DISCOUNTING AND WELFARE EVALUATION OF POLICIES

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Abstract. Policy discounting can be rationalised as an approximation to welfare in an overlapping generations model, even with heterogeneous consumers, when the status-quo is a balanced growth path — and only then: any form of discounting requires stationarity of the baseline. The implied intergenerationally fair discount rate for consumption under a relative utilitarian welfare function equals then the growth rate of per-capita consumption, say 2% for the U.S. This differs from the interest rate, even in the golden rule equilibrium unless population growth is null.

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1. Introduction

Recent debates on possible climate change [17, 18, 1, 3] sparkled a renewed interest in the normative foundations of net present value (NPV) evaluations, in particular, in the choice of "the right" discount rate, hence challenging the use of the interest rate for this purpose. Any such debate is inevitably reduced to formulating normative principles, thus raising the question when an NPV calculation (or *discounting*) is consistent with (utilitarian) welfare, the classical reference point in economic analysis.

If an NPV calculation is to correspond to any form of welfare evaluation, then, being a linear function of commodities, it must be a linearisation of a social welfare function, and hence its derivative. So the focus here is on small projects only.[1]

Our first result suggests, that if, further, following standard cost-benefit analysis practice ([19]), project evaluation is based on market prices, then the implied welfare weights are fully determined by those prices (sect. 1.3, also cor. 1).

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[1]For the use of differential approach to evaluate small policy changes, see, e.g., [8], [5].

Next we show that whenever more generally the derivative of welfare has a NPV form, the baseline allocation must be stationary (sect. 1.4).

Finally, as demonstrated in sect. 2, in a general overlapping generations (OG) model with exogenous growth, given a stationary baseline (i.e., a balanced growth equilibrium, BGE), the derivative of welfare does have the NPV form, at least under relative utilitarianism (RU), where individual utilities are 0–1-normalised.

With intergenerational equity (anonymity), the resulting discount rate on policies (expressed as the equivalent endowment perturbations) is the growth-rate of per-capita consumption, say 2% for the U.S. An independent argument based on the value of life confirms this conclusion.

The discount rate in our results can be used as well with endowment changes as with changes of final consumption, hence the conclusion is robust: even if one is to use a wrong re-equilibrating model, the discount rate should stay the same.

1.1. **Net present values.** So, the main goal is to describe the restrictions imposed on welfare (W) by requiring that its derivative (δW) has a NPV form as used in cost benefit analysis (CBA).[2] There, for practical purposes[3] the full impact of a public project is first translated into an equivalent change in individual (private) goods, $\delta c_{i,t,\tau,s}$ (here i is the index for physical goods, t indexes time, τ and s denote type and age on an individual),[4] of which a weighted sum is then taken. If W is defined on the set of equivalent consumption changes, (and the derivative exists) the weights ($q_{t,s,\tau,i}$) in general are unrestricted:

$$\delta W = \sum_{t,s,\tau,i} q_{t,s,\tau,i} \delta c_{t,s,\tau,i} \tag{1}$$

Discounting (and NPV in particular) allows only particular weights, and we study implications these restrictions. To be more precise,

Definition 1. If the weights in equation 1 are not individual-specific, ($q_{t,s,\tau,i} = q_{t,i}$) the *evaluation is distribution free*, i.e., the value of the project depends only on the aggregates. If further those weights are the corresponding prevailing prices of the goods, ($q_{t,i} = p_{t,i}$) it is a *current price evaluation.*

Clearly, for long-term projects the goods are dated and so current price evaluation is equivalent to an NPV with the interest rate as discount rate.

[2] To avoid the risk of leaving out important contributions related to CBA, we will not review the literature here.

[3] See, e.g., [19].

[4] The change is assumed to include, in addition to the direct impact of the project, also the compensating variation (in goods) for the different external effects.

We allow for a *general interpretation of discounting*: it does not necessarily imply that the evaluation is distribution-free, neither does it require a constant discount factor. More precisely,

Definition 2. If the weights in equation 1 are the product of a time-independent factor ($q_{s,\tau,i}$) with a discount factor, which is at the minimum assumed to be type-independent ($H_{t,s,i}$), we say that the welfare differential, δW, has the *discounting form.*

1.2. **Equilibrium evaluation.** The welfare impact of a policy change (welfare differential) involves the equilibrium response of the economy. Evaluating the latter is a daunting task on its own, as it entails solving a complex fixed point problem in the infinite-dimensional space of equilibrium paths, and is therefore often avoided by considering instead a "representative agent" model, thus reducing it to an optimisation problem,[5] and losing thereby the potentially important equilibrium effects.

Moreover, if one is to address long-term projects, hence inevitably involving some form of intergenerational transfers, the model should contain (overlapping) generations. Using [13], one can overcome the difficulty of equilibrium evaluation and derive the discount rate on policies from the primitives of an OG economy, even with heterogeneous agents, in the neighbourhood of a BGE — allowing thus to meaningfully address the discount rate debate. Here we demostrate such an evaluation for a particular class of policies.

Note however that requiring the evaluation of welfare to be done under general equilibrium has its costs: we can only identify easily the "time component" of the derivative. The way the instantaneous endowment- (or policy-)change is aggregated across individuals and goods remains hidden in the aggregator (q, see prop. 2), which depends on the full specification of the economy and the balanced growth equilibrium considered, as well as on the welfare function.[6]

1.3. **Distribution-free discounting imposes welfare weights.** Let us first establish that, at least in a finite economy, calculating the monetary value of a project based on prevailing prices is not "welfare free", rather, it corresponds to a very particular utilitarian welfare function.

Claim 1. *In a finite economy with differentiable utilities, at a locally Lipschitz equilibrium*[7] *there are welfare weights such that the market value of an aggregate endowment change is the first order approximation to the welfare change. These weights are the reciprocals of the individual marginal utilities of income.*

[5] See, e.g., [2].

[6] [13] is developing (for a single-type case) a way to compute q explicitly.

[7] An equilibrium (allocation and prices) (c, y, p) of an economy with endowment ω is locally Lipschitz if $\exists K \exists \varepsilon > 0$: for any equilibrium (c', y', p') of the economy with endowment ω', $\|(c', y', p') - (c, y, p)\| \leq \varepsilon \Rightarrow \|(y', p') - (y, p)\| \leq K\|\omega - \omega'\|$, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. This is a weak form of regularity, requiring no smoothness.

Proof. The status-quo is a competitive equilibrium (consumption, production and prices), (c, y, p), of an economy with endowment $\omega \in \mathbb{R}^{IN}$ of I goods owned by N individuals. Construct a social welfare function (SWF) W as a weighted sum $\sum_{n=1}^{N} \lambda_n u_n$ of individual utilities, with weights chosen such as to equate individual marginal utilities of consumption, (for any good i consumed in strictly positive quantity) with the equilibrium price system p:

$$\lambda_n \frac{\partial u_n(\cdot)}{\partial c_{i,n}} = p_i \tag{2}$$

Hence, the welfare weights are the reciprocals of the individual marginal utilities of income. We show that the variation of the welfare function so constructed equals the market value of the endowment variation.

By construction of the welfare function, given a small enough endowment perturbation $\delta\omega$, the welfare variation is the market value of individual consumption variations:

$$\delta W = \sum_n \lambda_n \delta u_n = \sum_n \lambda_n \sum_i \frac{\partial u_n}{\partial c_{i,n}} \delta c_{i,n} = \sum_i p_i \sum_n \delta c_{i,n}$$

Since $\delta c = \delta\omega + \delta y$, the claim, $\delta W = \sum_i p_i \sum_n \delta\omega_{i,n}$, will follow if the value of the variation in production δy (induced by the endowment perturbation) is negligible, or simply, the *orthogonality condition*,[8]

$$\lim_{\|\delta\omega\| \to 0} \frac{|\sum_i p_i \delta y_i|}{\|\delta\omega\|} = 0 \tag{3}$$

This follows from the Lipschitz condition. Indeed, by profit maximisation, $\sum_i p_i \delta y_i \leq 0$, and similarly $\sum_i p'_i \delta y_i \geq 0$, so $0 \geq \sum_i p_i \delta y_i \geq \sum_i (p_i - p'_i) \delta y_i$. So, by the local Lipschitz condition (fn. 7), $|\sum_i p_i \delta y_i| \leq \varepsilon \|\delta\omega\|$ for $\|\delta\omega\|$ sufficiently small. ■

Clearly, with dated goods, prices incorporate the interest rate; so discounting future benefits at this rate is justified if one is ready to choose welfare weights in accordance with the prevailing prices, as in (2), meaning the status-quo is considered welfare-optimal. Observe the evaluation obtained here is *distribution-free*: the change in welfare depends only on the aggregates. It is also obvious that with any other welfare weights the derivative will not have the desired form, which implies that NPV is not even the first order approximation to the welfare change.

This first step of the argument might not be hard to reproduce in an (infinite) OG model, suitable to analyse long term projects.[9] However, to get to the final result, that the welfare change is the market value

[8]The condition does not hold everywhere, even for smooth economies, cf. app. A.

[9]The main difficulty is to ensure welfare and its derivative are well defined, cf. eq. 1.

of the *endowment* perturbations might be harder in the OG economy.[10] And this final step is crucial for the applicability of CBA: calculating the equivalent endowment change is clearly much easier than predicting the resulting consumption variation.

So even to rationalise market evaluation a new approach is desirable.

But it is definitely needed when the purpose is to investigate what discount rate corresponds to the concept of intergenerational equity.

1.4. **Discounting implies stationarity.** While market evaluation is so restrictive, the following simple model shows that any plausible form of discounting — short of *defining* it as the derivative of welfare — can be rationalised only at a stationary status-quo allocation.

The model here is very simple, and intentionally so — we want to flash out the main difficulty of making welfare differential to be of a discounting form.

Time is discrete, $t \in \mathbb{Z}$. Each period there are 2 individuals, living only that period, one of each of the 2 types $\tau \in \{1, 2\}$. There are 2 goods, $i \in \{1, 2\}$, and utilities U_τ are strictly concave, monotone and differentiable. Aggregate endowment $\Omega \in \mathbb{R}^2$ is constant over time, and endowments are Pareto-optimal, so endowment changes are equal to the change in consumption, and policy changes amount to a consumption perturbation $\delta c_t \in \mathbb{R}^4$ with finite support.[11] Status-quo consumption is $\bar{c}_t \in \mathbb{R}^4$, with $p > 0$ as supporting price vector.

Consider the classical utilitarian welfare function[12]

$$W(\delta c) = \sum\nolimits_{t=-\infty}^{\infty} e^{-\beta t} \sum_{\tau} \lambda_\tau (U_\tau(\bar{c}_{t,\tau} + \delta c_{t,\tau}) - U_\tau(\bar{c}_{t,\tau})) \tag{1}$$

Claim 2. *Assume the welfare function is differentiable at the status-quo. If the differential has a discounting form, the status-quo allocation is constant over time. If it is a distribution-free discounting, then in addition, the status-quo allocation is the welfare maximiser, so* $\delta W(\delta c) = \sum_{t=-\infty}^{\infty} e^{-\beta t} \sum_i \bar{p}_i \delta C_{t,i}$, *with* $\bar{p} = p_t$ *for all* t *and* $\delta C_{i,t} = \sum_\tau \delta c_{t,\tau,i}$.

Proof. By construction, the differential of W w.r.t. a change in policy is

$$\delta W(\delta c) = \sum\nolimits_{t,\tau,i} e^{-\beta t} \lambda_\tau \nabla U_\tau(\bar{c}_{t,\tau}) \delta c_{t,\tau,i} \tag{2}$$

where $\nabla U_\tau(\bar{c}_{t,\tau})$ is the gradient of U_τ at the status-quo allocation $\bar{c}_{t,\tau}$. Since agents cannot improve by re-trading, the marginal utility vector is proportional to the price system, $\nabla U_\tau(\bar{c}_{t,\tau}) = \mu_{t,\tau} p_{t,i}$, so the weight associated with $\delta c_{t,\tau,i}$ is $e^{-\beta t} \lambda_\tau \mu_{t,\tau} p_{t,i}$. Under our general interpretation

[10] If one is to follow the steps of the argument above, reproducing the orthogonality condition (3) would need a detailed proof. Expressions like $\langle p, \delta c \rangle$ are not necessarily well-defined, the sums being potentially infinite, and the Lipschitz condition is non-trivial, as the production set for capital-investment is not smooth, being a linear subspace with non-negativity constraints.

[11] Finite support for the associated change in welfare to be well-defined.

[12] Subtracting the status quo utility to make sure W is well-defined.

of discounting (sect. 1.1), since agents live only one period, this weight has to be multiplicatively separable in time and type.

Therefore, setting e.g. $i = 1$, there should exist measurable functions of type and time correspondingly, κ_τ, H_t, such that $\lambda_\tau \mu_{t,\tau} = \kappa_\tau H_t$, so $\lambda_\tau \kappa_\tau^{-1} \nabla U_\tau(\bar{c}_{t,\tau})$ is independent of τ for every t. This implies that the status-quo allocation at any t solves the welfare optimization problem with the same welfare weights, $\lambda_\tau \kappa_\tau^{-1}$, hence is constant in time, by strict concavity.

The second assertion follows (using definition 2) by setting κ_τ to unity. ■

Even if one wants to allow for formulas like in prop. 1, by altering definition 2, that the discount factors in the weights are independent of the commodity — so they are functions of time and agent (type and birth-date), — one would deduce similarly that $p_{t,i} = \pi_i H_t$: all $\bar{c}_t$ have the same supporting price π. This is compatible with a non-constant status-quo only in degenerate cases; e.g., for CES utility functions $U_\tau(c_{t,\tau}) = (\theta_\tau c_{t,\tau,1}^{\sigma_\tau} + (1-\theta_\tau) c_{t,\tau,2}^{\sigma_\tau})^{1/\sigma_\tau}$, a non-constant $\bar{c}_t$ requires the ratio $\frac{\theta_\tau}{1-\theta_\tau}\left(\frac{\Omega_1}{\Omega_2}\right)^{\sigma_\tau - 1}$ to be independent of τ; call it ρ. Indeed, if this holds the contract curve is the diagonal of the Edgeworth box and all its points are supported by the relative price ρ; and if not, any two different Pareto optima have different supporting prices.

In the next section we show that discounting is still consistent with balanced growth, even in a full-fledged model.

2. Discounting in an OG model with exogenous growth

Consider the general form of the classical exogenous-growth model in an overlapping generations setting.

2.1. **The Economy.**

Time is the real line, $\mathbb{R}$. Individuals differ by type $\tau \in \Theta$ (Θ is finite) and by birth-date, $x \in \mathbb{R}$. They have life-length T_τ, and population grows exponentially at a constant rate ν, such that the distribution of age-groups and types is stationary. Instantaneous consumption of any individual is a non-negative bundle of n consumption goods and h fractions of total time allocated to h different types of labour. Individual preferences over lifetime streams of time allocation and consumption bundles are described by a utility function U_τ, homogeneous[13] of degree $1 - \rho_\tau$ in consumption.

Instantaneous production transforms m capital goods and l types of effective labour inputs into n consumption goods and m types of investment goods, with constant returns to scale.[13] The fraction of time, $z_{\tau,j}(s,t)$, devoted at date t to activity j by an agent of type τ and age s is multiplied by a non-negative and integrable efficiency factor $\varepsilon_{\tau,j}(s)$,

[13]Homogeneity and constant returns to scale are necessary for balanced growth.

to form effective time.[14] Effective time devoted at date t to any activity is multiplied by $e^{\gamma t}$ to form effective labour input, $e^{\gamma t}\varepsilon_{\tau,j}(s)z_{\tau,j}(s,t)$, thus representing labour-saving technological progress.

There are m capital goods (K_j), and a corresponding investment good (I_j) for each, linked by the usual capital accumulation equation, $K'_j(t) = I_j(t) - \delta_j K_j(t)$.

2.2. **Balanced Growth Paths.** We focus on balanced growth equilibria (BGE).[15] On a balanced growth path, individual labour is independent of the birth-date, individual consumption grows at rate γ, and all aggregate inputs and outputs at rate $\gamma + \nu$, as in the standard (1 type, 1 good) case, e.g., [2, 9].

2.3. **Policies.** Assume as before that policies consist of lump-sum real taxes and subsidies (possibly interpreted as a private-goods equivalent of public goods and other external effects) that might vary over time.[16] For that define function ω of three real variables: time (t), age (s) and type (τ), with values in the space of consumption bundles, $\mathbb{R}^n$.[17]

A *status-quo* policy $\bar{\omega}$ is the one that is consistent with a balanced growth equilibrium and so, satisfies $\bar{\omega}(t,s,\tau) = e^{\gamma h}\bar{\omega}(t-h,s,\tau)$ for any $h \in \mathbb{R}$, as all individual quantities grow at rate γ at a BGE.

We will consider here only policies that are "small" deviations from the status-quo, and do not destroy an equilibrium, i.e., the policy (transfers) are bouned. So, we start by defining the space of feasible policy changes F^λ.

Definition 3. F^λ is the space of λ-*exponentially integrable* $\mathbb{R}^n$- valued functions: $\{f\colon \sum_\tau \int e^{\lambda t} f(t,s,\tau)e^{\nu(t-s)}dtds < \infty\}$:

Instantaneous transfers are bounded: $\mathsf{sup}_t|\sum_\tau \int f(t,s,\tau)e^{\nu(t-s)}ds| < \infty$
Life-time transfers are bounded: $\mathsf{sup}_x|\int f(x+s,s,\tau)ds| < \infty$

The first two restrictions will refer to an aggregate policy (transfer), and that is why the term $e^{\nu(t-s)}$ is needed to reflect the demographic structure of the population.[18]

Definition 4. The *set of policies* P^λ consists of $\mathbb{R}^n$-valued functions ω defined for any $(t,s,\tau) \in \mathbb{R}\times[0,T_\tau]\times\{1,..,\Theta\}$ such that such that the resulting policy change, $\delta\omega = \omega - \bar{\omega}$ is in F^λ.

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[14] For example, setting $\varepsilon_{\tau,j}(s) = 1$ in a first part of life and 0 thereafter (or vice-versa) corresponds to the classical 2-period models.

[15] Cf. sect. 1.4 for the justification.

[16] Cf. [13] for the corresponding results for general policies.

[17] Using a more traditional definition of endowment $\tilde{\omega}_x(s,\tau)$ of agent of type τ born at time x is $N_\tau^{-1}e^{-\nu x}\omega(x+s,s,\tau)$ at age s with N_τ the number of agents of type τ born at time 0.

[18] $t-s$ is the date of birth of an individual who is of age s at time t.

The restriction to a neighbourhood of the original policy is there to assure that the welfare differential is well-defined. The allowed set is rather wide, including temporary changes from the status-quo as well as reforms that start at a particular date and contunue on forever, provided the change is not "too far" for "too long".

To apply the results from [14] one has to show that the set of policy changes introduced here satisfies the corresponding definition in that paper, which is done in lemma 1 relegated to appendix B.

2.4. **Results.** In [14] we show that for the most classical model of this sort, generically on the parameter space there exists an equilibrium selection by local uniqueness around each BGE, so that "comparative statics" is well-defined. Moreover, the perturbed equilibrium converges to the status-quo exponentially fast at $\pm\infty$. Hence the two welfare functions introduced below are well-defined, and so are their derivatives, thus both "negative" and "positive" results that follow are non-vacuous.

2.4.1. *The negative results.* Consider first *traditional utilitarian* welfare.

Proposition 1. *If a traditional utilitarian welfare function W is Gâteaux-differentiable*[19] *at 0 on the set of policies $P^{-(\beta+\rho_\tau\gamma)}$ for all τ, then its differential equals, for some $q \in L_\infty$,*[20]

$$\sum\nolimits_\tau \int_{-\infty}^{\infty} e^{-(\beta+\rho_\tau\gamma)t} \sum\nolimits_i \int q_i(s,\tau)\omega_i(t,s,\tau)e^{\nu(t-s)}ds\,dt =$$
$$\int_{-\infty}^{\infty} e^{-\beta t} \sum\nolimits_\tau \int \sum\nolimits_i e^{-\rho_\tau\gamma t} q_i(s,\tau)\omega_i(t,s,\tau)e^{\nu(t-s)}ds\,dt$$

and hence the weight in the welfare function of the types with the smallest risk-aversion ρ goes to one as time goes to $+\infty$.

Proof. By lemma 1 (app. B) with $\lambda_\tau = -\beta + (1-\rho_\tau)\gamma$ for all τ and [14, cor. 3]. ■

Corollary 1. *Evaluation at current prices can not be rationalised using a traditional utilitarian welfare function in a growing* OG *economy with heterogeneous consumers.*

The "discount rate" is well-defined *only if* all individuals are identical, i.e., $\rho_\tau = \rho$, as in this case the time-dependence factors out in the welfare differential. Surprisingly, it conforms then to the usual formula $\beta + \rho\gamma$ (in terms of aggregate resources) for the discount rate for *final consumption* in a representative agent model, e.g., [2].

[19] I.e., it has directional derivatives in every direction, which form a continuous linear function of the direction. It is the weakest sense of differentiability.

[20] More precisely, $q \in L_\infty^{\mathbb{R}^n}(\cup_\tau([0,T_\tau]\times\{\tau\}))$, for simplicity we just refer to the space of bounded functions, L_∞ thereafter.

If, in addition, the utility weights (or the corresponding β) are chosen "just right", $\beta = r - \rho\gamma$, the social disount rate is the interest rate r, exactly as in claim 1. Clearly also in this case the "pure transfer" policy, i.e., the one that satisfies $\sum_\tau \int e^{-rt}\omega(t,s,\tau)e^{\nu(t-s)}dsdt = 0$ is welfare neutral. This conclusion is also derived in [11], with the same welfare weights (multiplying utilities by the accumulated interest and dividing by a marginal utility, which here is proportional to $e^{\rho\gamma t}$) but in a discrete-time OG model.

Alternatively, taking $\beta = 0$ and reasonable estimates for the other parameters, $\rho\gamma$ seems far too high for an intergenerationally fair discount rate ([19] suggests a range of 1–3%), thus leading, e.g., Stern [17] to force $\rho = 1$.

2.4.2. *The positive results.* Those negative results suggest to rather use a welfare criterion, apparently better suited for comparing different generations' welfare in a growing economy.

In the *relative utilitarian* (RU) welfare functional,[21] individual vNM utilities are normalised between 0 and 1 on an exogenously given set of "all feasible and just" alternatives (policies), and then summed. Assume that this set of acceptable policies is shift invariant, cf. assumption 3 in [14], so that any feasible and just policy, when shifted in time, remains feasible and just; and that each individual utility is bounded on this set.

We will use an extension of RU by discounting utilities with β.[22]

Proposition 2. *If the extended relative utilitarian welfare function W is Gâteaux-differentiable on the set of policies $P^{-(\beta+\gamma)}$ at 0, then its differential equals $\int e^{-(\beta+\gamma)t}\sum_i\sum_\tau\int q_i(s,\tau)\omega_i(t,s,\tau)e^{\nu(t-s)}dsdt$ for some $q \in L_\infty$.*

Proof. By lemma 1 (app. B) with $\lambda = \gamma - \beta$ and [14, cor. 7]. ■

Proposition 2 shows that, if — in the spirit of intergenerational equity, and following the anonymity axiom of relative utilitarianism — one imposes $\beta = 0$, then the discount rate is the growth rate of per capita consumption (around 2% for the U.S.), which does fall right into the range suggested in [19].

Corollary 2. *Under extended relative utilitarianism, current price evaluation can be rationalised in a growing OG economy only when choosing $\beta = r - \gamma$, where r is the real interest rate, assuming the W defined by this β is Gâteaux-differentiable at the status-quo.*

Observe indeed, that under the relative utilitarian normalisation the marginal utility of wealth is growing for each type at the rate $r - \gamma$, so becomes constant for each type when discounted at the rate $\beta = r - \gamma$.

[21] Introduced and axiomatised in [4] for a finite set of agents.

[22] This violates the anonymity axiom, used to derive RU. We conjecture it should be replaced, for this, by anonymity within generations plus a form of stationarity.

Hence, choosing appropriate normalisation weights for the types (by an appropriate shape of the set of alternatives) allows then to equalize those marginal utilities across individuals. Again, as seen in the proof of claim 1, this is the condition needed for distribution-free discounting.[23]

Recall that pure relative utilitarianism requires equal treatment of individuals of all generations, hence to set β to zero. But this can be consistent with market evaluation of projects only if the interest rate is equal to γ, which is impossible in a "golden rule" equilibrium, where $r = \gamma + \nu$, unless, of course, population is constant ($\nu = 0$).

An independent argument, [14, thm. 3], confirms the fair discount rate is γ:

Proposition 3. *In the* OG *model extended by variable life-times, γ is the only discount rate treating human lives equally.*

To conclude, observe that forcing $\rho = 1$, e.g., by Stern [17], amounted in his framework to choosing the best possible approximation to RU, in the sense that this too makes all marginal utilities of income inversely proportional to income along a balanced growth path.[24]

3. Discussion

If one is to rationalise a net present value calculation using a utilitarian welfare, or use a CBA approach to approximate a welfare change, only small projects can be considered, though we allowed even for permanent changes (close enough to the status-quo). Clearly, for big policy changes, the first order approximation will no longer be sufficient.

Stationarity of the status-quo allocation, as shown in a simple intertemporal model, is needed for a rather general form of discounting (common time component for all goods or all types of individuals) to be welfare-justifed. The main results above are, most likely, not valid beyond a balanced growth path as status-quo: [13] shows that while the derivative of welfare (in large neighbourhoods of a BGE) typically still exists, even in a simple model it will no longer have a NPV form — while it does have it at a BGE. So, to stress, in those cases (beyond the BGE) NPV is not even a first order approximation to the derivative of welfare.

Similar to the finite economy, we found that imposing evaluation at current prices requires the welfare weights (for different generations) to depend on the price system (interest rate) and hence the intergenerationally fair discount rate is typically distinct from the prevailing interest rate, even in an efficient equilibrium. Let us stress that the reason for this discrepancy is purely "distributional", and has no relation to the gap between prices and the "true social value" of resources

[23] For results in that direction see also [11].

[24] Of course cor. 1 shows how to obtain RU exactly using type-dependent β's and appropriate weights for each type. But such β's make no sense except by reference to relative utilitarianism, or to the equivalent 'value of life' argument.

that might be present in an economy with "distortions" (see [15] for the most recent generalisation of the related results and the overview).

Finally, pure theory (based on [14]) gave us γ as the intergenerationally fair discount rate, and it was confirmed by an independent argument based on the value of human life. It is then our hope that RU would also be a reliable tool in more general cases involving large reforms.

Appendix A. The "locally Lipschitz" condition in claim 1

This example[25] shows that even in smooth economies[26] the orthogonality condition (3) of claim 1 can be violated (at a critical point).

A.1. **Economy.** There are two goods and two individuals with utilities $u_1(x_{11}, x_{21}) = x_{11} - \frac{1}{2}x_{21}^{-2}$ and $u_2(x_{12}, x_{22}) = x_{22} - \frac{1}{2}x_{12}^{-2}$ and the corresponding endowments $(1, r)$ and $(r, 1)$, with $r > 0$. Both individuals share profits equally. The aggregate production set is $Y = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 + y_2 \leq \mathsf{min}\{0, \alpha y_1 y_2\}\}$, where $\alpha > 0$.

A.2. **Equilibria.** Let $p = \frac{p_1}{p_2}$.

A.2.1. *Profit maximization.* $\mathsf{max}_{y_1}\, py_1 + \frac{-y_1}{1-\alpha y_1}$. So, the optimal production plan satisfying $y_1+y_2 \leq 0$ is $y_1(p) = \frac{1}{\alpha}\big(1-\frac{1}{\sqrt{p}}\big)$, $y_2(p) = \frac{1}{\alpha}\big(1-\sqrt{p}\big)$. Hence, the profit is $\frac{(1-\sqrt{p})^2}{\alpha}p_2$.

A.2.2. *Consumer demand.* Given the budget constraints of the first and the second consumer, respectively,

$$\begin{aligned} px_{11} + x_{21} &= p + r + \frac{(1-\sqrt{p})^2}{2\alpha} \\ px_{12} + x_{22} &= 1 + rp + \frac{(1-\sqrt{p})^2}{2\alpha} \end{aligned}$$

their demands are

$$(1 + rp^{-1} + \frac{(\sqrt{p}-1)^2}{2p\alpha} - p^{-2/3}, p^{1/3})$$
$$(p^{-1/3}, 1 + rp + \frac{(\sqrt{p}-1)^2}{2\alpha} - p^{2/3})$$

We will only look at equilibria in the neighbourhood of $p = 1$, where all those quanties are strictly positive.

[25] Based on [12].

[26] Indifference curves hit the axes, but the equilibria considered are away from that.

A.2.3. *Market clearing.* Take the market for the second good:
$p^{1/3}+1+rp+\frac{\left(\sqrt{p}-1\right)^2}{2\alpha}-p^{2/3}=1+r+\frac{1}{\alpha}-\frac{\sqrt{p}}{\alpha}$
Let $z=p^{1/6}$, so the market clearing condition becomes

$$F(z,r)=(r+\frac{1}{2\alpha})(z^6-1)-z^2(z^2-1)=0$$

Note that $z=\pm 1$ solves the equation, for any r and α.
Hence the other equilibria are the positive (z) roots of
$G(z,r)=F(z,r)/(z^2-1)=(r+\frac{1}{2\alpha})(z^4+z^2+1)-z^2=0.$

A.3. **Perturbation of endowment (r) around a critical point.** The equilibrium graph can thus be described as the graph of the function $z^2+z^{-2}+1$ $(z>0)$, together with the vertical through $z=1$, thinking of the vertical axis as the parameter $(r+\frac{1}{2\alpha})^{-1}$ of the economy and of the horizontal axis as the parameter z of the equilibrium. Obviously this has 1 critical point: $z=1$, $r+\frac{1}{2\alpha}=\frac{1}{3}$.

So, consider e.g. $\alpha=3$, $r=1/6$, and perturbations of the form $r=\frac{1}{6}-\rho$, $\rho>0$.

Since for $z=p=1$ the optimal production is zero,

$$\begin{aligned}\lim_{\rho\to 0}\frac{1}{\rho}\langle p,\delta y\rangle &= \lim_{\rho\to 0}\frac{y_1(1+\delta p)+y_2(1+\delta p)}{\rho}\\ &= \lim_{\rho\to 0}\frac{-\left(\sqrt{1+\delta p}-1\right)^2}{3\rho\sqrt{1+\delta p}}\end{aligned}$$

Letting $z=1+\zeta$, express now $\sqrt{1+\delta p}$ as $(1+\zeta)^3$ and use the previous equation to express also ρ in terms of $1+\zeta$: $\rho=\frac{1}{3}-\frac{z^2}{z^4+z^2+1}=\frac{1}{3}\frac{(z^2-1)^2}{z^4+z^2+1}$, so $3\rho\sim\frac{4}{3}\zeta^2$ and thus:

$$\lim_{\rho\to 0}\frac{1}{\rho}\langle p,\delta y\rangle=-\lim_{\zeta\to 0}\frac{3(3\zeta+3\zeta^2+\zeta^3)^2}{4\zeta^2(1+\zeta)^3}=-\frac{27}{4}\neq 0$$

Appendix B. Space of policy changes satisfies the assumptions of [14]

Define an auxiliary measure on age-groups and types, $\psi\colon (t,S_\tau)\mapsto \sum_\tau\int_{S_\tau}\omega(t,s,\tau)ds$ for a set of Borel subsets S_τ of $\mathbb{R}$. Let also the status quo measure be defined in terms of status-quo policy, $\bar{\psi}(t,S_\tau)=\sum_\tau\int_{S_\tau}\bar{\omega}(t,s,\tau)ds$.

Let $\pi\colon t\mapsto\pi(t)=e^{-(\gamma+\nu)t}\psi(t,\cdot)\in L_1$, so $\pi(t)$ is an anonymous distribution (thus, in L_1) of consumption across age-groups and types, adjusted for population and economic growth, i.e., is an integrable function of age and type with values in the space of instantaneous consumption bundles, $\mathbb{R}^n$, this coincides then with the definition of policies in [14] for this particular case with the underlying Banach space being $L_1^{\mathbb{R}^n}(\cup_\tau([0,T_\tau]\times\{\tau\}))$.

By construction, for any $t \in \mathbb{R}$ the value $e^{-(\gamma+\nu)t}\bar{\psi}(t,\cdot)$ is independent of t, and so for any $\bar{\omega}$ (and hence for any $\bar{\psi}$) one can define $b_0\colon S_\tau \mapsto e^{-(\gamma+\nu)t}\bar{\psi}(t, S_\tau)$, and let $\bar{\pi}(t) = b_0$.

It follows then by definition 4 that $\pi - \bar{\pi}$ is in $F^{\lambda'}$, or is λ'-integrable, where $\lambda' = \lambda + \gamma$.

Definition 5. K_{L_1} is the space of infinitely differentiable functions $\varphi\colon \mathbb{R} \to L_1$ with compact support.

$\varphi_n \in K_{L_1}$ converges to 0 if it and its successive derivatives converge uniformly to 0 and $\exists h \in \mathbb{R}\colon\ |x| \geq h \Rightarrow \varphi_n(x) = 0$ for all n. $K^\star_{L_1}$ is the space of linear functionals ψ on K_{L_1} s.t. $\psi(\varphi_n) \to 0$ when $\varphi_n \to 0$ in K_{L_1}. $K[= K_{\mathbb{R}}]$ is defined in [16, 7].

Lemma 1. *K_{L_1} is dense in F^λ for any $\lambda \in \mathbb{R}$.*

Proof. The two steps imply the statement by transitivity.

Step 1. *K_{L_1} is dense in $H = \{t \mapsto \sum_i^N b_i \mathbb{1}_{S_i}(t)\}$, the space of simple functions, where $S_i \subset \mathbb{R}$ are Borel sets of finite measure, $b_i \in L_1$ are constants, and $\mathbb{1}_{S_i}$ are indicator functions.*

Define $\phi(x) = e^{-\frac{1}{a(1-x)x}}$ for $0 < x < 1$ and 0 otherwise. ϕ is infinitely differentiable (cf. [10, p. 243]) clearly has finite support, and so belongs to K. ϕ converges to the indicator $\mathbb{1}_{[0,1]}$ in L_1 by the (Lebesgue) dominated convergence theorem. This implies K is dense in the set of indicator function of $[0,1]$. The statement then follows by linearity and given that the set of indicator functions of finite disjoint intervals is dense (in L_1 norm) in the set of indicators of Borel sets of finite measure.

Step 2. *H is dense in F^λ.*

That the set of simple functions is a dense subset of integrable functions is a basic property of Bochner integral (sometimes taken as its definition, cf. e.g., [6, def. III.2.17]). Hence the statement is proved for $\lambda = 0$.

For an arbitrary $\lambda \in \mathbb{R}$, and any $\phi \in K_{L_1}$, define function $g(t) \stackrel{\text{def}}{=} e^{\lambda t}\phi(t)$, then $g \in K_{L_1}$. Given the commutative property, [13, lemma 8] the statement holds for any $\lambda \in \mathbb{R}$. ■

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