Economic Indices of Absolute and Relative Riskiness

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September 10, 2012

Abstract

Gambles and securities are two types of risky assets: while the returns of a gamble are absolute, the returns of a security are relative, i.e., proportional to the investment amount. Following Aumann and Serrano (2008) who characterize an index of riskiness of gambles by axioms, we characterize an index of riskiness of securities by similar axioms. It is important to emphasize that the two indices are not equivalent but relate to two different aspects of risk, namely, absolute and relative risk.

Keywords: Riskiness, Risk Aversion, Index of Riskiness.
JEL Classification: D81 G32

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The Author thanks Bob Aumann, Elchanan Ben-Porath, Florian Bierman, Dean Foster, Sergiu Hart, Moti Michaeli and Eyal Winter for useful discussions and suggestions.

1 Introduction

In many situations of decision making under risk, individuals take a decision in relation to some risky assets.[1] We distinguish between two types of risky assets: assets whose returns are absolute ("gambles") and assets whose returns are relative ("securities"). To clarify this distinction, note that accepting a gamble g at initial wealth w causes the wealth to be distributed as $w + g$, and investing w in a security r causes the wealth to be distributed as wr. We call the riskiness of gambles *absolute riskiness* and that of securities *relative riskiness*.[2]

In their seminal work Aumann and Serrano (2008) characterize an index of riskiness of gambles by two axioms; chief among them is a duality axiom that, roughly speaking, asserts that less absolute risk-averse individuals accept riskier gambles. Since the the Aumann–Serrano index is defined on gambles we call it an index of absolute riskiness. In the present paper we characterize an index of riskiness defined on securities by translating the Aumann and Serrano's axioms to relative terms. Since our index is defined on securities we call it an index of relative riskiness.

Although the two indices of riskiness are defined on two different objects, namely, gambles and securities, both of them can be used to measure the riskiness of the same investment. In this context, an *investment* is the possibility of exchanging an initial level of wealth for a wealth level that is distributed randomly. The mechanism behind an investment can be either a gamble or a security. We define the absolute riskiness of an investment as the riskiness of that gamble, measured by the Aumann–Serrano index, and its relative riskiness as the riskiness of that security, measured by our index of relative riskiness.

It is important to emphasize that **the indices are not ordinally equivalent**, i.e., they induce a different order of riskiness on the set of investments. In other words, given two investments, one investment may be riskier in absolute terms but less risky in relative terms, and vice versa. The fact that there exist two different orders of riskiness fits well with the idea of Arrow–Pratt who introduce two different orders of risk aversion (absolute and relative).

[1]Such decisions might be accepting or rejecting a gamble or the classic asset allocation problem.

[2]Our discussion deals with assets whose distribution is known. Michaeli (2012) is a related paper that deals with measuring the risk of situations in which the distributions of the random variables are unknown.

Indeed, the absolute riskiness relates to absolute risk aversion and the relative riskiness relates to relative risk aversion.

Among its many properties are some that make the index of relative riskiness an appropriate tool for measuring the riskiness of securities and portfolios. For instance, unlike the (inverse) Sharp ratio and the variance-mean ratio, our index is compatible with the stochastic-dominance order. In addition, on log-normal distribution, our index coincides with the variance-mean ratio.

The Aumann-Serrano index can be characterized in different ways. For instance, Hart (2011) extends the well-known *stochastic-dominance* order,[3] which is basically an incomplete order, in two different ways, one of them being equivalent to the order induced by the Aumman-Serrano index. Another approach is taken by Foster and Hart (2011) who suggest an alternative axiomatic characterization of the index. In addition, as Meilijson (2009) indicates, the reciprocal of the value of the Aumann-Serrano index was known in the insurance risk literature as the "adjustment coefficient" which has to do with the risk of going bankruptcy. This suggests another way of characterizing the Aumann-Serrano index. In the present paper we show that all these three approaches lead to our index of relative riskiness when applied to securities instead of gambles.

It is interesting to note that Aumann and Serrano (2008) themselves suggest that the riskiness of securities be measured in a different way from what we suggest here. Their order of riskiness of securities, despite its virtues, is incompatible with the principle of duality between risk and risk aversion, while ours is not. But even more important, their measure of riskiness of securities and their measure of riskiness of gambles induce exactly the same order of risk when applied to investments. By contrast, our order of riskiness of securities represents a new aspect of risk, namely, relative risk.

The paper is organized as follows. Section 2 is devoted to the basic axiomatic definition of the index of relative riskiness and its numerical characterization. In order to emphasize the similarities as well as the differences between the indices of absolute riskiness and relative riskiness, the Aumann–Serrano index is presented together with the new index. Section 3 sets forth some desirable properties of the index of relative riskiness. Section 4 defines the concept of investment opportunities. In particular, it shows how to apply

[3]For stochastic-dominance see Hadar and Russell (1969), Levy and Hanoch (1969), and Rothschild and Stiglitz (1970).

both indices to measure the risk that arises from general investments. One of the important ideas of Section 4 is that the two indices induce two different orders on the set of investment opportunities. Section 5 presents three alternative approaches towards risk that lead to the Aumann–Serrano index. It is shown that all these approaches lead to our index of riskiness when they are applied to securities instead of gambles. Section 6 concludes. The proofs are relegated to Appendix C. In addition, Appendix A shows that the two indices have a special role when measuring small risks. Appendix B deals with the Foster–Hart measure of riskiness in the setup of securities (instead of gambles).

2 Axiomatic Characterization

In this section we characterize our index of relative riskiness by two axioms which are extensions of the axioms that are used by Aumann and Serrano (2008) to define their index of riskiness. For convenience, we present the characterization of the Aumman-Serrano index together with the characterization of our index. The letters (AS) in the end of a statement indicate that the statement is taken from Aumann and Serrano (2008).

2.1 The Indices

Throughout this paper, a utility function is a von Neumann–Morgenstern utility function for money; it is strictly monotonic, strictly concave, and twice continuously differentiable.

We consider two types of risky assets, namely, gambles and securities. A *gamble* g is a random variable with finitely many real values—interpreted as dollar amounts—some of which are negative and that has positive expectation. Say that an agent with utility function u accepts a gamble g at wealth w if $\mathrm{E}\, u(w+g) > u(w)$ (E stands for expectation), that is, if she prefers taking the gamble at w to refusing it. Otherwise, she rejects it.

A *security* $r = [x_1, p_1; x_2, p_2; ... \, x_n, p_n]$ is a random variable with finitely many positive real values $x_1, x_2, ... \, x_n$, with respective probabilities $p_1, p_2, ... \, p_n$. The values of r are interpreted as (gross) returns, some of which are less than one, while its weighted geometric mean is greater than one, i.e., $\Pi x_i^{p_i} > 1$.[4]

[4]If the geometric mean of a security is less than one, we consider the riskiness of the security as infinity because investing repetitively in the security leads to bankruptcy with

We say that an investor with utility function u accepts (invests in) a security r at wealth w if $\mathrm{E}u(wr) > u(w)$, i.e., if she prefers investing all her wealth w in r to refusing the investment.[5]

We emphasize the distinction between the effect of gambles and securities on the wealth of agents who accept them. If an agent has an initial wealth w, accepting a gamble g causes the wealth to be distributed as $w + g$ and accepting a security r causes the wealth to be distributed as wr. Although gambles and securities are basically random variables they have different properties that make the set of gambles and the set of securities disjoint sets (recall that the values that a security can take are all positive. By contrast, any gamble takes at least one negative value with a positive probability).

Following Aumann and Serrano (2008) who define an (incomplete) order relation on the set of agents based on accepting or rejecting gambles, we define another (incomplete) order relation based on accepting or rejecting securities. The orders are defined as follows:

Definition 2.1.

1. *Agent i is uniformly no less absolute-risk averse than agent j, written $i \succeq_A j$, if whenever i accepts a gamble at some wealth, j accepts that gamble at any wealth. (AS)*

2. *Agent i is uniformly no less relative-risk averse than agent j, written $i \succeq_R j$, if whenever i accepts a security at some wealth, j accepts that security at any wealth.*

We call agent i uniformly more absolute- (relative-) risk averse than j, denoted by $i \succ_A j$ ($i \succ_R j$), if $i \succeq_A j$ ($i \succeq_R j$) but not $j \succeq_A i$ ($j \succeq_R i$).

Define an index as a positive real-valued function on risky assets (to be thought of as measuring riskiness). Given an index Q, we say that asset s_i is riskier than asset s_j if $Q(s_i) > Q(s_j)$. Aumann and Serrano (2008) characterize Q by two axioms that relate to gambles. Here, we extend the definition of these axioms to include securities as well.

Let Q_A and Q_R be indices of riskiness of gambles and securities, respectively. Let g and h be two gambles and let r and k be two securities. The

probability one; see Foster and Hart (2009).

[5] In Section 4 we consider the case where there exists a risk-free alternative in the economy. In this case, the condition for investing w in r is $\mathrm{E}u(wr) > u(wr_f)$, where r_f indicates the risk-free interest rate.

first axiom posits a kind of duality between riskiness and risk aversion, such that less risk-averse agents accept riskier assets.

Axiom of Duality.

1. *If $i \succ_A j$, i accepts g at w, and if $Q_A(g) > Q_A(h)$, then j accepts h at w. (AS)*

2. *If $i \succ_R j$, i accepts r at w, and if $Q_R(r) > Q_R(k)$, then j accepts k at w.*

Axiom of Scaling.

1. *$Q_A(tg) = tQ_A(g)$ for all positive numbers t. (AS)*

2. *$Q_R(r^t) = tQ_R(r)$ for all positive numbers t.*

As Aumann and Serrano (2008) explain, duality means that if the more risk-averse of two agents accepts the riskier of two assets, then a fortiori the less risk-averse agent accepts the less risky asset. The scaling axiom embodies the cardinal nature of riskiness. Accepting a risky asset twice is twice as risky as accepting it only once. If the asset is a gamble, say g, accepting g twice (dependently) gives absolute returns of $2g$. On the other hand, if the asset is a security, say r, accepting r twice (dependently) in a row gives relative returns of r^2. The axiom asserts that the riskiness of $2g$ is twice the riskiness of g, and the riskiness of r^2 is twice the riskiness of r.

We define now the two indices of riskiness. The first is the Aumman–Serrano index of absolute riskiness and the second is our proposed index of relative riskiness. If g is a gamble, $R(g)$ denotes the absolute riskiness of g, and if r is a security, $S(r)$ is the relative riskiness of r. $R(g)$ and $S(r)$ are defined implicitly as follows:

$$Ee^{-g/R(g)} = 1. \tag{1}$$

$$\mathrm{E}\, r^{-1/S(r)} = 1. \tag{2}$$

Although R and S are basically real-valued functions of random variables, R is not well defined on securities and S is not well defined on gambles.

The following theorem asserts that the indices just defined satisfy the two axioms.

Theorem 2.2.

1. *For each gamble g, there is a unique positive number R that solves for* (1)*. R satisfies duality and scaling, and any index of gambles satisfying these two axioms is a positive multiple of R. (AS)*

2. *For each security r, there is a unique positive number S that solves for* (2)*. S satisfies duality and scaling, and any index of securities satisfying these two axioms is a positive multiple of S.*

As Aumann and Serrano (2008) note in relation to R, duality and scaling are both essential: omitting either one of them results in admitting indices that are not positive multiples of R. But duality is the more central: together with certain weak conditions of continuity and monotonicity—but not scaling—it already implies that the index is ordinally equivalent to R. The same statement is correct in relation to S.

2.2 Log Returns

The indices R and S have different domains. While the domain set of R is the set of random variables that have the properties of gambles, the domain set of S is the set of random variables that have the properties of securities. R is not well defined on securities and S is not well defined on gambles. If one wants to use R for measuring the riskiness of securities, a transformation from the set of securities to the set of gambles is needed.

Aumann and Serrano (2008) define the riskiness of a security r, which we denote by $S^{AS}(r)$, as

$$S^{AS}(r) \equiv R(r-1).$$

Note that if r is a security $r-1$ is a gamble and hence S^{AS} is well defined. As an index of riskiness of securities, S^{AS} does not satisfy the duality axiom, which means that S^{AS} and S are not ordinally equivalent; i.e., they induce two different orders. As we will see in the next section, R and S^{AS} induce exactly the same order on investments. By contrast, our index S induces a different order.

Our approach for defining the riskiness of securities is quite different from that of Aumann and Serrano (2008). We started by characterizing an index of riskiness of securities, S, by the duality axiom. As a result, we derive the following relationship between R and S:

$$S(r) = R(\log r). \tag{3}$$

Indeed, if r is a security, $\log r$ is a gamble and hence S is well defined. It is easy to see that (3) follows from (1) and (2).

It is quite common in the finance literature to use the log returns of securities rather than the raw returns. In general, the reasons are tractability and simplification of algorithm complexity. By contrast, in our case we have an axiomatic justification for using the log returns.

In this context, it is interesting to examine how other measures of riskiness defined on gambles are applied to securities. For instance, the measure of riskiness of Foster and Hart (2009), denoted by R^{FH}, is basically defined on gambles. In their paper, Foster and Hart (2009) extend the domain of R^{FH} to securities by defining the riskiness of a security r as $R^{FH}(r-1)$. Indeed, Foster and Hart (2009) take the approach of Aumann and Serrano (2008) (but not ours) by defining the riskiness of a security to be the riskiness of the gamble that has exactly the same returns of the security minus one.

2.3 Risk Aversion and Duality

To understand the concept of uniform comparative risk aversion that underlies this treatment, recall first that Arrow (1965) and Pratt (1964) define two coefficients of risk aversion, one for absolute risk aversion (ARA) $\rho_i(w) \equiv \rho(w, u_i) \equiv -u_i''(w)/u_i'(w)$, and one for relative risk aversion (RRA) $\varrho_i(w) \equiv \varrho(w, u_i) \equiv -w\rho(w, u_i)$. As the following lemma asserts, there is a straightforward connection between the incomplete orders of uniform risk aversion defined above (Section 2.1) and the familiar notions of risk aversion of Arrow and Pratt.

Lemma 2.3.

1. *i is uniformly no less absolute-risk averse than j if and only if $\rho_i(w_i) \geq \rho_j(w_j)$ for all w_i and w_j. (AS)*

2. *i is uniformly no less relative-risk averse than j if and only if $\varrho_i(w_i) \geq \varrho_j(w_j)$ for all w_i and w_j.*

As Aumann and Serrano (2008) show, the Arrow–Pratt concept of absolute and relative risk aversion is a "local" concept in that it concerns i's attitude toward infinitesimally small risky assets at a specified wealth only. In contrast, the concepts of being uniformly no less absolute-risk averse and uniformly no less relative-risk averse are "global" in two senses: (1) they apply to risky assets of an arbitrary, finite size, which (2) may be taken at any wealth. However, these are only partial orders, whereas Arrow and Pratt define a numerical index (and hence a total order).

2.4 CARA and CRRA

An agent i is said to have constant absolute risk aversion (CARA) if her ARA is a constant α that does not depend on her wealth. In that case, i is called a CARA agent and her utility u a CARA utility, both with parameter α. There is an essentially unique CARA utility with parameter α, given by $u(w) = -e^{-\alpha w}$.

Similarly, an agent i is said to have constant relative risk aversion (CRRA) if the value of $\varrho_i(w)$ is constant for all w. CRRA expresses the idea that wealthier people are less risk averse. Here, wealth is assumed to be positive. There is an essentially unique[6] CRRA utility with parameter α, given by

$$u_\alpha(x) = \begin{cases} \frac{(x^{1-\alpha}-1)}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1 \end{cases}$$

While defined in terms of a local concept of risk aversion, CARA and CRRA may in fact be characterized in global terms, as follows.

Lemma 2.4.

1. *An agent i has CARA if and only if for any gamble g and any two wealth levels, i either accepts g at both levels or rejects g at both levels. (AS)*

2. *An agent i has CRRA if and only if for any security r and any two wealth levels, i either accepts r at both levels or rejects r at both levels.*

[6]Up to additive and positive multiplicative constants.

Just as CARA agents do not base their decision whether to accept or reject a gamble on their wealth level, CRRA agents do not base their decision whether to accept or reject a security on their wealth level. The independence of the wealth level enables us to use the parameter of the CARA agents for ranking riskiness of gambles and to use the parameter of CRRA agents for measuring the riskiness of securities.

Lemma 2.5.

1. *If a CARA agent accepts a gamble, then any CARA agent with a smaller parameter of CARA also accepts the gamble. Equivalently, if a CARA agent rejects a gamble, then any CARA agent with a larger parameter also rejects the gamble. (AS)*

2. *If a CRRA agent accepts a security, then any CRRA agent with a smaller parameter of CRRA also accepts the security. Equivalently, if a CRRA agent rejects a security, then any CRRA agent with a larger parameter also rejects the security.*

From Lemma 2.5 it follows that for each gamble g (security r), there is precisely one "cutoff" value of the parameter, such that g (r) is accepted by CARA (CRRA) agents with a smaller parameter and rejected by CARA (CRRA) agents with a larger parameter. The larger the parameter, the more absolute- (relative-) risk averse the agent, and so the duality axiom indicates that this cutoff might be a good inverse measure of absolute (relative) riskiness. And, indeed, we have the following theorem:

Theorem 2.6.

1. *The riskiness $R(g)$ of a gamble g is the reciprocal of the number α such that a CARA person with parameter α is indifferent between taking and not taking the gamble. (AS)*

2. *The riskiness $S(r)$ of a security r is the reciprocal of the number $\alpha - 1$ such that a CRRA person with parameter α is indifferent between investing and not investing in the security.*

Proof. Follows from (1) and (2) and the form of CARA and CRRA utilities. □

Lemma 2.7.

1. *If* $\rho_i(x) < 1/R(g)$ *for all* x *between* $w + \min g$ *and* $w + \max g$*, then* i *accepts* g *at* w*; if* $\rho_i(x) > 1/R(g)$ *for all such* x*, then* i *rejects* g *at* w*. (AS)*

2. *If* $\varrho_i(x) < 1/S(r) + 1$ *for all* x *between* $w \cdot \min r$ *and* $w \cdot \max r$*, then* i *accepts* r *at* w*; if* $\varrho_i(x) > 1/S(r) + 1$ *for all such* x*, then* i *rejects* r *at* w*.*

3 Properties

We focus here on the properties of relative riskiness. For the properties of absolute riskiness, see Aumann and Serrano (2008).

3.1 The Risk-Free Alternative

Our definition of relative riskiness implies that the relative riskiness of security r depends only on the distribution of r. A more general definition of riskiness might take into account the risk-free alternative available for investors. Let $r_f \geq 1$ be the risk-free (gross) return available for investors, such that investing w in r_f gives wr_f at the end of the investment period.

Given r_f, we define the relative riskiness of security r as[7]

$$S_f(r_f, r) = S(r/r_f). \tag{4}$$

It is easy to see that if $r_f = 1$ we return to the original setup and $S_f(r_f, r) = S(r)$. For any value of r_f, S_f satisfies the duality axiom.

It is reasonable to expect that a higher risk-free interest rate makes securities riskier. Let r be a security. In the range of values of r_f in which r/r_f still has the properties that characterize securities, we get the following:

Lemma 3.1. $S_f(r_f, r)$ *increases with* r_f*.*

Proof. If $r_{f1} > r_{f2}$, then r/r_{f2} stochastically dominates r/r_{f1}, which implies that the riskiness of r/r_{f2} is lower. □

We proceed now to study the properties of S.

[7] Assuming that r/r_f still has the properties that characterize securities, i.e., a geometric mean greater than one, and takes values less than one with positive probability.

3.2 Investing Only a Fraction of Wealth

If $\alpha < 1$, the investment of only αw in security r is less risky than investing w in r. Formally, let $r(\alpha) = 1 + \alpha(r - 1)$. Obviously, investing w in $r(\alpha)$ is equivalent to investing only $w\alpha$ in r.

Lemma 3.2. *$S(r(\alpha)) < S(r)$ for $0 < \alpha < 1$.*

Another interesting result shows the connection between the two types of riskiness. As the fraction of wealth that is invested in the security goes to zero, the value of the relative riskiness goes to the value of absolute riskiness:

Lemma 3.3. $\lim_{\alpha\to 0} S(r(\alpha))/\alpha = R(r - 1)$.

The lemma follows directly from (3).

3.3 Log-Normal Securities

If the security r has a log-normal distribution[8] with parameters μ and σ, then $S(r) = \sigma^2/2\mu$, where σ^2 is the variance of $\log r$ and μ is the expectation of $\log r$. Indeed, the density of r's distribution is $e^{(\ln x-\mu)^2/2\sigma^2}/x\sigma\sqrt{2\pi}$, so

$$\mathrm{E}r^{-1/(\sigma^2/2\mu)} = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x^{-1}e^{-(\ln x-\mu)^2/2\sigma^2}x^{-1/(\sigma^2/2\mu)}dx.$$

By substituting $y = \ln x$ we get

$$\begin{aligned} &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(y-\mu)^2/2\sigma^2}e^{-y/(\sigma^2/2\mu)}dy \\ &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(y+\mu)^2/2\sigma^2}dy = 1. \end{aligned}$$

3.4 Repeated Investments

Investing repeatedly in independent identically distributed (i.i.d.) securities is just as risky as investing once in one of these securities.

Lemma 3.4. *If $r_1, r_2, \ldots, r_n$ are i.i.d. securities with riskiness s, then Πr_i also has riskiness s.*

[8] By our earlier definition, a security has only finitely many values and so its distribution cannot be log-normal. We therefore redefine a security as a random variable r for which S is well defined.

Lemma 3.5. *If r and k are independent, then the riskiness of rk lies between the riskiness of r and the riskiness of k.*

Even without independence, we still have subadditivity:

Lemma 3.6. $S(rh) \leq S(r) + S(h)$ *for any security r and h.*

Recall that for a security r, $S(r) = R(\log r)$. Defining two gambles, $g = \ln r$ and $h = \ln k$, the proofs of Lemmas 3.4, 3.5, and 3.6 follow immediately from Section 4.H. in Aumann and Serrano (2008); see there.

To summarize: if two securities are identically distributed and hence have the same riskiness s, then if the securities are "totally" positively correlated (i.e., equal), the product security has riskiness $2s$. If they are independent, the product has the same riskiness s as each of the securities separately. When they are "totally" negatively correlated, the risk is minimal but need not vanish.

3.5 Sensitivity to High Moments

As Kadan and Liu (2011) show, the absolute riskiness of a gamble decreases with its odd-numbered moments and increases with its even-numbered moments. It follows from (3) that the relative riskiness of a security decreases with the odd-numbered moments of its log return and increases with the even-numbered moments of its log return.

3.6 A Benchmark

A security that results in the lowest return r_{min} with probability p where $r_{min} = p$ and a "very large" return with the remaining probability has riskiness 1. Moreover, if $r_{min} = p^{\alpha}$ ($\alpha > 0$), then the riskiness is α. For instance, if $r = [0.5, 0.5; 100, 0.5]$, then $S(r) \cong 1$ and if $r = [0.25, 0.5; 100, 0.5]$, then $S(r) \cong 2$. Formally, let $r_x = [r_m, p; x, 1-p]$. Then,

$$\lim_{x \to \infty} S(r_x) = \log_p(r_m).$$

3.7 Other Properties

Many other properties of our index, such as monotonicity in first- and second-order stochastic-dominance and continuity, can be derived directly from the equivalent properties of the index of absolute riskiness by using (3) or through similar proofs; see Aumann and Serrano (2008).

4 Investments

From the agents' perspective, the distinction between gambles and securities is quite artificial. Consider an agent whose initial wealth is \$100. Obviously, the agent is indifferent between accepting the gamble $[-\$10, 0.5; \$20, 0.5]$ or investing her initial wealth in the security $[0.9, 0.5; 1.2, 0.5]$. In both cases, she ends up with a wealth distributed as $[\$90, 0.5; \$120, 0.5]$. Since the agent cares only about her wealth, the decision whether to accept or reject an investment does not depend on whether it is a gamble or a security. In this section we measure the riskiness of such investments, regardless of whether the mechanism behind the investment is a gamble or a security.

Formally, an *investment* is a pair of elements $(w, \widetilde{w})$, where w is a real number, interpreted as an initial level of wealth, and $\widetilde{w}$ is a finite random variable whose values are interpreted also as levels of wealth. We say that an agent accepts the investment $(w, \widetilde{w})$ if she prefers $\widetilde{w}$ to w, i.e., if $u_i(\widetilde{w}) > u_i(w)$. It is easy to see that for an investor whose initial wealth is w, exchanging w for $\widetilde{w}$ is equivalent to accepting the gamble $g = \widetilde{w} - w$, and it is also equivalent to investing w in the security $r = \widetilde{w}/w$.[9] It is only natural to define the absolute riskiness of $(w, \widetilde{w})$ as $R(g)$ and the relative riskiness of $(w, \widetilde{w})$ as $S(r)$. Thus, any investment runs two kinds of risks, absolute risk and relative risk. Note that unlike the riskiness of gambles and securities, the riskiness of an investment depends on the initial wealth level of the decision maker (but not on her utility function).

4.1 Ordinal difference

When applied to investments, R and S are not ordinally equivalent. In other words, given two investments, one investment might be absolutely riskier but relatively less risky.

Consider the following example: let $(w, \widetilde{w}_1)$ and $(w, \widetilde{w}_2)$ be two investments, where $w = \$100$, $\widetilde{w}_1 = [\$90, 0.5; \$120, 0.5]$ and $\widetilde{w}_2 = [\$75, 0.25; \$135, 0.75]$. By definition, the absolute riskiness of $(w, \widetilde{w}_1)$ equals $R(g_1)$, where $g_1 = [-\$10, 0.5; \$20, 0.5]$, and the relative riskiness of $(w, \widetilde{w}_1)$ equals $S(r_1)$, where $r_1 = [0.9, 0.5; 1.2, 0.5]$. Similarly, the absolute riskiness of $(w, \widetilde{w}_2)$ equals

[9] we assume here that for a given investment $(w, \widetilde{w})$, the random variable $\widetilde{w}/w$ has the properties of a security, i.e., its geometric mean is greater than one, and it takes at least one value that is less than one, with some positive probability. Note that this assumption implies that the random variable $\widetilde{w} - w$ has the properties of a gamble.

$R(g_2)$, where $g_2 = [-\$25, 0.25; \$35, 0.75]$, and the relative riskiness of $(w, \widetilde{w}_2)$ equals $S(r_2)$, where $r_2 = [0.75, 0.25; 1.35, 0.75]$. Calculating the values of the riskiness of each of them results in the following "contradictory orders" of riskiness: $R(g_1) > R(g_2)$ and $S(r_1) < S(r_2)$.

One should not be surprised by the existence of two different orders of riskiness as we already know that there are two different orders of risk aversion (see Arrow (1965) and Pratt (1964)): absolute risk aversion and relative risk aversion. Since, according to our approach, riskiness is dual to risk aversion, the existence of two types of risk aversion implies the existence of two types of riskiness.

4.2 Absolute vs. Relative

The values of the absolute riskiness and the relative riskiness of an investment are connected in some mathematical way.[10]

If $(w, \widetilde{w})$ is an investment, we denote by $\widetilde{w}_M$ and $\widetilde{w}_m$ the maximal value and the minimal value that $\widetilde{w}$ takes, respectively.

Theorem 4.1. *Let $(w, \widetilde{w})$ be an investment such that $g \equiv \widetilde{w} - w$ is a gamble and $r \equiv \widetilde{w}/w$ is a security. Then,*

$$\widetilde{w}_m \leq \frac{R(g)}{\widehat{S}(r)} \leq \widetilde{w}_M, \tag{5}$$

where $\widehat{S}(r) = S(r)/(S(r) + 1)$.

$\widehat{S}(r)$ is a monotonic function of $S(r)$, taking values between zero and one. The value of $\widehat{S}(r)$ is itself significant. Recall that the value of $S(r)$ is the reciprocal of minus one plus the value of the parameter of the CRRA agent who is indifferent between accepting and rejecting r. By contrast, the value of $\widehat{S}(r)$ is exactly the reciprocal of the value of this parameter. Thus, $\widehat{S}(r)$ satisfies the duality axiom but not the scaling axiom.

[10] Two investments may have the same absolute riskiness but different values of relative riskiness. Similarly, they may have the same relative riskiness but different values of absolute riskiness. That means that one type of riskiness cannot be presented as a function of the other type of riskiness.

4.3 Riskiness and Rejection

Even if an investment is riskier in both absolute and relative terms, it does not imply that any decision maker who rejects the less risky investment will reject also the riskier one.[11] Here we show that if one investment is not only riskier but "much riskier" than another investment, there is quite a large set of utilities such that for each one of them, a rejection of the less risky investment implies the rejection of the riskier investment. This set of utilities, which Hart (2011) calls "regular utilities", includes all utilities whose absolute risk aversion (weakly) decreases with wealth (DARA), and whose relative risk aversion (weakly) increases with wealth (IRRA). We denote this set of utilities by U^*.

Recall that for a given investment $(w, \widetilde{w})$, we denote by $\widetilde{w}_M$ and $\widetilde{w}_m$ the maximal value and the minimal value that $\widetilde{w}_i$ takes. In addition, recall that $\widehat{S}(r) = S(r)/(S(r) + 1)$. We have the following result:

Theorem 4.2. *Let $(w, \widetilde{w}_1)$ and $(w, \widetilde{w}_2)$ be two investment opportunities and assume that $g_i \equiv \widetilde{w}_i - w$ is a gamble and that $r_i \equiv \widetilde{w}_i/w$ is a security, for $i = 1, 2$. If either*

$$\frac{R(g_1)}{R(g_2)} > \frac{\widetilde{w}_{M_1}}{\widetilde{w}_{m_2}} \tag{6}$$

or

$$\frac{\widehat{S}(r_1)}{\widehat{S}(r_2)} > \frac{\widetilde{w}_{M_1}}{\widetilde{w}_{m_2}}, \tag{7}$$

then, for every utility $u \in U^*$, if u rejects $(w, \widetilde{w}_1)$ then u rejects also $(w, \widetilde{w}_2)$.

5 Alternative Characterizations

In Section 2, we followed Aumann and Serrano (2008) and characterized the absolute riskiness and the relative riskiness by an axiom which asserts that the concept of riskiness and the concept of risk aversion should stand in a kind of dual relationship. In this section, we present three alternative approaches, all of which lead to the same orders of riskiness as R and S.

[11]It does imply that any CARA agent and any CRRA agent who rejects the less risky one will reject also the riskier investment.

5.1 Operational Characterization

The Aumman-Serrano index is the inverse of the "adjustment coefficient" of the insurance risk literature. Here, we characterize an order of riskiness of financial assets which relates to the concept of the adjustment coefficient.

Let g be a gamble and let $\{g_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. gambles, distributed as g. Similarly, let r be a security and let $\{r_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. securities, distributed as r.

Definition 5.1.

1. *Gamble g is riskier than gamble h with respect to bankruptcy, denoted by $g \succ_B h$, if there exists B such that for all $b < B$,*

$$P(\exists n \text{ s.t. } \Sigma_{i=1}^{n} g_i < b) < P(\exists n \text{ s.t. } \Sigma_{i=1}^{n} h_i < b).$$

2. *Security r is riskier than security k with respect to bankruptcy, denoted by $r \succ_B k$, if there exists B such that for all $b < B$,*

$$P(\exists n \text{ s.t. } \Pi_{i=1}^{n} r_i < b) < P(\exists n \text{ s.t. } \Pi_{i=1}^{n} k_i < b).$$

Going below some threshold can be interpreted as going bankruptcy (see Foster and Hart 2009).

Theorem 5.2.

1. *$g \succ_B h$ iff $R(g) > R(h)$.*

2. *$r \succ_B k$ iff $S(r) > S(k)$.*

The first part of Theorem 5.2 is a variant of Theorem 1 of Meilijson (2009). The second part of the theorem follows directly from its first part plus (3).

5.2 Wealth Uniform Dominance

Hart (2011) defines an order of riskiness of gambles that he calls *wealth uniform dominance*. He shows that wealth uniform dominance is equivalent to the order induced by the Aumann–Serrano index. Here, we define a similar order in relation to securities instead of gambles and show its equivalency to the order induced by our index of relative riskiness.

In defining wealth uniform dominance, Hart (2011) relates to a specific set of utilities, U^*, which he calls "regular utilities". The properties of utilities in U^* that are relevant to our discussion are that their absolute risk aversion decreases (weakly) with wealth (DARA) and that their relative risk aversion increases (weakly) with wealth (IRRA). The wealth uniform dominance order is defined as follows.

Definition 5.3. *A gamble g wealth uniformly dominates a gamble h, denoted $g \geq_{WU} h$, whenever:*

$$\text{if } g \text{ is rejected by } u \text{ at all } w > 0$$
$$\text{then } h \text{ is rejected by } u \text{ at all } w > 0,$$

for every utility $u \in U^*$.

Similarly, we define the stochastic order of wealth-uniform dominance in the setup of securities as follows.

Definition 5.4. *A security r wealth uniformly dominates a security h, denoted $r \geq_{WU} h$, whenever:*

$$\text{if } r \text{ is rejected by } u \text{ at all } w > 0$$
$$\text{then } h \text{ is rejected by } u \text{ at all } w > 0,$$

for every utility $u \in U^*$.

The following theorem cites the result of Hart (2011) in relation to gambles and adds the equivalent result in relation to securities:

Theorem 5.5.

1. *For any two gambles g and h, $g \geq_{WU} h$ if and only if $R(g) \leq R(h)$. (Hart 2011)*

2. *For any two securities r and k, $r \geq_{WU} k$ if and only if $S(r) \leq S(k)$.*

Note that the second part of the theorem cannot be derived directly from (3) in a trivial way.

Although absolute riskiness and relative riskiness are equivalent concepts, not every characterization of absolute riskiness can be translated directly to a reasonable characterization of relative riskiness. To demonstrate this, let us consider the following definition of absolute riskiness: one gamble is riskier than another gamble if there is a certain level of wealth such that for any decision maker whose wealth is above this threshold, if she rejects the less risky gamble she rejects also the riskier gamble. Formally:

Definition 5.6. *Gamble g is riskier than gamble h if there exists w_0, such that for all wealth levels $w > w_0$ and for all $u \in U^*$, if u rejects h at w, u rejects g at w.*

Theorem 5.7. *Gamble g is riskier than gamble h (according to 5.6) iff $R(g) > R(h)$.*

Unfortunately, we did not find any equivalent definition in relation to securities that would lead to S. Yet, we did find a similar characterization that leads to S^{AS} which is not the same as S.[12] As we denoted above, given a security r and a real number α, $r(\alpha) \equiv 1 + \alpha(r - 1)$, where $0 < \alpha < 1$. Investing w in $r(\alpha)$ is equivalent to investing only αw in r. It can be shown that for any two securities r and k, $S^{AS}(r) > S^{AS}(k)$ if and only if there is a number α^* such that for every $\alpha < \alpha^*$ and for every $u \in U^*$, a rejection of $k(\alpha)$ implies the rejection of $r(\alpha)$. The proof of this statement is given in the Appendix, together with the proof of Theorem 5.7.

5.3 The Four Axioms Approach

Foster and Hart (2011) provide axiomatic characterizations of two measures of riskiness of gambles: one is the Aumann–Serrano index of absolute riskiness and the other is the Foster-Hart measure of riskiness (Foster and Hart 2009). Here, we show that translating the axiomatic characterization of the Aumann–Serrano index from absolute terms (gambles) to relative terms (securities) leads to our index of riskiness. For each axiom, we present the original axiom followed by our translation to relative terms. Let g and h be

[12]Recall that $S^{AS}(r) \equiv R(r - 1)$ is the index of securities defined by Aumann and Serrano.

two gambles and let r and k be two securities. We denote by Q_A an index of riskiness of gambles and by Q_R an index of riskiness of securities.

Axiom of Distribution.

1. *If g and h have the same distribution then $Q_A(g) = Q_A(h)$.*

2. *If r and k have the same distribution then $Q_R(r) = Q_R(k)$.*

Axiom of Scaling.

1. *$Q_A(\lambda g) = \lambda Q(g)_A$ for every $\alpha > 0$.*

2. *$Q_R(r^\lambda) = \lambda Q_R(r)$ for every $\alpha > 0$.*

Axiom of Monotonicity.

1. *If $g \geq h$ and $r \neq h$ then $Q_A(g) < Q_A(h)$.*

2. *If $r \geq k$ and $r \neq h$ then $Q_R(r) < Q_R(k)$.*

Axiom of Wealth Independent Compound Asset.

1. *Let $f = g + \mathbf{1}_A h$ be a compound gamble, where A is an event such that g is constant on A, i.e., $g|_A \equiv s$ for some x and h is independent of A. If $Q_A(h) = Q_A(g)$ then $Q_A(f) = Q_A(g)$.*

2. *Let $f = r \times (1 + \mathbf{1}_A(k-1))$ be a compound security, where A is an event such that r is constant on A, i.e., $r|_A \equiv x$ for some x, and k is independent of A. If $Q_R(k) = Q_R(r)$ then $Q_R(f) = Q_R(r)$.*

Theorem 5.8.

1. *A function of gambles Q_A satisfies the four axioms if and only if it is a positive multiplication of R. (Foster and Hart 2011)*

2. *A function of securities Q_R satisfies the four axioms if and only if it is a positive multiplication of S.*

The proof of the theorem in relation to R appears in Foster and Hart (2011). The proof in relation to relative terms is based on the simple observation of (3), which says that for any security r, $S(r) = R(\log(r))$.

6 Conclusions

In this paper we introduced an index of riskiness which, like the Aumann-Serrano index, characterized by the dual relationship between risk and risk aversion. While the Aumann-Serrano index is defined on gambles and relates to absolute risk aversion, our index is defined on securities and relates to relative risk aversion. Although the two indices are defined on different objects, namely, gambles and securities, both can be used for measuring the risk that arises from the same investment. We showed that the orders induced by the two indices are not equivalent.

The two indices of riskiness reflect two different approaches towards measuring risk. Given an initial wealth w_0, there is a one-to-one mapping between the set of gambles and securities (i.e., $g \leftrightarrow r$, s.t. $w_0 + g = w_0 r$).[13] However, following Aumann and Serrano (2008), Foster and Hart (2009) and Hart (2011), we consider riskiness as something that is uniform in wealth, i.e., does not depend on a wealth level. This uniformity breaks the correspondence between gambles and securities: if a gamble g and a security r satisfy $w_0 + g = w_0 r$ for a wealth level $w_0 > 0$, the set of final outcomes obtained from varying the wealth and keeping the gamble fixed is very different from the set obtained by varying the wealth and keeping the security fixed. This explains why the approaches of Aumann and Serrano (2008) and that of the current paper are conceptually different. In fact, it might not be clear which one of the instances fits better than the other.

Appendix

A Riskiness in the Small

The measures of absolute and relative risk aversion of Pratt and Arrow are local indices based on the first and second derivatives of utility at a specific level of wealth. As such, they are applicable only to infinitesimal risks–those for which differential calculus is a suitable analytical tool. The same holds for absolute and relative indices of riskiness. Indeed, Schreiber (2012) shows that if the indices are used to measure the riskiness of financial assets whose

[13] In order for this mapping to take place, one should redefine a security as a random variable whose mean (rather than its geometric mean) is greater than one.

prices follow continuous-time random processes, the decision whether to accept or reject an asset depends only on the riskiness of the asset, for all utilities. Moreover, he shows that absolute and relative riskiness of investments coincide in that framework.[14]

Even if assets have small —but not infinitisimaly small— returns, the indices can be characterized by the decisions of a large set of utilities, U^*, the set of "regular utilities".

Definition A.1. *Gamble g is riskier than gamble h, if for every $w > 0$ there exists $\delta^* > 0$, such that, for every $0 < \delta < \delta^*$,*

$$\textit{if } u \textit{ rejects } \delta h \textit{ at } w \textit{ then } u \textit{ rejects } \delta g \textit{ at } w,$$

for every $u \in U^*$.

Definition A.2. *Security r is riskier than security k, if there exists $\lambda^* > 0$, such that, for every $0 < \lambda < \lambda^*$,*

$$\textit{if } u \textit{ rejects } r^\lambda \textit{ at } w \textit{ then } u \textit{ rejects } k^\lambda \textit{ at } w,$$

for every $u \in U^*$ and for every $w > 0$.

This yields the following Theorem:

Theorem A.3.

1. *Gamble g is riskier than gamble h (according to definition A.1) if and only if $R(g) > R(h)$.*

2. *Security r is riskier than security k (according to definition A.2) if and only if $S(r) > S(k)$.*

B Uniform Dominance

Foster and Hart (2009) propose another measure of riskiness of gambles, based on the critical wealth level below which it becomes "risky" to accept the gamble. Accepting gambles when the wealth is below the critical wealth

[14]Shorrer (2011) used a different method to show a similar idea.

level might lead to bankruptcy. According to Foster and Hart (2009), the riskiness of gamble g, denoted by $R^{FH}(g)$, is defined implicitly by the formula

$$\mathrm{E}\left[\log\left(1+\frac{1}{R^{FH}(g)}g\right)\right]=0. \tag{8}$$

Hart (2011) defines two orders of riskiness of gambles, called "wealth uniform dominance" and "utility uniform dominance". He shows that wealth uniform dominance is equivalent to the order induced by the Aumann–Serrano index of riskiness R, and that utility uniform dominance is equivalent to the order induced by the Foster–Hart index of riskiness R^{FH}.

Recall that if r is a security, the riskiness of r according to Aumann and Serrano is defined by

$$S^{AS}(r) \equiv R(r-1),$$

and the riskiness of r according to Foster and Hart is defined by

$$S^{FH}(r) \equiv R^{FH}(r-1).$$

Needless to say, S^{AS}, S^{FH}, and our index of relative riskiness S induce three different orders on the set of securities.

In this section we give an alternative characterization of S^{FH}. In the spirit of the analysis of Hart (2011), our characterization uses a specific set of utilities. In Section 4, "regular utilities" referred to all utilities whose absolute risk aversion weakly decreases (DARA) and whose relative risk aversion weakly increases (IRRA). Here, we add the assumption that as the value of the wealth goes to zero, the utility of the agent goes to minus infinity. Formally, the last assumption asserts that $\lim_{w\to 0} u(w) = -\infty$. We will denote by U^{**} the resulting class of utilities; i.e., U^{**} is the class of utilities that are both DARA and IRRA and satisfy the last assumption.

Given a security r, we define a new security $r(\alpha) = 1 + \alpha(r-1)$, for $0 < \alpha < 1$. As we said above, investing only αw in security r is equivalent to investing w in r.

Theorem B.1.

$$S^{FH}(k) > S^{FH}(r)$$

iff

for every fraction of wealth, $\alpha > 0$, and for every wealth level $w > 0$:

if $r(\alpha)$ is rejected by all $u \in U^{**}$ at w

then $k(\alpha)$ is rejected by all $u \in U^{**}$ at w

C Proofs

Many of the theorems in the present paper, especially in Section 2, have two statements, one in relation to absolute riskiness and one in relation to relative riskiness. Part of the statements in relation to absolute riskiness appear in Aumann and Serrano (2008). In this case we added the letters (AS) at the end of the statement, and the proof of the statement can be found in Aumann and Serrano (2008). Here we prove mostly the claims regarding relative riskiness.

In this section, investors i and j have utility functions u_i and u_j and Arrow–Pratt coefficients ϱ_i and ϱ_j of relative risk aversion. Since utilities may be modified by additive and positive multiplicative constants, we assume throughout that

$$u_i(1) = u_j(1) = 0 \text{ and } u_i'(1) = u_j'(1) = 1. \tag{9}$$

Note that any CRRA utility function satisfies (9).

Lemma C.1. *For some $\delta > 1$, suppose that $\varrho_i(w) > \varrho_j(w)$ at each w with $1/\delta < w < \delta$. Then $u_i(w) < u_j(w)$ whenever $1/\delta < w < \delta$ and $w \neq 1$.*

Proof. Let $1/\delta < y < \delta$. If $y > 1$ then by (9),

$$\log u_i'(y) = \log u_i'(y) - \log u_i'(1) = \int_1^y [\log u_i'(z)]' dz = \int_1^y \frac{u_i''(z)}{u'(z)} dz$$
$$= \int_1^y -(\varrho_i(z)/z) dz < \int_1^y -(\varrho_j(z)/z) dz = \log u_j'(y);$$

if $y < 1$ the reasoning is similar but the inequality is reversed. So if $w > 1$, then by (9),

$$u_i(w) = \int_1^w u_i'(y) dy < \int_1^w u_j'(y) dy = u_j(w);$$

and if $w < 1$, then

$$u_i(w) = \int_w^1 u_i'(y) dy < \int_w^1 u_j'(y) dy = u_j(w).$$

□

Corollary C.2. *If $\varrho_i(w) \leq \varrho_j(w)$ for all $w > 0$, then $u_i(w) \geq u_j(w)$ for all $w > 0$.*

Lemma C.3. *If r is a security, its riskiness $S(r)$ is well defined.*

Proof. Given r, we define the function f_r as follows:

$$f_r(\beta) \equiv \mathrm{E}\, r^\beta = \Sigma p_i r_i^\beta, \tag{10}$$

where β is a real number. The first and second derivatives of f_r are

$$f_r'(\beta) = \Sigma p_i r_i^\beta \log r_i, \tag{11}$$

$$f_r''(\beta) = \Sigma p_i r_i^\beta (\log r_i)^2. \tag{12}$$

Since by definition at least one of the values of r is greater than one and at least one of the values is less than one,

$$\lim_{\beta \to \pm\infty} f_r(\beta) = \infty. \tag{13}$$

In addition, since f_r'' is positive for all β, f_r' increases with β, which implies that f_r has a single minimum point. It follows from (10) that $f_r(0) = 1$. If $f_r'(0) \neq 0$, there should be another value of β, for which $f_r(\beta) = 1$. Based on this insight, we define β^* as follows:

1. If $f_r'(0) > 0$, then there is only one additional value of β, $\beta = \beta^*$, in which $f_r(\beta^*) = 1$ and $\beta^* < 0$.

2. If $f_r'(0) < 0$, then there is only one additional value of β, $\beta = \beta^*$, in which $f_r(\beta^*) = 1$ and $\beta^* > 0$.

3. If $f_r'(0) = 0$, then there is no other value of β, $\beta \neq 0$, in which $f_r(\beta) = 1$. In this case we set $\beta^* = 0$.

Since we assumed that the weighted geometric mean of any security is greater than one, $f_r'(0) = \Sigma p_i \log r_i > 0$ and we are in the first case where $\beta^* < 0$. Defining $S(r) = -1/\beta^*$ shows the existence of $S(r)$ and shows also that $S(r) > 0$. This completes the proof. □

Lemma C.4. *For any two portfolios r and k,*

$$S(k) > S(r) \Leftrightarrow f_r(-1/S(k)) < 1.$$

Proof. The lemma follows from the proof of (C.3). Since $f'_r(0) > 0$, $\beta^* = -1/S(r) < 0$ and the minimum point of f_r is between $\beta^* < 0$ and 0 (scenario 1 in the proof of (C.3)). This, together with the continuity of f_r, implies that for any $\beta^* < \beta < 0$, $f_r(\beta) < 1$. Since $\beta^* < -1/S(k) < 0$, defining $\beta = -1/S(k)$ completes the proof. □

Lemma C.5. *For any utility function u_α and value of $\delta > 1$ there is a security $r = r(\alpha, \delta)$, such that $u_\alpha(r) = 0$ and $\forall i,\ 1/\delta < r_i < \delta$, where r_is are the values r takes.*

Proof. Let $f(\epsilon)$ be defined as $f(\epsilon) = \epsilon u_\alpha(\sqrt{1/\delta}) + (1-\epsilon)u_\alpha(\sqrt{\delta})$. It is easy to see that if $\epsilon = 0$, then $f(\epsilon) > 0$, and if $\epsilon = 1$, then $f(\epsilon) < 0$. Since f is continuous in ϵ, $f(\epsilon^*) = 0$ for some ϵ^* between zero and one. The desired security is $r(\alpha, \delta) = [\epsilon^*, 1-\epsilon^*; \sqrt{1/\delta}, \sqrt{\delta}]$. □

The following lemma is equivalent to Lemma 4 in Aumann and Serrano (2008); however, another proof is needed.

Lemma C.6. *If $\varrho_i(w_i) > \varrho_j(w_j)$, then there is a security r that j accepts at w_j and i rejects at w_i.*

Proof. Without loss of generality, $w_i = w_j = 1$,and so $\varrho_i(1) > \varrho_j(1)$.[15] Let ϱ be a number between $\varrho_i(w)$ and $\varrho_j(w)$, $\varrho_i(w) > \varrho > \varrho_j(w)$. Since u_i and u_j are twice continuously differentiable, it follows that there is a number $h > 1$ such that $\varrho_i(w) > \varrho > \varrho_j(w)$ at each w with $1/h < w < h$. By Lemma (C.5), there is a security $r(\varrho, h)$ such that u_ϱ is indifferent between accepting or rejecting it. Therefore, by Lemma (C.1),

$$u_i(w) < u_\varrho(w) < u_j(w) \text{ whenever } 1/\delta < w < \delta \text{ and } w \neq 1, \tag{14}$$

implies that $u_i(r(\varrho, h)) < 0 < u_j(r(\varrho, h))$. Hence i rejects the security but j accepts it. □

Proof of Lemma 2.3. We have to show that $\varrho_i(w) \geq \varrho_j(w)$ for all wealth levels w if and only if i is no less relative-risk averse than j.
"If": Assume that there is a w with $\varrho_i(w) < \varrho_j(w)$. By Lemma (C.6), there

[15] For arbitrary w_i and w_j, define $u_i^*(x) = [u_i(xw_i) - u_i(w_i)]/(w_i u_i'(w_i))$ and u_j^* similarly, and apply the current reasoning to u_i^* and u_j^*. u_i^* and u_j^* accept or reject securities at $x = 1$, just as u_i and u_j accept or reject securities at w_i and w_j, respectively. In addition, $u_i^*(1) = u_j^*(1) = 0$ and $u_i^{*\prime}(1) = u_j^{*\prime}(1) = 1$.

is a security that i accepts at w and j rejects at w, thereby contradicting i being less relative-risk averse than j.

"Only if": Assuming that $\varrho_i(w) \geq \varrho_j(w)$ for all wealth levels w, we must show that for each wealth level w and security r, if i accepts r at w, then j accepts r at w. Without loss of generality, $w = 1$, and so we must show that

if i accepts r at 1, then j accepts r at 1.

From Corollary (C.2) (with i and j reversed), we conclude that $u_j(w) \geq u_i(w)$ for each w, and so $\mathrm{E}\, u_j(r) \geq \mathrm{E}\, u_i(r)$, which yields the above claim. □

Proof of Theorem 2.2. For $\alpha > 0$, let $u_\alpha(x)$ be the CRRA utility function with parameter α. The functions u_α satisfy (9), and so by Lemma C.1 (with δ arbitrarily large) their graphs are nested; that is,

$$\text{if } \alpha > \beta, \text{ then } u_\alpha(x) < u_\beta(x) \text{ for all } x > 0, x \neq 1. \tag{15}$$

The existence of $S(r)$ is proved in Lemma C.3.

To see that S satisfies the duality axiom, let i, j, r, h, and w be as in the hypothesis of that axiom; without loss of generality, $w = 1$. Set $\gamma \equiv 1 + 1/S(r)$,$\eta \equiv 1 + 1/S(h)$, $\alpha_i = \inf \varrho_i$ and $\alpha_j = \sup \varrho_j$. Thus

$$\mathrm{E}\, u_\gamma(r) = 0 \text{ and } \mathrm{E}\, u_\eta(h) = 0. \tag{16}$$

By hypothesis, $S(r) > S(h)$, so $\eta > \gamma$. By Corollary C.2,

$$u_i(x) \leq u_{\alpha_i}(x) \text{ and } u_{\alpha_j}(x) \leq u_j(x) \text{ for all x.} \tag{17}$$

Now assume $\mathrm{E}\, u_i(r) > 0$; we must prove that $\mathrm{E}\, u_j(h) > 0$. From $\mathrm{E}\, u_i(r) > 0$ and (17), it follows that $\mathrm{E}\, u_{\alpha_i}(r) > 0$. So by (16), $E_\gamma(r) = 0 < \mathrm{E}\, u_{\alpha_i}(r)$. So by (15), $\gamma > \alpha_i$. By Lemma 2.3 $\alpha_i \geq \alpha_j$ so $\eta > \gamma$ yields $\alpha_j < \eta$. Since (16),(15) and (17) yield $0 < \mathrm{E}\, u_\eta(h) < \mathrm{E}\, u_{\alpha_j}(h) < \mathrm{E}\, u_j(h)$, it follows that S satisfies the duality axiom.

That S satisfies the scaling axiom is immediate, and so, indeed, S satisfies the two axioms.

In the opposite direction, let Q be an index that satisfies the axioms. We first show that

$$\text{Q is ordinally equivalent to S.} \tag{18}$$

If this is not true, then there must exist r and h that are ordered differently by Q and R. This means either that the respective orderings are reversed, that is,

$$Q(r) > Q(h) \text{ and } S(r) < S(h), \tag{19}$$

or that the equality holds for exactly one of the two indices, that is,

$$Q(r) > Q(h) \text{ and } S(r) = S(h) \tag{20}$$

or

$$Q(r) = Q(h) \text{ and } S(r) > S(h). \tag{21}$$

If either (20) or (21) holds, then by the scaling axiom, replacing r by r^δ for sufficiently small $\delta > 1$ leads to reversed inequalities. So without loss of generality we may assume (19).

Now let $\gamma \equiv 1 + 1/S(r)$ and $\eta \equiv 1 + 1/S(h)$; then (16) holds. By (19), $\gamma > \eta$. Choose μ and ν so that $\gamma > \mu > \nu > \eta$. Then $u_\gamma(x) < u_\mu(x) < u_\nu(x) < u_\eta(x)$ for all $x \neq 0$. So by (16) $\mathrm{E}\, u_\mu(r) > \mathrm{E}\, u_\gamma(r) = 0$ and $\mathrm{E}\, u_\nu(h) < \mathrm{E}\, u_\eta(h) = 0$. So if i and j have utility functions u_μ and u_ν, respectively, then i accepts r and j rejects h. But from $\mu > \nu$ and Lemma (2.3), it follows that $i \succ j$, contradicting the duality axiom for Q. So (18) is proved.

To see that Q is a positive multiple of R, let r_0 be an arbitrary but fixed security and set $\lambda \equiv Q(r_0)/S(r_0)$. If r is any security and $t \equiv Q(r)/Q(r_0)$, then $Q(r_0^t) = tQ(r_0) = Q(r)$, and so $tS(r_0) = S(r_0^t) = S(r)$ by the ordinal equivalence between Q and S, and $S(r)/S(r_0) = t = Q(r)/Q(r_0)$, and $Q(r)/S(r) = Q(r_0)/S(r_0) = \lambda$, and $Q(r) = \lambda S(r)$. This completes the proof of Theorem A. □

Needless to say, both duality and scaling are essential to Theorem A. Thus the mean log $E \log r$ satisfies scaling but violates duality, while the index $[S(r)]$, where $[x]$ denotes the integer part of x, satisfies duality but violates scaling. Neither $E \log r$ nor $[S(r)]$ is even ordinally equivalent to S.

Proof of Lemma 2.4. Recall that all CRRA utility functions have the form

$$u_\alpha(x) = \begin{cases} \frac{(x^{1-\alpha}-1)}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1 \end{cases} \tag{22}$$

for $\alpha > 0$.

"Only if": Let $u_\alpha(x)$ be a CRRA utility with parameter α. u_α accepts r at w if and only if $\mathrm{E}\, u_\alpha(wr) > u_\alpha(w)$, that is, if and only if $\mathrm{E}\, u_\alpha(r) > u_\alpha(1)$.

"If": It follows from Lemma C.6; just take $j = i$. □

Proof of Lemma 2.5. Given a CRRA parameter α, we present the function u_α as follows:

$$u_\alpha(r) = \frac{f_r(\beta) - 1}{\beta},$$

where f_r is the function defined in (10), $\beta = 1 - \alpha$, and $\beta^* = 1 - \alpha^*$. It follows from the analysis of the behavior of $f_r(\beta)$ in the proof of Lemma C.3, that if $\alpha > \alpha^*$ then $u_\alpha(r) < 1$, which means rejecting the investment, and if $\alpha < \alpha^*$ then $u_\alpha(r) > 1$, which means accepting the investment. □

Proof of Lemma 2.7. Let u_i be i's utility and assume that $\varrho_i(x) < 1/S(r) + 1$ for all x between $w \min r$ and $w \max r$. Define a utility u_j as follows: when x is between $w \min r$ and $w \max r$, define $u_j(x) \equiv u_i(x)$; when $x \leq w \min r$, define $u_j(x)$ to equal a CRRA utility with parameter $\varrho_i(w \min r)$ and $u_j(w \min r) = u_i(w \min r)$ and $u'_j(w \min r) = u'_i(w \min r)$; when $x \geq w \max r$, define $u_j(x)$ to equal a CRRA utility with parameter $\varrho_i(w \max r)$ and $u_j(w \max r) = u_i(w \max r)$ and $u'_j(w \max r) = u'_i(w \max r)$. Let u_k be a CRRA utility with parameter $[1/S(r) + 1] - \epsilon$. Then

$$\min_x \varrho_k(x) > \max_x \varrho_j(x)$$

for positive ϵ sufficiently small. By Theorem 2.6, a CRRA person with parameter $[1/S(r)+1]$ is indifferent between taking and not taking g. Therefore, k, who is less risk averse, accepts g, and so by (2.5), j also accepts r. But between the minimum and maximum of wr, the utilities of i and j are the same. So i accepts r at w. The proof of the second part of Lemma (2.7) is similar. □

Proof of Lemma 3.2. Let $r = [x_1, p_1; x_2, p_2; ...; x_n, p_n]$ be a security. For simplicity, we denote $\epsilon_i = x_i - 1$. Recall that $S(r)$ is defined implicitly by $\Sigma p_i(1+\epsilon_i)^{-1/S(r)} = 1$ and $S(r(\alpha))$ is defined implicitly by $\Sigma p_i(1+\alpha\epsilon)^{-1/S(r(\alpha))} = 1$. We must show that for any $0 < \alpha < 1$, $S(r) > S(r(\alpha))$. Define a new function

$$K_r(\alpha) = \Sigma p_i(1 + \alpha\epsilon_i)^{-1/S(r)},$$

whose first and second derivatives are

$$K_r'(\alpha) = -1/S(r)\Sigma p_i(1+\alpha\epsilon_i)^{-(1/S(r)+1)}\epsilon_i$$

$$K_r''(\alpha) = (1/S(r))((1/S(r)+1)\Sigma p_i(1+\alpha\epsilon_i)^{-(1/S(r)+2)}\epsilon_i^2.$$

Note that $K_r(0) = 1$ and $K_r(1) = 1$. Since the second derivative is positive for any $0 < \alpha < 1$, it follows that for any $0 < \alpha < 1$, $K_r(\alpha) < 1$. Since $K_r(\alpha) = f_{r(\alpha)}(-1/S(r)) < 1$, it follows from (C.4) that $S(r) > S(r(\alpha))$. □

Proof of Theorem 4.1. Denote $r = \widetilde{w}/w$ and $g = \widetilde{w} - w$. The theorem says that

$$\min(\widetilde{w}) < \frac{R(g)}{\widehat{S}(r)} < \max(\widetilde{w}).$$

Assuming by contradiction that $1/(\min(\widetilde{w})\widehat{S}(r)) < 1/R(g)$, consider a CARA agent i whose parameter ρ satisfies $1/(\min(\widetilde{w})\widehat{S}(r)) < \rho_i < 1/R(g)$. From Lemma 2.7 we get: i rejects g since $\rho_i < 1/R(g)$, but i accepts r since $\min(\widetilde{w})\rho_i > 1/\widehat{S}(r)$, a contradiction. If $1/(\max(\widetilde{w})\widehat{S}(r)) > 1/R(g)$, the reasoning is similar. □

Theorem (4.1) implies two corollaries:

Corollary C.7. *Let g be a gamble and let the value of w be such that $1+g/w$ has the properties of securities. Then*

$$w + \min(g) \leq \frac{R(g)}{\widehat{S}(1+g/w)} \leq w + \max(g). \tag{23}$$

Corollary C.8. *Let r be a security. Then*

$$\min(r) \leq \frac{R(r-1)}{\widehat{S}(r)} \leq \max(r). \tag{24}$$

In Corollary C.8 we used the scaling axiom for gambles.

Proof of Theorem 4.2. If

$$\frac{R(g_1)}{R(g_2)} > \frac{\widetilde{w}_{M_1}}{\widetilde{w}_{m_2}}, \tag{25}$$

then

$$\frac{\widetilde{w}_{m_2}}{R(g_2)} > \frac{\widetilde{w}_{M_1}}{R(g_1)}. \tag{26}$$

Now assume an agent who accepts $(w, \widetilde{w}_1)$ but rejects $(w, \widetilde{w}_2)$. From Lemma 2.7 it follows that

$$\rho_i(\widetilde{w}_{M_1}) < 1/R(g_1) \Rightarrow \varrho_i(\widetilde{w}_{M_1}) < \widetilde{w}_{M_1}/R(g_1) \tag{27}$$

and

$$\rho_i(\widetilde{w}_{m_2}) \geq 1/R(g_2) \Rightarrow \varrho_i(\widetilde{w}_{m_2}) \geq \widetilde{w}_{m_2}/R(g_2). \tag{28}$$

Since we assumed IRRA, $\varrho_i(\widetilde{w}_{M_1}) \geq \varrho_i(\widetilde{w}_{m_2})$. Hence, from (27) and (28) it follows that $\widetilde{w}_{M_1}/R(g_1) \geq \widetilde{w}_{m_2}/R(g_2)$, in contradiction to (26). A similar reasoning holds for the second part of the theorem. □

The theorem we just proved implies the following.

Corollary C.9. *Given two securities, r and k, if either*

$$\frac{\widehat{S}(r)}{\widehat{S}(k)} > \frac{r_{max}}{k_{min}} \tag{29}$$

or

$$\frac{R(r-1)}{R(k-1)} > \frac{r_{max}}{k_{min}}, \tag{30}$$

then, for all $w > 0$ and for all $u_i \in U^*$, if u_i rejects k at w, u_i rejects also r at w.

To see this, for any given w, define $\widetilde{w}_1 = rw$ and $\widetilde{w}_2 = kw$. Equation 29 follows from (7). Define $g_1 = w(r-1)$ and $g_2 = w(k-1)$; equation 30 follows from (6) plus the scaling axiom.

Corollary C.10. *Given two gambles, g and h, if either*

$$\frac{R(g)}{R(h)} > \frac{w + h_{max}}{w + g_{min}} \tag{31}$$

or

$$\frac{\widehat{S}(1 + g/w)}{\widehat{S}(1 + h/w)} > \frac{w + h_{max}}{w + g_{min}}, \tag{32}$$

then, for all $u \in U^*$, if u rejects h at w then u rejects g at w.

To see this, for a given w, define $\widetilde{w}_1 = w + g$ and $\widetilde{w}_2 = w + h$. Equation 31 follows from (6). Define $r_1 = 1 + g/w$ and $r_2 = 1 + h/w$; equation 32 follows from (7).

Proof of Theorem 5.5. For the proof of the first part of Theorem 5.5, see Hart (2011). Here we prove the second part of the theorem. Let r and k be two securities such that $S(k) > S(r)$ and assume by contradiction that there exists i such that r is rejected by i at all w but i accepts k at w_0. From Lemma 2.7 it follows that $\varrho(w_0) < 1/S(k) + 1$, but from the same lemma it also follows that for every $w > 0$, $\varrho(w) \geq 1/S(r) + 1$, and in particular $\varrho(w_0) \geq 1/S(r) + 1$. So it must be that $1/S(k) > 1/S(r)$ but that contradicts the assumption that $S(k) > S(r)$. The opposite direction is proved as following. Assume that r wealth uniformly dominates k but that $S(r) > S(k)$. Let $x = (1/(-1 + S(r)) + 1/(-1 + S(k)))/2$. According to theorem 2.6, a CRRA agent with parameter x rejects r but accepts k at any $w > 0$, a contradiction. □

Proof of Theorem 5.7. Let g and h be two gambles such that $R(g) > R(h)$. From Corollary (C.10) it follows that for w large enough for all $u \in U^*$, if u rejects h at w then u rejects g at w. Assume now by contradiction that there exists $w^* > 0$ such that for all $w > w^*$ and for all $u \in U^*$, if u rejects h at w, then u rejects g at w but $R(h) > R(g)$. This contradicts the first part of the proof. □

Theorem C.11. *For any two securities r and k, $S^{AS}(r) > S^{AS}(k)$ if and only if there exists a number α^* such that for every $\alpha < \alpha^*$ and for every $u \in U^*$, if u rejects $k(\alpha)$ u rejects also $r(\alpha)$.*

Proof. Let r and k be two securities such that $S^{AS}(r) > S^{AS}(k)$. Recall that $S^{AS}(r) \equiv R(r-1)$. From the scaling axiom it follows that $S^{AS}(r)/S^{AS}(k) = S^{AS}(r(\alpha))/S^{AS}(k(\alpha))$ for all α. Since as α goes to zero, the ratio $r(\alpha)_{max}/k(\alpha)_{min}$ goes to 1, it follows from Corollary C.9 (particularly equation 30) that for α small enough for all $u \in U^*$, if u rejects $k(\alpha)$ at w then u rejects $r(\alpha)$ at w as well. The other direction is trivial. □

Proof of Theorem A.3. The first part follows from Corollary C.10, particularly from (31), and the scaling axiom. The second part of the theorem follows from Corollary C.9, particularly from (29) and the scaling axiom. □

Proof of Theorem B.1. For the proof of the first part of Theorem B.1, see Hart (2011). Here we prove the second part.

Assume $r \geq_{UU} k$. Recall that r and k are securities but $r-1$ and $k-1$ are gambles. The first step is to show that the two gambles $r-1$ and $k-1$ satisfy $(r-1) \geq_{UU} (k-1)$. To this end, we have to show that if the gamble $(r-1)$ is rejected by all u at w, then $(k-1)$ is also rejected by all u at w. Let w be such a level of wealth that $(r-1)$ is rejected by all u at w. For any agent, taking the gamble $(r-1)$ at w results in the same distribution of wealth as investing in a security $r(\alpha)$, where $\alpha = 1/w$. Hence, the security $r(\alpha)$ is rejected by all at w. From the assumption, the security $k(\alpha)$ is rejected by all at w. It follows that the gamble $k-1$ is rejected by all at w, which completes the proof that $(r-1) \geq_{UU} (k-1)$. But since, $S^{FH}(r) = R^{FH}(r-1)$, $R^{FH}(k-1) \geq R^{FH}(r-1) \Rightarrow S^{FH}(k) \geq S^{FH}(r)$.

Assume $S^{FH}(r) \leq S^{FH}(k)$. From the assumption and the definition of S^{FH} it follows that for any (positive) α and w, $R^{FH}(w\alpha(r-1)) \leq R^{FH}(w\alpha(k-1))$. It is easy to see that, for any α and w, the gamble $w\alpha(r-1)$ is rejected by all u at w if and only if the security $r(\alpha)$ is rejected by all $u \in U^*$ at w. But if the gamble $w\alpha(r-1)$ is rejected by all at w, it follows from the first part of the theorem that also $w\alpha(k-1)$ is rejected by all at w; hence, $k(\alpha)$ is also rejected by all at w. This completes the proof. □

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