Sequential All-Pay Auctions with Head Starts*

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Abstract

We study a sequential (Stackelberg) all-pay auction where heterogeneous contestants are privately informed about a parameter (ability) that affects their cost of effort. In the case of two contestants, contestant 1 (the first mover) makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins. This model is then generalized to any number of contestants where in each period of the contest, $1 \leq j \leq n$, a new contestant joins and chooses an effort. Contestant j observes the efforts of all contestants in the previous periods and then makes an effort in period j. He wins if his effort is larger than or equal to the efforts of all the contestants in the previous periods and strictly larger than the efforts of all the contestants in the following periods. We characterize the unique subgame perfect equilibrium of these sequential all-pay auctions and analyze the use of head starts to improve the contestants' performances.

Keywords: Stackelberg contests, sequential all-pay auctions, head starts.

JEL classification: D44, O31, O32

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1 Introduction

In many contest settings, effort choices are made sequentially rather than simultaneously. The differences between simultaneous and sequential contests have been addressed in the literature by several researchers.[1] Baik and Shogren (1992), Leininger (1993) and Morgan (2003) investigated the question of which form of contest, sequential or simultaneous, naturally arises in competitive situations. They studied two-player models where contestants compete in the (generalized) Tullock contest and each contestant is able to choose between two dates to make their efforts. If the contestants choose different dates, a sequential contest occurs, but if they choose the same date the contest will be a simultaneous one. They all showed that sequential contests may arise endogenously in equilibrium.[2] Despite these important findings, while numerous studies have dealt with simultaneous all-pay auctions (all-pay contests) only a few have focused on sequential all-pay auctions. The purpose of this paper is to fill this gap in the literature by studying a sequential all-pay auction with heterogeneous contestants under incomplete information.

In the all-pay auction each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. All-pay auctions have been studied either under complete information where each player's type (valuation for winning the contest or ability) is common knowledge[3] or under incomplete information where each player's type is private information and only the distribution from which the players' types is drawn is common knowledge.[4] Most studies dealing with sequential all-pay auctions assume a two-stage contest under complete information. Leininger (1991) modeled a patent race between an incumbent and an entrant as a sequential asymmetric all-pay auction under complete information, and Konrad and Leininger (2007) characterized the equilibrium

[1] Dixit (1987) studied a sequential Tullock contest and examined whether the ability to commit to an effort choice before other contestants choose their effort while assuming that they can then observe this choice is advantageous or not. Linster (1993) analyzed two-player sequential Tullock contests and showed that if the stronger player is the first (second) mover in the sequential contest the players' total effort is larger (smaller) than in the simultaneous contest.

[2] Hamilton and Slutsky (1990) Deneckere and Kovenock (1992) and Mailath (1993) studied sequential oligopoly games and showed that sequential choices of quantities in a Cournot competition can be the equilibrium outcome of non-cooperative play.

[3] All-pay auctions under complete information have been studied, among others, by Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1993, 1996), Che and Gale (1998) and Siegel (2009)).

[4] All-pay auctions under incomplete information have been studied, among others, by Hillman and Riley (1989), Amman and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006) and Moldovanu et al. (2010)).

of the all-pay auction under complete information in which a group of players choose their effort 'early' and the other group of players choose their effort 'late'. The assumption of incomplete information complicates the analysis of the sequential all-pay auction but also makes it more relevant and interesting.

In this work, we study a sequential all-pay auction under incomplete information where the ability of each contestant is private information. We consider first a sequential all-pay auction with two contestants where contestant 1 (the first mover) makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins.[5] This particular type of sequential contest where the players' outputs are observable in any stage of the contest has various applications, including sport contests such as athletics and gymnastics, political races in which the candidates confront each other by a sequence of speeches, and court trials when the lawyers of both sides make their final speeches. Moreover, in R&D and other market races it is sometimes the case that the incumbent observes the output of the leader and only then decides how much effort to put in. In all these cases, the players in the later stages have some advantage because they have observed their opponents' outputs in the previous stages. Similarly, in our model contestant 2 has an obvious advantage over contestant 1. For this reason contestant 1 exerts a relatively low effort and sometimes, depending on the distribution of his opponent's abilities, he might even prefer not to participate in the contest at all (it is worth noting that this feature of our model can explain why players sometimes choose to stay out of a contest). Given the low effort of contestant 1 in the first period as well as the rules of the contest according to which contestant 2 needs only to equalize the effort of contestant 1 in order to win, we have a relatively low expected total effort as well as a low expected highest effort. However, a designer who wishes to maximize the expected total effort or the expected highest effort can change the rules of the sequential all-pay auction to make it more profitable by explicitly or implicitly favoring contestant 1 over contestant 2. In other words, he can

[5] The concept of Stackelberg games in which players choose their strategies sequentially was introduced and analyzed also by computer scientists such as Garg and Narahari (2008), Luh et al. (1984) and others. All these authors impose a hierarchical decision-making structure on a simultaneous game to describe sequential choices of strategies. The solution concept they use is a Stackelberg equilibrium where the leaders use "secure strategies" that guarantees them a minimal payoff while the followers use an optimal response strategy.

give contestant 1 a head start.

There are numerous examples of real-life sequential contests in which the players who play in the first stage are given a head start. Suppose, for example, that Microsoft Corporation is the first company to produce a hardware product. Then, if Apple Inc. wants to produce a competitive product, in order to convince customers to buy this new product it has to be either better or cheaper than the Microsoft product. In that case, Microsoft is exogenously given a head start. However, head starts can also be given endogenously. For example, a common situation often occurs in the labor market when an applicant gets a job and then any new applicant is required to be better in order to win his place. Thus contests with head starts may raise the contestants' expected total effort or alternatively their expected highest effort. Kirkegaard (2009), for example, studied asymmetric all-pay auctions with head starts under incomplete information where players simultaneously choose their efforts. He showed that the total effort increases if the weak contestant is favored with a head start, but if the contestants are sufficiently heterogenous, then in some cases the weak contestant should be given both a head start and a handicap.[6] Corns and Schotter (1999) demonstrated by theoretical and empirical arguments that a head start in the form of a price preference policy that is given to a subset of the firms might not only benefit that subset but can actually lower the purchasing cost of the government. In our sequential all-pay auction therefore we wish to demonstrate that a head starts can not only benefit one of the players but can also enhance the overall expected performance of the players. Since in our setting, contestant 2 has an advantage over contestant 1 because of the timing of their play, we assume that contestant 1 is given a multiplicative head start which is exogenously determined . That is, contestant 2 will win the contest if his effort x_2 is larger or equal to tx_1, where x_1 is the effort of contestant 1 and t is a constant larger than 1.[7] We provide sufficient conditions under which by imposing a head start for contestant 1 the designer of the contest can significantly increase the expected efforts of both contestants, particularly the expected total effort as well as the expected highest effort. The optimal head start can be high enough such that several types of contestant 1 will win for sure since no type of contestant 2 will want

[6] Siegel (2010) provided an algorithm that constructs the unique equilibrium in simultaneous all-pay auctions with head starts in which players do not choose weakly-dominated strategies.

[7] This multiplicative head start was chosen for the sake of convenience and may not necessarily be the optimal form of a head start.

to participate. As such, head starts may also play the role of a winning bid in a sequential all-pay auction when contestant 1 has an incentive to participate independently of the distribution of his opponent's type. Finally, head starts improve the inherent inefficiency of the sequential all-pay auction. The probability that a low ability contestant wins against a high ability contestant in a contest with a head start is lower than in a contest without any head start.

We then turn to study a sequential all-pay auction with $n > 2$ players. In this model, in each period of the contest, $1 \leq j \leq n$, a new contestant joins and chooses an effort. Contestant j, $j = 1, ..., n$ observes the efforts of all contestants in the previous $j-1$ periods and then makes an effort in period j. Contestant j wins if his effort is larger than or equal to the efforts of all the contestants in the $j-1$ previous periods and strictly larger than the efforts of all the contestants in the following $n-j$ periods. We also study this n-player model with head starts.[8] The analysis of the sequential all-pay auction with n players and head starts turns out to be quite complicated since a head start which is given to the contestant in period k affects the equilibrium strategies of all the contestants in the following periods $j \geq k$. Furthermore, in contrast to the model with two players, the use of head starts in the sequential all-pay auction with $n > 2$ players may decrease the number of active periods since players may choose to withdraw, and therefore may lower the contestants' expected highest and total effort. However, we provide sufficient conditions under which there always exist some non-trivial head starts that increase the expected total effort. Furthermore, we show that using head starts for any subset of contestants who play in the first $n-1$ periods increases the expected highest effort. Hence, our analysis establishes a key role for head starts in sequential all-pay auctions, particularly in sequential contests under incomplete information.

The rest of the paper is organized as follows: Section 2 presents the two-player sequential all-pay auction. Section 3 presents the general form of the n-player sequential all-pay auction. Sections 2 and 3 also characterize the unique sub-game perfect equilibrium with and without head starts and provide conditions

[8] When the head starts are relatively large so that they play the role of a winning bid, our sequential all-pay auctions are related to sequential second price auctions with a buy price (see, e.g., Milgrom 2003) in which buyers arrive one after the other without knowing their place in the queue. When a bidder arrives, he can either buy the object at the publicly announced "buy price" and end the auction, or place a bid lower than the buy price. If no bidder takes the buy price a second price auction determines the outcome.

under which the use of head starts improves the contestants' performance. Section 4 concludes. All proofs are in the Appendix.

2 The two-player model

We consider first a sequential all-pay auction with two contestants where contestant 1 (the first mover) makes an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then makes an effort in the second period. Contestant 2 wins the contest if his effort (x_2) is larger than or equal to the effort of contestant 1 (x_1); otherwise, contestant 1 wins. Both contestants' valuation for the prize is 1. An effort x_i causes a cost $\frac{x_i}{a_i}$ where $a_i \geq 0$ is the ability (or type) of contestant i which is private information to i.[9] Contestants' abilities are drawn independently. Contestant i's ability is drawn from the interval $[0, 1]$ according to a distribution function F_i which is common knowledge. We assume that $F_i, i = 1, 2$ has a positive and continuous density function $F_i' > 0$.

We begin the analysis by considering the equilibrium effort function of contestant 2 in the second period. We assume that if both contestants make the same effort then contestant 2 is the winner. Therefore contestant 2 makes the same effort as contestant 1 as long as his type a_2 is larger than or equal to the effort of contestant 1; otherwise he stays out of the contest. Formally, the equilibrium effort of contestant 2 is given by:

$$b_2(a_2; a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < b_1(a_1) \\ b_1(a_1) & \text{if} \quad b_1(a_1) \leq a_2 \leq 1 \end{cases}$$

where we assume that contestant 1 uses a strictly monotonic equilibrium effort function $b_1(a_1)$. Contestant 1 with ability a_1 chooses to behave as an agent with ability s that solves the following optimization problem:

$$\max_s \left\{ F_2(b_1(s)) - \frac{b_1(s)}{a_1} \right\} \tag{1}$$

The F.O.C. is then

$$a_1 F_2'(b_1(s)) b_1'(s) - b_1'(s) = 0 \tag{2}$$

and the S.O.C. is

$$a_1 F_2''(b_1(s))(b_1'(s))^2 + a_1 F_2'(b_1(s)) b_1''(s) - b_1''(s) = a_1 F_2''(b_1(s))(b_1'(s))^2 < 0$$

[9] An equivalent interpretation is that a_i is player's i valuation for the prize and his cost is equal to his bid.

Note that if F_2 is convex, the S.O.C does not hold and then $b_1(a_1) = 0$ for all a_1 is the solution of the maximization problem (1). Thus, in the following we assume that F_2 is concave (F_1 is not necessarily concave). Then the S.O.C. holds and in equilibrium, the maximization problem (1) must be solved by $s = a_1$. Thus we obtain that the equilibrium effort of contestant 1 with type a_1 is

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 \leq \tilde{a} \\ (F_2')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad \tilde{a} \leq a_1 \leq 1 \end{cases} \tag{3}$$

where the cutoff $\tilde{a}$ is defined by $\max\left\{\frac{1}{F_2'(0)}, 0\right\}$. This cutoff depends on the distribution of the second player's ability. If $F_2'(0)$ is a finite number then types $0 \leq a_1 \leq \tilde{a}$ do not find it optimal to exert a positive effort. As was mentioned above, for the class of convex distribution functions we have $\tilde{a} = 1$ such that all types of contestant 1 choose to stay out of the contest (in the following we will solve this problem by providing an incentive, a head start, for contestant 1 to participate in the contest). However, if contestant 2's distribution function F_2 is concave, we have a real competition in the sequential all-pay auction even without head starts.

The expected efforts of contestants 1 and 2 are

$$\begin{aligned} TE_1 &= \int_{\tilde{a}}^{1} b_1(a_1)\, F_1'(a_1)da_1 = \int_{\tilde{a}}^{1} (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \\ TE_2 &= \int_{\tilde{a}}^{1} \left(\int_{b_1(a_1)}^{1} b_2(a_2; a_1)\, F_2'(a_2)da_2 \right) F_1'(a_1)da_1 \\ &= \int_{\tilde{a}}^{1} \left[1 - F_2((F_2')^{-1}(\frac{1}{a_1}))\right] (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \end{aligned}$$

Note that contestant 2 makes the same effort as contestant 1 or else makes an effort of zero. Therefore the expected highest effort is equal to the expected effort of contestant 1 and is given by

$$HE = \int_{\tilde{a}}^{1} (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \tag{4}$$

The expected total effort is given by[10]

$$TE = TE_1 + TE_2 = \int_{\tilde{a}}^{1} \left[2 - F_2((F_2')^{-1}(\frac{1}{a_1}))\right] (F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 \tag{5}$$

Example 1 *Consider a sequential all-pay auction with two contestants whose abilities are distributed according to the distribution functions $F_1(x) = F_2(x) = x^{0.5}$. By (3), the equilibrium effort function of contestant*

[10] We assume that the contest designer does not discount the effort in the second period. We discuss this generalization and its implication in Section 5.

1 *in the sequential all-pay auction is*

$$b_1(a_1) = \frac{a_1^2}{4} \quad \textit{for all } a_1 \geq 0$$

Therefore by (4) the expected highest effort is given by

$$HE = \int_0^1 \frac{a_1^2}{4} \frac{1}{2\sqrt{a_1}} da_1 = 0.05$$

and by (5) the expected total effort is

$$TE = \int_0^1 \left(2 - \sqrt{\frac{a_1^2}{4}}\right) \frac{a_1^2}{4} \frac{1}{2\sqrt{a_1}} da_1 = \frac{23}{280} \approx 0.0821$$

In Example 1, the contestants' expected highest effort as well as their expected total effort are significantly lower than in the standard all-pay auction where both contestants simultaneously choose their efforts. In the next subsection we change the rules of the sequential all-pay auction by adding head starts to improve the contestants' performance in the contest.

2.1 Head starts

In our sequential all-pay auction, contestant 2 has an advantage over contestant 1 because of the timing of their play. Thus, contestant 1's effort is relatively low and sometimes, depending on the distribution of contestant 2's abilities, will choose to stay out of the contest. In that case there is no real competition. Thus we examine whether the players' performance can be enhanced by using a head start for contestant 1. By introducing a head start we may also improve the inherent inefficiency of the sequential all-pay contest. The probability that contestant 1 with a high ability wins against contestant 2 with a low ability is higher with a head start. We want the head start to be independent of the contestant's effort and therefore we introduce a multiplicative head start. We therefore assume that contestant 2 will win the contest if his effort x_2 is larger than or equal to tx_1 where x_1 is the effort of contestant 1 and t is a constant larger than 1. The equilibrium effort of contestant 2 is then given by

$$\beta_2(a_2; a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < t\beta_1(a_1) \\ t\beta_1(a_1) & \text{if} \quad t\beta_1(a_1) \leq a_2 \leq 1 \end{cases}$$

where we assume that contestant 1 uses a strictly monotonic equilibrium effort function $\beta_1(a_1)$. Contestant 1 with ability a_1 chooses to behave as an agent with ability s that solves the following optimization problem:

$$\max_s \left\{ F_2(t\beta_1(s)) - \frac{\beta_1(s)}{a_1} \right\} \tag{6}$$

The F.O.C. is

$$a_1 F_2'(t\beta_1(s))t\beta_1'(s) - \beta_1'(s) = 0$$

and the S.O.C. is

$$aF_2''(t\beta_1(s))(t\beta_1'(s))^2 + a_1 F_2'(t\beta_1(s))t\beta_1''(s) - \beta_1''(s) = a_1 F_2''(t\beta_1(s))(t\beta_1'(s))^2 < 0$$

Thus, if F_2 is concave, in equilibrium, the above maximization problem must be solved by $s = a_1$. Then we obtain the following condition

$$a_1 F_2'(t\beta_1(a_1))t - 1 = 0$$

and the equilibrium effort of contestant 1 with type a_1 is

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 \leq \widehat{a} \\ \frac{1}{t}\left(F_2'\right)^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \widehat{a} \leq a_1 \leq a^* \\ \frac{1}{t} & \text{if} \quad a^* \leq a_1 \leq 1 \end{cases} \tag{7}$$

where $\widehat{a}$ is defined as $\max\left\{\frac{1}{tF_2'(0)}, 0\right\}$ and a^* is the minimum between 1 and the solution to the following equation

$$t\beta_1(a) = 1 \Rightarrow a^* = \min\left\{1, \frac{1}{tF_2'(1)}\right\}$$

Note that $a^* \geq \hat{a}$ since $a^* = 1$ or $a^* = \frac{1}{tF_2'(1)}$, while $\hat{a}$ is either zero or $\hat{a} = \frac{1}{tF_2'(0)}$ and F_2' is a decreasing function. Furthermore, if $1 \leq t \leq \frac{1}{F_2'(1)}$, then $a^* = 1$ and only when $t > \frac{1}{F_2'(1)}$ does there exist a cutoff type $0 < a^* < 1$ and an interval of types $a^* \leq a_1 \leq 1$ who exert the effort $b_1(a^*) = \frac{1}{t}$ and win for sure (this serves as a winning bid).

The expected efforts of contestants 1 and 2 are given by

$$\begin{aligned} TE_1(t) &= \int_{\hat{a}}^{a^*} \frac{1}{t}(F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1 + \int_{a^*}^{1} \frac{1}{t}F_1'(a_1)da_1 \\ TE_2(t) &= \int_{\hat{a}}^{a^*} \left[1 - F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right](F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1 \end{aligned}$$

The expected total effort is therefore

$$\begin{aligned}TE(t) &= TE_1(t)+TE_2(t) \qquad (8)\\ &= \int_{\hat{a}}^{a^*}\left[\frac{1}{t}+1-F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right](F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1+\int_{a^*}^{1}\frac{1}{t}F_1'(a_1)da_1\end{aligned}$$

Note that the expected effort of contestant 1 is not always higher than the expected effort of contestant 2 as was the case without a head start and therefore the expected highest effort is not equal to the expected effort of contestant 1. The expected highest effort is given by

$$\begin{aligned}HE(t) &= \int_0^1\int_0^1 \max\{\beta_1(a_1),\beta_2(a_2;a_1)\}F_2'(a_2)\,da_2F_1'(a_1)\,da_1 \qquad (9)\\ &= \int_{\hat{a}}^{a^*}\left[F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right]\frac{1}{t}(F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1 \qquad (10)\\ &\quad +\int_{\hat{a}}^{a^*}\left[1-F_2((F_2')^{-1}(\frac{1}{a_1 t}))\right](F_2')^{-1}(\frac{1}{a_1 t})F_1'(a_1)da_1+\int_{a^*}^{1}\frac{1}{t}F_1'(a_1)da_1\end{aligned}$$

The first term describes those types of contestant 2 who choose to stay out of the contest $(0\leq a_2<t\beta_1(a_1))$ in which case the highest effort is equal to that of contestant 1, $\beta_1(a_1)=\frac{1}{t}(F_2')^{-1}(\frac{1}{a_1 t})$. The second term describes those types of contestant 2 who equalize the effort of contestant 1 multiplied by t in which case the highest effort is equal to $t\beta_1(a_1)=(F_2')^{-1}(\frac{1}{a_1 t})$. The last term describes those types of contestant 1 who win for sure by choosing the winning bid.

Below we discuss the equilibrium behavior of the contestants when the distribution function of contestant 2's types is convex rather than concave (again, there is no restriction on the distribution of contestant 1's types). When F_2 is convex and a head start $t>1$ is given to contestant 1 then the equilibrium effort of contestant 2 is once again

$$\beta_2(a_2;a_1)=\begin{cases}0 & \text{if} \quad 0\leq a_2<t\beta_1(a_1)\\ t\beta_1(a_1) & \text{if} \quad t\beta_1(a_1)\leq a_2\leq 1\end{cases}$$

while the equilibrium effort of contestant 1 is given by

$$\beta_1(a_1)=\begin{cases}0 & \text{if} \quad 0\leq a_1<\frac{1}{t}\\ \frac{1}{t} & \text{if} \quad \frac{1}{t}\leq a_1\leq 1\end{cases}$$

Note that when F_2 is convex and a head start is given to contestant 1 some of contestant 1's types participate in the contest. In this case the expected total effort and the expected highest effort are the same and are both equal to contestant 1's expected effort. The optimal head start it then t that maximizes $\frac{1}{t}\left(1-F_1\left(\frac{1}{t}\right)\right)$.

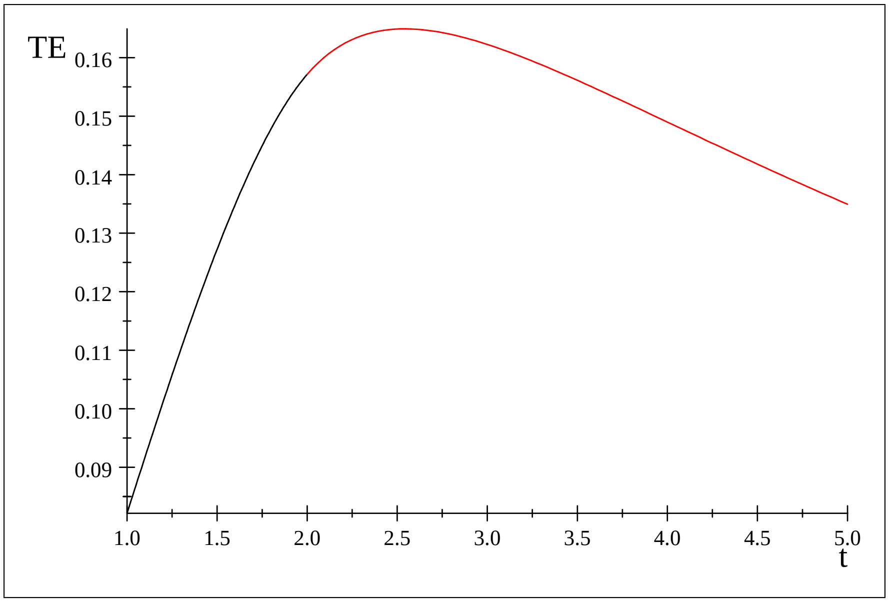
Example 2 *Consider a sequential all pay auction with two contestants where* $F_1(x) = F_2(x) = x^{0.5}$*. By (7), the equilibrium effort function of contestant* 1 *is given by*

$$\beta_1(a_1) = \begin{cases} \frac{1}{t}(F_2^{'})^{-1}(\frac{1}{a_1 t}) = \frac{ta_1^2}{4} & \text{if} \quad 0 \le a_1 \le \min\left\{\frac{2}{t}, 1\right\} \\ \frac{1}{t} & \text{if} \quad \min\left\{\frac{2}{t}, 1\right\} < a_1 \le 1 \end{cases}$$

The expected total effort is given by

$$\begin{aligned} TE &= \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\frac{a_1^2 t}{4}\right) \frac{1}{2\sqrt{a_1}} da_1 + \int_{\min\left\{\frac{2}{t},1\right\}}^1 \left(\frac{1}{t}\right) \frac{1}{2\sqrt{a_1}} da_1 \\ &+ \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\int_{\frac{a_1^2 t^2}{4}}^1 \left(\frac{a_1^2 t^2}{4}\right) \frac{1}{2\sqrt{a_2}} da_2\right) \frac{1}{2\sqrt{a_1}} da_1 \end{aligned}$$

The following figure presents the expected total effort as a function of t*.*



The optimal head start that yields the highest expected total effort in the sequential all-pay auction is therefore

$$t_{total} = \frac{7}{4}\left(199 - 5\sqrt{1561}\right) = 2.541\,9$$

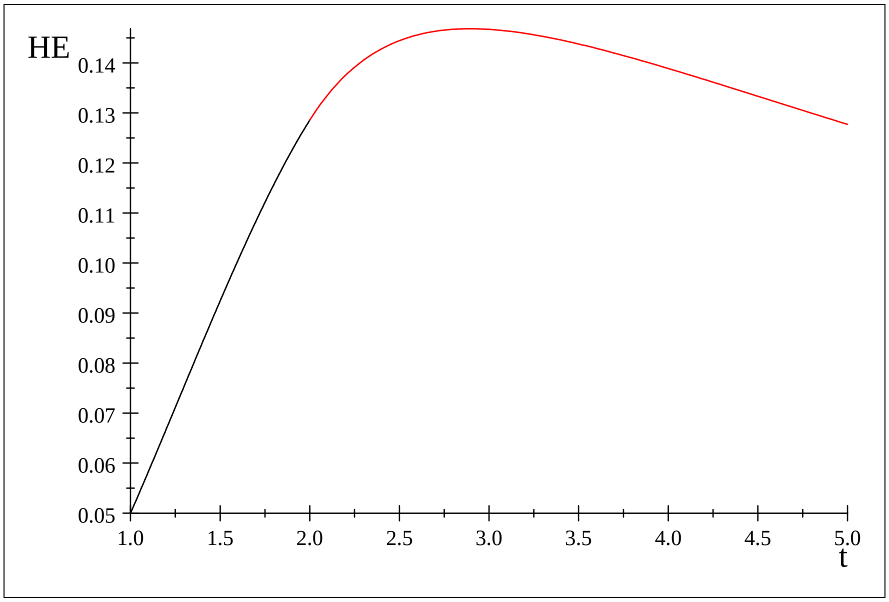
and the expected total effort is then

$$TE(t_{total}) = 0.16492$$

The expected highest effort is

$$\begin{aligned} HE &= \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\int_0^{\frac{a_1^2 t^2}{4}} \left(\frac{a_1^2 t}{4}\right) \frac{1}{2\sqrt{a_2}} da_2 + \int_{\frac{a_1^2 t^2}{4}}^1 \left(\frac{a_1^2 t^2}{4}\right) \frac{1}{2\sqrt{a_2}} da_2\right) \frac{1}{2\sqrt{a_1}} da_1 \\ &+ \int_{\min\left\{\frac{2}{t},1\right\}}^1 \frac{1}{t} \frac{1}{2\sqrt{a_1}} da_1 \end{aligned}$$

The following figure presents the expected highest effort as a function of t.



The optimal head start that yields the highest expected highest effort in the sequential all-pay auction is therefore

$$t_{high} = \frac{1}{\left(\frac{1}{180}\sqrt{10}\sqrt{317} + \frac{7}{36}\sqrt{2}\right)^{2}} = 2.8945$$

and the expected highest effort is then

$$HE(t_{high}) = 0.1468$$

From Examples 1 and 2 we can see that the optimal head start significantly increases the contestants' expected highest effort as well as their expected total effort.

We now turn to examine the conditions under which a head start is efficient in the sequential all-pay auction, namely, those conditions on the distribution of the contestants' abilities that ensure that a head start increases the expected highest effort or the expected total effort. The following condition is required for establishing the effects of a head start on contestant 1's equilibrium effort. Let $h(a_1) = (F_2')^{-1}\left(\frac{1}{a_1}\right)$ represent the equilibrium effort function of contestant 1 in the sequential all-pay auction without a head start, when $\tilde{a} \leq a_1 \leq 1$.

Condition 1 *The function $h(a_1)$ is strictly convex for all $\tilde{a} \leq a_1 \leq 1$.*

If Condition 1 is satisfied[11], any head start t close to 1 increases the expected effort of contestant 1 since then, for $t > 1$ and $\tilde{a} \leq a_1 \leq 1$ we have $b_1(a_1) < \frac{1}{t} b_1(ta_1) = \beta_1(a_1)$. Given that without any head start, the expected highest effort is equal to the expected effort of contestant 1, we obtain the following result about the positive effect of head starts on the expected highest effort in the contest.

Proposition 1 *If Condition 1 holds, then the expected highest effort in the two-player sequential all-pay auction with a head start* $1 < t \leq \frac{1}{F_2'(1)}$ *is higher than the expected highest effort in the sequential all-pay auction without any head start.*

Proof. See Appendix. ■

Now we examine the effect of head starts on the expected effort of contestant 2. On the one hand, the effort of every type of contestant 1 increases when a head start is given and therefore contestant 2 should also increase his effort if he wants to win the contest. But, on the other hand, by giving a head start to contestant 1, low types of contestant 2 will prefer to stay out of the contest since the minimal effort which is required from them in order to win is relatively high.

The following conditions are required for establishing the effect of a head start on the effort of contestant 2.

Condition 2 *The function* $G(x) = (1 - F_2(x))\, x$ *is concave.*[12]

Condition 3 *The highest equilibrium effort of contestant 1 (the effort of type* $a_1 = 1$*) in the contest without a head start is lower than* $x^* = \arg\max_{x \in [0,1]} G(x)$*. Formally,*

$$b_1(1) = (F_2')^{-1}(1) < x^*$$

Using conditions 1, 2 and 3 we obtain a positive effect of relatively small head starts on the expected effort of contestant 2 as well.

[11] Note that if this condition holds then the density function $F_2'(x)$ is convex. This follows by taking the derivative w.r.t. a of both sides of the equality $F_2'(h(a)) = \frac{1}{a}$. We get $-a^2 F_2''(h(a))\, h'(a) = 1$. Taking the derivative w.r.t. a of both sides of this equality and rearranging yields the following equality $h''(a) = \frac{\left(2F_2''(h(a)) + aF_2'''(h(a))h'(a)\right)h'(a)}{-aF_2''(h(a))}$ and since by our assumptions $F_2''(h(a)) < 0$ and $h'(a) > 0$ we conclude that $h''(a) > 0 \Rightarrow F_2'''(h(a)) > 0$.

[12] The *failure (or hazard) rate* of F is given by the function $\lambda(x) \equiv F'(x) / [1 - F(x)]$. F is said to have an increasing failure rate (IFR) if $\lambda(x)$ is increasing in x. The IFR condition implies Condition 2.

Proposition 2 *If Conditions 1,2 and 3 hold, then for* $t > 1$ *sufficiently close to* 1*, the expected effort of contestant 2 increases in* t.

Proof. See Appendix. ■

Note that all the three conditions 1,2 and 3 hold for a large class of distribution functions including, for example, every concave distribution function of the form $F(x) = x^{\gamma}, 0 < \gamma < 1$. The combination of Proposition 1 and Proposition 2 yields the result that the use of a head start in the sequential all pay auction is efficient for a designer who wishes to maximize the expected total effort.

Proposition 3 *If Conditions 1,2 and 3 hold, then the expected total effort in the two-player sequential all-pay auction with a head start* $t > 1$ *which is sufficiently close to 1 is higher than the expected total effort in the two-player sequential all-pay auction without any head start.*

By Proposition 3, a head start $t > 1$ that is sufficiently close to 1 increases the expected highest effort as well as the expected total effort. However, we cannot conclude that the optimal head start for a designer who wishes to maximize the expected highest or total effort is close to 1. Note that for $1 < t \leq \frac{1}{F_2'(1)}$ the effort of every type of contestant 1 is higher than in the contest without a head start. However, for $t > \frac{1}{F_2'(1)}$ the effort of low types of contestant 1 is higher than in the contest without a head start but the effort of the high types in the contest with a head start is not necessarily higher than their efforts in the contest without a head start. In this case, the head start serves as a winning bid and therefore some high types will choose the winning bid but not any bid above it as they might have done without the head start. Nevertheless, as we can see from Example 2, the optimal head starts (those that imply the highest expected total effort and the highest expected highest effort) might be obtained for a head start satisfying $t > \frac{1}{F_2'(1)}$ although such a head start does not necessarily increase the effort of all possible contestants' types.

3 The n-player model

We consider now a generalized sequential all-pay auction with $n > 2$ contestants with a head start $t \geq 1$ (the case of $t = 1$ will be referred to as a contest without a head start). In this generalized model, contestant $j, 1 \leq j \leq n$, observes the efforts of contestants $1, 2, ..., j-1$ in the previous periods and then makes an effort

x_j at period j. Contestant j wins a prize equal to 1 iff $x_j \geq tx_i$ for all $i < j$ and $tx_j > x_i$ for all $i > j$.[13] In the case without head starts ($t = 1$) contestant j wins if his effort is larger than or equal to the efforts of all the contestants in the previous periods and his effort is larger than the efforts of all the contestants in the following periods.

An effort x_i causes a cost $\frac{x_i}{a_i}$ for contestant i, where $a_i \geq 0$ is the ability (or type) of contestant i which is private information to i. As previously, contestant i's ability is drawn (independently of the other contestants' abilities) from the interval $[0, 1]$ according to a distribution function F_i which is common knowledge. We assume that F_i has a positive and continuous density $F_i' > 0, i = 1, 2, ..., n$.

Note that contestant n faces the same problem as that of contestant 2 in the two-player model. Let $a_{-n} = (a_1, ..., a_{n-1})$ then the equilibrium effort of contestant n is given by

$$\beta_n(a_n; a_{-n}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_n < \max_{j<n} t\beta_j(a_j, t; a_1, ..., a_{j-1}) \\ \max_{j<n} t\beta_j(a_j, t; a_1, ..., a_{j-1}) & \text{if} \quad \max_{j<n} t\beta_j(a_j, t; a_1, ..., a_{j-1}) \leq a_n \leq 1 \end{cases}$$

We assume that contestant i, $i = 2, ..., n-1$ uses a strictly monotonic equilibrium effort function $\beta_i(a_i, t; a_1, ..., a_{i-1})$. If contestant i observes an effort $\beta_j(a_j)$ for some $j < i$ and $t\beta_j(a_j)$ is higher than his type, he will stay out of the contest. Otherwise, player i, $i = 2, ..., n-1$ with ability a_i chooses to behave as an agent with ability s to solve the following optimization problem:

$$\max_s \left\{ H_i(t\beta_i(s)) - \frac{\beta_i(s)}{a_i} \right\} \tag{11}$$
$$s.t \quad \beta_i(s) \geq t\beta_j(a_j) \;\; \text{for all } j < i$$

where

$$H_i(x) = \Pi_{j=i+1}^n F_j(x)$$

Then, all types that find it optimal to participate (namely, $a_i \geq t\beta_j(a_j)$ for all $j < i$), but for whom the constraint in the above maximization problem is binding, will exert the effort of $\max_{j<i} t\beta_j(a_j)$. Let $\gamma_i(t) = \gamma_i(t, a_1, ..., a_{i-1}) = \max_{j<i} t\beta_j(a_j, t; a_1, ..., a_{i-1})$. If $H_i(x)$ is concave and $\gamma_i(t) \leq \frac{1}{t}$ then contestant

[13] If none of these coditions holds (i.e. there exists no player j such that $x_j \geq tx_i$ for all $i < j$ and $tx_j > x_i$ for all $i > j$) then there is no winner in the contest.

i's equilibrium effort, $i = 2, ..., n-1$ is given by

$$\beta_i\left(a_i, t; a_1, ..., a_{i-1}\right) = \begin{cases} 0 & \text{if} \quad 0 \leq a_i < \frac{\gamma_i}{H_i(t\gamma_i)} \\ \gamma_i & \text{if} \quad \frac{\gamma_i}{H_i(t\gamma_i)} \leq a_i < \overline{a}_i\left(t\right) \\ \frac{1}{t}\left(H_i'\right)^{-1}\left(\frac{1}{ta_i}\right) & \text{if} \quad \overline{a}_i\left(t\right) \leq a_i < \overleftrightarrow{a}_i\left(t\right) \\ \frac{1}{t} & \text{if} \quad \overleftrightarrow{a}_i \leq a_i \leq 1 \end{cases} \tag{12}$$

where $\overline{a}_i\left(t\right)$ is the solution to the following equation

$$\gamma_i\left(t\right) = \frac{1}{t}\left(H_i'\right)^{-1}\left(\frac{1}{t\overline{a}_i}\right)$$

and $\overleftrightarrow{a}_i\left(t\right)$ is the minimum between 1 and the solution to the following equation

$$(H_i')^{-1}(\frac{1}{\overleftrightarrow{a}_i t}) = 1$$

Note that $a_i = \frac{\gamma_i}{H_i(t\gamma_i)}$ is the lowest type who gets a non negative expected payoff if she bids γ_i. Moreover $\frac{\gamma_i}{H_i(t\gamma_i)} < \overline{a}_i\left(t\right)$ since for type $\overline{a}_i\left(t\right)$ the expected payoff if she bids γ_i is given by $H_i(t\gamma_i) - \frac{\gamma_i}{\overline{a}_i}$ and it is positive since $\gamma_i = \frac{1}{t}\left(H_i'\right)^{-1}\left(\frac{1}{t\overline{a}_i}\right)$ is the solution to this type's maximization problem. However it is not necessarily true here that $\overline{a}_i\left(t\right) < \overleftrightarrow{a}_i\left(t\right)$, and then the third range of (12) does not exist. Moreover it is also not necessarily true that $\overleftrightarrow{a}_i\left(t\right) < 1$ and then the fourth range of (12) does not exist. Finally, if $\frac{1}{t} < \gamma_i$ then contestant i's equilibrium effort, $i = 2, ..., n-1$ is given by

$$\beta_i\left(a_i, t; a_1, ..., a_{i-1}\right) = \begin{cases} 0 & \text{if} \quad 0 \leq a_i < \gamma_i \\ \gamma_i & \text{if} \quad \gamma_i \leq a_i < 1 \end{cases}$$

Contestant 1 solves the same maximization problem as in the two-player model and therefore

$$\beta_1(a_1, t) = \begin{cases} 0 & \text{if} \ \ 0 \leq a_1 \leq \overline{a}_1\left(t\right) \\ \frac{1}{t}\left(H_1'\right)^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \overline{a}_1\left(t\right) \leq a_1 \leq \overleftrightarrow{a}_1\left(t\right) \\ \frac{1}{t} & \text{if} \ \ \overleftrightarrow{a}_1\left(t\right) \leq a_1 \leq 1 \end{cases} \tag{13}$$

where $\overline{a}_1\left(t\right)$ is defined as previously as $\max\left\{0, \frac{1}{tH_1'(0)}\right\}$ and $\overleftrightarrow{a}_1\left(t\right)$ is defined as $\min\left\{1, \frac{1}{tH_1'(1)}\right\}$.

The expected effort of contestant 1 is then given by

$$TE_1\left(t\right) = \int_0^1 \beta_1(a_1, t) F_1'(a_1) da_1$$

and the expected effort of contestant i, $i=2,...,n$ is given by

$$TE_i\left(t\right)=\int_0^1\int_0^1...\int_0^1\int_{\gamma_i}^1\beta_i(a_i,t;a_1,...,a_{i-1})F_i'(a_i)da_iF_{i-1}'(a_{i-1})da_{i-1}...F_2'(a_2)da_2F_1'(a_1)da_1$$

The expected total effort is therefore

$$TE\left(t\right)=\sum_{i=1}^{n}TE_i\left(t\right) \tag{14}$$

For a given realization of the players' abilities $a_1,..,a_n$ we define

$$HE\left(t,a_1,..,a_n\right)=\max_{1\leq i\leq n}\beta_i\left(a_i,t;a_1,...,a_{i-1}\right)$$

Then, the expected highest effort is given by

$$HE\left(t\right)=\int_0^1\int_0^1...\int_0^1\int_0^1 HE\left(t,a_1,..,a_n\right)F_n'(a_n)da_nF_{n-1}'(a_{n-1})da_{n-1}...F_2'(a_2)da_2F_1'(a_1)da_1 \tag{15}$$

The following example illustrates the effects of head starts in a three-player sequential all-pay auction.

Example 3 *Consider a sequential all-pay auction with three contestants and $F_i(x)=x^{0.5}, i=1,2,3$. Assume that contestants 1 and 2 are given a head start $t\leq 4$. We have*

$$H_1\left(x\right)=x^{\frac{1}{4}}$$

By (13), the equilibrium effort of the first contestant is given by

$$\beta_1(a_1,t)=\frac{1}{t}\left(H_1'\right)^{-1}\left(\frac{1}{ta_1}\right)=\frac{t^{\frac{1}{3}}a_1^{\frac{4}{3}}}{4^{\frac{4}{3}}}\quad \textit{if } 0\leq a_1\leq 1$$

By (12), the equilibrium effort of the second contestant for $1\leq t\leq 2$ is given by

$$\beta_2\left(a_2,t;a_1\right)=\begin{cases} 0 & \textit{if} \quad 0\leq a_2<\frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}} \\ \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} & \textit{if} \quad \frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}}\leq a_2<\frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}} \\ \frac{ta_2^2}{4} & \textit{if} \quad \frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}}\leq a_2\leq 1 \end{cases}$$

For $2\leq t\leq 2^{\frac{8}{7}}=2.208\,2$ it is given by

$$\beta_2\left(a_2,t;a_1\right)=\begin{cases} 0 & \textit{if} \quad 0\leq a_2<\frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}} \\ \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} & \textit{if} \quad \frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}}\leq a_2<\frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}} \\ \frac{ta_2^2}{4} & \textit{if} \quad \frac{t^{\frac{1}{6}}a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}}\leq a_2\leq \frac{2}{t} \\ \frac{1}{t} & \textit{if} \quad \frac{2}{t}\leq a_2\leq 1 \end{cases}$$

For $2^{\frac{8}{7}} = 2.208\,2 \le t \le 4$ *we have several cases. If* $0 \le a_1 \le \frac{4}{t^{\frac{7}{4}}}$ *then*

$$\beta_2(a_2,t;a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < \frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}} \\ \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} & \text{if} \quad \frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}} \le a_2 < \frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}} \\ \frac{ta_2^2}{4} & \text{if} \quad \frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}} \le a_2 \le \frac{2}{t} \\ \frac{1}{t} & \text{if} \quad \frac{2}{t} \le a_2 \le 1 \end{cases}$$

and if $\frac{4}{t^{\frac{7}{4}}} \le a_1 \le 1$ *we have*

$$\beta_2(a_2,t;a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} \\ \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} & \text{if} \quad \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} \le a_2 \le 1 \end{cases}$$

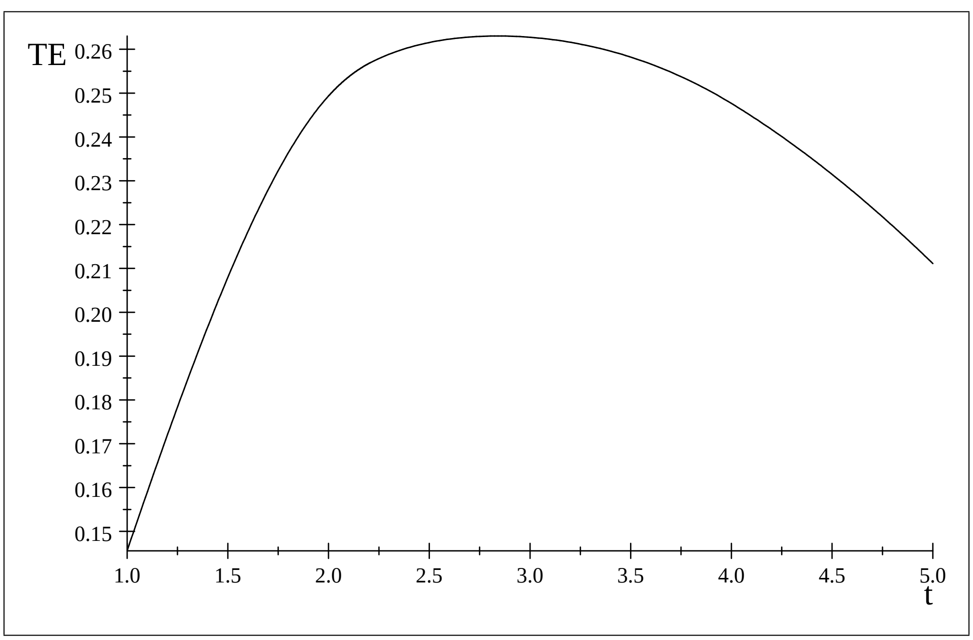
By (12) we also derive the equilibrium effort of the third contestant. We have many different conditions. For example, if $1 \le t \le 2$ *and* $0 \le a_1 \le 1$ *and* $0 \le a_2 < \frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}}$ *then*

$$\beta_3(a_3,t;a_1,a_2) = \begin{cases} 0 & \text{if} \quad 0 \le a_3 < \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} \\ \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} & \text{if} \quad \left(\frac{ta_1}{4}\right)^{\frac{4}{3}} \le a_3 \le 1 \end{cases}$$

while if $1 \le t \le 2$ *and* $0 \le a_1 \le 1$ *and* $\frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{2}{3}}} \le a_2 < \frac{t^{\frac{1}{6}} a_1^{\frac{2}{3}}}{4^{\frac{1}{6}}}$ *then*

$$\beta_3(a_3,t;a_1,a_2) = \begin{cases} 0 & \text{if} \quad 0 \le a_3 < \frac{t^{\frac{7}{3}} a_1^{\frac{4}{3}}}{4^{\frac{4}{3}}} \\ \frac{t^{\frac{7}{3}} a_1^{\frac{4}{3}}}{4^{\frac{4}{3}}} & \text{if} \quad \frac{t^{\frac{7}{3}} a_1^{\frac{4}{3}}}{4^{\frac{4}{3}}} \le a_3 \le 1 \end{cases}$$

and many other sub cases as those. The following figure presents the total effort as a function of the head start t*.*



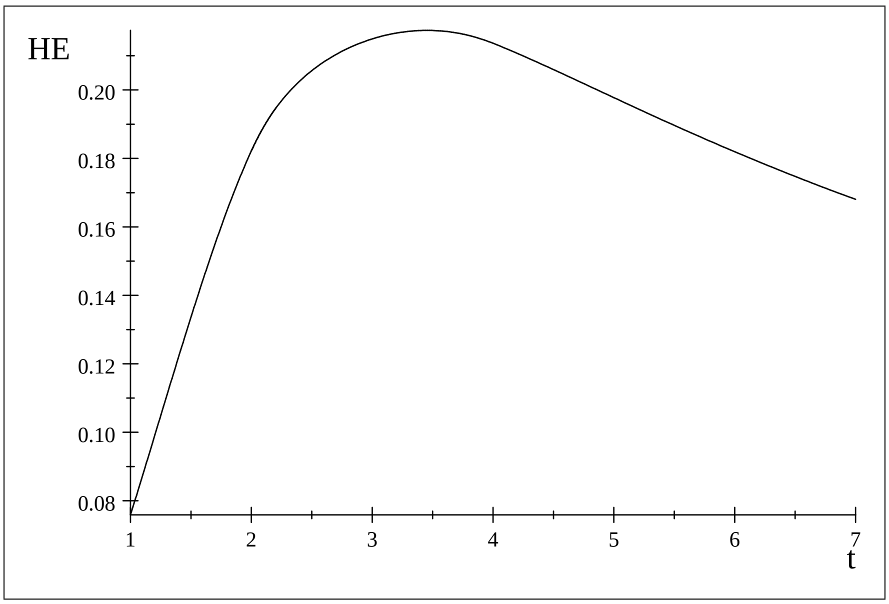
Thus, if we give a head start to contestants 1 *and* 2*, the optimal head start that maximizes the expected total effort is*

$$t_{total} = 2.8401$$

and the highest expected total effort is then

$$TE(t_{total}) = 0.263\,03$$

The following figure presents the expected highest effort as a function of the head start t*.*



Thus, if we give a head start to contestants 1 *and* 2*, the optimal head start that maximizes the expected highest effort is*

$$t_{highest} = 3.454\,4$$

and the highest expected highest effort is then

$$HE(t_{highest}) = 0.217\,40$$

The analysis of the expected total effort as well as the expected highest effort in this model with a head start $t > 1$ is quite complicated since as we can see from the equilibrium analysis a head start which is given to the contestant in period k affects the equilibrium strategies of all consecutive contestants $j \geq k$. Furthermore, in contrast to the model with two contestants, the use of a head start in the sequential all-pay auction with $n > 2$ contestants may decrease the number of active contestants (those who choose to submit a positive bid) and therefore may decrease the contestants' expected total effort and the expected highest effort. However, as we show in the following there are sufficient conditions on the distribution functions of the contestants' types according to which the use of head starts is profitable for a designer who wishes to maximize the expected highest effort as well as the expected total effort in the sequential all-pay auction with any number of contestants.

We first need a generalization of Condition 1.

Condition 4 *The equilibrium effort function of contestant* i, $i = 1, ..., n-1$ *in the sequential all-pay auction without a head start* ($t = 1$) *given by (13) and (12) is strictly convex for all* $\overline{a}_i(1) \leq a_i < \overleftrightarrow{a}_i(1)$ *i.e. the function* $h_i(a_i) = (H_i')^{-1}\left(\frac{1}{a_i}\right)$ *is strictly convex for all* $\overline{a}_i(1) \leq a_i < \overleftrightarrow{a}_i(1)$ *and all* $1 \leq i \leq n$.

Using Condition 4 we can show our main result.

Theorem 1 *If Condition 4 holds, then the expected highest effort of the contestants in the sequential all-pay auction with* n *players and a head start* $t > 1$ *sufficiently close to 1 is higher than the expected highest effort in the sequential all-pay auction without a head start.*

Proof. See Appendix. ■

According to Theorem 1, if every contestant is given a head start with respect to his next opponent then the expected highest effort in the sequential contest with head starts is higher than in the sequential contest without head starts. Moreover, by the proof of Theorem 1 this result holds even if a head start is given only for a subset of the contestants. In the following, we will assume that the head start is not necessarily given to all the contestants. In particular, we only give a head start to player $n-1$. Then similarly to the two-player model we assume the following conditions.

Condition 5 *The function* $G_n(x) = (1 - F_n(x))\,x$ *is concave.*

Condition 6 *The equilibrium highest effort of contestant* $n-1$ *(the effort of type* $a_{n-1} = 1$*) in the contest without a head start is lower than* $x_n^* = \arg\max_{x \in [0,1]} G_n(x)$. *Formally,*

$$\beta_{n-1}(1) = (F_n')^{-1}(1) < x_n^*$$

By the same arguments as in the proof of Proposition 3 we obtain

Proposition 4 *If Conditions 4,5 and 6 hold, then the expected total effort in the sequential all-pay auction with a head start to contestant $n-1$, $t>1$ which is sufficiently close to 1 is higher than the expected total effort in the sequential all-pay auction without a head start.*

By Proposition 4, if a head start is given to contestant $n-1$ only, the expected total effort increases, but obviously this is not the optimal allocation of head starts that maximizes the expected total effort. Furthermore, the optimal allocation of head starts may include different head starts for contestants according to their timing of play.

4 Concluding remarks

We presented a model of sequential all-pay auctions in which contestants arrive one by one and where each contestant observes the effort of the previous contestants before making his effort. We characterized the equilibrium behavior of the contestants and derived expressions for the expected total and highest efforts. Then we analyzed the implications of using a head start mechanism in which early contestants are favored over later ones. These head starts, on the one hand, encourage early contestants to exert higher efforts but, on the other, may cause later contestants to withdraw from the contest. We demonstrated that in our model the allocation of head starts increases the expected highest effort as well as the expected total effort. We assumed throughout that the contestants have asymmetric distribution functions for their types $F_i, i = 1, ..., n$ but are given the same head start t. It can be easily verified that all the results in this paper hold for asymmetric head starts $t_i, i = 1, ..., n$ as long as they are sufficiently close to 1.

The question of the optimal head start for the designer who wishes to maximize either the expected total effort or the expected highest effort is a challenging problem but one that cannot be addressed in the environment we deal with. The optimal head start can be explicitly calculated for only specific distribution functions of the contestants' abilities, but not in general for all of them.

It is important to note that if we assume that the contest designer incurs some cost for each contestant, either because of the discount of time or because of the cost of adding a new contestant, then the results in this paper would still hold. In the two-player model, for example, the contest designer values more the effort made in the first period and therefore he would want to increase the first mover's expected effort by increasing the head start. Thus, we can show that every head start that increases the expected total effort or the expected highest effort without any discount of time will also increase these terms when the second mover's effort is discounted by some fixed factor between zero and one.

One direction for future research could consider the option to stop the sequential auction at any point of time. The use of a such a "stopping rule" becomes especially crucial in contests in which adding a new contestant is costly for the contest designer (either because time is costly or because bringing in a new contestant involves a cost). In this context, one of the more complex research goals would be to find the optimal stopping rule that maximizes the contestants' performance in an n-player sequential all-pay auction.

A Appendix

A.1 Proof of Proposition 1

The expected highest effort in the two-player model without a head start is equal to contestant 1's expected effort, while the expected highest effort in the two-player model with a head start is larger than or equal to contestant 1's expected effort. Thus, in order to prove that a head start increases the expected highest effort it is sufficient to show that a head start increases contestant 1's expected effort. However, what we actually show is even stronger. In that for every type of contestant 1 who made a positive effort when there was no head start, this effort increases when a head start is given. Therefore we show that

$$\beta_1(a_1) \geq b_1(a_1) \text{ for all } 0 \leq a_1 \leq 1 \text{ and } 1 \leq t \leq \frac{1}{F_2'(1)}$$

Note that if Condition 1 holds then since $b_1(a_1)$ is increasing in a_1 and $\tilde{a} \geq 0$ then for all $t > 1$,

$$\beta_1(a_1) = \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_1(a_1)$$

Likewise, the lowest type of contestant 1 who is active in the two-player model with a head start is lower than the lowest active type of contestant 1 in the two-player model without any head start. Formally,

$\widehat{a} = \max\left\{\frac{1}{tF_2'(0)}, 0\right\} \leq \widetilde{a} = \max\left\{\frac{1}{F_2'(0)}, 0\right\}$ for any $t \geq 1$. Thus, we have

$$\begin{aligned}
\beta_1(a_1) &= \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_1(a_1) \quad \text{for all } \widetilde{a} \leq a_1 \leq 1 \\
\beta_1(a_1) &= \frac{1}{t}(F')^{-1}\left(\frac{1}{ta_1}\right) > b_1(a_1) = 0 \quad \text{for all } \widehat{a} \leq a_1 \leq \widetilde{a} \\
\beta_1(a_1) &= b_1(a_1) = 0 \quad \text{for all } 0 \leq a_1 \leq \widehat{a}
\end{aligned}$$

and the expected effort of contestant 1 with a head start t is higher than his expected effort without any head start. *Q.E.D.*

A.2 Proof of Proposition 2

The expected effort of contestant 2 given an effort $\beta_1(a_1, t) > 0$ of contestant 1 is

$$E_2(t, a_1) = (1 - F_2(t\beta_1(a_1, t)))\, t\beta_1(a_1, t)$$

The expected effort of contestant 2 is then

$$TE_2(t) = \int_0^1 E_2(t, a_1)F_1'(a_1)da_1 = \int_{\widehat{a}}^1 E_2(t, a_1)F_1'(a_1)da_1 > 0$$

The function $t\beta_1(a_1, t) = (F_2')^{-1}\left(\frac{1}{a_1 t}\right)$ is increasing in a_1 as well as in t. By Condition 3 we know that $(F_2')^{-1}(1) < x^*$. Therefore we obtain that, for $t > 1$ close enough to 1 and for all $a_1 \leq 1$,

$$t\beta_1(a_1, t) \leq t\beta_1(a_1 = 1, t) = (F_2')^{-1}\left(\frac{1}{t}\right) < x^*$$

Thus by Condition 2 we have

$$\frac{dE_2(t, a_1)}{dt} > 0$$

So far we showed that for all types $\widehat{a} \leq a_1 \leq 1$ for which contestant 1 exerts a positive effort the expected effort of contestant 2 increases in t as long as t is sufficiently close to 1. By Condition 1, the interval of types of contestant 1 who exert a positive effort increases in t, i.e., $\frac{d\widehat{a}}{dt} = \frac{d}{dt}\max\left\{\frac{1}{tF_2'(0)}, 0\right\} \leq 0$ and therefore, if t is sufficiently close to 1 we established that

$$\frac{d}{dt}TE_2(t) = \frac{d}{dt}\int_0^1 E_2(t, a_1)F_1'(a_1)da_1 = \frac{d}{dt}\int_{\widehat{a}}^1 E_2(t, a_1)F_1'(a_1)da_1 > 0$$

Q.E.D.

A.3 Proof of Theorem 1

By Condition 4, the function $h_i(a_i) = (H_i')^{-1}\left(\frac{1}{a_i}\right)$ is strictly convex and therefore, for $t > 1$ and $i = 1, ..., n-1$ we have

$$\frac{1}{t}(H_i')^{-1}\left(\frac{1}{ta_i}\right) > (H_i')^{-1}\left(\frac{1}{a_i}\right) \tag{16}$$

Denote the equilibrium effort of contestant i with a type a_i in the contest without any head start by $\beta_i(a_i, t=1; a_1, ..., a_{i-1}) = b_i(a_i; a_1, ..., a_{i-1})$ for $i = 2, ..., n$ and by $\beta_1(a_1, t=1) = b_1(a_1)$, then we have

Lemma 1 *For $t > 1$ close enough to 1 if the equilibrium effort of contestant i with a type a_i is positive, then this equilibrium effort is higher than or equal to his equilibrium effort in the contest without a head start ($t = 1$). Formally, for $i = 1$ if $a_1 \geq \overline{a}_1(t)$ then*

$$\beta_1(a_1, t) \geq b_1(a_1)$$

and for $i = 2, ..., n$ if $a_i \geq \frac{\gamma_i}{H_i(t\gamma_i)}$ then

$$\beta_i(a_i; t; a_1, ..., a_{i-1}) \geq b_i(a_i; a_1, ..., a_{i-1})$$

Proof: By (13) and (12) if $t = 1$ contestant i's equilibrium efforts $i = 2, ..., n-1$ are given by

$$b_i(a_i; a_1, ..., a_{i-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_i < \frac{\gamma_i(1)}{H_i(\gamma_i(1))} \\ \gamma_i(1) & \text{if} \quad \frac{\gamma_i(1)}{H_i(\gamma_i(1))} \leq a_i < \overline{a}_i(1) \\ (H_i')^{-1}\left(\frac{1}{a_i}\right) & \text{if} \quad \overline{a}_i(1) \leq a_i \leq 1 \end{cases} \tag{17}$$

Note that $\gamma_i(1) = \max_{j<i} b_j(a_j; a_1, ..., a_{i-1})$. And

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 \leq \overline{a}_1(1) \\ (H_1')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad \overline{a}_1(1) \leq a_1 \leq 1 \end{cases}$$

While for t close enough to $1, \gamma_i(t) \leq \frac{1}{t}$ and

$$\beta_i(a_i; t; a_1, ..., a_{i-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq a_i < \frac{\gamma_i(t)}{H_i(t\gamma_i(t))} \\ \gamma_i(t) & \text{if} \quad \frac{\gamma_i(t)}{H_i(t\gamma_i(t))} \leq a_i < \overline{a}_i(t) \\ \frac{1}{t}(H_i')^{-1}\left(\frac{1}{ta_i}\right) & \text{if} \quad \overline{a}_i(t) \leq a_i < 1 \end{cases}$$

and

$$\beta_1(a_1;t) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 \le \overline{a}_1(t) \\ \frac{1}{t}(H_1')^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \overline{a}_1(t) \le a_1 \le 1 \end{cases}$$

For $i=1$ since $\overline{a}_1(t) \le \overline{a}_1(1)$ the result follows from (16). We prove the rest of the lemma by induction on i. For $i=2$, if $b_1(a_1)=0$ then $\gamma_2(1)=0$ and

$$b_2(a_2;a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < \overline{a}_2(1) \\ (H_2')^{-1}\left(\frac{1}{a_2}\right) & \text{if} \quad \overline{a}_2(1) \le a_2 \le 1 \end{cases}$$

where $\overline{a}_2(1) = \max\left\{0, \frac{1}{H_2'(0)}\right\}$ and if $b_1(a_1) = (H_1')^{-1}\left(\frac{1}{a_1}\right)$ then $\gamma_2(1) = (H_1')^{-1}\left(\frac{1}{a_1}\right)$ and

$$b_2(a_2;a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < \frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left((H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \\ (H_1')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad \frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left((H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \le a_2 < \overline{a}_2(1) \\ (H_2')^{-1}\left(\frac{1}{a_2}\right) & \text{if} \quad \overline{a}_2(1) \le a_2 \le 1 \end{cases}$$

where $\overline{a}_2(1) = \frac{1}{H_2'\left((H_1')^{-1}\left(\frac{1}{a_1}\right)\right)}$. For $t>1$ if $\beta_1(a_1;t)=0$ then $\gamma_2(t)=0$ and

$$\beta_2(a_2,t;a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < \overline{a}_2(t) \\ \frac{1}{t}(H_2')^{-1}\left(\frac{1}{ta_2}\right) & \text{if} \quad \overline{a}_2(t) \le a_2 < 1 \end{cases}$$

where $\overline{a}_2(t) = \max\left\{0, \frac{1}{tH_2'(0)}\right\}$ and if $\beta_1(a_1;t) = \frac{1}{t}(H_1')^{-1}\left(\frac{1}{ta_1}\right)$ then $\gamma_2(t) = (H_1')^{-1}\left(\frac{1}{ta_1}\right)$

$$\beta_2(a_2,t;a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < \frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \\ (H_1')^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \le a_2 < \overline{a}_2(t) \\ \frac{1}{t}(H_2')^{-1}\left(\frac{1}{ta_2}\right) & \text{if} \quad \overline{a}_2(t) \le a_2 < 1 \end{cases}$$

where $\overline{a}_2(t) = \frac{1}{tH_2'\left(t(H_1')^{-1}\left(\frac{1}{ta_1}\right)\right)}$.

We need to show that if $a_2 \ge \frac{\gamma_2(t)}{H_2(t\gamma_2(t))}$ then

$$\beta_2(a_2,t;a_1) \ge b_2(a_2,t;a_1)$$

If $\gamma_2(t)=0$ then since $\overline{a}_1(t) \le \overline{a}_1(1)$ if $\beta_1(a_1;t)=0$ then $b_1(a_1)=0$ and then the lemma follows as before from the fact that $\max\left\{0, \frac{1}{tH_2'(0)}\right\} \le \max\left\{0, \frac{1}{H_2'(0)}\right\}$ and (16). Otherwise $\gamma_2(t)>0$.

Note that indeed it is always the case that

$$(H_1')^{-1}\left(\frac{1}{ta_1}\right) > (H_1')^{-1}\left(\frac{1}{a_1}\right)$$

since the bid function is increasing in a. However we might have either of the following inequalities

$$\bar{a}_2(1) < \bar{a}_2(t) \text{ or } \bar{a}_2(1) > \bar{a}_2(t)$$

We need to show that for $a_2 \geq \frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)}$ we have $\beta_2(a_2, t; a_1) > b_2(a_2; a_1)$. We consider two cases. Case 1) If $\frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \leq a_2 < \bar{a}_2(t)$ then either $\frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \leq a_2 < \bar{a}_2(1)$ or $\bar{a}_2(1) \leq a_2 \leq 1$. If $\frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \leq a_2 < \bar{a}_2(1)$ then the lemma follows from $\beta_2(a_2, t; a_1) = (H_1')^{-1}\left(\frac{1}{ta_1}\right) > (H_1')^{-1}\left(\frac{1}{a_1}\right) = b_2(a_2; a_1)$. If $\bar{a}_2(1) \leq a_2 \leq 1$ then the lemma follows since in this range the constraint for player 2 is binding and therefore $\beta_2(a_2, t; a_1) = (H_1')^{-1}\left(\frac{1}{ta_1}\right) > \frac{1}{t}(H_2')^{-1}\left(\frac{1}{ta_2}\right) > (H_2')^{-1}\left(\frac{1}{a_2}\right) = b_2(a_2; a_1)$ where the last inequality follows from (16).

Case 2) If $\bar{a}_2(t) \leq a_2 \leq 1$ then either $\bar{a}_2(1) \leq a_2 \leq 1$ or $\frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \leq a_2 < \bar{a}_2(1)$. If $\bar{a}_2(1) \leq a_2 \leq 1$ then the lemma follows from (16). If $\frac{(H_1')^{-1}\left(\frac{1}{a_1}\right)}{H_2\left(t(H_1')^{-1}\left(\frac{1}{a_1}\right)\right)} \leq a_2 < \bar{a}_2(1)$ then in this range $\beta_2(a_2, t; a_1) = \frac{1}{t}(H_2')^{-1}\left(\frac{1}{ta_2}\right) \geq (H_1')^{-1}\left(\frac{1}{ta_1}\right) > (H_1')^{-1}\left(\frac{1}{a_1}\right) = b_2(a_2; a_1)$.

Assume by induction that the lemma is true for all $i = 2, ..., l-1$. We need to show that for all $a_l \geq \frac{\gamma_l(t)}{H_l(t\gamma_l(t))}$ we have $\beta_l(a_l; t; a_1, ..., a_{l-1}) > b_l(a_l; a_1, ..., a_{l-1})$. By the induction assumption, we know that $t\beta_i(a_i; t; a_1, ..., a_{i-1}) > b_i(a_i; a_1, ..., a_{i-1})$ for $i = 1, .., l-1$ and a_i such that $\beta_i(a_i; t; a_1, ..., a_{i-1}) > 0$, therefore $\gamma_l(t) > \gamma_l(1)$. The inequality might be an equality only in the case where $\beta_i(a_i; t; a_1, ..., a_{i-1}) = 0$ for all $i < l$ but in this case the proof is similar to the proof of the second player. Thus we assume $\gamma_l(t) > \gamma_l(1)$.

Similarly to the case of $l = 2$ we have two cases. Case 1) If $\frac{\gamma_l(1)}{H_l(\gamma_l(1))} \leq a_l < \bar{a}_l(t)$ then either $\frac{\gamma_l(1)}{H_l(\gamma_l(1))} \leq a_l < \bar{a}_l(1)$ or $\bar{a}_l(1) \leq a_l \leq 1$. If $\frac{\gamma_l(1)}{H_l(\gamma_l(1))} \leq a_l < \bar{a}_l(1)$ then the lemma follows from $\beta_l(a_l; t; a_1, ..., a_{l-1}) = \gamma_l(t) > \gamma_l(1) = b_l(a_l; a_1, ..., a_{l-1})$. If $\bar{a}_l(1) \leq a_l \leq 1$ then the lemma follows since in this range the constraint for player l is binding and therefore $\beta_l(a_l; t; a_1, ..., a_{l-1}) = \gamma_l(t) > \frac{1}{t}(H_l')^{-1}\left(\frac{1}{ta_;}\right) > (H_l')^{-1}\left(\frac{1}{a_l}\right) = b_l(a_l; a_1, ..., a_{l-1})$ where the last inequality follows from (16). Case 2) If $\bar{a}_l(t) \leq a_l \leq 1$ then either $\bar{a}_l(1) \leq a_l \leq 1$ or $\frac{\gamma_l(1)}{H_l(\gamma_l(1))} \leq a_l < \bar{a}_l(1)$. If $\bar{a}_l(1) \leq a_l \leq 1$ then the lemma follows from (16). If $\frac{\gamma_l(1)}{H_l(\gamma_l(1))} \leq a_l < \bar{a}_l(1)$ then in this range $\beta_l(a_l; t; a_1, ..., a_{l-1}) = \frac{1}{t}(H_l')^{-1}\left(\frac{1}{ta_;}\right) \geq \gamma_l(t) > \gamma_l(1) = b_l(a_l; a_1, ..., a_{l-1})$.■

We use Lemma 1 to prove the theorem. For a given realization of the contestants' abilities: $a_1, ..., a_n$ we denote by $HE(a_1, ..., a_n)$ the highest effort when $t = 1$. Notice that this effort can be made by more than one contestant. Therefore we denote by $j_0 = j_0(a_1,...,a_n)$ the first (i.e. the lowest indexed) contestant that makes this highest effort. Formally, if $l \in \arg\max_{1\leq i\leq n} b_i(a_i;a_1,...,a_{i-1})$ then $l \geq j_0$.

It is sufficient to prove that when a head start t sufficiently close to 1 is given to the contestants, then for any given realization the highest bid increases i.e.,

$$HE(t;a_1,...,a_n) = \max_{1\leq i\leq n} \beta_i(a_i;t;a_1,...,a_{i-1}) > b_{j_0}(a_i;a_1,...,a_{j_0-1}) = HE(a_1,...,a_n)$$

Since $HE(t;a_1,...,a_n) \geq \beta_{j_0}(a_{j_0};t;a_1,...,a_{j_0-1})$ it is enough to show that

$$\beta_{j_0}(a_{j_0};t;a_1,...,a_{j_0-1}) > b_{j_0}(a_{j_0};a_1,...,a_{j_0-1})$$

This last inequality was proved in Lemma 1 but only if $\beta_{j_0}(a_{j_0};t;a_1,...,a_{j_0-1}) > 0$. Thus, it remains to show that $\beta_{j_0}(a_{j_0};t;a_1,...,a_{j_0-1}) > 0$ or equivalently that $a_{j_0} \geq \frac{\gamma_{j_0}(t)}{H_{j_0}(t\gamma_{j_0}(t))}$. First note that we must have

$$b_{j_0}(a_{j_0};a_1,...,a_{j_0-1}) = \left(H'_{j_0}\right)^{-1}\left(\frac{1}{a_{j_0}}\right) \text{ and } a_{j_0} > \overline{a}_{j_0}(1)$$

Otherwise, either $b_{j_0}(a_{j_0};a_1,...,a_{j_0-1}) = 0$ (but then obviously this cannot be the highest bid), or $b_{j_0}(a_{j_0};a_1,...,a_{j_0-1}) = \gamma_{j_0}(1)$ which contradicts the definition of j_0 as the lowest indexed contestant who submits the highest effort. By (17)

$$\frac{\gamma_{j_0}(1)}{H_{j_0}\left(\gamma_{j_0}(1)\right)} \leq \overline{a}_{j_0}(1) < a_{j_0} \tag{18}$$

and from Lemma 1

$$\frac{\gamma_{j_0}(1)}{H_{j_0}\left(\gamma_{j_0}(1)\right)} < \frac{\gamma_{j_0}(t)}{H_{j_0}\left(t\gamma_{j_0}(t)\right)} \tag{19}$$

Moreover, since

$$\lim_{t\to 1} \frac{\gamma_j(t)}{H_j\left(t\gamma_j(t)\right)} = \frac{\gamma_j(1)}{H_j\left(\gamma_j(1)\right)}$$

then by (18) and (19) we can find $t > 1$ close enough to 1 such that

$$\frac{\gamma_{j_0}(1)}{H_{j_0}\left(\gamma_{j_0}(1)\right)} < \frac{\gamma_{j_0}(t)}{H_{j_0}\left(t\gamma_{j_0}(t)\right)} \leq \overline{a}_{j_0}(1) < a_{j_0}$$

Note that it might be the case that for all $i < j_0$ we have $b_i(a_i;a_1,...,a_{i-1}) > 0$ while $\beta_i(a_i;t;a_1,...,a_{i-1}) = 0$ but in this case $\gamma_{j_0}(t) = 0$ and obviously $a_{j_0} \geq \frac{\gamma_{j_0}(t)}{H_{j_0}(t\gamma_{j_0}(t))}$. *Q.E.D.*

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