Authority Measure for Opinion Dynamics

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Abstract

We propose a formal framework for measuring the authority of individuals in a setting of opinion dynamics. Our framework is based on an axiomatic approach in the tradition of coalitional game theory. We show that the unique solution concept satisfying our axioms is a generalization of the "authority distribution" notion of Hu and Shapley [9]. A novelty of our approach is that it is based on the dynamics of opinion exchange rather than on network topology. However, for any fixed opinion dynamics, our approach associates a network centrality measure with any given network topology. Previous as well as new network centrality measures are obtained in this way.

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1 Introduction

The term 'opinion dynamics' is used in this paper as a metaphor for a mapping from the n-dimensional cube, or only its vertices, to itself. The terminology reflects the following interpretation.

Initially, each of n persons has a certain opinion regarding the answer to a specific question. Depending on the context, the opinion may be binary, either 0 (No) or 1 (Yes), or it may be any number in the unit interval, reflecting for example the person's confidence that the answer is affirmative. Then, following one or more rounds of opinion exchanges, deliberation, persuasion, or the like, the same n persons reach their final opinions. Even in the case of binary initial opinions, we allow the final opinions to be fractions, with the possible interpretation that each of these fractions expresses the probability that the corresponding final opinion is Yes. Our model is thus more general than that of Grabisch and Rusinowska [7], for example, where both the initial and final opinions are assumed binary.

A fair number of concrete models of opinion dynamics fit the above description. A good survey of such models is Grabisch and Rusinowska's paper [8]. Typically, the connections among the n individuals are described by a network, and their opinions are updated synchronously in a manner reflecting the network structure. Our model may refer, in this case, either to the opinions after one or any other specified number of rounds of updating, or to the (potentially, time-average) limit resulting from repeated updating. The latter possibility, limit opinions, may be relevant also in the case of asynchronous updating, where in each step only one person's opinion changes.

An authority measure for a given opinion dynamics is an n-tuple of numerical scores serving as measures of each person's influence on the final opinions. It is comparable to centrality measure in networks [3], where the score is meant to reflect the importance of each person's position in the network. For opinion dynamics with an underlying network structure, the position is only one determinant of authority. The other component is the process by which people's opinions are influenced by their neighbors' opinions.

A concrete example of opinion dynamics in networks is DeGroot's [4] learning model (see also [2]). The opinions here are numbers in $[0, 1]$, and the network is a weighted directed graph where the weights are those individuals attach to each other's opinions. The weights each person i attaches to the n opinions sum up to 1, and in each period, i's opinion is replaced by the corresponding weighted average of the n opinions. It is well known that, if the graph is strongly connected and aperiodic, then the n-tuple of opinions converges over time. Moreover, the limit opinion is the same for all individuals, a case we refer to as *consensual dynamics*.

We abstract away from the DeGroot and similar models by assuming the following about the opinion dynamics. If the initial opinions are all 0 or are all 1, then the final opinions are also identical. In the former case, they are all 0, but in the latter case, we allow the final opinions to have a common value that is not necessarily 1. We also assume monotonicity: increasing any of the initial opinions can only increase each of the final opinions or leave it unchanged. Finally, we assume a property akin to, but essentially weaker than, the strong connectivity assumption in the DeGroot model. Specifically, we restrict attention to dynamics that are *simple* in the sense that there are no two disjoint sets of people with the property that the final opinions of the people in the set are unaffected by the initial opinions of those outside it (an *autonomous* set). If there *are* two or more disjoint autonomous sets – if the dynamics is *compound* – then we maintain that each set should be considered separately. For we do not see any meaningful sense in which the authority of a person in one autonomous set can be compared with that of a person in another, disjoint such set.

1.1 Axiomatic approach

Conceptually, there is much similarity between the specific notion of authority measure for opinion dynamics and the general notion of solution concept in cooperative game theory. A solution concept assigns to each game in a particular class of n-player cooperative, or coalitional, games an n-tuple of real numbers, which reflect each player's "value" in the game. In opinion dynamics, the dependence of the final opinion of a specific individual on the totality of initial opinions may be viewed as a monotonic transferable utility coalitional game. The class of consensual dynamics is thus essentially the class of monotonic games recast as a special case of opinion dynamics, while the general case involves n "simultaneous" games. An authority measure may be interpreted as expressing the players' value in these n games, taken together.

There is a long line of tradition in cooperative game theory of deriving solution concepts from a small set of plausible axioms. Several such works are especially relevant to our own.

Dubey [5] addressed the question, first raised by Lloyd Shapley, of whether an axiomatic foundation could be obtained for a value in the context of the subclass of simple superadditive games. He showed that the Shapley value itself can be obtained by replacing the usual linearity axiom with one that is meaningful for the subclass in question, which is not a linear subspace of the class of all games.

In the class of *all* games, Young [16] showed that axiomatic characterization for the Shapley value is obtained by replacing the additivity and the null player axioms with the single axiom of strong monotonicity. The axiom states that if a player's marginal contribution to the worth of every coalition in one game is the same or higher than in a second game, then the player's value is also at least as high in the first game as in the second game.

Neyman [13] extended Young's result to a large family of classes of games. In each class, the strong monotonicity requirement, which Neyman refers to as strong positivity, together with the axioms of efficiency and symmetry uniquely characterize the Shapley value.

It was shown by Monderer and Neyman [12] that strong positivity can replace additivity also in the axiomatization of the Aumann-Shapley value in the context of smooth nonatomic games. The Aumann-Shapley value is given by the so-called diagonal formula. The diagonal is the set of all "fuzzy coalitions" in which all players have the same partial membership $0 \le t \le 1$. Fuzzy coalitions and partial membership are also relevant in our setting (which has finitely many agents, not a continuum), as they represent uncertain initial opinions (see above). It will therefore not be surprising to find that a version of the diagonal formula is relevant too.

The Banzhaf value is a familiar solution concept similar to the Shapley value except that it does not satisfy the efficiency axiom. Lehrer [11] proposed axiomatizations for the Banzhaf value that replace efficiency with alternative axioms, the central of which is called amalgamation.

1.2 Hu-Shapley authority measure

Of central importance to our work is a specific authority measure that we call the Hu-Shapley authority measure. It is a straightforward generalization of the "authority distribution" concept put forward by Hu and Shapley [9] in a context that concerned only simple games (which corresponds to opinion dynamics where both the domain and the image are the vertices of the n-dimensional cube; in other words, only binary vectors are involved). Similarly to the Shapley non-transferable utility (NTU) value [15], the Hu-Shapley authority measure is defined in a seemingly circular way. First, the (regular, TU) Shapely value is calculated for each of the games returning the final opinion of a specific person as a function of everyone's initial opinions. Then, the Hu-Shapley authority measure is defined as the weighted average of these n Shapley values, with weights given by (or, more generally, proportional to) the Hu-Shapley authority measure itself. That this implicit definition actually has a solution follows from the theory of eigenvalues of stochastic matrices. Under the restrictions on the opinion dynamics that are detailed above, the solution is unique, making the authority distribution a well-defined authority measure.

1.3 Main results

In the setting of opinion dynamics described above, the Hu-Shapley authority measure is uniquely picked up by three axioms, which we call efficiency, symmetry and marginality. The marginality axiom is an adaptation of Young's strong monotonicity axiom from TU games to the more involved present setting. This is the content of our Theorem 3.1, which concerns the case of binary initial opinions.

In the case of fractional initial opinions, Theorem 5.1 shows that a unique authority measure is determined by four axioms. Three are similar to those determining the Hu-Shapley authority measure. The fourth axiom, called the replication axiom, connects dynamics with n and $n-1$ agents, with the latter constructed by replacing one agent with two replicas. This authority measure is defined similarly to the Hu-Shapley authority measure but employs the so-called diagonal formula from the Aumann-Shapley value rather than the Shapley value.

With binary initial opinions, an alternative to the Hu-Shapley authority measure can be obtained by employing the Banzhaf value instead of the Shapley value. As Theorem 7.3 shows, the Banzhaf authority measure is characterized by five axioms, one of which is an adaptation of amalgamation. This characterization builds on Lehrer's [11] characterization of the Banzhaf value.

The closest paper to ours is Karos and Peters [10]. These authors provide an axiomatic foundation for a family of authority measures defined on a class of opinion dynamics. These authority measures include the Hu-Shapley authority measure as a special case. Their axioms are analogous to Dubey's [5] axioms for the Shapley-Shubik power index. The main differences between the model studied by Karos and Peters [10] and our model are that (i) their setting only allows for binary final opinions, (ii) the set of axioms does not single out a unique solution, and (iii) these axioms are in the spirit of Shapley's original axioms while we replace linearity with the marginality axiom.

2 Basic model

There is a finite set of players $N = \{1, 2, \ldots, n\}$, whose power set is denoted 2^N. An *opinion dynamics*, or simply *dynamics*, on N is a function $d = (d_1, d_2, \ldots, d_n) : 2^N \longrightarrow \mathbb{R}^n$ such that, for all $i \in N$ and $A, B \subseteq N$ with $A \subseteq B$,

$$0 = d_i(\varnothing) \le d_i(A) \le d_i(B) \le d_i(N) = \|d\|,$$

where $\|d\| > 0$ is a constant, called the *norm* of d.

The interpretation is that $A \subseteq N$ represents the subset of players whose initial opinion is 1, Yes, with the opinion of everyone else being 0, No. For each $i \in N$, d_i returns the player's final opinion, which as discussed in the introduction, may also be a noninteger. In the case of unanimity of the initial opinions ($A = \varnothing$ or N), we require the final opinions to be identical as well. This requirement may be viewed as a (very weak) symmetry assumption: identical, uniform input results in identical, uniform output.

A set $A \subseteq N$ is called *autonomous* if it is nonempty and satisfies

$$d_i(T) = d_i(T \cap A), \quad i \in A, T \subseteq N.$$

In words: the opinions of the members of A are not influenced by the opinions of those outside A. Note that if A and B are autonomous, then the intersection $A \cap B$ is either empty or is also autonomous. Therefore, minimal autonomous sets are pairwise disjoint. A dynamics is called *simple* if there are no two disjoint autonomous sets or, equivalently, if there is a unique minimal autonomous set. Otherwise, the dynamics is *compound*.

As explained in the introduction, we restrict attention to simple dynamics, and view compound dynamics as comprising several simple ones. The class of all simple dynamics on N is denoted $\mathcal{D}(N)$.

A *measure* on N is an additive mapping $\mu: 2^N \longrightarrow \mathbb{R}_+$. The set of all nonzero measures on N is denoted $\mathcal{M}(N)$. For $\mu \in \mathcal{M}(N)$ and $x \in \mathbb{V}^n$, where $\mathbb{V}$ is any real vector space, we call the following quantity x *weighted by* μ:

$$\mu \odot x := \frac{1}{\mu(N)} \sum_{i \in N} \mu(\{i\}) x_i.$$

An *authority measure* on $\mathcal{D}(N)$ is a mapping $\psi: \mathcal{D}(N) \longrightarrow \mathcal{M}(N)$, $d \mapsto \psi d$. We use $\psi_i d$ as short for $\psi d(\{i\})$, the *authority* of player i in the dynamics d. This allows us to view ψd also as an n-tuple, an element of $\mathbb{R}^n_+ \setminus \{0\}$, with ith component $\psi_i d$.

We view the *ratios* between the players' authorities as having the primary significance. These ratios do not change when all authorities are scaled by the same positive factor, which gives us the freedom to make any choice regarding the sum of the authorities. A useful, and quite natural, choice that we impose on authority measures is the following.

Axiom 1 (Efficiency). For all $d \in \mathcal{D}(N)$,

$$\psi d(N) = \sum_{i \in N} \psi_i d = \|d\|.$$

A second natural axiom asserts that players are not treated differently based on their indices; if we permute the players, their authority is correspondingly permuted. Formally, consider the group S_N of all permutations on N. This group acts on n-tuples in a natural way: for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\pi \in S_N$, πx is defined by

$$(\pi x)_{\pi(i)} = x_i, \quad i \in N.$$

Permutations also act on dynamics. For $d \in \mathcal{D}(N)$ and $\pi \in S_N$, the dynamics $\pi d \in \mathcal{D}(N)$ is defined by

$$\pi d(\pi(A)) = \pi(d(A)), \quad A \subseteq N.$$

Axiom 2 (Symmetry). For all $d \in \mathcal{D}(N)$ and $\pi \in S_N$,

$$\psi(\pi d) = \pi(\psi d).$$

A corollary of symmetry is that, if two players are substitutes, then they have the same authority. Players i and j are *substitutes* in a dynamics d if $\pi_{ij} d = d$, where π_{ij} is the transposition interchanging i and j.

Our third, and most significant, axiom concerns the individual players' effect on the dynamics. The *marginal contribution* of player $i \in N$ to dynamics $d \in \mathcal{D}(N)$ is given by the function $\Delta_i d: 2^N \longrightarrow \mathbb{R}^n_+$ defined by

$$\Delta_i d(A) = d(A \cup \{i\}) - d(A).$$

The player's *weighted marginal contribution* to d is the marginal contribution weighted by ψd,

$$\psi d \odot \Delta_i d = \frac{1}{\sum_{j \in N} \psi_j d} \sum_{j \in N} \psi_j d \cdot \Delta_i d_j,$$

where $\Delta_i d_j$ is the jth component of $\Delta_i d$, that is, the marginal contribution of i's initial opinion to the final opinion of j. We require that, if a player i has the same weighted marginal contribution in two dynamics d and d', then the player's authority is also the same. This requirement reflects the self-referential idea that a person's authority is measured by his influence on people of authority.

Axiom 3 (Marginality)**.** For all $d, d' \in \mathcal{D}(N)$ and $i \in N$,

$$\psi d \odot \Delta_i d = \psi d' \odot \Delta_i d' \Longrightarrow \psi_i d = \psi_i d'.$$

Authority measures satisfying the next axiom are determined by their restriction to dynamics with norm 1, or normalized dynamics. For any dynamics d, the authorities are the same as in the normalized version of d, except that they are multiplied by $\|d\|$. As indicated, we view only the ratios between the players' authorities as significant. From this perspective, the last property entails that scaling dynamics by any positive factor leaves the authorities essentially unchanged.

Axiom 4 (Homogeneity)**.** For all $d \in \mathcal{D}(N)$ and $\lambda > 0$, $\psi(\lambda d) = \lambda\, \psi d$.

In certain settings (see below), the following property may also be desirable.

Axiom 5 (Null player)**.** For all $d \in \mathcal{D}(N)$, ψd is supported in the minimal autonomous set of d. That is, for every player i outside this set, $\psi_i d = 0$.

Axioms of the same names as Axioms 1–5 are known in the context of solution concepts for transferable utility (TU) coalitional games. A natural way to derive the TU analogs of our axioms is as follows. For any non-zero monotonic TU game $v: 2^N \longrightarrow \mathbb{R}_+$ there is a corresponding simple dynamics $\bar{v} := (v, \ldots, v) \in \mathcal{D}(N)$. We call a dynamics of this form *consensual.* Restriction to the set of consensual dynamics induces the TU-games analog of each axiom.

3 Basic results

Hu and Shapley [9] proposed a particular authority measure, which they called the "authority distribution". We call it the *Hu-Shapley authority measure* and denote it by Ψ.

There are two equivalent definitions of Ψ. The first, which is the one used by Hu and Shapley, presents it as the stationary measure of a certain Markov chain. The second defines Ψ as the unique solution to a Shapley value consistency condition.

For $d \in \mathcal{D}(N)$, consider the $n \times n$ matrix P_d whose ith row is $\mathrm{Sh}(d_i)$, the Shapley value of the TU game d_i. Thus,

$$P_d(i, j) = \mathrm{Sh}_j(d_i), \quad i, j \in N.$$

By the simplicity of d, the matrix $\frac{1}{\|d\|} P_d$ is a stochastic matrix with a unique stationary distribution. The authority measure Ψd is defined as the stationary distribution multiplied by $\|d\|$. Equivalently, it is the unique measure $\mu \in \mathcal{M}(N)$ satisfying

$$\mu(\{j\}) = \frac{1}{\mu(N)} \sum_{i \in N} \mu(\{i\}) \mathrm{Sh}_j(d_i), \quad j \in N. \tag{1}$$

Note that, by the linearity of the Shapley value, the expression on the right-hand side of Eq. (1) is the Shapley value of the TU game

$$\mu \odot d = \frac{1}{\mu(N)} \sum_{j \in N} \mu(\{j\}) d_j.$$

Therefore, Ψd can be defined more succinctly as the unique solution $\mu \in \mathcal{M}(N)$ to the consistency condition

$$\mu = \mathrm{Sh}(\mu \odot d). \tag{2}$$

Our main result is the following theorem.

Theorem 3.1. *The unique authority measure on $\mathcal{D}(N)$ satisfying the efficiency, symmetry and marginality axioms is the Hu-Shapley authority measure.*

It follows immediately from the characterization (2) that the Hu-Shapley authority measure also satisfies the homogeneity axiom. It satisfies the null player axiom, too, and in fact, has an even stronger property. Namely, the players outside the minimal autonomous set are the *only* players with zero authority.

Proposition 3.2. *For $d \in \mathcal{D}(N)$ and $i \in N$, $\Psi_i d > 0$ if and only if i belongs to the minimal autonomous set of d.*

4 Network centrality

A central application of authority measures is measuring centrality in networks. Here, the opinion dynamics reflects the topology of a network whose vertices are the players.

For example, the 0-*biased majority dynamics*[1] for a connected undirected graph works as follows. At time 0, each player i holds an opinion $x_0(i) \in \{0,1\}$. Then, at time $t = 1, 2, \ldots$, player i updates his opinion to match the majority opinion of its neighbors, that is, the players connected with i in the graph. In case of a tie, the opinion is set to 0. Formally, the majority updating rule is defined by

$$x_t(i) = \begin{cases} 1 & \left|\{j \in N_i \mid x_{t-1}(j) = 1\}\right| > \frac{1}{2}|N_i| \\ 0 & \left|\{j \in N_i \mid x_{t-1}(j) = 1\}\right| \leq \frac{1}{2}|N_i| \end{cases},$$

where N_i is the set of neighbors of i.

There are several opinion dynamics that can be derived from a given majority dynamics for a graph G. For example, we may view the majority updating rule $d^G : x_0 \mapsto x_1$ as an opinion dynamics. Alternatively, any $t \geq 2$, we may consider the t-fold composition of d^G,

$$(d^G)^t : x_0 \mapsto x_t.$$

Another alternative is the limit opinions,[2]

$$(d^G)^\infty := x_0 \mapsto \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} x_t \qquad \Big(= \lim_{t\to\infty} \frac{x_t + x_{t+1}}{2} \Big).$$

Let us compute Ψd^G. The matrix whose ith row is $\mathrm{Sh}(d_i^G)$ is exactly the Markov matrix of a simple random walk on G. Therefore, the authority measure Ψd^G is the stationary distribution of a simple random walk on G, which is proportional to the vertices' degrees.

The degrees of vertices in a network is an example of a *network centrality measure* (see section 2.2 in [3]). Such measures assign to each agent in a social network a centrality that is based on the topology of the network and possibly other factors. One example is the *degree centrality*, which measures the number of neighbors each agent has. Another is the *eigenvector centrality*, which assigns to each agent a centrality that depends on the centrality of his neighbors.

As shown above, we can derive the degree centrality as the authority measure of the dynamics d^G.[3] This derivation can be generalized. Network opinion dynamics comprise two components: a network and an opinion-updating rule. Fixing the updating rule while letting the network vary allows us to obtain a network centrality measure from an opinion dynamics authority measure. Then, by varying the updating rule, we get a whole family of network centrality measures from the same opinion dynamics authority measure. This way, established as well as new network centrality measures can be obtained.

[1] This also applies to the 1-biased majority dynamics.

[2] Goles and Olivos [6] showed that x_t is eventually periodic with a period of at most 2.

[3] The measures $\Psi(d^G)^t$, $t \geq 2$, and $\Psi(d^G)^\infty$ are also examples of network centrality measures. However, computing these measures is an open problem.

5 Smooth dynamics

A different kind of opinion dynamics concerns initial opinions that can be fractions rather than only 0 and 1. A *smooth dynamics* on N is a continuously differentiable function $r = (r_1, r_2, \ldots, r_n) : [0,1]^n \longrightarrow \mathbb{R}^n$ such that, for all $i \in N$ and $a, b \in [0,1]^n$ with $a \le b$ (here, and elsewhere in this paper, inequality between vectors means component-wise inequality), and with $\bar{0}$ and $\bar{1}$ denoting the vectors of all 0's and all 1's respectively,

$$0 = r_i(\bar{0}) \le r_i(a) \le r_i(b) \le r_i(\bar{1}) = \|r\|,$$

where $\|r\| > 0$ is a constant, called the *norm* of r.

A set $A \subseteq N$ is *autonomous* if it is nonempty and satisfies

$$\big(\forall i \in A \; a_i = b_i\big) \implies \big(\forall i \in A \; r_i(a) = r_i(b)\big), \quad a, b \in [0,1]^n.$$

A smooth dynamics is *simple* if it has a unique minimal autonomous set. The set of all simple smooth dynamics on N is denoted $\mathcal{R}(N)$.

An *authority measure* on $\mathcal{R}(N)$ is a mapping $\varphi: \mathcal{R}(N) \longrightarrow \mathcal{M}(N)$, $r \mapsto \varphi r$. We use $\varphi_i r$ as short for $\varphi r(\{i\})$, the *authority* of player i in r, and view φr also as an n-tuple.

The efficiency axiom is naturally adapted for smooth dynamics as follows.

Axiom 1′ (Efficiency)**.** For all $r \in \mathcal{R}(N)$,

$$\varphi r(N) = \sum_{i \in N} \varphi_i r = \|r\|.$$

For symmetry, we first define the action of permutations on smooth dynamics similarly to the definition for dynamics. For $r \in \mathcal{R}(N)$ and $\pi \in S_N$, $\pi r \in \mathcal{R}(N)$ is defined by

$$\pi r(\pi x) = \pi(r(x)), \quad x \in [0,1]^n.$$

Axiom 2′ (Symmetry)**.** For all $r \in \mathcal{R}(N)$ and $\pi \in S_N$,

$$\varphi(\pi r) = \pi(\varphi r).$$

In the context of smooth dynamics, the marginality axiom is replaced by a somewhat stronger requirement, which is similar to Young's [16] axiom of strong monotonicity. Loosely speaking, it asserts that a player's authority is a *monotonic* function function of his weighted marginal contributions rather than just a function. Formally, define the *marginal contribution* of player $i \in N$ to a smooth dynamics $r \in \mathcal{R}(N)$ as the function $\partial_i r: [0,1]^n \longrightarrow \mathbb{R}_+^n$ given by

$$\partial_i r(a) = \frac{\partial r}{\partial a_i}(a).$$

Strong monotonicity is the requirement that, if player i's weighted marginal contribution in smooth dynamics r is the same or higher than in r', then the player's authority is also the same or higher. In other words, higher weighted marginal contribution entails greater authority.

Axiom 3′ (Strong monotonicity)**.** For all $r, r' \in \mathcal{R}(N)$ and $i \in N$,

$$\varphi r \odot \partial_i r \ge \varphi r' \odot \partial_i r' \implies \varphi_i r \ge \varphi_i r'.$$

It turns out, however, that even with this stronger form of Axiom 3, the above three axioms are not sufficient for determining a unique authority measure on $\mathcal{R}(N)$. Here are two examples of authority measures that satisfy the axioms.

The first authority measure is a naive derivative of the Hu-Shapley authority measure Φ on $\mathcal{D}(N)$. It is defined as the composition $\Phi \circ \iota$, where $\iota: \mathcal{R}(N) \to \mathcal{D}(N)$ is the restriction of a smooth dynamics to the cube's vertices $\{0,1\}^n$ (which is identified with the power set of N), $\iota: r \mapsto r|_{2^N}$.

The second authority measure is conceptually similar to the Hu-Shapley authority measure but replaces the Shapley value with what may be referred to as the Aumann-Shapley value [1]. The Aumann-Shapley value of a smooth function $f:[0,1]^n \longrightarrow \mathbb{R}$ is the vector

$$\mathrm{AS}(f) := \int_0^1 \nabla f(t,\dots,t)\,\mathrm{d}t,$$

where the nabla symbol ∇ denotes the gradient.

For $r \in \mathcal{R}(N)$, Φr is defined as the unique solution $\mu \in \mathcal{M}(N)$ of the equation

$$\mu = \mathrm{AS}(\mu \odot r). \tag{3}$$

The existence and uniqueness of such μ is proved by the same arguments applied to the authority distribution in Section 3 (and the linearity of the Aumann-Shapley value), that is, by considering a corresponding stationary distribution for the $n \times n$ matrix whose i's row is $\mathrm{AS}(r_i)$, $i \in N$. We call Φ the *Aumann-Shapley authority measure.*

To pick up a unique authority measure for smooth dynamics, an additional axiom is required. This axiom concerns dynamics with different numbers of players, n and $n-1$. It is a consistency requirement for authority measures that are defined both on the space $\mathcal{R}(N)$ and on $\mathcal{R}(N \setminus \{n\})$.

For a smooth dynamics $r \in \mathcal{R}(N \setminus \{n\})$, consider the dynamics $\hat{r} \in \mathcal{R}(N)$ obtained by replicating the $(n-1)$th player in r and averaging the initial opinion of the player's two copies, which are players $n-1$ and n in $\hat{r}$. That is,

$$\hat{r}_i(a_1,\dots,a_n) = r_{\min\{i,n-1\}}(a_1,\dots,a_{n-2},\tfrac{1}{2}(a_{n-1}+a_n)), \quad i \in N, a \in [0,1]. \tag{4}$$

From the perspective of the other $n-2$ players, there is not much that tells $\hat{r}$ apart from r. It therefore seems natural to require that these players' authorities be the same in the two dynamics.

Axiom 6 (Replication)**.** For all $n \geq 3$ and $r \in \mathcal{R}(N \setminus \{n\})$, $\varphi_i \hat{r} = \varphi_i r$ for $i = 1,\dots,n-2$.

With the notation $\mathcal{R} := \bigcup_{n=1}^{\infty} \mathcal{R}(\{1,\dots,n\}))$ and $\mathcal{M} := \bigcup_{n=1}^{\infty} \mathcal{M}(\{1,\dots,n\})$, an *authority measure* on $\mathcal{R}$ is a graded mapping $\varphi: \mathcal{R} \longrightarrow \mathcal{M}$, that is, a mapping satisfying $\varphi(\mathcal{R}(\{1,\dots,n\}) \subseteq \mathcal{M}(\{1,\dots,n\})$ for all $n \geq 1$.

Theorem 5.1. *The unique authority measure on $\mathcal{R}$ satisfying the efficiency, symmetry, strong monotonicity and replication axioms is the Aumann-Shapley authority measure.*

Like the Hu-Shapley authority measure, the Aumann-Shapley authority measure Φ satisfies the additional axiom of homogeneity (which follows immediately from the characterization (3)) and the stronger version of the null player axiom.

6 Network centrality from smooth dynamics

Smooth dynamics give rise to additional kinds of centrality measures.

Consider, for example, the opinion dynamics derived from the DeGroot [4] learning model for a connected graph G, which is defined as follows. At time 0, each player i holds an opinion $x_0(i) \in [0,1]$. Then, at time $t = 1,2,\dots$, player i updates his opinion to the average opinion of its neighbors in G, the set N_i. Formally, the DeGroot updating rule is defined by

$$x_t(i) = \tfrac{1}{|N_i|} \sum_{j \in N_i} x_{t-1}(j).$$

More generally, a weighted DeGroot updating rule is defined by taking a weighted average of the opinions of the neighbors,

$$x_{t+1} = Px_t,$$

where P is an irreducible stochastic matrix.

The irreducibility of P implies that the smooth opinion dynamics $r : x_0 \mapsto x_1$ is simple. Let us compute its Aumann-Shapley authority measure. The matrix whose ith row is $\mathrm{AS}(r_i)$ is here P itself. Therefore, the meaning of (3) is that μ is a left eigenvector of P whose components sum up to 1. Thus, Φr is the stationary distribution of the Markov chain defined by P. This network centrality measure is known as the *eigenvector centrality.* Since P and P^t have the same eigenvectors, we get that Ψr^t is the same as Ψr, for all $t \geq 2$.

7 Variations

7.1 Imbalanced dynamics

The assumption that $d_i(N) = d_j(N)$ for all $i, j \in N$ may not hold for all applications. However, it is possible to extend our framework to include such *imbalanced* dynamics. This requires some modifications which we present in this subsection.

The class $\mathcal{ID}(N)$ of *simple imbalanced dynamics* is a superclass of $\mathcal{D}(N)$ that is defined in a similar way except that the assumption that there is a number $\|d\| > 0$ such that $d_i(N) = \|d\|$ for all $i \in N$ is replaced by the weaker assumption that $d_i(N) > 0$ for all $i \in N$.

An authority measure on $\mathcal{ID}(N)$ is a mapping $\psi{:}\mathcal{ID}(N) \longrightarrow \mathcal{M}(N)$. The symmetry and marginality axioms for $\mathcal{ID}(N)$ are defined similarly to their definition for $\mathcal{D}(N)$. The efficiency axiom is extended to

$$\min_{i \in N} d_i(N) \leq \psi d(N) \leq \max_{i \in N} d_i(N).$$

We say that an authority measure ψ is *more concentrated* than another one ψ' if for all $d \in \mathcal{ID}(N)$ the support of ψd is contained in that of $\psi' d$.

Theorem 7.1. *There is a unique authority measure Ψ on $\mathcal{ID}(N)$ that satisfies the efficiency, symmetry and marginality axioms and is more concentrated than any other authority measure satisfying these axioms. For all $d \in \mathcal{ID}(N)$, the support of Ψd is the minimal autonomous set of d.*

We call the authority measure Ψ identified by Theorem 7.1 the Hu-Shapley authority measure, as it is obviously an extension of the authority measure on $\mathcal{D}(N)$ bearing the same name. The reason for the more indirect characterization here is that, for $d \in \mathcal{ID}(N)$, while $\mu = \Psi d$ solves Eq. (2), it is not necessarily the only solution. However, by invoking Axiom 5, we get as an immediate corollary of Theorem 7.1 a result more similar to Theorem 3.1.

Proposition 7.2. *The unique authority measure on $\mathcal{ID}(N)$ satisfying the efficiency, symmetry, marginality and null player axioms is the Hu-Shapley authority measure.*

7.2 Other power indices

Hu and Shapley's formulation of their authority measure concerned only dynamics with 0 or 1 values. Correspondingly, it invoked the Shapley-Shubik power index that concerns only simple games. The Shapley-Shubik power index is generalized by the Shapley value. Other power indices or values could be used instead. An example is the Banzhaf value B, for which the analog of Eq. (2) is

$$\mu = \mathrm{B}(\mu \odot d). \tag{5}$$

Eq. (5) may have multiple solutions in $\mathcal{M}(N)$. However, we show in Section 8.4 that exactly one solution is supported in the minimal autonomous set of d. Thus, there is a unique authority measure ψ on $\mathcal{D}(N)$ satisfying the null player axiom such that, for all $d \in \mathcal{D}(N)$, $\mu = \psi d$ solves Eq. (5). We call it the *Banzhaf authority measure.*

The Banzhaf authority measure cannot be expected to satisfy efficiency, for the reason that the Banzhaf value itself is not efficient. The efficiency axiom is replaced in the present context by two axioms that are inspired by Lehrer's [11] characterization of the Banzhaf value.

For $d \in \mathcal{D}(N)$ and $i, j \in N$, the *amalgamation of player* j *into* i is the dynamics $d^{i \leftarrow j} \in \mathcal{D}(N)$ defined by

$$d^{i \leftarrow j}(A) = \begin{cases} d(A \cup \{j\}) & i \in A, \\ d(A \smallsetminus \{j\}) & i \notin A. \end{cases}$$

We say that $d \in \mathcal{D}(N)$ is a *dictatorship* and that $i \in N$ is a *dictator* if $d = d^{i \leftarrow j}$ for every $j \in N$, or equivalently,

$$d(A) = \begin{cases} (\|d\|, \dots, \|d\|) & i \in A, \\ (0, \dots, 0) & i \notin A. \end{cases}$$

Axiom 1a (Dictatorship)**.** If d is a dictatorship, then $\sum_{i \in N} \psi_i d = \|d\|$.

Axiom 1b (Amalgamation)**.**

$$\psi_i d^{i \leftarrow j} = \psi_i d + \psi_j d, \quad d \in \mathcal{D}(N), \; i, j \in N.$$

Theorem 7.3. *The unique authority measure on* $\mathcal{D}(N)$ *satisfying the null player, symmetry, marginality, dictatorship and amalgamation axioms is the Banzhaf authority measure.*

Remark. Lehrer [11] defines amalgamation somewhat differently. His definition involves two players who are replaced by a single player whose marginal contribution is that of the two players together. Thus, the resulting game has one fewer player than the original game. We chose to slightly modify Lehrer's definition so as to keep the set of players intact, which allows us to discuss authority measures on $\mathcal{D}(N)$ rather than $\bigcup_{n=1}^{\infty} \mathcal{D}(\{1, \dots, n\})$. We could alternatively maintain Lehrer's formulation. Doing so would allow us to limit the dictatorship axiom to single-player games only.

8 Proofs

The proofs of our theorems rely on reductions to TU games.

8.1 TU games preliminaries

The class of all TU games on the set of players N is denoted $\mathcal{G}(N)$. The sub-classes of additive and monotonic games are denoted $\mathcal{A}(N)$ and $\mathcal{G}^+(N)$ respectively. Their intersection is denoted $\mathcal{A}^+(N)$. It includes the zero game o, where the worth of every coalition is 0. A *point solution concept* on a class of games $\mathcal{K} \subseteq \mathcal{G}(N)$ is a function $\alpha : \mathcal{K} \longrightarrow \mathcal{A}(N)$. We use $\alpha_i v$ as short for $\alpha v(\{i\})$. This allows us to view αv also as an element of $\mathbb{R}^n_+$, with ith component $\alpha_i v$.

With the interpretation of additive games as signed measures and the identification of a consensual dynamics $\bar{v} = (v, v, \dots, v)$ with the TU game v, restriction of an authority measure to consensual dynamics induces a point solution concept on the class $\mathcal{G}^+(N) \smallsetminus \{o\}$ of non-zero monotonic games. Any axiom formulated for authority measures can thus be interpreted as also pertaining to point solution concepts. To make this interpretation explicit and to extend it to all games (not only monotonic ones), we spell out below the meaning of these axioms regarding TU games.

The counterpart of Axiom 1 for TU games is *efficiency* of a point solution concept α, which means

$$\sum_{i \in N} \alpha_i(v) = v(N), \quad v \in \mathcal{K}.$$

For Axiom 2, the action of a permutation $\pi \in S_N$ on a game $v \in \mathcal{G}(N)$, denoted πv, is given by

$$\pi v(\pi(A)) = v(A), \quad A \subseteq N.$$

A class of TU games $\mathcal{K}$ is *closed under permutations* if $\pi v \in \mathcal{K}$ for every $\pi \in S_N$ and $v \in \mathcal{K}$. A point solution concept α on a class of games $\mathcal{K}$ is *symmetric* if $\mathcal{K}$ is closed under permutations and

$$\pi\alpha(v) = \alpha(\pi v), \quad \pi \in S_N, v \in \mathcal{K}.$$

The counterpart of Axiom 5 is the following one. A player $i \in N$ is called a *null player* in a game $v \in \mathcal{G}(N)$ if

$$v(A) = v(A \cup \{i\}), \quad A \subseteq N.$$

A point solution concept α on a class of games $\mathcal{K}$ satisfies the *null player axiom* if $\alpha_i(v) = 0$ whenever i is a null player in a game $v \in \mathcal{K}$.

In the context of opinion dynamics, Axiom 3 uses the notion of weighted marginal contributions, which is well defined only when the weights are nonnegative and not all zero. However, for *consensual* dynamics the weights are irrelevant, and so we can do without them. This enables the counterpart of Axiom 3 for TU games to be relevant also for solution concepts that return signed, rather than only unsigned, measures.

For $v \in \mathcal{G}(N)$ and $i \in N$, the *marginal contribution* of i to v at $A \subseteq N$ is the difference $v(A \cup \{i\}) - v(A)$. A point solution concept α satisfies *marginality* if

(I) its domain satisfies $\mathcal{K} = \mathcal{K} + \mathcal{A}^{+}(N)$, and

(II) for every $v, u \in \mathcal{K}$ and $i \in N$, if the marginal contributions of player i to v and u are equal at every $A \subseteq N$ (equivalently, i is a null player in $v - u$), then $\alpha_i(v) = \alpha_i(u)$.

For $v \in \mathcal{G}(N)$ and $i, j \in N$, the *amalgamation* of j into i is the game $v^{i \leftarrow j} \in \mathcal{G}(N)$ defined by

$$v^{i \leftarrow j}(A) = \begin{cases} v(A \cup \{j\}) & i \in A, \\ v(A \setminus \{j\}) & i \notin A. \end{cases}$$

A game $v \in \mathcal{G}(N)$ is a *dictatorship*, with $i \in N$ being a dictator, if $v^{i \leftarrow j} = v$ for every $j \in N$, equivalently, if $v = a e_i$, where e_i is the unanimity game with carrier $\{i\}$ and a is any real number.

The counterpart of Axiom 1a is the following *dictatorship axiom*: if v is a dictatorship, then $\sum_{i \in N} \alpha_i(v) = v(N)$.

For the counterpart of Axiom 1b, we require the domain $\mathcal{K}$ to be closed under amalgamations. A point solution concept α on $\mathcal{K}$ satisfies the *amalgamation axiom* if for every $v \in \mathcal{K}$ and $i, j \in N$

$$v^{i \leftarrow j} \in \mathcal{K}, \text{ and}$$
$$\alpha_i(v^{i \leftarrow j}) = \alpha_i(v) + \alpha_j(v).$$

Two additional properties of a point solution concept α on $\mathcal{K} \subseteq \mathcal{G}(N)$, which do not have counterparts in the context of opinion dynamics, are the following. It has the *dummy player property* if for every $i \in N$ and $v \in \mathcal{K}$ the equalities

$$v(A \cup \{i\}) - v(A) = v(\{i\}), \quad A \subseteq N$$

imply $\alpha_i(v) = v(\{i\})$. It is *translation covariant* if $\alpha(v) - \alpha(u) = v - u$ whenever $v - u \in \mathcal{A}(N)$.

The proof of the following proposition, which is by direct verification, is given in the appendix.

Proposition 8.1. *Let α be a translation covariant point solution concept on a class $\mathcal{K} \subseteq \mathcal{G}(N)$. There exists a unique translation covariant extension of α to $\mathcal{K} + \mathcal{A}(N)$.*

We say that a *property* $\mathbf{P}$ of point solution concepts is translation covariant if the following holds: whenever a translation covariant point solution concept α on a class of games $\mathcal{K}$ has the property $\mathbf{P}$, the translation covariant extension of α to $\mathcal{K} + \mathcal{A}(N)$ also has property $\mathbf{P}$.

The proof of the following proposition, which is by direct verification, is given in the appendix.

Proposition 8.2. *The dummy player, efficiency, symmetry, marginality and amalgamation axioms are translation covariant.*

A *smooth game* is a continuously differentiable function $f\colon [0,1]^N \longrightarrow \mathbb{R}$ satisfying $f(\bar{0}) = 0$. The class of all nonzero monotonic smooth games on the set of players N is denoted $\Gamma(N)$. A (positive) point solution concept on $\mathcal{K} \subseteq \Gamma(N)$ is a function $\beta\colon \mathcal{K} \longrightarrow \mathcal{M}(N)$.

The following lemma facilitates a reduction of the proofs of Theorems 3.1 and 7.3 to TU games, and that of Theorem 5.1 to smooth TU games.

Lemma 8.3. *Let ψ be an authority measure satisfying marginality on $\mathcal{D}(N)$, $\mathcal{ID}(N)$ or $\mathcal{R}(N)$. Let f be a point solution concept on $\mathcal{G}^+(N) \smallsetminus \{o\}$ in the first two cases and on $\Gamma(N)$ in the third case. If every consensual $\bar{v} = (v,\dots,v)$ in the domain of ψ satisfies $\psi\bar{v} = f(v)$, then all elements d in that domain satisfy*

$$\psi d = f(\psi d \odot d). \tag{6}$$

Proof. For d in the domain of ψ, consider the game $v = \psi d \odot d$ and the consensual element $\bar{v} = (v,\dots,v)$. It is easy to see that d and $\bar{v}$ have the same marginal contributions weighted by ψd and $\psi\bar{v}$ respectively. Therefore, the marginality assumption gives that $\psi d = \psi\bar{v}$. If ψ satisfies the stated condition, then $\psi\bar{v} = f(v) = f(\psi d \odot d)$, which gives Eq. (6). □

8.2 Proofs of Theorem 3.1 and Proposition 3.2

The proof of Theorem 3.1 uses an earlier result of Young [16]. We state here a variant of the original theorem that substitutes marginality for Young's (slightly stronger) strong monotonicity axiom, which is similar to our Axiom 3′ in the context of smooth dynamics. We remark, however, that our Theorem 3.1 would remain true if the marginality axiom were replaced by strong monotonicy.

Theorem 8.4 (Young [16], Theorem 2)**.** *The unique point solution concept on $\mathcal{G}(N)$ satisfying the efficiency, symmetry and marginality axioms is the Shapley value.*

Following is a variation of Theorem 8.4 that refers to the class of non-zero monotonic games rather than the class of all games.

Theorem 8.5. *The unique point solution concept on $\mathcal{G}^+(N) \smallsetminus \{o\}$ satisfying the efficiency, symmetry and marginality axioms is the Shapley value.*

Proof. Let $\# \in \mathcal{A}(N)$ be the counting measure, that is $\#(A)$ is the cardinality of A, for every $A \subseteq N$. Since for every $v \in \mathcal{G}(N)$, there exists $M > 0$ such that $v + M\#$ is strictly monotonic, we have $\mathcal{G}^+(N) + \mathcal{A}(N) = \mathcal{G}(N)$.

Let α be a point solution concept on $\mathcal{G}^+(N) \smallsetminus \{o\}$ satisfying the efficiency, symmetry and marginality axioms. The next step is to show that it is translation covariant.

For $v \in \mathcal{G}^+(N) \smallsetminus \{o\}$, $i \in N$ and $a \geq 0$, consider the game $u := v + ae_i$, where e_i denotes the unit unanimity game with carrier $\{i\}$. We claim that

$$\alpha(u) = \alpha(v) + ae_i.$$

The marginal contribution of any player $j \neq i$ at any coalition $A \subseteq N$ is the same in u and v. Therefore, by marginality of α, we have $\alpha_j(u) = \alpha_j(v)$. It follows that

$$\alpha_i(u) - \alpha_i(v) = \sum_{j \in N} \alpha_j(u) - \sum_{j \in N} \alpha_j(v) = v(N) + a - v(N) = a,$$

where the second equality holds by efficiency.

By repeated application of the last claim, it follows that $\alpha(v + l^+) = \alpha(v) + l^+$ for every $l^+ \in \mathcal{A}^+(N)$. Every $l \in \mathcal{A}(N)$ can be represented as $l = l^+ - l^-$, for some $l^+, l^- \in \mathcal{A}^+(N)$. If $v + l \in \mathcal{G}^+(N) \setminus \{o\}$, then

$$\alpha(v + l) - \alpha(v) = \big(\alpha(v + l) - \alpha(v + l^+)\big) + \big((\alpha(v + l^+) - \alpha(v)\big) = -l^- + l^+ = l.$$

This proves the translation covariance of α.

By Propositions 8.1 and 8.2, α can be extended to a point solution concept on $\mathcal{G}^+(N) + \mathcal{A}(N) = \mathcal{G}(N)$ satisfying efficiency, symmetry and marginality. By Theorem 8.4, we have that α is the Shapley value. □

Proof of Theorem 3.1. To see that Ψ satisfies axioms 1–3, consider $\mu \in \mathcal{M}(N)$ and $d \in \mathcal{D}(N)$ such that $\mu = \mathrm{Sh}(\mu \odot d)$, which means that $\mu = \Psi d$. By the efficiency of the Shapley value, $\mu(N) = (\mu \odot d)(N) = \|d\|$, which proves that Ψ satisfies Axiom 1. Let $\pi \in S_N$. Since $\pi(\mu \odot d) = (\pi\mu) \odot (\pi d)$ and the Shapley value satisfies symmetry, we have

$$\pi\mu = \pi(\mathrm{Sh}(\mu \odot d)) = \mathrm{Sh}(\pi(\mu \odot d)) = \mathrm{Sh}((\pi\mu) \odot (\pi d)),$$

which means that $\pi\mu = \Psi(\pi d)$. Since $\mu = \Psi d$, the last equality proves that Ψ satisfies Axiom 2. Lastly, consider also $\mu' \in \mathcal{M}(N)$ and $d' \in \mathcal{D}(N)$ such that $\mu' = \mathrm{Sh}(\mu' \odot d')$, which means that $\mu' = \Psi d'$, and suppose that $i \in N$ is such that

$$\mu \odot \Delta_i d = \mu' \odot \Delta_i d'.$$

The last equality can also be written as $\Delta_i(\mu \odot d) = \Delta_i(\mu' \odot d')$, which, by the marginality property of the Shapley value, gives $\mathrm{Sh}_i(\mu \odot d) = \mathrm{Sh}_i(\mu' \odot d')$, hence $\mu_i = \mu'_i$. Since $\mu = \Psi d$ and $\mu' = \Psi d'$, this gives $\Psi_i d = \Psi_i d'$. This proves that Ψ satisfies Axiom 3.

We turn to show uniqueness. Let ψ be an authority measure on $\mathcal{D}(N)$ satisfying efficiency, symmetry, and marginality. For $d \in \mathcal{D}(N)$, Ψd is defined as the unique solution of Eq. (2), which is a special case of Eq. (6) with f being the Shapley value Sh. Therefore, by Lemma 8.3, to prove that ψ is the Hu-Shapley authority measure, it suffices to show that the mapping $\alpha : v \mapsto \psi(v, \dots, v)$ agrees with the Shapley value on $\mathcal{G}^+(N) \setminus \{o\}$.

Indeed, since the point solution concept α inherits from ψ the properties of efficiency, symmetry and marginality on $\mathcal{G}^+(N) \setminus \{o\}$, Theorem 8.5 ensures that α agrees with the Shapley value.

□

Proposition 3.2 follows immediately from the characterization of the Hu-Shapley authority measure as the unique solution of Eq. (1) and from the following general lemma.

Lemma 8.6. *For d in $\mathcal{ID}(N)$ (in particular, in $\mathcal{D}(N)$), with minimal autonomous set $A \subseteq N$, let M be an $n \times n$ non-negative matrix such that, for all $i, j \in N$, $M(i,j) = 0$ if and only if j is a null player in the TU game d_i. Then, every $\mu \in \mathcal{M}(N)$ that solves the equations*

$$\mu(\{j\}) = \frac{1}{\mu(N)} \sum_{i \in N} \mu(\{i\}) M(i,j), \quad j \in N \tag{7}$$

satisfies $\mu(\{j\}) > 0$ for all $j \in A$, and there is a unique such solution that also satisfies $\mu(\{j\}) = 0$ for all $j \notin A$.

Proof. The support of any solution of Eq.(7) is an autonomous set, and must therefore include the unique minimal one A. It remains to show that there is a unique solution μ for which the inclusion holds as equality. The meaning of the equality is that the nonzero entries of μ constitute a nonnegative left eigenvector of the submatrix of M obtained by retaining only the rows and columns corresponding to the elements of A, and that their sum $\mu(N)$ is equal to the eigenvalue. As the submatrix is irreducible, the uniqueness of such an eigenvector follows from the Perron–Forbenius theorem. □

8.3 Proof of Theorem 5.1

The proof follows a similar line to that of the proof of Theorem 5 in [12], together with the proof of Theorem 3.1.

Recall that the class of all nonzero monotonic smooth games on the set of players $\{1,\dots,n\}$ is denoted $\Gamma(\{1,\dots,n\})$. Let $\Gamma \coloneqq \bigcup_{n=1}^{\infty} \Gamma(\{1,\dots,n\})$ be the union of these classes. A (positive) point solution concept on Γ is a graded mapping $\beta\colon \Gamma \longrightarrow \mathcal{M}$.

Axioms $1'$, $2'$, $3'$ and 6 have natural interpretations as properties of point solution concepts on classes of smooth games. Explicitly, let β be a point solution concept on a class $\mathcal{K} \subseteq \Gamma(N)$ or on $\mathcal{K} = \Gamma$. The natural embedding $\iota : f \mapsto (f,\dots,f)$ maps $\mathcal{K}$ to a subset of $\mathcal{R}(N)$ or $\mathcal{R}$ respectively. On this subset $\iota(\mathcal{K})$, the mapping $\varphi^{\beta} : (f,\dots,f) \mapsto \beta(f)$ is a "partial" authority measure. It is not defined on the entire $\mathcal{R}(N)$ or $\mathcal{R}$, yet there is a natural sense in which each of the above axioms may be said to hold for φ^{β}, which is that the corresponding requirement is satisfied for elements of $\iota(\mathcal{K})$. In this case, we say that β satisfies the axiom.

For a smooth game $f\colon [0,1]^N \longrightarrow \mathbb{R}$, the restriction of f to indicator vectors defines a TU game $\mathring{f}\colon 2^N \longrightarrow \mathbb{R}$ by $\mathring{f}(A) = f(\mathbf{1}_A)$. Denote by $\Lambda(N)$ the subclass of $\Gamma(N)$ consisting of all nonzero monotonic smooth games that are multilinear.

Lemma 8.7. *There is a unique point solution concept β on $\Lambda(N)$ that satisfies Axioms $1'$, $2'$ and $3'$, which is given by*

$$\beta(f) = \mathrm{Sh}\big(\mathring{f}\big). \tag{8}$$

Proof of Lemma 8.7. Since the Shapley value satisfies efficiency and symmetry, the point solution concept β defined by Eq. (8) satisfies Axioms $1'$ and $2'$. For $f \in \Lambda(N)$, the marginal contributions in $\mathring{f}$ coincide with the partial derivatives of f. Axiom $3'$ therefore follows from the strong monotonicity of the Shapley value (see the discussion at the beginning of Section 8.2).

To prove uniqueness, consider any point solution concept β on $\Lambda(N)$ that satisfies the above three axioms. Then, the solution concept α on $\mathcal{G}^{+}(N) \setminus \{o\}$ defined by

$$\alpha(v) = \beta(f),$$

where f is the multilinear extension of v (and so $v = \mathring{f}$), satisfies efficiency, symmetry and marginality. By Theorem 8.5, α is the Shapley value on $\mathcal{G}^{+}(N) \setminus \{o\}$, which gives (8). □

For a smooth game $f \in \Gamma(N)$, let $\|f\| \coloneqq \sum_{i\in N} \|\partial_i f\|_{\infty} = \sum_{i\in N} \sup_{x\in[0,1]^n} |\partial_i f(x)|$ and let $\hat{f} \in \Gamma(N)$ be the multilinear extension of the TU game $\mathring{f}$. For $k \in \mathbb{N}$, define f_k as the population game with n populations of size k that is derived from f by replacing each player i with k copies of that player, whose initial opinions are averaged. Specifically,

$$f_k \in \Gamma(\{(1,1),\dots,(1,k),\dots,(n,1),\dots,(n,k)\})$$

and is defined by

$$f_k(a_{1,1},\dots,a_{1,k},\dots,a_{n,1},\dots,a_{n,k}) = f\Big(\frac{1}{k}\sum_{j=1}^{k} a_{1,j},\dots,\frac{1}{k}\sum_{j=1}^{k} a_{n,j}\Big).$$

Lemma 8.8. *For every $f \in \Gamma(N)$, $\lim_{k\to\infty} \|\hat{f}_k - f_k\| = 0$.*

Proof of Lemma 8.8. We need to show that

$$\lim_{k\to\infty} \sum_{j=1}^{nk} \sup_{x\in[0,1]^{nk}} |\partial_j(\hat{f}_k - f_k)(x)| = 0.$$

Fix $k \in \mathbb{N}$, $x \in [0,1]^{nk}$ and player $p = (i,j)$ of the game f_k. Let $X_{-p} \in [0,1]^n$ be the following random variable. For each of the $nk - 1$ players $(i',j') \neq p$, take an independent Bernoulli random variable with parameter $x_{i',j'}$. Each coordinate $i' \in \{1,\dots,n\}$ of X_{-p} is the sum of the Bernoulli variables whose indices are of the form (i',j') divided by k.

Since $\hat{f}_k$ is multi-linear, we have

$$\partial_p \hat{f}_k(x) = \hat{f}_k(x_{-p},1) - \hat{f}_k(x_{-p},0) = \mathbb{E}[f(X_{-p} + \tfrac{1}{k}e_i) - f(X_{-p})],$$

where e_i denotes the ith standard basis vector in $\mathbb{R}^n$ and x_{-p} is x with the pth coordinate omitted. By Lagrange's mean value theorem, there is a random variable $z \in (0,1/k)$, such that

$$f(X_{-p} + \tfrac{1}{k}e_i) - f(X_{-p}) = \frac{\partial_i f(X_{-p} + ze_i)}{k}.$$

Since

$$\partial_p f_k(x) = \tfrac{1}{k}\partial_i f(\mathbb{E}[X_{-p}] + \tfrac{x_p}{k}e_i),$$

We get

$$\partial_p(\hat{f}_k - f_k)(x) = \tfrac{1}{k}\Big(\mathbb{E}[\partial_i f(X_{-p} + ze_i)] - \partial_i f(\mathbb{E}[X_{-p}] + \tfrac{x_p}{k}e_i)\Big).$$

Since f is continuously differentiable on a compact set, it is uniformly continuously differentiable. Therefore, it is sufficient to show that $X_{-p} - \mathbb{E}[X_{-p}]$ converges to $\bar{0}$ in probability as k goes to ∞ uniformly in p and x. To this end, it is sufficient to show that $\mathrm{Var}[X_{-p}] \xrightarrow{k\to\infty} 0$ uniformly in p and x. Indeed, the coordinates of X_{-p} are mutually independent and each coordinate is the sum of at most k independent Bernoulli trials divided by k. Therefore, the variance of each coordinate is bounded from above by $\frac{1}{4k}$, and therefore, $\mathrm{Var}[X_{-p}] \le \frac{n}{4k}$. □

Proof of Theorem 5.1. The easy part of the proof is to show that the Aumann-Shapley authority measure Φ satisfies the axioms. Efficiency follows from the definition of Φ in Eq. (3) and the fact that, for every smooth function f (in particular, for $f = \mu \odot r$),

$$\sum_{i=1}^{n} \int_0^1 \partial_i f(t,\dots,t)\,\mathrm{d}t = f(\bar{1}) - f(\bar{0}).$$

Symmetry follows from the symmetry of the Aumann-Shapley value AS, as in the proof of Theorem 3.1. Strong monotonicity holds because, for $\varphi = \Phi$, the assumption in Axiom 3′ gives (by integration along the diagonal)

$$\mathrm{AS}_i(\Phi r \odot r) \ge \mathrm{AS}_i(\Phi r' \odot r'),$$

which by Eq. (3) gives $\Phi_i r \ge \Phi_i r'$.

To prove that Axiom 6 holds, consider any $r \in \mathcal{R}(\{1,\dots,n-1\})$. Eq. (3), which can be written as

$$\mu_i = \int_0^1 \partial_i(\mu \odot r)(t,\dots,t)\,\mathrm{d}t, \quad i = 1,\dots,n-1,$$

has a unique solution in $\mathcal{M}(\{1,\dots,n-1\})$, which is $\mu = \Phi r$. With this specific μ, define $\hat{\mu} \in \mathcal{M}(\{1,\dots,n\})$ by $\hat{\mu}_i = \mu_i$ for $i = 1,\dots,n-2$ and $\hat{\mu}_{n-1} = \hat{\mu}_n = \frac{1}{2}\mu_{n-1}$. The smooth dynamics $\hat{r} \in \mathcal{R}(\{1,\dots,n\})$, which is defined by (4), satisfies $\partial_i(\hat{\mu}\odot\hat{r})(t,\dots,t) = \partial_i(\mu\odot r)(t,\dots,t)$ (with the number of t's being n on the left-hand side and $n-1$ on the right-hand side) for $i = 1\ldots, n-2$, and $\partial_{n-1}(\hat{\mu} \odot \hat{r})(t,\ldots,t) = \partial_n(\hat{\mu} \odot \hat{r})(t,\ldots,t) = \frac{1}{2}\partial_{n-1}(\mu \odot r)(t,\ldots,t)$. It follows that $\hat{\mu} = \mathrm{AS}(\hat{\mu} \odot \hat{r})$, and therefore $\hat{\mu} = \Phi\hat{r}$. By the relation between μ and $\hat{\mu}$, this proves that Axiom 6 holds for Φ.

The harder part of the proof is showing uniqueness.

Let φ be an authority measure satisfying Axioms $1'$, $2'$, $3'$ and 6. It follows immediately from the definition of the Aumann-Shapley authority measure that, for $v \in \Gamma(N)$, the consensual dynamics $\bar{v} = (v,\ldots,v) \in \mathcal{R}$ satisfies $\Phi\bar{v} = \mathrm{AS}(v)$. If, for every such v, $\varphi\bar{v} = \mathrm{AS}(v)$, then it follows from Lemma 8.3 (with $f = \mathrm{AS}$), that φ agrees with Φ on all smooth dynamics $r \in \mathcal{R}$. Thus, to prove that $\varphi = \Phi$, it is sufficient to show that φ and Φ agree on consensual dynamics.

The restriction of φ to the class of consensual dynamics induces a point solution concept β on Γ (specifically, $\beta = \varphi\iota$; see definition of ι above) that satisfies axioms $1'$, $2'$, $3'$ and 6. We need to show that $\beta(f) = \mathrm{AS}(f)$ for all $f \in \Gamma(N)$. As β and AS are both efficient, the equality is equivalent to the condition that $\beta_i(f) \leq \mathrm{AS}_i(f)$ for all $i \in N$.

We first deduce three useful properties of β.

First, for $\ell \in \mathbb{N}$, consider the game f_k, where $k = 2^\ell$. By the efficiency, symmetry and replication properties of β, we have $\beta_i(f) = \beta_{(i)}(f_k)$ for $i = 1,\ldots,n$, where $(i) \subseteq [nk]$ denotes the ith population $\{(i,1),\ldots,(i,k)\}$ and $\beta_{(i)} = \sum_{j\in(i)} \beta_j$.

Second, since in f_k there are k identical copies of each player, we have $\|\partial_j \hat{f}_k - \partial_j f_k\|_\infty \leq \frac{1}{k}\|\hat{f}_k - f_k\|$ for each player j. Therefore, by strong monotonicity, $\beta_j(f_k) \leq \beta_j(\hat{f}_k + \frac{1}{k}\|\hat{f}_k - f_k\|I)$, where I is the additive smooth game defined by $I(x) = \sum_\ell x_\ell$. It follows that $\beta_{(i)}(f_k) \leq \beta_{(i)}(\hat{f}_k + \frac{1}{k}\|\hat{f}_k - f_k\|I)$, and similarly $\beta_{(i)}(\hat{f}_k) \leq \beta_{(i)}(f_k + \frac{1}{k}\|\hat{f}_k - f_k\|I)$.

Third, by Lemma 8.7, $\beta_{(i)}(\hat{f}_k) = \mathrm{Sh}_{(i)}(\mathring{f}_k)$.

The above three properties of β hold for AS, too, as the latter also satisfies Axioms $1'$, $2'$, $3'$ and 6. Together with the linearity of the Aumann-Shapley operator, they give the following:

$$\begin{aligned}
\beta_i(f) &= \beta_{(i)}(f_k) \leq \beta_{(i)}(\hat{f}_k + \tfrac{1}{k}\|\hat{f}_k - f_k\|I) = \mathrm{AS}_{(i)}(\hat{f}_k + \tfrac{1}{k}\|\hat{f}_k - f_k\|I) \\
&= \mathrm{AS}_{(i)}(\hat{f}_k) + \|\hat{f}_k - f_k\| \leq \mathrm{AS}_{(i)}(f_k + \tfrac{1}{k}\|\hat{f}_k - f_k\|I) + \|\hat{f}_k - f_k\| \\
&= \mathrm{AS}_{(i)}(f_k) + 2\|\hat{f}_k - f_k\| = \mathrm{AS}_i(f) + 2\|\hat{f}_k - f_k\| \xrightarrow{k\to\infty} \mathrm{AS}_i(f).
\end{aligned}$$

The last step uses Lemma 8.8. This concludes the proof of Theorem 5.1. □

8.4 Proofs of Theorems 7.1 and 7.3

Proof of Theorem 7.1. For any $d \in \mathcal{ID}(N)$, with minimal autonomous set A, let M be the $n \times n$ matrix whose ith row is $\mathrm{Sh}(d_i)$. By Lemma 8.6, Eq. (2) has a unique solution with support equal to A. We denote this solution Ψd. By the properties of the Shapley value, this defines an authority measure Ψ on $\mathcal{ID}(N)$ that satisfies efficiency, symmetry, and marginality.

Consider now any authority measure ψ on $\mathcal{ID}(N)$ that satisfies these axioms. By Theorem 3.1, ψ coincides with Ψ on $\mathcal{D}(N)$. Lemma 8.3 shows that ψd solves Eq. (2), for every $d \in \mathcal{ID}(N)$. Therefore, by Lemma 8.6, the support of ψd contains the minimal autonomous set of d. □

Before we turn to prove Theorem 7.3, we note that the Banzhaf authority measure introduced in Section 7.2 is well defined. That is, for every $d \in \mathcal{D}(N)$, Eq. (5) has a unique solution that is supported in the minimal autonomous set of d. Moreover, the support is *exactly* that set. This follows easily from Lemma 8.6.

In the proof of Theorem 7.3, we use the following classic result.

Theorem 8.9 (Nowak [14])**.** *The unique point solution concept on $\mathcal{G}(N)$ satisfying the dummy player, symmetry, marginality and amalgamation axioms is the Banzhaf value.*

Proof of Theorem 7.3. Let ψ be an authority measure on $\mathcal{D}(N)$ satisfying the null player, symmetry, marginality, dictatorship and amalgamation axioms. The restriction of ψ to consensual dynamics induces a point solution concept α on the class of TU games $\mathcal{G}^+(N) \smallsetminus \{o\}$ satisfying the null player, symmetry, marginality, dictatorship and amalgamation axioms.

We explain why it remains to prove that α is the Banzhaf value. If α is the Banzhaf value then ψd solves Eq. (5) for every $d \in \mathcal{D}(N)$, by Lemma 8.3. Furthermore, by Lemma 8.6, the null player axiom implies that ψd is the unique solution whose support is the minimal autonomous set of d, which means that it coincides with the Banzhaf authority measure.

To show that α is the Banzhaf value, we first show that it is translation covariant. As in the proof of Theorem 8.5, it is sufficient to show that

$$\alpha(v + ae_i) = \alpha(v) + ae_i, \quad v \in \mathcal{G}^+(N) \smallsetminus \{o\},\ i \in N,\ a \in \mathbb{R}_+. \tag{9}$$

Indeed, for $j \neq i$, player j has the same marginal contributions in $v + ae_i$ and in v, and therefore by marginality $\alpha_j(v + ae_i) = \alpha_j(v)$. For i, the proof is by induction on the number k of non-null players other than i in v. If $k = 0$, the result follows from dictatorship. Suppose that $k > 0$ and the result holds for $k - 1$. Let $j \neq i$ be a non-null player. We consider the game $v^{i \leftarrow j}$. In $v^{i \leftarrow j}$, there are $k - 1$ non-null players other than i and therefore, by the induction hypothesis,

$$\alpha_i(v^{i \leftarrow j} + ae_i) - \alpha_i(v^{i \leftarrow j}) = a.$$

Note that $v^{i \leftarrow j} + ae_i = (v + ae_i)^{i \leftarrow j}$, so by amalgamation and since $\alpha_j(v + ae_i) = \alpha_j(v)$, we have

$$\begin{aligned} a = \alpha_i((v + ae_i)^{i \leftarrow j}) - \alpha_i(v^{i \leftarrow j}) &= \alpha_i(v + ae_i) + \alpha_j(v + ae_i) - \alpha_i(v) - \alpha_j(v) \\ &= \alpha_i(v + ae_i) - \alpha_i(v), \end{aligned}$$

which completes the proof of Eq. (9).

We claim that α has the dummy player property. Let $i \in N$ be a dummy player in a game $v \in \mathcal{G}^+(N) \smallsetminus \{o\}$. That is, there is some $a \geq 0$ such that $v(A \cup \{i\}) - v(A) = a$ for every $A \subseteq N \smallsetminus \{i\}$. We have to show that $\alpha_i(v) = a$. If $a = 0$, the assertion follows directly from the null player property of α. Suppose that $a > 0$. In the game $v - ae_i$, all the players other than i have the same marginal contributions as in v and player i is a null player. It follows that $v - ae_i \in \mathcal{G}^+(N)$. If $v - ae_i = o$, then v is dictatorial and the claim holds by the dictatorship properties. Otherwise, we have $\alpha_i(v - ae_i) = 0$ by the null player axiom, and $\alpha_i(v) = \alpha_i(v - ae_i) + ae_i$, by the translation covariance property, which proves the claim.

By Propositions 8.1 and 8.2, α can be extended to a point solution concept on $\mathcal{G}(N)$ satisfying the dummy player, symmetry, marginality and amalgamation axioms. By Theorem 8.9, this extension is the Banzhaf value. □

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A Appendix

Proof of Proposition 8.1. Let α be a translation covariant point solution concept on $\mathcal{K} \subseteq \mathcal{G}(N)$. Any game in $\mathcal{K} + \mathcal{A}(N)$ has the form $v + l$, where $v \in \mathcal{K}$ and $l \in \mathcal{A}(N)$. The formula

$$\alpha(v+l) = \alpha(v) + l \tag{10}$$

well defines an extension of α to $\mathcal{K} + \mathcal{A}(N)$. This is because, if $v' \in \mathcal{K}$ and $l' \in \mathcal{A}(N)$ are such that $v + l = v' + l'$, then $v - v' = l' - l \in \mathcal{A}(N)$, and so by the translation covariance property

$$\alpha(v) + l = \alpha(v') + l'.$$

By definition, a point solution concept on $\mathcal{K} + \mathcal{A}(N)$ is translation covariant if and only if Eq. (10) holds for all $v \in \mathcal{K} + \mathcal{A}(N)$ and $l \in \mathcal{A}(N)$. However, it is easy to see that this is so if and only if the formula holds for $v \in \mathcal{K}$. Thus, the above extension is the unique translation covariant one. □

Proof of Proposition 8.2. Let α be a translation covariant point solution concept on $\mathcal{K} \subseteq \mathcal{G}(N)$. For each of the axioms in the statement of the theorem separately, we will suppose that α satisfies it on $\mathcal{K}$ and prove that its translation covariant extension satisfies the axiom on $\mathcal{K} + \mathcal{A}(N)$.

Dummy player. Let $i \in N$ be a dummy player in $u \in \mathcal{K} + \mathcal{A}(N)$. Write $u = v + l$, where $v \in \mathcal{K}, l \in \mathcal{A}(N)$. Since i is a dummy player in l, it is a dummy player in $v = u - l$ as well. By translation covariance, we have $\alpha_i(u) = \alpha_i(v) + l(\{i\})$ and, by the dummy player property $\alpha_i(v) = v(\{i\})$. It follows that $\alpha_i(u) = v(\{i\}) + l(\{i\}) = u(\{i\})$.

Efficiency. Let $u = v + l$, where $v \in \mathcal{K}, l \in \mathcal{A}(N)$. By translation covariance, we have

$$\sum_{i \in N} \alpha_i(u) = \sum_{i \in N} \big(\alpha_i(v) + l(\{i\})\big) = \sum_{i \in N} \alpha_i(v) + l(N) = v(N) + l(N) = u(N),$$

where the third equality uses the assumed efficiency of α on $\mathcal{K}$.

Symmetry. Let $u = v + l$, where $v \in \mathcal{K}, l \in \mathcal{A}(N)$ and $\pi \in S_N$. Clearly $\pi u = \pi v + \pi l$, and $\pi v \in \mathcal{K}, \pi l \in \mathcal{A}(N)$. By translation covariance, we have $\alpha(\pi u) = \alpha(\pi v) + \pi l$. By symmetry, we have $\alpha(\pi v) = \pi\alpha(v)$. It follows that

$$\alpha(\pi u) = \pi\alpha(v) + \pi l = \pi(\alpha(v) + l) = \pi\alpha(u).$$

Marginality. Let $i \in N$ and $u, u' \in \mathcal{K} + \mathcal{A}(N)$ be such that i has the same marginal contributions to u and u'. There exist $v, v' \in \mathcal{K}, l, l' \in \mathcal{A}(N)$ such that $u = v + l$ and $u' = v' + l'$. For $a = l'(\{i\}) - l(\{i\})$, player i has the same marginal contributions to v and to $v' + ae_i$, where e_i is the unanimity game with carrier $\{i\}$. We may assume w.l.o.g. that $a \geq 0$ (otherwise, transpose u and u').

By translation covariance, we have $\alpha(u) = \alpha(v) + l$, $\alpha(u') = \alpha(v') + l'$, and $\alpha(v' + ae_i) = \alpha(v') + ae_i$. By definition of marginality, $v' + ae_i \in \mathcal{K}$ and $\alpha_i(v) = \alpha_i(v' + ae_i)$. It follows that

$$\alpha_i(u) = \alpha_i(v) + l(\{i\}) = \alpha_i(v') + a + l(\{i\}) = \alpha_i(v') + l'(\{i\}) = \alpha_i(u').$$

Amalgamation. Let $u = v + l$, where $v \in \mathcal{K}, l \in \mathcal{A}(N)$, and $i, j \in N$. Since $u^{i \leftarrow j} = v^{i \leftarrow j} + l^{i \leftarrow j}$ and $l^{i \leftarrow j} \in \mathcal{A}(N)$, by translation covariance, we have $\alpha(u^{i \leftarrow j}) = \alpha(v^{i \leftarrow j}) + l^{i \leftarrow j}$. By the amalgamation property of α on $\mathcal{K}$, we have $\alpha_i(v^{i \leftarrow j}) + \alpha_j(v^{i \leftarrow j}) = \alpha_i(v) + \alpha_j(v)$. It follows that

$$\begin{aligned}\alpha_i(u^{i \leftarrow j}) + \alpha_j(u^{i \leftarrow j}) &= \alpha_i(v^{i \leftarrow j}) + \alpha_j(v^{i \leftarrow j}) + l^{i \leftarrow j}(\{i\}) + l^{i \leftarrow j}(\{j\}) \\ &= \alpha_i(v) + \alpha_j(v) + l(\{i, j\}) = \alpha_i(u) + \alpha_j(u).\end{aligned}$$

The last equality follows from the translation covariance property of α on $\mathcal{K} + \mathcal{A}(N)$. □