Differentiation Games

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Abstract

We consider a class of games in which players with private information are motivated to differ in their actions. Two related questions are studied: (1) the existence of a "collision-free" equilibrium, in which no two players choose the same action; (2) the maximal social welfare. We give exact answers for some specific information structures, and a lower bound for the general case.

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1 Introduction

In many situations players might be better off by taking different actions than the other players. For instance, if two store owners sell the same kind of products, then while each wishes to choose a good location for setting her business, both of them also prefer to have their stores in two different neighborhoods. Similarly, scientists wish to work on "good" scientific questions, and may also prefer not to work on the same questions as others (see, e.g., Chen et al. (2015).

Our paper analyzes situations characterized by such a motivation of the players to differ in their actions, and yet make good choices. While the equilibria of such games may be relatively simple when players are fully informed, the analysis becomes more involved when players have asymmetric information, which is the case we consider. We give sufficient conditions for the existence of a "collision-free" equilibrium (i.e., where all players choose different actions), and we further study the highest social welfare that can be achieved in equilibrium.[1]

We describe the incomplete information games using the knowledge partitions model of Aumann (1976). In our model, there is a given set Ω of choices available to the players (let us call these choices "locations"). One of these

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[1] In fact, the restriction to equilibria turns out to be inconsequential in this sense, see below.

locations contains a prize. If more than one player chooses the location that contains the prize, the prize is equally shared among the players who "found" it. Ex-ante, the common prior of the players is that the prize may be in any of the locations with equal probability. Each player is associated with a partition of Ω, and her private information consists of the element of her partition that contains the actual location of the prize ("the real state of the world"). Thus, each player knows that the prize is located in a subset of Ω, but that subset may differ from one player to another.

A strategy of a player consists of choosing, for each subset in her partition, a location within that subset. Since the prior probability is uniform over Ω, her posterior (i.e., after being informed of the subset) is uniform over that subset. Thus, if there were no other players, all strategies would have been equal.

A player can only be harmed by the presence of other players, since she may have to share the prize with them. Now, if a profile of strategies ensures that such collisions never occur, then this profile gives each player her maximal expected payoff, and therefore: (a) it is an equilibrium; (b) it yields the optimal (expected) social payoff.

In Section 3.1 we show that in any differentiation game, even in those games where collisions cannot be avoided, the optimal social payoff is always supported by some (pure) equilibrium.[2] This serves as motivation to study not only collision-freeness, but the optimal social payoff in general.

The finer the partitions of the players, the higher the social payoff. Of course, fine partitions also increase the chances of collisions. In Section 3.2 we give a few sufficient conditions on the size of the elements of the partitions that either guarantee that the game is collision-free or guarantee that the prize will be found. A special case is where the size of all elements of each of the n players is n: in this case the game is collision-free, and the prize is always found.

These results are in fact special cases of our main theorem (Theorem 3.6), which gives a lower bound for the optimal social payoff. First, note that had there been a geometry underlying the partitions of the players, we could have said, as a rule of thumb, that within "areas" in which information is abundant (i.e., partitions are fine) the chances of finding the prize are high, and within areas in which information is scarce the chances are low (and also collisions are easier to avoid). But such a geometry need not exist, since we consider arbitrary partitions. However, we define a "local" property of states (the *index* of a state), by which we can identify some parts of the world within which the prize is going to be found, and parts within which collisions can be avoided.

In Section 3.4 we consider the special case of independent signals. Then any non-degenerate game is collision-free.

Differentiation games bare only a remote similarity to Congestion Games (Rosenthal, 1973), which are complete information games in which each player

[2] As a simple example, suppose there are two players and let $\Omega = \{a, b, c\}$. Player 1's partition is $\{\{a\}, \{b, c\}\}$ and her strategy chooses a from the first element and b from the second. Player 2's partition is $\{\{a, b\}, \{c\}\}$ and she chooses b from the first element and c from the second. Although the players will have to share the prize if it lies in b, this is an equilibrium, and it is of course socially optimal, since it guarantees that the prize is found.

chooses a set of items to use, and players may prefer less loaded items. The scenario of privately informed scientists who choose projects is studied by Chen et al. (2015). They consider mechanism designs that allow the players to cooperate, i.e., share information. Kleinberg and Oren (2011) study mechanisms for allocating the credit for research by scientists.

Some popular puzzles about information and hats are in fact differentiation games, see Section 4.

In Section 2 we define the model, and in Section 3 we present the results and a few examples. Section 4 contains some further remarks, and Section 5 contains the proofs.

2 Model

Let N be a set of players, $|N| = n$, and let Ω be a finite set of locations. A "treasure" worth \$1 is hidden in one of the locations. The private information of the players is represented by partitions (see, e.g., Aumann (1976)): for each player i there is a partition $\mathcal{K}_i$ of Ω, namely a list of disjoint subsets of Ω whose union is the whole space Ω. When the true state of the world (i.e., the location of the treasure) is $\omega \in \Omega$, player i knows[3] $P_i(\omega)$ (i's "ken"), which is the element of her partition that includes ω.[4] The common prior belief of all players about the state of the world is a uniform distribution over Ω.

Each player makes a guess about the location of the treasure, and thus her set of actions is Ω. Therefore, a pure strategy for player i is a function $f_i : \mathcal{K}_i \to \Omega$ from i's kens to states, and w.l.o.g. we assume that for every $k \in \mathcal{K}_i$, $f_i(k) \in k$ (player i knows that the true state is within k, and would be foolish to choose anything outside k). Similarly, a mixed strategy for player i is a function $f_i : \mathcal{K}_i \to \Delta(\Omega)$ from i's kens to probability distributions over states, where for every $k \in \mathcal{K}_i$, $f_i(k) \in \Delta(k)$. The payoff for a player who found the treasure is $1/m$, where m is the number of players who found the treasure, and zero if she did not find it.

A profile of strategies $f = (f_1, \ldots, f_n)$ is an *equilibrium* if for any player i and any ken $k \in \mathcal{K}_i$, i's expected payoff does not strictly increase by altering $f_i(k)$, given that the other players play according to f^{-i}.

For a profile of pure strategies f, we say that a state ω is *covered* if there exists a player i and a ken $k \in \mathcal{K}_i$ such that $f_i(k) = \omega$. Note that it must then be the case that $k = P_i(\omega)$. Therefore, when the state of the world is indeed ω, player i will choose ω (and possibly other players may choose it as well). The expected *social payoff* in this game, i.e., the expected sum of the players' payoffs, equals the probability that the treasure will be found. By the above, the social payoff equals the proportion of the covered states out of the whole space Ω (and of course the social payoff can never be more than 1).

[3]I.e., i knows that the true state is one the states in $P_i(\omega)$, and that is all she knows.

[4]As is usual in these models, the partitions themselves are common knowledge among all players.

Since a pure strategy is a function from kens to states, the number of covered states is bounded by $\sum_{i\in N}|\mathcal{K}_i|$. The actual number of covered states may be much lower, depending on the information structure of all players. The above bound can be achieved only if "collisions" can be avoided.

Definition 2.1. A profile of pure strategies f is *collision-free* if for any two players i, j and any $k_i \in \mathcal{K}_i, k_j \in \mathcal{K}_j$, $f_i(k_i) \neq f_j(k_j)$. A game is *collision-free* if it admits a collision-free profile of strategies.

In general, a player may only lose from the presence of others, because of collisions. Under a collision-free profile f each player has her best possible expected payoff, as player i then covers $|\mathcal{K}_i|$ states, and expects $|\mathcal{K}_i| / |\Omega|$. Therefore, the optimal social payoff is also achieved under f, and f is also an equilibrium. We denote by x_i the number of kens of i. More generally, for a subset $E \subset \Omega$, $x_i(E)$ denotes the number of kens of i that are included in E. Hence, $x_i = x_i(\Omega)$.

The following "local" definition will turn out to have implications for the overall number of states that can be covered.

Definition 2.2. The *index* of ω, denoted $i(\omega)$, is defined by $i(\omega) = \sum_{i\in N} 1/|P_i(\omega)|$.

In words, $i(\omega)$ is the sum of the probabilities that players relate, after learning their kens, to the state of the world being ω, when the real state is indeed ω.

3 Results

3.1 Preliminary

The following proposition connects between equilibria and the optimal social payoff.

Proposition 3.1. *There exists a pure equilibrium in which the sum of expected payoffs of all players equals the socially optimal payoff. We call such an equilibrium a "socially optimal equilibrium".*

Thus, whenever we state that some level of social payoff can be achieved, it implies that there exists a (pure) equilibrium that achieves it.

Not all equilibria are socially optimal. For example, consider the following differentiation game. There are 2 players, Ω is a 2×2 matrix. Let (i, j) denote the location in row i and column j. Player I's partition consists of the rows (i.e., I's partition is $\{\{(1,1),(1,2)\},\{(2,1),(2,2)\}\}$) and player II's partition consists of the columns.

First, here is an equilibrium, depicted in Figure 1, that is socially optimal, since it yields a social payoff of 1. $f_1(\text{row } 1) = (1,1)$, $f_1(\text{row } 2) = (2,2)$, and $f_2(\text{column } 1) = (2,1)$, $f_2(\text{column } 2) = (1,2)$.

Now, the following pair of strategies is another equilibrium, that is not socially optimal. $f_1(\text{row } 1) = (1,1)$, $f_1(\text{row } 2) = (2,1)$, and $f_2(\text{column } 1) = (1,1)$, $f_2(\text{column } 2) = (1,2)$.

	1	2
1	I	II
2	II	I

Figure 1: Socially optimal equilibrium

3.2 Optimal equilibria

Note that the optimal social payoff never exceeds one. It equals one if and only if the whole space can be covered. The following results identify some games in which the optimal social payoff is found out, either by collision-freeness or by covering the whole space.

The first result asserts that if there are enough players with enough knowledge then the socially optimal payoff is one:

Proposition 3.2. *If there are at least k players whose kens are at most of size k, then the socially optimal payoff equals* 1*.*

If the kens of all players are at least of size n there is no general simple algorithm to find a profile of strategies that is collision-free. However, as the following result asserts, such a profile of strategies does exist (and hence is also a socially optimal equilibrium, as mentioned above).

Proposition 3.3. *If all the kens of all players are at least of size n, then the game is collision-free.*

Another proposition is as follows:

Proposition 3.4. *If each player has kens of uniform size, then the socially optimal payoff equals* $\min\left\{1, \sum_{i \in N} 1/q_i\right\}$*, where q_i is the size of i's kens.*

An interesting case is when all kens of all players are of size n.

Corollary 3.5. *If all the kens of all players are exactly of size n, then the game is collision-free and the socially optimal payoff equals 1.*

3.3 Main Theorem

In the previous section we considered games in which the information structures took some "restrained" forms. The following theorem generalizes the previous ideas to general games. It uses the "local" index property (Definition 2.2) to get a lower bound on the overall number of states that can be covered.

Theorem 3.6. *The socially optimal payoff is at least* $\left(|\Omega_H| + \sum_{i \in N} x_i(\Omega_L)\right) / |\Omega|$*, where* $\Omega_L = \{\omega \in \Omega | i(\omega) \leq 1\}$ *and* $\Omega_H = \Omega \setminus \Omega_L$*.*

A dual of this theorem, with $<$ in the definition of Ω_L, also holds, with the same proof.

3.4 Independent signals

Here we consider the special case of differentiation games in which the signals (i.e., the kens) of all the players are independent (as random variables). We will see that in this case the game is collision-free, unless it has a very small state space.

Since the real state of the world ω is a random variable, so are the kens of the players $P_1(\omega), \ldots, P_n(\omega)$. Let us now suppose that they are independent. Since the common prior is uniform, this is equivalent to assuming that for any tuple $k_1, \ldots, k_n$ where each $k_j \in \mathcal{K}_j$ is some possible ken of player j, and for any player i,

$$\left|\cap_{j \neq i} k_j\right| \cdot \frac{|k_i|}{|\Omega|} = \left|\cap_{i=1}^{n} k_i\right| . \tag{1}$$

An example of independent signals is when the state of the world is a vector of n coordinates, and each player i knows the i-th coordinate. But a game with independent signals need not have this form.

Proposition 3.7. *If the signals are independent then exactly* $\min\left\{|\Omega|, \sum_{i=1}^{n} x_i\right\}$ *states can be covered.*

Independent signals have the property that any tuple of kens $k_1, \ldots, k_n$ ($k_j \in \mathcal{K}_j$) has a nonempty intersection, as this follows from (1). The above proposition holds, in fact, not just for the case of independent signals, but for any game with this property.

Lemma 3.8. *If any tuple of kens $k_1, \ldots, k_n$ ($k_j \in \mathcal{K}_j$) has a nonempty intersection, then exactly* $\min\left\{|\Omega|, \sum_{i=1}^{n} x_i\right\}$ *states can be covered.*

4 Concluding Remarks

We analyzed a Bayesian game in which players are motivated to differentiate. Our analysis focused on two questions with respect to equilibria in the game: first, what is the maximal social welfare that players can get in equilibrium. Second, whether or not there is a collision free equilibrium.

Following Aumann (1976) we assumed that "kens" of players are elements of partitions where the partitions themselves are common knowledge. A natural extension is to assume other structures of knowledge. The most general structure of knowledge that one can think of is the case were signals of players are any subsets of Ω rather than elements of common-known partitions of Ω. However, this assumption is too general as it is easy to see that in this case, even with two players and $|\Omega| = 3$ there is no collision-free equilibrium. However, a collision-free equilibrium does exist if we assumed that players do not know much about the world, i.e., the size of their kens is greater than a lower bound L. In this case we have the following result.

Proposition 4.1. *A collision-free equilibrium exists if and only if*

$$L \geq \frac{n-1}{n}|\Omega| + 1, \tag{2}$$

where n is the number of players. The proof of the proposition is relegated to the appendix.

Interestingly, some popular puzzles can be viewed as treasure finding games. One of those puzzles can be described as follows. n agents are wearing hats which can be of n different colors (each hat has one color; there can be multiple hats with the same color). Each agent can see the colors of all hats except his own. The agents must simultaneously call out a color; they win if at least one agent calls the color of his own hat. They can agree on a strategy beforehand; the set of possible colors is known. The question is whether there is a winning strategy. In our terms, "locations" are n-dimension vectors of colors, where Ω contains all such locations. Each player has a partition with n^{n-1} elements, and the size of each element is n (the number of possible colors of his hat). The treasure is located in the true vector of colors. Obviously, if there is a collision-free equilibrium then the answer to the puzzle is affirmative as they agree on the collision-free equilibrium strategies. Indeed, it follows from corollary 3.5 that there is a collision-free equilibrium. It is easy to see that the information of players is independent in the sense of Gossner, Kalai and Weber (2009). For such information structure a collision-free profile of strategies can be described explicitly.[5]

5 Proofs

It will be useful to recall Hall's Marriage theorem.

Halls marriage theorem. Assume that each woman in W has a set of men which is a subset of M that she likes. A matching between women and men that they like exists if and only if, for every subset $X \subseteq W$ of women, the set of men $Y \subseteq M$ liked by women in X is at least as big as X.

Proof of Proposition 3.1. The idea: we should choose among the socially optimal settings one that is also an equilibrium, i.e., no player can move from a crowded state to a less crowded one. This is a "minimal" object, and it exists because after a few iterations we will reduce all the inequality that can be reduced. □

Proof of Theorem 3.6. The proof relies on the following two lemmata. The first lemma states that areas with high index can be fully covered.

[5]For instance, in the puzzle, players can agree on the following strategy. The set of players and the set of colors are indexed (separately) arbitrarily by numbers between zero to $n-1$, a different index for each element. The "guess" of a player number i is a color that makes the sum of the colors of all players' hats, including the color of i's hat, equal i mod n. It is easy to see that this strategy solves the puzzle. Many puzzles of this kind may be found on web and in severl books. One of these books that we find especially interesting is Winkler (2004).

Lemma 5.1. *Let $E \subset \Omega$. If for any $\omega \in E$, $i(\omega) \geq 1$, then E can be covered.*

(Of course, this means that E is covered by the kens that intersect E, as we assumed that you never choose anything unless it is in your ken, hence other kens will not go into E).

If Ω is covered then the treasure will be found for sure and therefore the optimal social payoff equals 1.

The second lemma states, roughly, that within an area with low index collisions can be avoided.

Lemma 5.2. *Let $E \subset \Omega$. If for any $\omega \in E$, $i(\omega) \leq 1$, then there exists a profile of strategies under which the kens contained in E do not collide.*

To prove the theorem, let $\mathcal{K}$ be the set of all the kens of all the players, and let $\mathcal{F} \subset \mathcal{K}$ be the kens that are contained in Ω_L. $\cup\mathcal{F} \subset \Omega_L$ denotes the union over all the members of $\mathcal{F}$. By Lemma 5.2 we can cover $|\cup\mathcal{F}| = \sum_{i \in N} x_i(\Omega_L)$ points in Ω_L by the kens in $\mathcal{F}$. The remaining kens are exactly those that intersect Ω_H, and by Lemma 5.1 these kens can cover all the points in Ω_H. $\square$

Proof of Lemma 5.1. We will show that for any subset $F \subset E$, the number of kens that intersect F is $\geq |F|$. By Hall's marriage theorem it then follows that for any $\omega \in E$, we can choose a ken that intersects it and no ken is used twice, as required.

Let $\mathcal{K}_i$ denote the kens of player i, and $\mathcal{K} = \cup_{i \in N} \mathcal{K}_i$ be all the kens. For F as above,

$$|F| = \sum_{\omega \in F} 1 \leq \sum_{\omega \in F} i(\omega) = \sum_{i \in N} \sum_{\omega \in F} 1/\left|P_i(\omega)\right| = \sum_{i \in N} \sum_{K \in \mathcal{K}_i} |K \cap F| / |K|,$$

because the contribution by $K \in \mathcal{K}_i$ to the sum $\sum_{\omega \in F} 1/\left|P_i(\omega)\right|$ is $1/|K|$ for those ω where $K = P_i(\omega)$, and 0 for other ω; thus, it contributes exactly $|K \cap F|$ times. Now, this is

$$\sum_{i \in N} \sum_{K \in \mathcal{K}_i} \mathbb{1}\{K \cap F \neq \emptyset\} = \sum_{K \in \mathcal{K}} \mathbb{1}\{K \cap F \neq \emptyset\} = \left|\{K \in \mathcal{K} : K \cap F \neq \emptyset\}\right|,$$

where $\mathbb{1}$ denotes the indicator function, i.e., $\mathbb{1}V = 1$ where the event V occurs, and 0 elsewhere. $\square$

Proof of Lemma 5.2. Let $\mathcal{F}$ denote the set of all kens that are contained in E. We want to show that there exists a selection of distinct representatives of $\mathcal{F}$, i.e., there exists a function $g : \mathcal{F} \to \cup\mathcal{F}$ that is injective and s.t. for any $f \in \mathcal{F}$, $g(f) \in f$.

Let $\mathcal{Q} \subset \mathcal{F}$ be any subset of $\mathcal{F}$.

$$|\mathcal{Q}| = \sum_{q \in \mathcal{Q}} 1 = \sum_{q \in \mathcal{Q}} \sum_{\omega \in q} 1/|q|$$

By changing the order of summation

$$= \sum_{\omega \in \cup \mathcal{Q}} \sum_{q \in \mathcal{Q}: w \in q} 1/|q|$$

For any $\omega \in \Omega$, the set of all kens (in $\mathcal{K}$) that include ω is $\{P_i(\omega)\}$. Since $\mathcal{Q} \subset \mathcal{K}$, $\{q \in \mathcal{Q} : w \in q\} \subset \{P_i(\omega)\}$. Therefore, $\sum_{q \in \mathcal{Q}: w \in Q} 1/|q| \leq \sum_{i \in N} 1/P_i(\omega) = i(\omega)$. Thus,

$$\leq \sum_{\omega \in \cup \mathcal{Q}} i(\omega) \leq \sum_{\omega \in \cup \mathcal{Q}} 1 = |\cup \mathcal{Q}|.$$

Thus, we got that for any $\mathcal{Q} \subset \mathcal{F}$, $|\mathcal{Q}| \leq |\cup \mathcal{Q}|$. By Hall's theorem, this ensures the existence of a function g as above. □

Proof of Propositions 3.2, 3.3, and 3.4. (i) The corollary follows from Theorem 3.6: by assumption for all ω, $i(\omega) \geq 1$ and therefore Ω can be covered, implying that the optimal social payoff equals 1.

(ii) To see that, note that if all the kens of all players are at least of size n then in terms of Theorem 3.6 $\Omega_L \equiv \Omega$. Therefore the number of covered places is the total number of kens. Hence, the equilibrium is collision free.

(iii) To see that, the index of any $\omega \in \Omega$ is constant and equals $\sum_{i \in N} 1/k_i$. If this sum is greater than one, than $\Omega_H = \Omega$ and therefore the corollary follows from Theorem 3.6. If is is smaller than one then $\Omega_L = \Omega$. In this case $k_i = x_i/|\Omega|$ and therefore the corollary is a result of Theorem 3.6.

□

Proof of Proposition 4.1. In the first direction, assume that there is a collision-free equilibrium and we have to show that $L \geq (n-1)/n|\Omega| + 1$. This claim follows from the idea that the minimal number of different locations that might be chosen by an equilibrium strategy of a player is $|\Omega| - L + 1$. To see that, assume w.l.o.g. that $f_1(\{\omega_1, \omega_2, ...\omega_L\}) = \omega_1$. If we replace ω_1 by ω_{L+1} in the set $\{\omega_1, \omega_2, ...\omega_L\}$, the strategy should choose another location which is not ω_1. We can repeat on this action of replacing the representative (as chosen by the strategy) by a new location until we finish all locations in Ω. This procedure gives us $|\Omega| - L + 1$ different locations that were chosen by the strategy. Denote this set by l_i. In order to avoid collision, the intersection between l_i and l_j for all i, j should be empty. Therefore, $n[|\Omega| - L + 1] \leq |\Omega|$ should be satisfied, implying the desired result.

In the opposite direction, we have to show that if L satisfies the condition, there is a collision-free equilibrium. We do it by showing a profile of strategies of such equilibrium. Recall that a strategy is basically a function that chooses a representative of any possible ken. Define n sets of representatives from the size $\lfloor \Omega/n \rfloor$, one for each player. Since any ken is of size greater than L, any ken of any player contains at least one location of the player's set of representatives. Now index each player's set of representative. The strategy for player i is defined by $f(S) = \omega_k$ where ω_k is the highest indexed location in the set of representatives of player i that is contained in S.

□

Proof of Lemma 3.8. TBD. □

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