add private signals to agents' information sets. We instead incorporate level-k thinking and study central bank balance sheet policies.

the home and foreign countries, respectively, and E_t is the nominal exchange rate. The fact that the law of one price holds is a necessary no-arbitrage equilibrium condition. Without it, the household problem does not have a solution. The exchange rate is defined as the quantity of home currency bought by one unit of foreign currency. Consequently, an increase in E_t corresponds to a depreciation of home currency. For convenience, we let $e_t \equiv \log E_t$, $p_t \equiv \log P_t$, and $p_t^* \equiv \log P_t^*$.

There are several assets in this world. Households in the home country can hold money issued by their own country's central bank,[36] one-period nominal bonds issued by both countries—which pay, respectively, continuously compounded interest rates i_t and i_t^*—and riskless real assets, available in perfectly elastic supply, that pay off a real gross return $R \equiv 1 + r > 1$. Similarly, households in the foreign country can hold money issued by their own country, nominal bonds issued by both countries, and the riskless real assets. For simplicity, we do not consider private risky assets and long-term public bonds.

There are two sources of risk in the world economy. Both home- and foreign-country money supplies follow processes given by $\log M_{t+1} = \log \overline{M} + \epsilon_t^m$ and $\log M_{t+1}^* = \log \overline{M}^* + \epsilon_t^{m*}$, where the disturbances ϵ_t^m and ϵ_t^{m*} are assumed to be independent from each other, independent over time, and normally distributed with zero mean and standard deviations σ_m, and σ_m^*, respectively. Note that these money supply processes differ from the one assumed in Section 3.1, where, for simplicity of exposition, the specific process eliminated any risk in one-period nominal bonds. Since, for simplicity, we do not introduce long-term bonds in this section, it is necessary to ensure that short-term bonds are risky.

We focus on home-country households, foreign-country households are symmetric. In each period, there is a mass ω of "young" households and a mass ω of "old" households, where $\omega \in (0, 1)$ represents the size of the home country. The size of the foreign country is $1 - \omega$. Households maximize preferences (20) by choosing real bonds s_{t+1}, nominal home bonds $b_{H,t+1}$ (expressed in period-t consumption goods), nominal foreign bonds $b_{F,t+1}$ (expressed in period-t consumption goods), home-country real money balances m_{t+1}, and consumption c_{t+1}, subject to the current budget constraint

$$P_t \left(s_{t+1} + b_{H,t+1} + m_{t+1} \right) + E_t P_t^* b_{F,t+1} \leq P_t w, \tag{A.12}$$

and the future budget constraint

$$P_{t+1}(c_{t+1} + T_{t+1}) \leq P_{t+1} R s_{t+1} + P_t (e^{i_t} b_{H,t+1} + m_{t+1}) + E_{t+1} P_t^* e^{i_t^*} b_{F,t+1}. \tag{A.13}$$

In period t, households form their expectations about the vector of endogenous variables $Z_{t+1} \equiv (p_{t+1}, i_{t+1}, T_{t+1}, p_{t+1}^*, i_{t+1}^*, T_{t+1}^*, e_{t+1})$. As before, the conditional one-period-ahead distribution of Z_{t+1} is denoted by $\widetilde{\phi}_t$. As in the simple model, we define the beliefs of level-k agents recursively starting from the status quo that corresponds to the linear REE before the policy intervention. We explicitly derive these beliefs in the proof of Proposition 4.

We specify the behavior of the consolidated public sector in each country. The home-country government controls real per capita taxes $\{T_{t+1}\}$, nominal money supply $\{M_{t+1}\}$, the *real* amount of home-currency public nominal bonds $\{B_{H,t+1}\}$, and the *real* amount of foreign-currency public bond purchases $\{B_{F,t+1}\}$. We let $\Pi_t \equiv \{T_{t+1}, M_{t+1}, B_{H,t+1}, B_{F,t+1}\}$. At time 0, the government announces the path of purchases of foreign and domestic bonds $\{B_{H,t+1}, B_{F,t+1}\}$. We assume that the purchases are fully financed by selling domestic bonds. Using the law of one price, the latter implies $B_{H,t+1} = B_{F,t+1}$. A policy of foreign-

[36] We follow the international economics literature and make the simplifying assumption that households in a country can only hold the money of the country they live in. It is easy to extend the analysis to the case where households can hold money of both countries.

bond purchases financed with the issuance of home-country bonds will be referred to as "(sterilized) FX intervention."

The consolidated budget constraint of the home-country government is

$$e^{i_{t-1}} P_{t-1} B_{H,t} + E_t P_t^* B_{F,t+1} + M_t = E_t e^{i_{t-1}^*} P_{t-1}^* B_{F,t} + \omega P_t T_t + P_t B_{H,t+1} + M_{t+1}. \tag{A.14}$$

The left-hand side represents government's nominal outlays consisting of repayment of home-country nominal bonds $e^{i_{t-1}} P_{t-1} B_{H,t}$, purchases of foreign bonds, and repayment of money liabilities. The right-hand side is government's income.

We will only consider the case in which balance sheet policies are implemented by the government in the home country. The analysis for the foreign country is symmetric. For simplicity, we assume that the government in the foreign-country sets money supply and taxes so as to keep a constant level of real bonds. Formally, the foreign government sets $\Pi_t^* \equiv \{T_{t+1}^*, M_{t+1}^*, B^*\}$, where B^* is the constant level of real foreign-country bonds (positive B^* represents outstanding debt), that satisfy the budget constraint

$$e^{i_{t-1}^*} P_{t-1}^* B^* + M_t^* = P_t^* B^* + (1-\omega) P_t^* T_t^* + M_{t+1}^*. \tag{A.15}$$

It is again straightforward to adapt the definition of reflective equilibrium to this environment.

Definition (Reflective Equilibrium)**.** A reflective equilibrium is a collection of beliefs $\{\widetilde{\Phi}_t^k\}_k$, household choices $\{c_t^k, b_{H,t+1}^k, b_{F,t+1}^k, m_{t+1}^k, s_{t+1}^k\}$ and $\{c_t^{*,k}, b_{H,t+1}^{*,k}, b_{F,t+1}^{*,k}, m_{t+1}^{*,k}, s_{t+1}^{*,k}\}$, government policies $\{\Pi_t, \Pi_t^*\}$, and prices $\{i_t, p_t, i_t^*, p_t^*, e_t\}$ such that

1. Given beliefs and prices, households choose consumption and portfolio optimally for all t;
2. Bonds and money markets clear for all t;
3. Government budget constraints (**??**) and (**??**) are satisfied and $B_{H,t+1} = B_{F,t+1}$ for all t;
4. Beliefs are generated recursively, starting from $\widetilde{\Phi}_t^1 = \Phi_t^{SQ}$, for all t.

As in section 3.1, we are interested the cashless limit of the reflective equilibrium in which constants $\overline{m}$, $\overline{M}$, and $\overline{M}^*$ approach zero, such that the ratios $\omega\overline{m}/\overline{M}$ and $(1-\omega)\,\overline{m}/\overline{M}^*$ approach one. The following section proves proposition 4 from the main text, which contains the main results of this extension.

A.3.1 Proof of Proposition 4

As in the proof of Proposition 3, we proceed in five steps.

Step 1: household behavior. Again, we work under the conjecture that one-period ahead forecasts of endogenous variables take the following form:

$$Z_{t+1} = \alpha_t + \beta_t \epsilon_{t+1}^m + \xi_t \epsilon_{t+1}^{m*}, \tag{A.16}$$

where α_t, β_t, and ξ_t are (vectors of) deterministic functions of time. As in the other settings we have considered, this conjecture will be verified in the equilibria we consider below. Another property that will be satisfied in all the equilibria is that households expect the law of one price to hold in any future period. For example, in period $t+1$, we have

$$p_{t+1} = p_{t+1}^* + e_{t+1}.$$

We focus on the household problem in the home country; the problem in the foreign country is analogous. Let $\mathcal{R}_{t+1} \equiv (i_t - \pi_{t+1} - r, i_t^* - \pi_{t+1}^* - r, -r - \pi_{t+1})'$ be the vector of realized real excess returns on home public bonds, foreign public bonds, and money. The prime denotes the transpose. As usual, we use a tilde to denote the moments computed under the conditional distribution $\widetilde{\phi}_t$.

To derive closed-form solutions, after combining the budget constraints (**??**) and (**??**) into a single intertemporal budget constraint, we assume that r is close to zero, take a first-order approximation around $i_t = r$, $\pi_{t+1} = 0$, and then treat the resulting budget constraint as exact:

$$c_{t+1} = Rw + (b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1} - T_{t+1}. \tag{A.17}$$

As before, we follow Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006) in making this assumption. This approximation can be avoided if we assume mean-variance preferences.

Equation (A.17) is a linear transformation of jointly distributed Normal variables. Thus, standard properties of CARA preferences imply that the household maximization problem can be equivalently rewritten as

$$\max_{\substack{b_{H,t+1}, b_{F,t+1}, \\ m_{t+1}, c_{t+1}}} \quad \widetilde{\mathbb{E}}_t c_{t+1} - \frac{\gamma}{2} \widetilde{\mathbb{V}}_t c_{t+1} - \frac{m_{t+1} \left[\log\left(m_{t+1}/\overline{m}\right) - 1\right]}{v},$$

subject to (A.17). In particular, we can use (A.17) to rewrite the first two terms explicitly:

$$\widetilde{\mathbb{E}}_t c_{t+1} = Rw + \widetilde{\mathbb{E}}_t \left[(b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}\right] - \widetilde{\mathbb{E}}_t T_{t+1},$$
$$\widetilde{\mathbb{V}}_t c_{t+1} = \widetilde{\mathbb{V}}_t \left[(b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}\right] + \widetilde{\mathbb{V}}_t \left(T_{t+1}\right) - 2(b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \widetilde{cov}_t(\mathcal{R}_{t+1}, T_{t+1}).$$

We can then use (A.16) to rewrite the terms in the last equation as follows:

$$\widetilde{\mathbb{V}}_t \left[(b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}\right] = (b_{H,t+1}, b_{F,t+1}, m_{t+1}) \, \widetilde{\Sigma}_t \, (b_{H,t+1}, b_{F,t+1}, m_{t+1})',$$
$$\widetilde{\mathbb{V}}_t \left(T_{t+1}\right) = (\beta_{T,t})^2 \sigma_m^2 + (\xi_{T,t})^2 (\sigma_m^*)^2,$$

where $\widetilde{\Sigma}_t \equiv \widetilde{\mathbb{V}}_t(\mathcal{R}_{t+1})$ is such that $(\widetilde{\Sigma}_t)_{1,1} = (\widetilde{\Sigma}_t)_{3,1} = (\widetilde{\Sigma}_t)_{1,3} = (\widetilde{\Sigma}_t)_{3,3} = \beta_{p,t}^2 \sigma_m^2 + \xi_{p,t}^2 (\sigma_m^*)^2$, $(\widetilde{\Sigma}_t)_{2,1} = (\widetilde{\Sigma}_t)_{1,2} = (\widetilde{\Sigma}_t)_{3,2} = (\widetilde{\Sigma}_t)_{2,3} = \beta_{p,t} \beta_{p^*,t} \sigma_m^2 + \xi_{p,t} \xi_{p^*,t} (\sigma_m^*)^2$, and $(\widetilde{\Sigma}_t)_{2,2} = \beta_{p^*,t}^2 \sigma_m^2 + \xi_{p^*,t}^2 (\sigma_m^*)^2$. Similarly, for the covariance,

$$\widetilde{cov}_t \left(\mathcal{R}_{t+1}, T_{t+1}\right) = -\begin{pmatrix} \beta_{p,t} \beta_{T,t} \sigma_m^2 + \xi_{p,t} \xi_{T,t} (\sigma_m^*)^2 \\ \beta_{p^*,t} \beta_{T,t} \sigma_m^2 + \xi_{p^*,t} \xi_{T,t} (\sigma_m^*)^2 \\ \beta_{p,t} \beta_{T,t} \sigma_m^2 + \xi_{p,t} \xi_{T,t} (\sigma_m^*)^2 \end{pmatrix}.$$

The first-order conditions with respect to $(b_{H,t+1}, b_{F,t+1}, m_{t+1})$ give

$$\widetilde{\Sigma}_t \begin{pmatrix} b_{H,t+1} \\ b_{F,t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{\gamma} \widetilde{\mathbb{E}}_t \mathcal{R}_{t+1} + \frac{1}{\gamma} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{v} \log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + \widetilde{cov}_t \left(\mathcal{R}_{t+1}, T_{t+1}\right). \tag{A.18}$$

Note that $\widetilde{\Sigma}_t$ is not invertible because money and one-period ahead return on home bonds have the same risk profile. Using this fact and taking the difference of the first and third lines of (A.18), we obtain the money demand function

$$m_{t+1} = \overline{m}e^{-vi_t}.$$

Step 2: temporary equilibrium. First, the market-clearing conditions in period t are

$$\begin{aligned} \omega b_{H,t+1} + (1-\omega) b^*_{H,t+1} &= B_{H,t+1}, \\ \omega b_{F,t+1} + (1-\omega) b^*_{F,t+1} &= B^* - B_{F,t+1}, \\ \omega \overline{m} e^{-vi_t} &= \frac{M_{t+1}}{P_t}, \\ (1-\omega)\, \overline{m} e^{-vi^*_t} &= \frac{M^*_{t+1}}{P^*_t}, \end{aligned}$$

where $b^*_{H,t+1}$ and $b^*_{F,t+1}$ are holdings of home- and foreign-country bonds by foreign households, respectively. In the cashless limit, the market-clearing conditions in the money market in the two countries imply

$$\begin{pmatrix} p_t \\ p^*_t \end{pmatrix} = v \begin{pmatrix} i_t \\ i^*_t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \epsilon^m_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \epsilon^{m*}_t, \tag{A.19}$$

Second, using the optimal portfolio choice in equation (A.18) and an analogous expression for foreign country, imposing market-clearing conditions, and taking the cashless limits, we obtain

$$(\widetilde{\Sigma}_t)_{1:2,1:2} \begin{pmatrix} B_{H,t+1} \\ B^* - B_{F,t+1} \end{pmatrix} = \frac{1}{\gamma} \widetilde{\mathbb{E}}_t (\mathcal{R}_{t+1})_{1:2} + \widetilde{cov}_t \left((\mathcal{R}_{t+1})_{1:2}, \omega T_{t+1} + (1-\omega) T^*_{t+1} \right). \tag{A.20}$$

We next solve equations (A.19) and (A.20) together for i_t and i^*_t:

$$\begin{aligned} \begin{pmatrix} i_t \\ i^*_t \end{pmatrix} =& \frac{1}{1+v} \begin{pmatrix} \alpha_{p,t} + r \\ \alpha_{p^*,t} + r \end{pmatrix} - \frac{1}{1+v} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \epsilon^m_t - \frac{1}{1+v} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \epsilon^{m*}_t \\ &+ \frac{\gamma}{1+v} (\widetilde{\Sigma}_t)_{1:2,1:2} \begin{pmatrix} B_{H,t+1} \\ B^* - B_{F,t+1} \end{pmatrix} - \frac{\gamma}{1+v} \widetilde{cov}_t \left((\mathcal{R}_{t+1})_{1:2}, \omega T_{t+1} + (1-\omega) T^*_{t+1} \right), \end{aligned} \tag{A.21}$$

where

$$\begin{aligned} &\widetilde{cov}_t \left((\mathcal{R}_{t+1})_{1:2}, \omega T_{t+1} + (1-\omega) T^*_{t+1} \right) \\ &= - \begin{pmatrix} [\omega \beta_{T,t} + (1-\omega)\beta_{T*,t}]\, \beta_{p,t} \sigma^2_m + [\omega \xi_{T,t} + (1-\omega) \xi_{T*,t}]\, \xi_{p,t} (\sigma^*_m)^2 \\ [\omega \beta_{T,t} + (1-\omega)\beta_{T*,t}]\, \beta_{p^*,t} \sigma^2_m + [\omega \xi_{T,t} + (1-\omega) \xi_{T*,t}]\, \xi_{p^*,t} (\sigma^*_m)^2 \end{pmatrix}. \end{aligned}$$

Third, taxes in the home country are

$$\omega T_t = \left(i_{t-1} + p_{t-1} - p_t - i^*_{t-1} - p^*_{t-1} + p^*_t \right) B_{F,t}, \tag{A.22}$$

while those in the foreign country are

$$(1-\omega)\, T^*_t = (i^*_{t-1} + p^*_{t-1} - p^*_t) B^*. \tag{A.23}$$

Step 3: rational expectations equilibrium. As usual, to compute the REE, we equalize expectations with equilibrium distributions. For the interest rates, when we set expectations (A.16) and equilibrium values (A.21) equal to each other, we obtain $\beta_{i,t}^{REE} = \xi_{i^*,t}^{REE} = -1/(1+v)$, $\xi_{i,t}^{REE} = \beta_{i^*,t}^{REE} = 0$. For the price levels, we use equation (A.19) to obtain $\beta_{p,t}^{REE} = \xi_{p^*,t}^{REE} = 1/(1+v)$ and $\xi_{p,t}^{REE} = \beta_{p^*,t}^{REE} = 0$. The sensitivities of taxes satisfy

$$\omega\beta_{T,t}^{REE} = (\beta_{p^*,t}^{REE} - \beta_{p,t}^{REE})B_{F,t+1} = -\frac{B_{F,t+1}}{1+v},$$
$$\omega\xi_{T,t}^{REE} = (\xi_{p^*,t}^{REE} - \xi_{p,t}^{REE})B_{F,t+1} = \frac{B_{F,t+1}}{1+v},$$

and

$$(1-\omega)\,\beta_{T^*,t}^{REE} = -\beta_{p^*,t}^{REE}B^* = 0,$$
$$(1-\omega)\,\xi_{T^*,t}^{REE} = -\xi_{p^*,t}^{REE}B^* = -\frac{B^*}{1+v}.$$

Again, we omit time subscripts from those coefficients that do not depend on time, thus,

$$(\beta_i^{REE}, \beta_{i^*}^{REE}, \beta_p^{REE}, \beta_{p^*}^{REE}, \beta_{T,t}^{REE}, \beta_{T^*}^{REE}) = \left(-\frac{1}{1+v}, 0, \frac{1}{1+v}, 0, -\frac{B_{F,t+1}}{\omega(1+v)}, 0\right),$$
$$(\xi_i^{REE}, \xi_{i^*}^{REE}, \xi_p^{REE}, \xi_{p^*}^{REE}, \xi_{T,t}^{REE}, \xi_{T^*}^{REE}) = \left(0, -\frac{1}{1+v}, 0, \frac{1}{1+v}, \frac{B_{F,t+1}}{\omega(1+v)}, -\frac{B^*}{(1-\omega)(1+v)}\right).$$

In the REE, the top-left sub-matrix of the variance-covariance matrix, which we denote with Σ^{REE}, is given by

$$(\Sigma^{REE})_{1:2,1:2} = \begin{pmatrix} (\beta_p^{REE})^2\sigma_m^2 & 0 \\ 0 & (\xi_{p^*}^{REE})^2(\sigma_m^*)^2 \end{pmatrix} = \frac{1}{(1+v)^2}\begin{pmatrix} \sigma_m^2 & 0 \\ 0 & (\sigma_m^*)^2 \end{pmatrix}.$$

Also, the covariance of taxes and returns is

$$cov_t^{REE}\left((\mathcal{R}_{t+1})_{1:2}, \omega T_{t+1} + (1-\omega)T_{t+1}^*\right) = \begin{pmatrix} \omega\beta_{T,t}^{REE}\beta_p^{REE}\sigma_m^2 \\ \left[\omega\xi_{T,t}^{REE} + (1-\omega)\xi_{T*}^{REE}\right]\xi_{p^*,t}^{REE}(\sigma_m^*)^2 \end{pmatrix}$$
$$= \frac{1}{(1+v)^2}\begin{pmatrix} -B_{F,t+1}\sigma_m^2 \\ (B_{F,t+1} - B^*)(\sigma_m^*)^2 \end{pmatrix}.$$

Finally, the constant parts of the endogenous variables satisfy the following system of equations:

$$\alpha_{i,t}^{REE} = \frac{\alpha_{p,t+1}^{REE} + r}{1+v},$$
$$\alpha_{i^*,t}^{REE} = \frac{\alpha_{p^*,t+1}^{REE} + r}{1+v},$$
$$\alpha_{p,t}^{REE} = v\alpha_{i,t}^{REE},$$
$$\alpha_{p^*,t}^{REE} = v\alpha_{i^*,t}^{REE},$$
$$\omega\alpha_{T,t}^{REE} = \left(i_t + p_t - i_t^* - p_t^* + \alpha_{p^*,t}^{REE} - \alpha_{p,t}^{REE}\right)B_{F,t+1},$$
$$(1-\omega)\,\alpha_{T^*,t}^{REE} = (i_t^* + p_t^* - \alpha_{p^*,t,t+1}^{REE})B^*,$$

which has a solution:

$$\begin{aligned}
\alpha_i^{REE} &= r, \\
\alpha_{i^*}^{REE} &= r, \\
\alpha_p^{REE} &= vr, \\
\alpha_{p^*}^{REE} &= vr, \\
\omega\alpha_{T,t}^{REE} &= (i_t + p_t - i_t^* - p_t^*)\, B_{F,t+1}, \\
(1-\omega)\,\alpha_{T^*,t}^{REE} &= (i_t^* + p_t^* - vr)B^*.
\end{aligned}$$

Therefore, in the REE,

$$\begin{aligned}
i_t &= r - \frac{1}{1+v}\epsilon_t^m, \\
i_t^* &= r - \frac{1}{1+v}\epsilon_t^{m*}, \\
p_t &= vr + \frac{1}{1+v}\epsilon_t^m, \\
p_t^* &= vr + \frac{1}{1+v}\epsilon_t^{m*}.
\end{aligned}$$

Finally, the exchange rate is obtained from the law of one price:

$$e_t = p_t - p_t^* = \frac{\epsilon_t^m - \epsilon_t^{m*}}{1+v}.$$

Step 4: level-*k* beliefs. Equations (A.19) and (A.21)-(A.23) define a mapping $\Psi(\cdot)$ from beliefs about policy actions and future endogenous variables into distributions over the endogenous variables $Z_t = (p_t, i_t, T_t, p_t^*, i_t^*, T_t^*)$. Since it does not influence the derivations below, we dropped the exchange rate e_t from Z_t to simplify the exposition. Once we derive expressions for the price levels, the nominal exchange rate follows immediately from the law of one price: $e_t = p_t - p_t^*$.

To obtain level-k beliefs, we iterate on the mapping $\Psi(\cdot)$ starting from the status quo corresponding to the REE before the policy intervention. As in the case of Section 3.1, it is straightforward to see that some of the sensitivities coincide with their REE counterparts. Specifically, $\beta_i^k = -1/(1+v)$, $\beta_{i^*}^k = 0$, $\beta_p^k = 1/(1+v)$, $\beta_{p^*}^k = 0$, $\beta_{T^*}^k = 0$, and $\xi_i^k = 0$, $\xi_{i^*}^k = -1/(1+v)$, $\xi_p^k = 0$, $\xi_{p^*}^k = 1/(1+v)$, $\xi_{T^*}^k = -B^*/[(1-\omega)(1+v)]$, for all $k \geq 1$. In particular, the latter imply that $(\widetilde{\Sigma}_t^k)_{1:2,1:2} = (\Sigma^{REE})_{1:2,1:2}$, for all $k \geq 1$.

We are left to derive $\beta_{T,t}^k$, $\xi_{T,t}^k$, and the α^k's, for all $k \geq 1$. For home-country taxes, the sensitivity of beliefs to shocks depend on the level of sophistication

$$\omega\beta_{T,t}^k = \begin{cases} 0, & k = 1, \\ -\frac{B_{F,t+1}}{1+v}, & k > 1, \end{cases}$$

and

$$\omega\xi_{T,t}^k = \begin{cases} 0, & k = 1, \\ \frac{B_{F,t+1}}{1+v}, & k > 1, \end{cases}$$

where the case with $k = 1$ follows from the fact that level-1 agents do not update their beliefs after the policy intervention.

We next summarize the constant part of the endogenous variables. We start with the interest rates and use equations (A.21) and (A.19) to obtain

$$\alpha_{i,t}^{k} = \begin{cases} r, & k = 1 \\ \frac{v\alpha_{i,t+1}^{k-1}+r}{1+v} + \frac{\gamma\sigma_m^2}{(1+v)^3} B_{F,t+2}, & k = 2, \\ \frac{v\alpha_{i,t+1}^{k-1}+r}{1+v}, & k \geq 3, \end{cases}$$

$$\alpha_{i^*,t} = \begin{cases} r, & k = 1, \\ \frac{v\alpha_{i^*,t+1}^{k-1}+r}{1+v} - \frac{\gamma(\sigma_m^*)^2}{(1+v)^3} B_{F,t+2}, & k = 2, \\ \frac{v\alpha_{i^*,t+1}^{k-1}+r}{1+v}, & k \geq 3, \end{cases}$$

Note that there are three cases corresponding to $k = 1$, $k = 2$, and $k \geq 3$. The reason we have three separate cases is that, following a policy announcement, *(i)* level-1 agents do not update their beliefs; *(ii)* level-2 agents form their beliefs based on the equilibrium outcomes under level-1 beliefs, which imply non-neutrality of policy interventions; *(iii)* level-k agents, with $k \geq 3$, form their beliefs from level-$k-1$ agents, who understand the fiscal consequences of balance sheet policies, which in turn implies neutrality of such interventions.

We rewrite the equation for $\alpha_{i,t}^{k}$, in the case corresponding to $k \geq 3$, as follows:

$$\begin{aligned} \alpha_{i,t}^{k} - r &= \frac{v}{1+v}\left(\alpha_{i,t+1}^{k-1} - r\right) \\ &= \left(\frac{v}{1+v}\right)^{k-2}\left(\alpha_{i,t+k-2}^{2} - r\right) \\ &= \left(\frac{v}{1+v}\right)^{k-2} \frac{\gamma\sigma_m^2}{(1+v)^3} B_{F,t+k} = \left(\frac{v}{1+v}\right)^{k} \frac{\gamma\sigma_m^2}{v^2\,(1+v)} B_{F,t+k}. \end{aligned}$$

As a result,

$$\alpha_{i,t}^{k} - r = \begin{cases} 0, & k = 1, \\ \left(\frac{v}{1+v}\right)^{k} \frac{\gamma\sigma_m^2}{v^2(1+v)} B_{F,t+k}, & k \geq 2. \end{cases}$$

The equation for $\alpha_{i^*,t}^{k}$ is analogous. Using (A.19), we can derive the constant part of the price levels. For example, in the case of the home country,

$$\alpha_{p,t}^{k} = v\alpha_{i,t}^{k} = \begin{cases} vr, & k = 1, \\ vr + \left(\frac{v}{1+v}\right)^{k} \frac{\gamma\sigma_m^2}{v(1+v)} B_{F,t+k}, & k \geq 2. \end{cases}$$

Step 5: reflective equilibrium. We first solve for the interest rates in the reflective equilibrium. To do this, we aggregate the demand for public bonds from all households in each country and then impose market clearing. Formally, we aggregate equation (A.20) over all households in the economy

$$\sum_{k=1}^{\infty} f(k)(\widetilde{\Sigma}_t^k)_{1:2,1:2} \begin{pmatrix} B_{H,t+1} \\ B^* - B_{F,t+1} \end{pmatrix} = \frac{1}{\gamma} \sum_{k=1}^{\infty} f(k)\widetilde{\mathbb{E}}_t^k \begin{pmatrix} i_t - \pi_{t+1} - r \\ i_t^* - \pi_{t+1}^* - r \end{pmatrix} + \sum_{k=1}^{\infty} f(k)\widetilde{cov}_t^k((\mathcal{R}_{t+1})_{1:2}, \omega T_{t+1} + (1-\omega)T_{t+1}^*).$$

Using the fact that the variance-covariance matrix does not vary with the level of sophistication $(\widetilde{\Sigma}_t^k)_{1:2,1:2} = (\Sigma^{REE})_{1:2,1:2}$, that $\widetilde{\mathbb{E}}_t^k \pi_{t+1} = \alpha_{p,t}^k - p_t$ (with an analogous expression in the foreign country), and that

$$\widetilde{cov}_t^k((\mathcal{R}_{t+1})_{1:2}, \omega T_{t+1} + (1-\omega)T_{t+1}^*) = -\frac{1}{1+v} \begin{pmatrix} \omega \beta_{T,t}^k \sigma_m^2 \\ \left(\omega \xi_{T,t}^k - \frac{B^*}{1+v}\right)(\sigma_m^*)^2 \end{pmatrix},$$

we can solve the market-clearing equation for i_t and i_t^*:

$$\begin{pmatrix} i_t \\ i_t^* \end{pmatrix} = \begin{pmatrix} r - \frac{\epsilon_t}{1+v} \\ r - \frac{\epsilon_t^*}{1+v} \end{pmatrix} + \frac{\gamma}{v(1+v)^2} \begin{pmatrix} \sigma_m^2 \\ -(\sigma_m^*)^2 \end{pmatrix} \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{F,t+k}.$$

The price levels are then

$$\begin{pmatrix} p_t \\ p_t^* \end{pmatrix} = \begin{pmatrix} vr + \frac{1}{1+v}\epsilon_t \\ vr + \frac{1}{1+v}\epsilon_t^* \end{pmatrix} + \frac{\gamma}{(1+v)^2} \begin{pmatrix} \sigma_m^2 \\ -(\sigma_m^*)^2 \end{pmatrix} \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{F,t+k},$$

and the nominal exchange rate satisfies

$$e_t = p_t - p_t^* = \frac{\epsilon_t - \epsilon_t^*}{1+v} + \frac{\gamma}{(1+v)^2}[\sigma_m^2 + (\sigma_m^*)^2] \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B_{F,t+k}.$$

A.4 Extensions

A.4.1 Rational Expectation Agents

To investigate whether the presence of households with rational expectations can undo the non-neutrality result of balance sheet policies, we add a mass of rational-expectations agents to the simple model. Specifically, we assume that fraction $\tau \in [0,1]$ of agents form their expectations rationally, while the remaining fraction $1-\tau$ uses the level-k thinking process. Moreover, the fraction $1-\tau$ is split into groups with different levels k, where groups have mass given by the distribution function $f(k)$.

In the reflective equilibrium augmented with rational-expectations agents, market-clearing in the risky asset market requires

$$\tau \left(\frac{r^x + q_{t+1} - q_t R}{\gamma \sigma_x^2} - X_{t+1} \right) + (1-\tau) \sum_{k=1}^{\infty} f(k) \left(\frac{r^x + \alpha_{q,t}^k - q_t R}{\gamma \sigma_x^2} + \beta_{T,t}^k \right) = \overline{X} - X_{t+1}.$$

As in the simple model, the price q_t is deterministic. The first term on the left-hand side is the demand for risky assets by the rational-expectations agents, while the second term represents the demand of level-k thinkers. Conditional on the beliefs of level-k thinkers that that we computed in Section 2, we look for price q_t. We can rewrite the last equation to the equation for q_t as follows:

$$q_t - q^{REE} = \tau \frac{q_{t+1} - q^{REE}}{R} + (1-\tau)\,\gamma\sigma_x^2 \sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{R^k}. \tag{A.24}$$

To solve for price q_t, we introduce the following new variables:

$$G_t \equiv \gamma\sigma_x^2 \sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{R^k},$$
$$\widehat{q}_t \equiv q_t - q^{REE}.$$

Using this notation, we rewrite equation (A.24) as follows:

$$\widehat{q}_t = \tau \frac{\widehat{q}_{t+1}}{R} + (1-\tau)\, G_t,$$

and iterate it forward (imposing a non-bubble condition) to get

$$q_t = q^{REE} + (1-\tau) \sum_{s=0}^{\infty} \left(\frac{\tau}{R}\right)^s G_{t+s},$$

or, using the definitions of G_t and $\widehat{q}_t$,

$$q_t = q^{REE} + (1-\tau)\,\gamma\sigma_x^2 \sum_{k=1}^{\infty} \frac{f(k)}{R^k} \sum_{s=0}^{\infty} \left(\frac{\tau}{R}\right)^s X_{t+s+k}. \tag{A.25}$$

Comparing (A.25) to the price of risky assets in the reflective equilibrium in the simple model, equation (19), we note the following. First, the presence of the term $(1-\tau)$ implies that a higher fraction of rational expectations agents must reduce the effectiveness of the risky assets purchases. At the same time, however, the term $(\tau/R)^s$ implies that a higher fraction of rational-expectations agents leads to a lower discounting of future policies and, thus, to a stronger effect of future purchases. These two opposing effects echo the discussion on the implications for balance sheet policies of a higher average level of sophistication in Section 2.5. To make this more explicit, we compute price q_t in two special cases.

Example 1. Consider an example in which $f(k) = (1-\lambda)\,\lambda^{k-1}$ and $X_{t+k} = X_{t+1}\mu^{k-1}$. In this case,

$$G_t = \gamma\sigma_x^2 X_{t+1} \sum_{k=1}^{\infty} (1-\lambda)\,\lambda^{k-1} \frac{\mu^{k-1}}{R^k}$$
$$= \gamma\sigma_x^2 X_{t+1} \frac{1-\lambda}{R-\lambda\mu}.$$

The price is

$$q_t = q^{REE} + \gamma\sigma_x^2 \frac{1-\lambda}{R-\lambda\mu} X_{t+1} \frac{R\,(1-\tau)}{R-\tau\mu}.$$

The last expression has its maximum at $\tau = 0$ and monotonically declines to zero at $\tau = 1$. In this example, the first effect dominates and a higher fraction of rational-expectations agents makes balance sheet policies weaker.

Example 2. Consider now the following path of risky assets purchases $\{X_{t+1}\} = \{0, X_{t+2}, 0, 0, \ldots\}$. In this case,

$$G_t = \gamma\sigma_x^2 f(2)\frac{X_{t+2}}{R^2},$$
$$G_{t+1} = \gamma\sigma_x^2 f(1)\frac{X_{t+2}}{R^1},$$
$$G_{t+2} = 0,$$

and the price is

$$q_t = q^{REE} + \frac{\gamma\sigma_x^2}{R^2}(1-\tau)(\lambda+\tau)(1-\lambda)X_{t+2}.$$

The price is now a non-monotonic function of τ. The derivative of the price with respect to τ is:

$$\frac{dq_t}{d\tau} = \frac{\gamma\sigma_x^2}{R^2}(1-\lambda)X_{t+2}[1-\lambda-2\tau],$$

we have that, when the fraction of rational-expectations agents is low enough, $\tau < (1-\lambda)/2$, then $dq_t/d\tau > 0$. That is, the second effect dominates and balance sheet policies become stronger as τ grows and as long as $\tau < (1-\lambda)/2$.

A.4.2 Unraveling

In this section, we modify our simple model and allow for a simple version of "learning" or "dynamic equilibrium unraveling." In particular, we assume that the level of sophistication of agents changes over time. We introduce this assumption in the simplest possible way to highlight a number of qualitative results. Specifically, we assume that a current level-k thinker becomes level-$(k+h)$ in the subsequent period, where h is a constant non-negative integer number. One interpretation of this assumption is as follows. At the time a new policy is announced, an agent can compute k deductive iterations to form the expectations about future endogenous variables. In every subsequent period, the agent uses the already computed beliefs and performs h additional deductive iterations.

Formally, we assume that the distribution of levels of sophistication changes over time according to

$$f_t(k) = \begin{cases} f(k-ht), & k \geq 1+ht, \\ 0, & k < 1+ht, \end{cases} \tag{A.26}$$

where $f(k)$ is the distribution at the time the policy is announced.

We can compute the price effect of the intervention by evaluating the results in Proposition 2 with the distribution (A.26). Specifically, if we focus, for simplicity, on a permanent intervention of the size X, we obtain

$$q_t = q^{REE} + \gamma\sigma_x^2 X \sum_{k=1}^{\infty} f_t(k)\frac{1}{R^k}.$$

Now the distribution of level-k agents changes over time. Using (A.26) and assuming that the initial distribution is exponential, we get

$$q_t = q^{REE} + \gamma\sigma_x^2 \frac{X}{\bar{k}(R-1)+1} R^{-ht}. \tag{A.27}$$

Equation (A.27) shows that, over time, the price approaches q^{REE} at rate $1/R^h$. Thus, h determines the speed of convergence. The key implication of equation (A.27) is that a central bank cannot stimulate the economy forever by keeping the size of its balance sheet at a constant level.

To counteract the dampening forces coming from equilibrium unraveling, it is crucial that the size of the intervention increases over time. In the specific example, to keep the price q_t at a constant higher level, the central bank needs to increase asset purchases exponentially at the rate $\mu = R^{h/(1+h)} > 1$. Formally, with $X_{t+1} = \mu^t X_1$, where $\mu \le (R/\mu)^h$ (to keep the price bounded), we get

$$q_t - q^{REE} = \gamma\sigma_x^2 \cdot \frac{1}{\bar{k}(R-\mu)+\mu} X_1 \left(\frac{\mu^{1+h}}{R^h}\right)^t,$$

so that, if $\mu^{1+h}/R^h = 1$, then q_t does not depend on time.

An empirical prediction of equilibrium unraveling is that new rounds of balance sheet policies tend to be weaker than initial rounds. For example, after controlling for the size of the intervention, the first round of quantitative easing implemented by the Federal Reserve in 2009 should have had stronger effects than the second round implemented in 2010.

B Data Sources

All variables used in the empirical part of the paper are monthly and identical to those in Fieldhouse et al. (2018) (FMR) except for forecasts of mortgage returns. For convenience we list all of the data sources here.

- **Agency purchase commitments** are computed by FMR in summing purchases by Fannie Mae, Freddie Mac, and the Federal Reserve.
- **The noncyclical narrative policy indicator** m_t is computed in FMR.
- **Personal income** is from NIPA (series PI in the FRED database).
- **The core PCE price index** is from NIPA (series PCEPILFE in the FRED database).
- **Nominal house price index** is the Freddie Mac house price index.
- **Total mortgage debt** are from the Financial Accounts of the United States and additional computations in FMR.
- **Residential mortgage originations** are computed by FMR from various sources and available from the authors.
- **Housing starts** are from the Census Bureau (series HOUST in the FRED database).
- **The 3-month T-bill rate** is from the Federal Reserve Release (FRSR) H.15 (series TB3MS in the FRED database).
- **The 10-year Treasury rate** is from the Federal Reserve Release (FRSR) H.15 (series GS10 in the FRED database).

- **The BAA-AAA corporate bond spread** is obtained by taking the difference in the Moody's seasoned BAA and AAA yields (series BAA and AAA in the FRED database).
- **The conventional mortgage rate** is the 30-year fixed-rate conventional conforming mortgage rate. It is measured as monthly average commitment rate from the Freddie Mac primary mortgage market survey.
- **Mortgage rate forecast** is the Blue Chip Forecasts of home mortgage rate which is defined as the 30-year fixed-rate conventional conforming mortgage rate. The Blue Chip reports note that "Interest rate definitions are the same as those in FRSR H.15."

C Extra Tables and Figures

	Forecast horizon in months				
	1-3	4-6	7-9	10-12	mean
$\widehat{\bar{k}}$	1.23	1.03	1.35	1.03	1.17

Table A1: The average level of sophistication of agents in the economy obtained from equation (30) for different horizons and the mean value across all horizons. Because we use the BCFF dataset where the participants are asked to forecast variables for the future *calendar* quarters, we do not have fixed forecasting horizons when we use monthly data. As a result, we introduce the notation where "1-3" denotes the forecast for the next calendar quarter, "4-6" denotes the forecast for the quarter after the next calendar quarter, and so on.