Informational Limitations of Ascending Combinatorial Auctions

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Abstract

We study the inherent limitations of natural widely-used classes of ascending combinatorial auctions. Specifically, we show that ascending combinatorial auctions that do not use both non-linear prices and personalized prices can not achieve social efficiency with general bidder valuations. This casts doubt on the performance that can be achieved using the simpler auctions suggested, e.g., by Kwasnica et al. (2005),Porter et al. (2003) and Wurman and Wellman (2000) and justifies the added complexity in the auctions suggested by, e.g., Parkes and Ungar (2000) and Ausubel and Milgrom (2002).

Our impossibility results are robust in several senses: they allow the analysis of all the information that was aggregated during the auction rather than considering only the final price level; they hold for any price update system or payment determination rule; they do not depend on strategic aspects, on computational limitations or on communication capacities. We also show that the loss of efficiency is severe and that only a diminishing fraction of the social welfare may be captured.

Keywords: Auctions, Combinatorial auctions, Package auctions, Ascending Auctions, non-linear prices, personalized prices.

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1 Introduction

Combinatorial auctions are a general name given to auctions in which multiple heterogeneous items are concurrently sold and in which bidders may place bids on *combinations* of items rather than just on single items. Such *combinatorial bidding* is desired whenever items sold are complements or substitutes of each other, at least for some of the bidders. In such cases, the combinatorial bidding allows the bidders to better express their complex preferences, allowing the auction to achieve higher social welfare, and often (but not necessarily) higher revenue as well. Combinatorial auctions have been used in many settings such as truckload transportation (Ledyard et al. (2002); Sheffi (2004)), airport slot allocation (Rassenti et al. (1982); Cramton (2002a)), industrial procurement (Bichler et al. (2006)), and, prominently, spectrum auctions (Cramton (2002b); FCC Auctions (2006)). Additionally, combinatorial auctions serve as a common abstraction for many resource allocation problems in decentralized computerized systems such as the Internet, and may serve as a central building block of future electronic commerce systems.

The design of combinatorial auctions faces multiple types of complexities: informational, cognitive, computational, and strategic. Indeed, the design of combinatorial auctions is still part art and part science. While many aspects have been analyzed mathematically or empirically, many other aspects remain an art form. In many cases the design is ad-hoc for a given application, and it is usually not clear how well the existing design performs relative to the other non-implemented alternatives. Indeed, when the US Federal Communications Commission held a series of workshops addressing the intended design of their multi-billion dollar combinatorial auctions for radio spectrum (see, e.g., FCC Combinatorial Bidding Conference (2003)), there has been very little agreement among the participants. We refer the reader to the recent tomes (Cramton et al. (2006),Milgrom (2004)) that elaborate on various aspects, applications and suggestions for combinatorial auctions.

This paper concerns a large class of combinatorial auction designs which contains the vast majority of implemented or suggested ones: ascending auctions. In this class of auctions, the auctioneer publishes prices, initially set to zero (or some other minimum prices), and the bidders repeatedly respond to the current prices by bidding on their most desired bundle of goods under the current prices. The auctioneer then repeatedly updates the prices by increasing some of them in some manner, until a level of prices is reached where the auctioneer can declare an allocation. (Intuitively, prices related to over-demanded items are increased until the demand equals supply.) There are several reasons for the popularity of ascending auctions, including their intuitiveness, the fact that private information need only be partially revealed, that they increase the trust in the auctioneer as bidders see the prices gradually emerging, that it is clear that they will terminate and that they may sometimes reduce the winner's curse and increase the seller's revenue (Milgrom and Weber (1982)). Another major advantage of ascending auctions is that they afford a price-discovery process in markets where bidders do not know their exact valuations for every possible bundle, and determining these values is costly in practice. Ascending auctions, however, guide the attention of the bidders to bundles that are relevant to the allocation determination. Although their equilibrium analysis is not always clear, ascending auctions are usually preferred over sealed-bid VCG auctions since the latter auctions suffer from several severe weaknesses, such as low seller revenue and vulnerability to collusion and false-name bidding. A survey by Cramton (1998) describes in more details the advantages and disadvantages of ascending auctions, and papers by Rothkopf et al. (1990) and Ausubel and Milgrom (2006) discuss the practical flaws of VCG auctions.

Ascending auctions may vary from each other in the bidding rules, in the price update scheme, in the termination condition, etc. The most notable difference is in the types of prices used. Some auctions attach a price to each item, and the price of each bundle of items is the sum of the item prices. Such auctions are termed *item-price* auctions or linear-price auctions. A more general class of auctions maintains a separate arbitrary price for each bundle of items. These are called *bundle-price* auctions or non-linear price auctions. Some auctions present the same set of prices to all bidders – these are called *anonymous-price* auctions. Others maintain a separate set of prices for each bidder – these are called *personalized* price auctions (or *non-anonymous* price auctions). It is clear that item-price auctions are preferable to bundle-price ones in terms of simplicity, and similarly that anonymous-price ones are simpler than personalized-prices ones. This simplicity is important in many respects, including the cognitive, computational, and communication burden placed on the bidders and on the auctioneer. In particular, such auctions tend to be simpler to bid on, will run faster, and will require less communication and computation and thus will be feasible for a larger number of items. The question is really whether the added expressiveness of the more complex types of auctions offers benefits that overcome the cost in complexity. Indeed, presentations at the 2003 conference of the U.S. Federal Communications Commission (FCC Combinatorial Bidding Conference (2003)) reveal an interesting debate along these lines between the suggestions of David Porter, Stephen Rassenti and Vernon Smith (on the simplicity side) and of Larry Ausubel, Peter Cramton and Paul Milgrom (on the complexity side).

1.1 Ascending Auctions and Equilibrium Prices

Most of the literature on iterative combinatorial auctions centered on the existence of competitive equilibria. It is known that both bundle prices and non-anonymous prices are required for guaranteeing the existence of such equilibria, see, e.g., Bikhchandani and Ostroy (2002); Milgrom (2000a); Scarf (1960). Otherwise, strict restriction on the preferences are necessary for obtaining competitive equilibria. In particular, item-price equilibria are known to exist when the bidders have substitutes valuations, see Kelso and Crawford (1982), and anonymous bundle-price equilibria exist when the bidders have complementarities, i.e., when all bidders have super-additive valuations[1],

[1] A valuation v is super-additive if for every two disjoint bundles S, T we have that $w(S) + w(T) \leq w(S \cup T)$.

see the work of Parkes (2001). However, in most setting one would not expect such homogeneity of the preferences of the bidders. A bidder may consider some of the items as substitutes and some as complements, or some bidders may have substitute valuations and the preferences of the other may possess complementarities. In many of these cases, an equilibrium typically does not exist, but ascending auctions may still be in use. This paper analyzes auctions that run in environments where an equilibrium does not exist, and measures whether such auctions can guarantee efficiency, or approximate efficiency. We consider a general model where the decisions can be made using *all the information that was collected during the course of the auction* and not only based on demand at the final price level.

Consider, for example, one of the most successful family of combinatorial auctions – Simultaneous Ascending Auctions (SAA). These auctions have been running by the U.S. FCC beginning in 1994, and they have been adopted for dozens of spectrum auctions worldwide. They were proposed to the FCC by Paul Milgrom, Robert Wilson and Preston McAfee and they are a natural extension of "Deferred-Acceptance Mechanisms" from the literature on matching (see the survey by Roth and Sotomayor (1992)). One of the main reasons for their success is their simplicity: all items are sold at the same time, and the bidders can bid on any item; the auction increases the price of over-demanded items until every item is demanded by at most one bidder. SAA were theoretically analyzed in the work of Kelso and Crawford (1982), Demange et al. (1986), Gul and Stacchetti (1999) and Milgrom (2000a). The basic theorem shows that if all bidders have (gross) substitutes valuations, then this converges to a competitive (Walrasian) equilibrium and thus leads to social efficiency. The restriction to having (gross) substitutes valuations is known to be critical; for example, Gul and Stacchetti (1999) show that for any bidder whose preferences fail the substitutes condition we can add a set of unit-demand bidders such that the resulting economy has no Walrasian equilibrium. However, empirical results by Ausubel et al. (1997) show that in the US spectrum market there are clear evidences that bidders have synergies (or complementarities) for neighboring licenses, proving that the substitutes condition does not hold. Therefore, Simultaneous Ascending Auctions are not guaranteed to end up in any sort of equilibrium in the FCC auctions. Our work studies whether such auctions, or their variants, can be efficient despite the lack of efficient equilibria, and whether they can provide a reasonable approximation for the efficiency loss.

Another prominent family of ascending auctions was recently introduced in the work of Parkes and Ungar (2000) and Ausubel and Milgrom (2002). These auctions always end up with a socially-efficient allocation and use personalized bundle prices. The main idea here is that the auctioneer computes, at each stage, an optimal tentative allocation, and then losers in this tentative allocation are allowed to increase their bids. The basic theorem states that when no loser wants to increase his bid, then an optimal allocation has been reached. This holds for arbitrary bidder valuations.

1.2 Our Contribution

The fundamental question that we address is whether the added complexities of bundle prices and of personalized prices are indeed necessary for achieving efficient results by ascending-price auctions. We present a strong affirmative answer on both counts. We prove that no ascending item price auction (using anonymous or personalized prices) can always reach a socially-efficient allocation among arbitrary bidder valuations. Similarly, we prove that no ascending anonymous-price auction (using either item prices or bundle prices) can always reach the socially optimal allocation. Our basic theorems are proved by analyzing two very simple scenarios in which we show that the appropriate type of auction can simply not gather sufficient information from the bidders.

We then prove several stronger variants of our theorems showing that our impossibility results are very robust is several senses. We show that not only is it impossible for an ascending item-price auction to obtain the social optimum, but even if we allow multiple, sub-exponentially many, "ascending paths" (e.g., as used in Ausubel (2006)), then the impossibility remains. We also show that the loss of welfare is extreme both for item price auctions (even non-anonymous) and for anonymous-price auctions (even with bundle prices), and that only a vanishingly small fraction of the social welfare may be captured[2]. This last pair of results is proved using a sophisticated combinatorial construction of valuations that are "hard to elicit" by these restricted types. We also show that our examples are not "unusual" by showing that for any set of substitutes bidders, it is possible to add a single extra bidder making it impossible to find the social optimum by item-price auctions. Recall that in environments with substitutes preferences, item-price Simultaneous Ascending Auctions are known to be able to determine the efficient allocation. Actually, our results contribute to the notion, originate in the work of Arrow et al. (1959) and Kelso and Crawford (1982), that item-price ascending auction essentially work only with bidder valuations that satisfy the (gross) substitutes property. Our results are stronger than the existing results, as they allow using all the information that is elicited in the course of the auction, study multiple ascending prices paths, and analyze how the rate in which inefficiency intensify.

All of our results are in a very general setting: they do not rely on any incentive constraints and hold even if bidders simply bid "as told". As long as their response at every stage is just a function of their desired bundles at the current prices, or any subset of those bundles, the impossibilities hold. In particular the impossibilities do not rely on any inter-dependencies between the bidders' valuations and hold for simple private values. Our impossibility results do not assume that any particular type of equilibrium will be achieved upon termination, and hold whether or not any equilibrium is achieved – they allow taking into account the whole amount of information obtained during the auction. The results do not rely on any computational limitations or limitations on the amount of communication that is transmitted, and hold even if unbounded (and unrealistic) computation and communication capabilities are available to the

[2]Formally, we show that no better than a $4/\sqrt{m}$ fraction of welfare may be captured by each auction type, where m is the number of items.

auctioneer and bidders. Our analysis works for all price increments, even infinitesimal.

The bottom line of our paper is a formal analysis showing that simple combinatorial auction schemes that use only item prices or that use only anonymous prices do have severe informational limitations. This will not allow them to match the performance guarantees of the more complex schemes. The exact tradeoff between these limitations and the significant costs of the more complex scheme remains part of the "art" of combinatorial auction construction.

1.3 More Related Work

While most previous work on combinatorial auctions has actually studied specific types of auctions, a few other impossibility results have been shown that should be compared to ours. First, are the known theorems (see, e.g., Bikhchandani and Ostroy (2002); Scarf (1960); Kelso and Crawford (1982); Bikhchandani and Mamer (1997); Milgrom (2000b); de Vries et al. (2005)) that for general, non-substitutes valuations, certain types of competitive equilibria cannot be found without personalized bundle prices. Note that item-price auctions with *non*-ascending prices *can* obtain the social optimum, despite the lack of any equilibrium (Blumrosen and Nisan (2005)). Other related results were proved by Nisan and Segal (2003) showing that exponential communication is required by any type of combinatorial auction for obtaining the optimum. These results are quantitative and are not delicate enough to qualitatively distinguish between different types of auctions, as we do here. Additionally, such lower bounds on the amount of the transmitted communication cannot be applied in our setting, as we show in the paper's body that an amount of information that is exponentially greater than the number of items can be elicited by ascending auctions, even with item prices.

Probably the closest result to ours, in spirit, is by Gul and Stacchetti (2000) who showed that ascending anonymous item-price auctions can not come up with VCG prices even for (gross)-substitutes valuations, despite the fact that the social optimum can be achieved in such cases. In contrast, our impossibility is for just finding the optimum, or even a reasonable approximation, rather than calculating a particular set of prices. Additionally, in contrast to our results, the impossibility in Gul and Stacchetti (2000) is very delicate, and stops holding if multiple ascending rounds are allowed, as in Ausubel (2006). Another close result is the recent paper by Mishra and Parkes (2005), who presented a class of efficient bundle-price non-anonymous ascending auctions that compute VCG payments. Their work allows the payments made by each bidder to be different from the final clearing prices, but their auction still aims to terminate with equilibrium prices (they call *Universal Competitive Equilibrium*), and therefore our definition of an ascending auction is broader.

The structure of the rest of the paper is as follows: in section 2 we formally present our model and definitions. Section 3 gives the impossibility results for item-price auctions, while section 4 gives the impossibility results for anonymous-price auctions. In the body of the paper we provide the full (and simple) proofs of the basic impossibility theorems; the proofs of the stronger variants are postponed to the appendix. Appendix A contains some definition to be used in proofs that appear later in Appendices B and C.

2 The Model

A seller wishes to sell a set M of m heterogeneous indivisible items to a set of n bidders. Each bidder i has a valuation function $v_i : 2^M \to \mathbb{R}_+$ that attaches a non-negative real value $v_i(S)$ for any bundle $S \subseteq M$. We assume two conventional assumptions on the preferences: (i) Free disposal (monotonicity), i.e., if $S \subset T$ then $v_i(S) \leq v_i(T)$. (ii) Normalization, i.e., $v_i(\emptyset) = 0$ for every bidder i.

The goal of the auctioneer is to find an *efficient allocation* of the items, that is, to find a partition $S_1, ..., S_n$ of the items that maximizes the *social welfare*, $\sum_{i=1}^{n} v_i(S_i)$. We do not study revenue maximization in this paper.

In this work, we concentrate on iterative auctions where, at each stage, the auctioneer publishes a set of prices p for the bundles, and each bidder responds with her *demand* given the published prices, that is, a bundle S that maximizes her (quasi linear) utility $u_i(S,p) = v_i(S) - p(S)$, where $p(S)$ denotes the price of S under the price level p.[3] The stages of the auction are ordered by time, and at each stage, a single set of prices is presented to each bidder. The prices can be presented in different ways. For example, the seller can explicitly publish a price for each bundle, or use a succinct representation for the prices (e.g., by only publishing item prices). We touch on several common representations below.

The specific *auction* is determined by the method that the auctioneer determines which prices will be presented to the bidders at each stage. The seller can determine the prices adaptively, i.e., as a function of the history of the published prices and responses. The seller can also use information gained from the responses of one bidder for determining the future prices for other bidders. At the end of the auction, the auctioneer analyzes the information received during the auction, and determines the final allocation accordingly. That is, the data that is available to the auctioneer at the end of the auction is exactly $\{(p_i^t, S_i^t)|$ for every bidder i and every stage $t\}$, where S_i^t denotes the demand of bidder i at stage t under the price vector p_i^t. To strengthen our results, and as opposed to most of the existing literature, we consider a general model where the allocation can be determined by all the information gathered during the auction, and not only according the demands at the final stage of the auction. Note that, to strengthen our results, we do not assume any limitations on the power of the participants, except for information limitations. In particular, the auctioneer may be computationally unbounded (including, e.g., the ability to solve hard problems classified as "NP-hard" in the computer-science terminology).

This paper centers on auctions with non-decreasing prices:

Definition 1. (Ascending auctions) In an *ascending auction*, each bidder responds

[3] All our results hold for any consistent tie-breaking rule by the bidders or by the auctioneer. Moreover, our result will also hold if every bidder i reports, at each stage, all the bundles that maximize her utility, i.e., her whole demand set $\{S \subseteq M \mid v_i(S) - p(S) \geq v_i(T) - p(T)$ for every $T \subseteq M\}$. An equivalent model allows the bidders to raise their "bids" on their desired bundles.

with his demand under every price level presented to him, and prices presented to the same bidder can only increase in time. Formally, let p be a set of prices presented to bidder i, and q be the prices for bidder i at a later stage in the protocol. Then, for all sets $S \subseteq M$, we have $q(S) \geq p(S)$.

Two highly important factors in the design of ascending combinatorial auctions concern the representation of the prices. First, the seller might choose to present only prices for the individual items, or, with greater expressiveness, publish a price per every possible bundle. Another pricing decision is whether to present personalized prices for each bidder, or present every price level to all bidders.

Definition 2. (Item/Bundle prices) An auction uses *item prices (or linear prices)*, if, at each stage, the auctioneer presents a price p_j for each item j, and the price of a set S is additive: $p(S) = \sum_{j \in S} p_j$. We say that an auction uses *bundle prices (or non-linear prices)* if each bundle S may have a different price $p(S)$ (which is not necessarily equal to the sum of the prices of the items in S).

Definition 3. (Anonymous/Non-Anonymous prices) An auction uses *anonymous* prices, if the prices seen by the bidders at any stage in the auction are the same, i.e., whenever a set of prices is presented to some bidder, the same set of prices is also presented to all other bidders. In auctions with *non-anonymous (personalized)* prices, each bidder i is presented with personalized prices for the bundles denoted by $p_i(S)$.[4]

Observe that with bundle prices, the number of distinct prices presented by the seller in each stage may be exponentially greater than the number of items (a price per every subset of items). Consequently, such auctions may be practically infeasible when selling more than few items.

[4]Note that a non-anonymous auction can clearly be simulated by n parallel anonymous auctions.

3 Item-Price Ascending Auctions

Before describing their limitations, we would like to demonstrate that item-price ascending auctions are not trivial in their power. The most prominent example is their ability to end up with a Walrasian equilibrium (which is, in particular, efficient) for environments with (gross) substitutes valuations, see Kelso and Crawford (1982) and Gul and Stacchetti (1999).

We would also like to point out that despite using a linear number of item prices, ascending auctions may elicit a very large amount of information from the bidders. In particular, if small enough increments are allowed, such auctions can determine the optimal allocation in cases where this task requires exchanging an amount of information which exceeds the number of items by an exponential factor. This is shown in Example 1 in Appendix B. Example 1 actually shows that our results are incomparable with the hardness results of Nisan and Segal (2003), as item-price ascending auctions in our model can elicit an exponential amount of information.

Without restricting the prices to be ascending, analyzing the demand of the bidders under different price levels enables the auctioneer to easily determine the efficient allocation in any combinatorial auction (see Blumrosen and Nisan (2005)). However, as we show in this section, this is no longer true when the prices are restricted to be ascending, even for settings with only two items and two bidders. After proving this negative result, we strengthen it in several directions: in Theorem 1a, we show that the number of ascending trajectories of prices that are required for finding the efficient allocation is exponentially larger than the number of items; Then, in Theorem 1b, we show that a single item-price ascending auction can only guarantee a small fraction of the optimal welfare, a fraction that diminishes with the number of items. Finally, Theorem 1c indicates that inefficiencies may rise for every profile of bidders with substitutes preferences following an addition of a single bidder.

Our basic hardness result is given using the combinatorial auction setting in Figure 1. In this example, for determining the efficient allocation, the auctioneer has to know

	$\mathbf{v(ab)}$	$\mathbf{v(a)}$	$\mathbf{v(b)}$
Bidder 1	2	$\alpha \in (0,1)$	$\beta \in (0,1)$
Bidder 2	2	2	2

Figure 1: This example shows that no item-price ascending auction can always determine the optimal allocation: no such auction can tell whether α is greater than β or vice versa.

which one of the two singleton bundles has a greater value for Bidder 1. However, an ascending auction can only elicit information about one of the singletons, so the efficient outcome cannot be obtained. The basic idea is that in order to gain any information about one of the singletons, the price of the *other* item must be increased significantly, otherwise the bidder will continue demanding the whole bundle. Since the prices cannot decrease, it follows that the demand of Bidder 1 will be independent of his value for the latter item.

Theorem 1. *No item-price ascending auction can determine the efficient allocation for all profiles of bidder valuations.*

Proof. Consider the two valuations described in Figure 1. All the values are known to the auctioneer, except for the values α and β (between $(0,1)$) that Bidder 1 attaches to the singletons a and b, respectively. For such preferences, the only way to achieve a welfare greater than 2 is to allocate one singleton to Bidder 1 and the other to Bidder 2. Therefore, the identity of the efficient allocation depends on which of the two singletons gains a greater value for Bidder 1. We prove that a single ascending trajectory of item prices can reveal information only on one of these values. We first claim that no information is elicited as long as both prices are low.

Claim 1. As long as p_a and p_b are both below 1, Bidder 1 demands the whole bundle $\{ab\}$.

Proof. For every price level p in which both prices are smaller than 1, Bidder 1's utility from the bundle ab will be strictly greater than the utility from either a or b separately. For example, we show that $u_1(ab,p) > u_1(a,p)$ (the same statement for the singleton

b can be similarly shown):

$$u_1(ab,p) = 2 - (p_a + p_b) \tag{3.1}$$
$$= 1 - p_a + 1 - p_b \tag{3.2}$$
$$> v_A(a) - p_a + 1 - p_b \tag{3.3}$$
$$> u_1(a,p) \tag{3.4}$$

Where Equation 3.1 is due to the linearity of the prices, Inequality 3.3 holds since $v_A(a)$ is smaller than 1, and Inequality 3.3 follows from the assumption that p_b is smaller than 1. □

Thus, in order to gain *any* information about the unknown values α and β, the auctioneer must arbitrarily (i.e., without any new information) choose one of the items (w.l.o.g., a) and increase its price to be greater than 1. But then, since the prices are ascending, the singleton a will not be demanded by Bidder 1 throughout the auction, thus no information at all will be gained about α. Hence, the auctioneer will not be able to identify the efficient allocation.

Since the valuation of one of the bidders is fully known in advance to the auctioneer, the theorem holds even for *non-anonymous* item-price ascending auctions. □

The proof of Theorem 1 describes a profile of preferences for which no ascending trajectory of prices can elicit enough information for determining the optimal allocation. This would hold even if the auctioneer had some exogenous information (or a *good guess*) telling him what is the "right" way to increase the prices.[5] Similar arguments show that this hardness result also holds for the similar family of *descending-price* auctions (otherwise, the "reversed" price trajectory would be an ascending auction that finds the optimal allocation).

[5]Protocols that allow the usage of an exogenous data are often named "non-deterministic" protocols in the computer-science literature.

Theorem 1 is proved as a worst-case result, but it also holds for a wide range of probability distributions. For example, for any distribution of α and β between $[0, 1]$, and also when we draw any number of additional players from such distributions. It is easy to see that for the uniform distribution (on α and β) the optimal ascending auction with two bidders will raise the price of one of the items (say, a) until $v_1(b)$ is determined, and then allocate b to bidder 1 if and only if $v_1(b) \geq 1/2$. Although the expected inefficiency of such an auction is relatively low (less than 2 percent), we expect the inefficiency to get worse as the number of items increase and the informational difficulty of the seller becomes more severe. Later, in Theorem 1b, we will show an extreme scenario where this inefficiency is very significant.

While Theorem 1 proved that a single ascending trajectory of prices cannot guarantee finding the efficient allocation, it does not rule out the possibility that a small number of trajectories will achieve this goal. For instance, a similar question was studied regarding the number of ascending auctions that are required for calculating VCG prices for bidders with substitutes preferences: A negative result by Gul and Stacchetti (2000) showed that the VCG payments for substitutes valuations cannot be found by a single ascending-price trajectory; However, Ausubel (2006) presented an $(n + 1)$-trajectory ascending auction that achieved this task. Below, we extend the result presented in Theorem 1 and show that for guaranteeing that an efficient allocation will be discovered, for all profiles of valuations, an exponential number (in the number of items) of ascending-price trajectories is required.

We define a *k-trajectory ascending auction* as an auction in which the price vectors presented to the bidders at the different stages of the auction can be partitioned into up to k sets, ordered according to the time they were published, where the prices published within each set only increase in time (for a formal definition, see Definition 7 in Appendix B). Note that we use a general definition; It allows the trajectories to run in parallel or sequentially, and to use information elicited in some trajectories for determining the future queries in other trajectories.

The theorem is proved by presenting preferences for two bidders, where the efficient allocation depends on the identity of a particular $\frac{m}{2}$-sized bundle that gains one of the bidders a high value. For eliciting information about the value of some $\frac{m}{2}$-sized bundle S, the prices of all the items that are not in S should be very high, otherwise a larger bundle would be demanded. Therefore, every ascending auction can only reveal information on a *single* $\frac{m}{2}$-sized bundle. Since an exponential number of such bundles exists, the theorem follows. The proof can be found in Appendix B.

Theorem 1a. *The number of ascending item-price trajectories needed for revealing the efficient allocation, for every profile of bidder valuations, must be exponentially greater than the number of items.*

Implicit in the proof of Theorem 1a is that an exponential number of ascending item-price trajectories is necessary for guaranteeing more than a $\frac{2}{3}$-fraction of the optimal welfare. Our next result presents a much stronger bound on the rate in which the welfare in any single-trajectory ascending auction diminishes as the number of items and players grow. Formally, no item-price ascending auction can guarantee a fraction of the efficient welfare that is greater than $\max\{\frac{4}{n}, \frac{4}{\sqrt{m}}\}$. We emphasize that this result even holds for non-anonymous item-price ascending auctions, that is, auctions with a personalized ascending trajectory of prices per each bidder.

A sketch of the proof: we create a profile of valuations for the n bidders with certain combinatorial properties that make them hard to be elicited by any ascending auction. This is done by defining a set of bundles that form a special combinatorial structure: we divide the items to several partitions; Every two bundles from different partitions intersect (*"mutually-intersecting partitions"*), and therefore achieving the optimal allocation is possible only by partitioning the items according to one of these partitions. The values that each bidder attaches for these bundles are unknown to the auctioneer and are either 0 or 1. To gain any information about one of these bundles, the prices of *every* bundle from all the other partitions must exceed 1 (since the bidders have a value of 2 for some larger bundles). It follows that the bundles from the other partitions will not be demanded any more during the ascending-price auction. This way, the auctioneer can elicit information about bundles from at most one partition for each bidder. This is shown to be insufficient for achieving a reasonable approximation for the social welfare. The proof appears in Appendix B.

Theorem 1b. *No item-price ascending auction (even with non-anonymous prices) can guarantee better than a fraction of* $\max\{\frac{4}{n}, \frac{4}{\sqrt{m}}\}$ *of the efficient welfare for all profiles of bidder valuations.*

Our final result regarding item-price ascending auctions illustrates that inefficiencies may occur even for preferences that are slightly away from having the substitutes property. Substitutes preferences are, informally, preferences with the property that when a bidder demands a certain bundle, and some of the prices in this bundle increase, then the bidder will still demand the other items in this bundle (an exact definition is presented in Definition 6 in Appendix B). As mentioned, it is well known that item-price ascending auctions can determine the efficient allocation for substitutes valuations. We show that for every profile of players with substitutes valuations, the efficient outcome cannot be found after an addition of a single player. The proof takes advantage of the fact that the aggregate demand of n substitutes valuations also has the substitutes property. Therefore, the marginal contributions of bundles to the welfare of the n players must exhibit complementarities. We construct a valuation for the new player that obtains a greater value than the marginal values for some of the bundles. Due to the presence of complementarities, we argue that an ascending auction will not be able to determine which bundle obtains the highest additional gain.

This result applies for every profile of substitutes valuations, except for the degenerate case where the aggregation of these players forms an additive valuation (i.e., where for every two disjoint bundles S, T, the aggregate valuation exhibits exactly $v(S) + v(T) = v(S \cup T)$).[6]

[6]A valuation w is called the aggregation of the valuations $v_1, ..., v_n$ if for every bundle S, $w(S)$ equals the optimal welfare achieved by allocating the items in S over the n players.

Theorem 1c. *For every n, and for every profile of n substitutes valuations that their aggregation is not an additive valuation, there exists an additional bidder such that no item-price ascending auction can determine the efficient allocation among the $n+1$ bidders.*

4 Anonymous Ascending Auctions

All the ascending auctions in the literature that are proved to find the optimal allocation for unrestricted valuations are non-anonymous bundle-price auctions (e.g., iBundle(3) by Parkes and Ungar (2000) and the "Proxy Auction" by Ausubel and Milgrom (2002)). Yet, several *anonymous* ascending auctions with bundle prices have been suggested (e.g., AkBA by Wurman and Wellman (2000), the PAUSE auction by Kelly and Steinberg (2000), and iBundle(2) by Parkes and Ungar (2000)). The power of such anonymous auctions is not trivial, as they can reach an efficient outcome for super-additive preferences (Parkes (2001)). We first show that no anonymous ascending auction can always find the efficient solution for general valuations, even for environments with only two bidders and four items, and even if it is allowed to use bundle prices. Later in this section, we extend this negative result and show that such auctions can only guarantee a diminishing fraction of the social welfare.

In Figure 2, we present a class of valuations for which the efficient allocation cannot be determined by any anonymous bundle-price ascending auction. The basic idea: In the example, Bidder 1 and Bidder 2 have unknown values for some bundles S_1 and S_2, respectively. However, Bidder 1 also has a high value for S_2 and bidder 2 has a high value for S_1. Therefore, in order to reveal information about $v_1(S_1)$, the price of S_2 must be increased significantly, and thus "hide" the value $v_2(S_2)$. Similarly, for gaining information about $v_2(S_2)$ the price of S_1 must increase, "hiding" the value $v_1(S_1)$. This stems from the anonymity of the auction – the bidders face the same ascending trajectory of prices. Consequently, the auctioneer will only be able to attain information about both values, what will prevent him from identifying the

Bidder 1	$v_1(ac) = 2$	$v_1(bd) = 2$	$v_1(cd) = \alpha \in (0, 1)$
Bidder 2	$v_2(ab) = 2$	$v_2(cd) = 2$	$v_2(bd) = \beta \in (0, 1)$

Figure 2: This example shows that anonymous ascending auctions cannot always determine the efficient allocation. The value of every bundle that is not explicitly specified equals to the maximal value of a bundle it contains.

efficient allocation.

Theorem 2. *No anonymous bundle-price ascending auction can determine the efficient allocation for all profiles of bidder valuations.*

Proof. Consider the pair of valuations described in Figure 2. Each bidder has a value of 2 for two 2-item bundles, and some unknown value, between 0 and 1, for a third 2-item bundle. The values of the other bundles equals the maximal value of a bundle that they contain. For finding the optimal allocation the auctioneer must know whether α is greater than β or vice versa: If $\alpha > \beta$, the optimal allocation will allocate *cd* to Bidder 1 and *ab* to bidder 2. Otherwise, it should allocate *bd* to bidder 2 and *ac* to Bidder 1. Notice that since each item can be allocated only once, at most one bidder can gain a value of 2.

In an anonymous ascending auction, however, one can only elicit information on one of the values α and β: as long as the prices of both *cd* and *bd* are below 1, both bidders will clearly demand their high-valued bundles (that gain them utilities greater than 1). Therefore, in order to elicit any information, the auctioneer must raise one of these prices to be greater than 1, w.l.o.g., the price of *bd*. Thus, since the prices cannot decrease, no information will be gained about β. □

We now strengthen the impossibility result above by showing that anonymous auctions, even with bundle prices, cannot guarantee more than a vanishing fraction of the social welfare, namely, at most a $\frac{4}{\sqrt{m}}$-fraction of the efficient welfare. Using bundle prices may be appealing when each bidder is interested in a small number of bundles, but this pricing method may become impractical due the exponential number of potential prices. Note that a similar fraction of the optimal welfare, $O(\frac{1}{\sqrt{m}})$, can be achieved using a significantly smaller amount of prices (that is, with polynomial-sized communication - see, e.g., Blumrosen and Nisan (2005)).

For proving the limitations of anonymous auctions, we build a profile of valuations that, due to certain combinatorial properties, cannot be solved by anonymous ascending auctions. These preferences are different than those used in Theorem 1b. Nevertheless, we use the same combinatorial construction of *mutually-intersecting partitions* that was introduced in the proof for Theorem 1b. Recall that *mutually-intersecting partitions* are a set of partitions of the items with the property that every two bundles from different partitions have at least one item in common. We show that for the class of valuations that we build, before the auctioneer elicits any information, the prices of *all the bundles* from some partition should exceed 1. Since all the unknown values are below 1, an anonymous ascending auction will gain no information about the values that the bidders have for the bundles in this partition. Allocating bundles from this partition to different bidders may form an efficient allocation, but the auctioneer will not have enough information to correctly match those bundles to the bidders. We refer the reader to the full proof in Appendix C.

Theorem 2a. *No anonymous ascending auction can guarantee better than a fraction of* $\max\{\frac{4}{n}, \frac{4}{\sqrt{m}}\}$ *of the efficient welfare for all profiles of bidder valuations, even when it uses bundle prices.*

A slight variation of the preferences in the proof of Theorem 4 – when only one of the low-valued bundles has a positive value – shows an instance where at least n ascending bundle-price trajectories are required in order to find the efficient allocation. This gives an easy bundle-price equivalent to Theorem 3. This result is tight, as there exist efficient non-anonymous bundle-price auctions; such auctions are clearly composed of n price trajectories.

5 Conclusion

This article considered ascending-price auctions for combinatorial auctions. It presented several impossibility results, providing insights about the power of different pricing models for such auctions. The paper showed that both bundle prices and personalized prices are necessary in order to achieve efficient, or even approximately efficient, outcomes by ascending combinatorial auctions. Proposals for other kinds of ascending auctions carry the burden of proof for showing that good results can occur in their particular settings.

References

Arrow, K. J., Block, H. D., Hurwicz, L., 1959. On the stability of the competitive equilibrium ii. Econometrica 27.

Ausubel, L., 2006. An efficient dynamic auction for heterogeneous commodities. American Economic Review 96 (3).

Ausubel, L. M., Cramton, P., McAfee, R. P., McMillan, J., 1997. Synergies in wireless telephony: Evidence from the broadband pcs auctions. Journal of Economics & Management Strategy 6 (3), 497–527.

Ausubel, L. M., Milgrom, P., 2006. In P. Cramton and Y. Shoham and R. Steinberg (Editors), Combinatorial Auctions. Chapter 1. The Lovely but Lonely Vickrey Auction. MIT Press.

Ausubel, L. M., Milgrom, P. R., 2002. Ascending auctions with package bidding. Frontiers of Theoretical Economics 1, 1–42.

Bichler, M., Davenport, A., Hohner, G., Kalagnanam, J., 2006. In P. Cramton and Y. Shoham and R. Steinberg (Editors), Combinatorial Auctions. Chapter 22. Industrial Procurement Auctions. MIT Press.

Bikhchandani, S., Mamer, J. W., 1997. Competitive equilibrium in an exchange economy with indivisibilities. Journal of economic theory 74, 385 – 413.

Bikhchandani, S., Ostroy, J. M., 2002. The package assignment model. Journal of Economic theory 107, 377–406.

Blumrosen, L., Nisan, N., 2005. On the computational power of iterative auctions I: demand queries. Discussion paper no. 381, The Center for the Study of Rationality, The Hebrew University.

Cramton, P., 1998. Ascending auctions. European Economic Review 42 (3–5), 745–756.

Cramton, P., 2002a. Comments on alternative policy options for managing capacity and mitigating congestion and delay at laguardia airport. U.S. Department of Transportation. FAA-2001-9852, FAA-2001-9854.

Cramton, P., 2002b. In Martin Cave, Sumit Majumdar, and Ingo Vogelsang, eds., Handbook of Telecommunications Economics. Chapter 14: Spectrum auctions. Elsevier Science B.V.

Cramton, P., Shoham, Y., (Editors), R. S., 2006. Combinatorial Auctions. MIT Press.

de Vries, S., Schummer, J., Vohra, R. V., 2005. On ascending vickrey auctions for heterogeneous items. Forthcoming in J. Econ. Theory.

Demange, G., Gale, D., Sotomayor, M., 1986. Multi-item auctions. Journal of Political Economy 94, 863–872.

FCC Auctions, 2006. Fcc auctions home page. `http://wireless.fcc.gov/auctions`.

FCC Combinatorial Bidding Conference, 2003. Combinatorial bidding conference. See Conferences page in `http://wireless.fcc.gov/auctions`.

Gul, F., Stacchetti, E., 1999. Walrasian equilibrium with gross substitutes. Journal of Economic Theory 87, 95 – 124.

Gul, F., Stacchetti, E., 2000. The english auction with differentiated commodities. Journal of Economic Theory 92 (3), 66 – 95.

Kelly, F., Steinberg, R., 2000. A combinatorial auction with multiple winners for universal service. Management Science 46, 586–596.

Kelso, A., Crawford, V., 1982. Job matching, coalition formation, and gross substitute. Econometrica 50, 1483–1504.

Kwasnica, A., Ledyard, J., Porter, D., DeMartini, C., 2005. A new and improved design for multi-object iterative auctions. Management Science 51 (3), 419 – 434.

Ledyard, J. O., Olson, M., Porter, D., Swanson, J. A., Torma, D. P., 2002. The first use of a combined-value auction for transportation services. Interfaces 32 (5), 4 – 12.

Lehmann, B., Lehmann, D., Nisan, N., 2006. Combinatorial auctions with decreasing marginal utilities. Games and Economic Behavior 55 (2), 270–296.

Milgrom, P., 2000a. Ascending auctions with package bidding.

Milgrom, P., 2000b. Putting auction theory to work: the simultaneous ascending auction. Journal of Political Economy 108 (2), 245 – 272.

Milgrom, P., 2004. Putting auction theory to work. Cambridge University Press.

Milgrom, P. R., Weber, R. J., 1982. A theory of auctions and competitive bidding. Econometrica 50 (5), 1089–1122.

Mishra, D., Parkes, D., July 2005. Ascending price vickrey auctions for general valuations. Working paper, Harvard University.

Nisan, N., Segal, I., 2003. The communication requirements of efficient allocations and supporting prices. Forthcoming in the Journal of Economic Theory.

Parkes, D. C., 2001. Iterative combinatorial auctions: Achieving economic and computational efficiency. Ph.D. Thesis, Department of Computer and Information Science, University of Pennsylvania.

Parkes, D. C., Ungar, L. H., 2000. Iterative combinatorial auctions: Theory and practice. In: AAAI/IAAI. pp. 74–81.

Porter, D., Rassenti, S., Roopnarine, A., Smith, V., 2003. Combinatorial auction design. Proceedings of the National Academy of Sciences 100, 11153–11157.

Rassenti, S., Smith, V., Bulfin, R., 1982. A combinatorial auction mechanism for airport time slot allocation. Bell Journal of Economics 13 (2), 402–417.

Roth, A. E., Sotomayor, M., 1992. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Cambridge University Press.

Rothkopf, M. H., Teisberg, T. J., Kahn, E. P., 1990. Why are vickrey auctions rare? Journal of Political Economy 98 (1), 94–109.

Scarf, H., 1960. Some examples of global instability of the competitive equilibrium. International Economic Review 1 (3), 157 – 172.

Sheffi, Y., 2004. Combinatorial auctions in the procurement of transportation services. Interfaces 34 (4), 245 – 252.

Wurman, P., Wellman, M., 2000. Akba: A progressive, anonymous-price combinatorial auction. In: Second ACM Conference on Electronic Commerce. pp. 21 – 29.

A Critical Price Levels

In this subsection we give a simple, formal argument, to be used in the proofs of the impossibility results, saying that if an auction does not give an opportunity for a bidder to demand some bundle S, by presenting relevant levels of prices ("critical price levels"), then the auction reveals no information at all about the value of S.

Some notations that describe the uncertainty of the auctioneer regarding the bidders: Denote the set of all the possible valuations for bidder i by V_i. Also denote the set of all possible values for the bundle S in V_i by $Q_i(S) = \{v_i(S) \mid v_i \in V_i\}$. Finally, denote the set of the possible values for the bundle S, given that the realization of the value of some other bundle T is c_T, by $Q_i(S \mid v_i(T){=}c_T) = \{v_i(S) \mid v_i \in V \text{ and } v_i(T){=}c_T\}$.

First we define informationally-independent classes of valuations – valuations where obtaining information regarding any set of bundles adds no new information about the possible values of other bundles.

Definition 4. We say that a set V_i of valuations for bidder i is *informationally independent*, if for any bundle S, and any realization of the values of the other bundles $\{c_T\}_{T \neq S}$, the set of possible values for S remains unchanged. Namely, for every $S \subseteq M$,

$$Q_i(S) \quad = \quad Q_i \left(S \mid v_i(T) = c_T \text{ for every } T \neq S \right)$$

Definition 5. Denote the class of all possible valuations of bidder i by V_i. We say that the price level p is *critical for Bidder i with respect to the bundle S*, if for some $v_i \in V_i$, Bidder i demands the bundle S under the price level p.

The next easy proposition implies that if no critical price vector is presented to a bidder regarding some bundle S, then no information at all will be elicited on the value of this bundle when the valuations are informationally independent. The proposition also holds for non-ascending auctions, and for all pricing schemes.

Proposition 1. *Consider a bidder i, with an informationally-independent set of possible valuations V_i. If an auction reaches no critical price level for Bidder i with respect to a bundle S, then, at the end of the auction, no information is revealed on the value of S, that is, the set of possible values for S remains $Q_i(S)$.*

Proof. The proof is straightforward: Since no critical price level with respect to the bundle S is presented to Bidder i, then the data accumulated throughout the auction is completely independent of the value $v_i(S)$. Since the demands of the other bidders are also unchanged, and these demands are the only data that is available to the auctioneer, the auctioneer will not be able to differentiate between different values of $v_i(S)$. Therefore, no value of $v_i(S)$ can be ruled out. □

B Item-Price Ascending Auctions

Example 1. This example shows that a single item-price auction can elicit an exponential amount of information. Consider two bidders in a combinatorial auction with preferences of the following type: $v(S) = 1$ for every bundle S with more than $\frac{m}{2}$ items, $v(S) = 0$ if $|S| < \frac{m}{2}$ and every S such that $|S| = \frac{m}{2}$ has an unknown value of either 0 or 1. As proved by Nisan and Segal (2003), for determining the efficient allocation in this environment, the bidders may be required to communicate an amount of information which is exponentially larger than the number of items. However, using small enough increments, it is easy to determine the values of all the bundles of size $\frac{m}{2}$ by an ascending auction.[7] This information clearly suffices for determining the optimal allocation.

Definition 6. (Kelso and Crawford (1982)) A valuation v is said to satisfy the *substitutes (or gross-substitutes)* property if for every pair of item-price vectors $\overrightarrow{q} \geq \overrightarrow{p}$ (coordinatewise comparison), if $S = \{j \in M | p_j = q_j\}$ and A maximizes the bidder's utility under the price vector $\overrightarrow{p}$, then there exists a bundle B that maximizes the bidder's utility under the price vector $\overrightarrow{q}$ such that $S \cap A \subseteq B$.

[7]This can be done by enumerating on all the different bundles of size $\frac{m}{2}$, and for each bundle S set the prices of the items in S to some value λ and set the prices of the items not in S to $\lambda + \epsilon$ for sufficiently small ϵ. Clearly, the bundle S will be demanded if and only if $v_i(S) = 1$. Using exponentially small increments, we can construct such vectors of prices during a single ascending path of prices.

Definition 7. (k-trajectory ascending auctions) Consider an auction $\mathcal{A}$, and denote the set of all the price vectors presented to bidder i in $\mathcal{A}$ by $\mathcal{P}_i$.[8] We say that $\mathcal{A}$ is a *k-trajectory ascending auction* if for every bidder i, the set $\mathcal{P}_i$ can be divided into k ascending trajectories of prices $\mathcal{P}_i(1), ..., \mathcal{P}_i(k)$. Formally, $\cup_{j=1}^{k}\mathcal{P}_i(j) = \mathcal{P}_i$ and for every $j \in \{1, .., k\}$, and for every two price vectors $p, q \in \mathcal{P}_i(j)$ such that q was presented to bidder i at a later stage in $\mathcal{A}$ than p, and for every bundle $S \subseteq M$, we have that $q(S) \geq p(S)$.

Proof of Theorem 1a:

Proof. Consider a single agent whose valuation has the following properties: for every bundle S such that $|S| > \frac{m}{2}$ we have $v(S) = 2$, and for every $|S| \leq \frac{m}{2}$ we have $v(S) = 0$, except for a single unknown bundle T of size $\frac{m}{2}$ that either has a value of $1 - \delta$ (for some small $\delta > 0$) or 0. We first show that finding the hidden bundle T requires an exponential number of ascending item-price *trajectories*, even if the auctioneer knows these properties of the valuations.

Recall that under a "critical" price level with respect to the bundle S, the player demands S for some realization of his valuation (see Definition 5 in Appendix A). We first prove the following claim:

Claim 2. In an ascending auction, if the bidder is presented with a critical price vector for some bundle S of size $\frac{m}{2}$, then no critical price vector will be published at later stages of the ascending auction with respect to any other $\frac{m}{2}$-sized bundle.

Proof. Let $\overrightarrow{p}$ be a critical price vector presented to the bidder with respect to some bundle S, $|S| = \frac{m}{2}$. Thus, for some possible value of $v(S)$ and for any item $x \in M \setminus S$, the bidder (weakly) prefers the bundle S over the bundle $\{S \cup x\}$, i.e., $v(S \cup x) - p(S \cup x) \leq v(S) - p(S)$. Since the prices are linear, and since $v(S)$ is always smaller than 1, it follows that: $p_x \geq v(S \cup x) - v(S) > 2 - v(S) > 1$. Thus, the price of *any item*

[8]Recall that each price vector p specifies a price $p(S)$ for every bundle $S \subseteq M$.

in $M \setminus S$ is strictly greater than 1. Since the prices are ascending, it follows that the bidder will not demand any bundle of size $\frac{m}{2}$ containing an item from $M \setminus S$ at later stages of the auction. (Clearly, the only bundle of size $\frac{m}{2}$ that does not contain any item from $M \setminus S$ is S.) □

Due to Claim 2, an ascending path of prices can only contain critical price levels with respect to one of the $\frac{m}{2}$-sized bundles. Therefore, this ascending trajectory will be independent of the values of all the other $\frac{m}{2}$-sized bundles, and no new information will be elicited on them (this holds since the valuations are informationally independent – see Proposition 1). It follows that in each ascending trajectory, the auctioneer has to arbitrarily decide which $\frac{m}{2}$-sized bundle will be checked. An adversary (or "nature") may choose a valuation such that the last (or before last) bundle to be checked is the bundle T. Since the number of $\frac{m}{2}$-item bundles is exponential in m,[9] an exponential number of ascending trajectories is required for finding the hidden bundle.

Now, consider a second bidder that has a value of 2 for every bundle of size $\frac{m}{2}$ or more. The optimal allocation will clearly allocate the bundle T to Bidder 1, and the other $\frac{m}{2}$ items to the second bidder. Finding the efficient allocation for these two bidders is equivalent to finding the bundle T. The theorem follows. □

Proof of Theorem 1b:

Proof. Consider n bidders and n^2 items for sale, and assume that n is prime.[10] We construct a total of n^2 distinct bundles with the following properties: for each bidder i ($1 \leq i \leq n$), we define a partition $S^i = (S^i_1, ..., S^i_n)$ of the n^2 items to n bundles, such that any two bundles from different partitions intersect (i.e., for every two bidders $i \neq j$, and every k, l we have $S^i_k \cap S^j_l \neq \emptyset$). We call this combinatorial structure

[9]According to Stirling's formula, the number of distinct bundles of size $\frac{m}{2}$, out of m distinct items, is approximately $\sqrt{\frac{2}{\pi m}} \cdot 2^m$.

[10]Due to the celebrated Bertrand Conjecture from 1845 (proved by Chebyshev in 1850), for every natural number n there exists at least one prime number between n and $2n$. Therefore, we can assume that n is prime, where the number of items is at most twice the original number. This will result in an additional factor of 2 in our approximation result.

mutually-intersecting partitions. In Appendix D, we show an explicit construction of mutually-intersecting partitions using the properties of linear functions over finite fields. The rest of the proof is independent of the specific construction.

We now build a set of valuations for the bidders, and prove that they are hard to elicit by item-price ascending auctions. Each bidder i will have a value of 2 for every bundle that contains a union of two bundles from different partitions, and an unknown value of either 0 or $1-\delta$ (for some small $\delta > 0$) for bundles that contain only a single bundle from a partition (henceforth, the "low-valued" bundles). More formally, each bidder will have the following valuation (the value of any other bundle is the maximal value of a bundle that it contains):

- A value of 2 for the bundle $S_k^{j'} \cup S_l^j$, for every k, l and every $j' \neq j$.
- A value of either 0 or $1-\delta$ (unknown to the seller) for the bundle S_k^j, for every j, k.

Note that at most one bidder can gain a value of 2, since every two 2-valued bundles contain bundles from different partitions and thus must intersect. Therefore, for achieving more than a welfare of 2, we must allocate low-valued bundles. However, as the following claim shows, the demand of a bidder during a single ascending auction can only reveal information about his values for bundles from a single partition.

Claim 3. If a bidder is presented with a critical price vector with respect to a bundle from one partition, no critical price levels will be presented to this bidder with respect to bundles from *other* partitions at later stages of the ascending auction.

Proof. Let p be a critical price level for Bidder i with respect to his low-valued bundle S_k^j. Then, for every bundle S_k^l from a different partition (i.e., $l \neq k$), we have:

$$v(S_k^j) - p(S_k^j) \quad \geq \quad v(S_k^j \cup S_k^l) - p(S_k^j \cup S_k^l)$$

Since the prices are linear, it follows that:

$$p(S_k^l) \quad \geq \quad p(S_k^j \cup S_k^l) - p(S_k^j) \quad \geq \quad v(S_k^j \cup S_k^l) - v(S_k^j) \quad > \quad 1$$

where the final inequality holds since $v(S_k^j) < 1$. Hence, the bundle S_k^l will not be demanded before the auction concludes. □

It follows from the claim above that every ascending trajectory of prices will be independent of the values of every bidder to bundles from all the partitions, except at most one partition. Hence, for each bidder, the auctioneer will gain information about at most one partition of the n partitions. Therefore, for every ascending auction, there must exist a partition j (i.e., $S_1^j, ..., S_n^j$) for which *at most* one bidder revealed some information. An adversary ("nature") can set the values of the bundles in all the other partitions such that any way of allocating them will result in a total value of at most 2. In addition, the total value of the bidders to bundles in partition j may be arbitrary close to n (that is, $n - n\delta$) – each bidder will have a value of $1 - \delta$ for one distinct bundle from this partition. The auctioneer does not have any information on the values that the bidders (except, maybe, one) have for bundles in this partition, and therefore the auctioneer will not be able to correctly match the bundles in this partition to the bidders; The auctioneer can only guarantee a value of 2 by allocating all items to a single bidder, as opposed to the optimal welfare that can be arbitrarily close to n (and here, $n = \sqrt{m}$). The theorem follows (as mentioned, we lose an additional factor of 2 since we assumed that n is prime). □

Proof of Theorem 1c:

Proof. Let w be the valuation that aggregates the preferences of the n original players. Since all the original valuations hold the substitutes property, then their aggregation, w, also has the substitutes property, (e.g., Lehmann et al. (2006)). Substitutes valuations are, in particular, sub-additive – that is, for every two bundles S, T we have that

$w(S) + w(T) \geq w(S \cup T)$. Due to the assumption that the w is not additive, there are two bundles S and T for which the inequality is strict,

$$w(S) + w(T) > w(S \cup T) \tag{B.1}$$

Substitute valuations are also submodular, and thus exhibit diminishing marginal valuations (see, e.g., Lehmann et al. (2006)). Therefore, the marginal contribution of $M \setminus (S \cup T)$ in Inequality B.1 is greater for T than for $S \cup T$, thus,

$$w(S) + w(M \setminus S) > w(M) \tag{B.2}$$

Denote $\epsilon = w(S) + w(M \setminus S) - w(M)$. Now, consider the "dual" valuation to w denoted by $\overline{w}$, i.e., for every bundle X, $\overline{w}(X) = w(M) - w(M \setminus X)$. The dual valuation specifies the contribution of the bundle X to the welfare of the n players, given that they already hold the other items. Clearly, if an additional player has a value for S that exceeds $\overline{w}(S)$, allocating this bundle to her will increase the total welfare. Using Inequality B.2, we thus have that the bundles S and $M \setminus S$ are complements with respect to $\overline{w}$ (i.e., the value of their union is smaller than the sum of the separate values),

$$\begin{aligned}
& \overline{w}(S) + \overline{w}(M \setminus S) && \text{(B.3)}\\
= \; & w(M) - w(M \setminus S) + w(M) - w(S) && \text{(B.4)}\\
= \; & w(M) - (w(M \setminus S) + w(S) - w(M)) && \text{(B.5)}\\
= \; & \overline{w}(M) - \epsilon && \text{(B.6)}
\end{aligned}$$

We define an additional bidder k with the valuation $v_k(\cdot)$ for which $v_k(M) = \overline{w}(M)$ (which also equals $w(M)$), and the values $v_k(S)$ and $v_k(M \setminus S)$ are unknown to the auctioneer and may take the following values: $v_k(S) \in \{\ \overline{w}(S),\ \ \overline{w}(S) + \frac{\epsilon}{6},\ \ \overline{w}(S) + \frac{\epsilon}{3}\ \}$ and

$v_k(M \setminus S) \in \{\ \overline{w}(M \setminus S),\ \ \overline{w}(M \setminus S) + \frac{\epsilon}{6},\ \ \overline{w}(M \setminus S) + \frac{\epsilon}{3}\ \}$.

The values of all the other bundles is the maximal value of a bundle, from the above bundles, that they contain.

An efficient auction clearly has to determine which of the bundles S and $M \setminus S$ adds more value for the new bidder with respect to $\overline{w}$. We will show that an ascending item-price auction will not be able to find this bundle using the following claim. (The concept of critical price levels is defined in Definition 5 in Appendix A.)

Claim 4. If a critical price level is presented to player k with respect to the bundle S, no critical price levels will be presented with respect to the bundle $M \setminus S$ at later stages of the ascending auction.

Proof. Let p be a critical price level with respect to the bundle S. Then for some value of $v_k(S)$ the player will prefer this bundle to the whole bundle: $v_k(S) - p(S) \geq v_k(M) - p(M)$. Due to the linearity of the prices and the definition of $v_k(\cdot)$ it follows that:

$$p(M \setminus S) \quad \geq \quad v_k(M) - v_k(S) \quad \geq \quad \overline{w}(M) - \overline{w}(S) - \frac{\epsilon}{3} \tag{B.7}$$

$$> \quad \overline{w}(M \setminus S) + \epsilon - \frac{\epsilon}{3} \quad > \quad v_k(M \setminus S) \tag{B.8}$$

Where Inequality B.8 follows from Equation B.6. The price of the bundle $M \setminus S$ is greater than all its possible values, and this bundle will not be demanded at future stages since the prices are ascending. □

Similarly, we can also show that if a critical price is presented with respect to $M \setminus S$, then all future price levels will be independent of the value of S. Therefore, the auctioneer will be able to elicit information only on one of the bundles S and $M \setminus S$, and the optimal allocation will remain unknown. □

C Anonymous Ascending Auctions

Proof of Theorem 2a:

Proof. Consider n bidders and n^2 items, and assume that n is prime.[11] Consider n^2 distinct bundles defined by mutually-intersecting partitions (see Theorem 1b), that is, for each bidder, we define a partition $S^i = (S^i_1, ..., S^i_n)$ of the n^2 items to n bundles, such that any two bundles from different partitions intersect. (As mentioned, an explicit construction is given in Appendix D.)

Using these n^2 bundles we define the following distribution over the players' preferences. The preferences are drawn uniformly at random from the following class of valuations with the following properties:

- Each bidder i has a value 2 for every bundle S^i_j in his partition.
- There exists one player k such that all the elements S^k_j in his partition gain the other players a value of $1-\epsilon$ for some small ϵ.
- All the players $i \neq k$ gain a zero value from the bundles in player i's partition, i.e., $v_i(S^k_j) = 0$ for every j.

Using these n^2 bundles we construct the following valuations. We will define the values that the bidders have for each one of these n^2 bundles, and again, the value of any other bundle is the maximal value of a bundle that it contains. A bidder i has a value of 2 for any bundle S^i_j in his partition (i.e., the i'th partition). For all the bundles in the other partitions, he has a value of either 0 or of $1-\delta$ (for some small $\delta > 0$), and these values are unknown to the auctioneer. Since every pair of bundles from different partitions intersect, at most one bidder can receive a bundle with a value of 2. Nonetheless, for some realizations of the bidders' preferences, we

[11] We can assume this and lose a factor of two in the approximation ratio. See the proof of Theorem 1b.

may allocate the bundles of a particular partition, one bundle per each bidder, such that one bidder gains a value of 2 and all the others receive a value of $1-\delta$.

Consider the valuations described above. In every anonymous ascending auction, a bidder will not demand one of his low-valued bundle as long as the price of at least one of his high-valued bundles is below 1 (which gains him a utility greater than 1 for this bundle). Therefore, for eliciting any information about low-valued bundles, the auctioneer should first arbitrarily choose a bidder (w.l.o.g., Bidder 1) and raise the prices of *all* the bundles $S_1^1, ..., S_n^1$ to be greater than 1. Since the prices cannot decrease, no critical price level (see Definition 5) will be presented with respect to any of these bundles at later stages of the auction for any bidder. Since the valuations are informationally independent, no information at all will be gained by the auctioneer on the values of these bundles (see Definition 4 and Proposition 1). It might happen that the low values of all the bidders for the bundles not in Bidder 1's partition are zero (i.e., $v_i(S_j^k) = 0$ for every bidder i and any partition $k \neq 1$ and every bundle j in it). However, allocating each bidder a different bundle from Bidder 1's partition, might achieve a welfare of $n+1-(n-1)\delta$ (Bidder 1's valuation is 2, and $1-\delta$ for all other bidders); The auctioneer has no information on the values that the other bidders have for these bundles. Therefore, for every decision the auctioneer makes about the allocation, an adversary ("nature") may choose a profile of valuations for which no more than a welfare of 2 is achieved (2 for Bidder 1's high-valued bundle, 0 for all other bidders). We conclude that no anonymous bundle-price ascending auction can guarantee a welfare greater than 2 for this class, where the optimal welfare can be arbitrarily close to $n+1$. The theorem follows. □

D Constructing Mutually-Intersecting Partitions

We now present an explicit construction for the combinatorial structure used in Theorem 1b.We also use this combinatorial structure when we prove the inefficiency of anonymous bundle-price ascending auctions in Theorem 2a. We assume that there

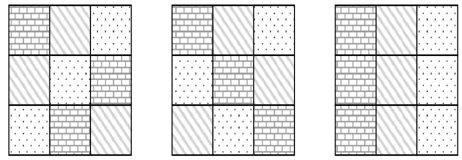

Figure 3: Mutually-intersecting partitions for 3 bidders and 9 items. Each large square defines a partition of the items (small squares with the same color in the same large square form a bundle). Indeed, every two bundles from different partitions intersect. The partition are defined by parallel linear functions over the relevant finite field.

are n bidders and n^2 items (n is prime). For every bidder i, we define a partition $S^i = (S^i_1, ..., S^i_n)$ of the n^2 items to n bundles of size n, such that any two bundles from different partitions intersect (i.e., $S^i_j \cap S^k_l \neq \emptyset$ for every $i \neq k$ and every l, j). Figure 3 describes such a construction for 3 bidders and 9 items.

We use the properties of linear functions over finite fields (for that, we denote the bidders by $0, ..., n-1$):

Recall that $Z_n = \{0, ..., n-1\}$ is a field if (and only if) n is prime. Denote the n^2 items for sale by pairs of numbers in Z_n. Each linear function $ax + b$ over the finite field Z_n denotes an n-item bundle (a total of n^2 bundles where $a, b \in Z_n$). The items in each bundle are the pairs $(x, ax + b)$ for every $x \in Z_n$. The bundles assigned to Bidder i are the n bundles $ix + b$ where $b \in Z_n$ (that is, all the parallel linear functions with a slope i). We need to show that the bundles assigned to Bidder i form a partition, and indeed the functions $ix + b_1$ and $ix + b_2$ cannot intersect when $b_1 \neq b_2$. It is also easy to see that every two bundles that are assigned to different bidders do intersect: consider the functions $ix + b_1$ and $jx + b_2$. Since z_n is a field, clearly an x exist such that $x(j - i) = (b_1 - b_2)$ when $j \neq i$ for any b_1, b_2. The jth bundle of Bidder i is therefore, $S^i_j = \{(0, i \cdot 0 + j), ..., (n-1, i \cdot (n-1) + j)\}$.