Economic Indices of Absolute and Relative Riskiness

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Abstract

Following the paper of Aumann and Serrano (2008) who characterize by axioms an index of riskiness defined on absolute returns, we characterize a new index of riskiness defined on relative returns. Both indices are characterized by a similar principle of duality between risk and risk aversion, but while the index of absolute riskiness refers to absolute risk aversion, the index of relative riskiness refers to relative risk aversion. The similarities and differences between the two indices are studied.

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1 Introduction

One of the key determining factors in the formation of investments decisions is the riskiness of the investments. Hence, *measures (indices) of riskiness* are of a great theoretical and practical interest. A measure of riskiness is essentially a real-valued function defined on random returns of investments. Examples of such common measures include variance, VaR (Value at Risk) and semi-variance.

In the present paper we emphasize the idea that investments returns can be viewed either as absolute or relative, and that this distinction implies different concepts of risks. Consider, for instance, the investment in a security s. If the security's current price is s_0 and its random future value is s_1, $s_1 - s_0$ is considered to be the *absolute return* of s, and s_1/s_0 to be its *relative return*. Absolute returns are additive; if an agent with initial wealth w buys security s, her wealth will be distributed as $w + s_1 - s_0$. By contrast, relative returns are multiplicative; if the agent invests all her initial wealth w in the security, her wealth will be distributed as ws_1/s_0. Functions of returns may induce different orders on the set of securities once they are applied to absolute returns or relative returns. For instance, given two securities, it might be that the expected absolute return of one security is greater but that its expected relative return is smaller (or vice versa). The same is correct in relation to variance and many other functions of random variables.

The renewed interest of the economic literature in risks[1] focuses on the riskiness of *additive gambles* rather than securities. Additive gambles are basically random variables whose values are interpreted as absolute returns. If an agent with initial wealth w accepts such an additive gamble g, her wealth will be distributed as $w + g$. Hence, the absolute return of a security is by definition an additive gamble. In their paper, Aumann and Serrano (2008) characterize an index of riskiness of such additive gambles by a "dual" relationship between risk and risk aversion. According to their approach, if "riskiness" refers to the same concept of risk that risk averters dislike, it is natural to expect that agents who are less averse to risk will be willing to accept riskier gambles.[2] This principle, together with the homogeneity of riskiness, yields a unique index of riskiness denoted by A^*. This index was

[1] This literature includes Aumann and Serrano (2008), Foster and Hart (2009), Foster and Hart (2013), and Hart (2011).

[2] Risk aversion here relates to "wealth uniform" risk aversion; see Section 2.1.

first presented by Palacios-Huerta et al. (2004).[3]

In this paper we propose two indices of riskiness of securities: A, an index of absolute riskiness, and R, an index of relative riskiness.[4] The A index is simply the index A^* applied to absolute returns of securities. On the other hand, R is a new index developed in the present paper. The A and R indices are both functions of random returns but while A is a function of absolute returns, R is a function of relative returns. Like A, R is also characterized by the duality principle, but while the riskiness as measured by A is dual to absolute risk aversion, the riskiness measured by R is dual to relative risk aversion. Needless to say, A and R are not ordinally equivalent; i.e., one security might be absolutely more risky but relatively less risky.

Obviously, one could think of alternative definitions for an index of relative riskiness. In fact, Aumann and Serrano (2008) themselves propose to measure the riskiness of a multiplicative gamble as the absolute riskiness of its net return.[5] Applying this index of multiplicative gambles to (net) relative returns of securities will yield an alternative index of relative riskiness. However, some characteristics that are unique to R make it the best candidate to measure the relative riskiness. Moreover, these characteristics are equivalent to those of A but are recast in relative terms. Some of the main characteristics of absolute and relative riskiness measured by A and R are:

1. (i) Agents whose absolute risk aversion is uniformly[6] lower are willing to accept securities that are absolutely riskier.

   (ii) Agents whose relative risk aversion is uniformly lower are willing to accept securities that are relatively riskier.

2. (i) The higher the absolute riskiness of a security is, the higher the probability that a sum of such i.i.d. absolute returns will go below a threshold.

   (ii) The higher the relative riskiness of a security is, the higher the

[3]In the insurance risk literature, the reciprocal of A^* is knwon as the "adjustment coefficient" ; see Meilijson (2009).

[4]We assume that distributions are known. A generalization to sets of distributions and non-expected-utility models can be done using the approach of Michaeli (2012).

[5]A multiplicative gamble is a random variable whose values are interpreted as relative returns. If r is a multiplicative gamble, investing w in r will yield wr at the end of the period. Aumann and Serrano define the riskiness of r as the absolute riskiness of $r - 1$.

[6]"Uniformly" is used here in the same sense as in Section 2.1.

probability that a multiplication of such i.i.d. relative returns will go below a threshold.

3. (i) Any decision maker with a "standard"[7] utility who rejects the absolute return of a security at any level of wealth will also reject the absolute return of an absolutely riskier security at any level of wealth.

   (ii) Any decision maker with a "standard" utility who rejects the relative return of a security at any level of wealth will also reject the relative return of a relatively riskier security at any level of wealth.

The first item is the duality axiom of Aumann and Serrano (2008), the second item appears in Melijson (2010), and the third appears in Hart (2011). Our index of relative riskiness has the advantage of being the only index of relative returns that satisfies these properties in relation to relative returns.

The paper is organized as follows. Section 2 is devoted to the basic axiomatic definitions of the indices. The A index is characterized by two axioms, namely, absolute duality and absolute scaling, which are a simple translation of the Aumann–Serrano axioms from the environment of gambles to the environment of securities. However, the axioms that characterize the R index, namely, relative duality and relative scaling, are new. Section 3 relates the indices to Arrow–Pratt risk aversion. Section 4 sets forth some desirable properties of the indices. Section 5 proposes two alternative characterizations (other than "duality") for the two indices and Section 6 concludes. Proofs are relegated to the Appendix.

2 The Indices

In this section we give an axiomatic characterization of the indices of absolute and relative riskiness of securities. The index of absolute riskiness is simply the application of the index A^*, originally defined on additive gambles, to absolute returns of securities. However, the index of relative riskiness is a new index that induces a different order on the set of securities.

[7] Here "standard" refers to the definition of Hart (2011); see Section 5.2.

2.1 Axiomatic Characterization

Throughout this paper, a utility function is a von Neumann–Morgenstern utility function for money; it is strictly monotonic, strictly concave, and twice continuously differentiable. A decision maker (agent) is characterized by a utility function.

We consider a two-period model in which investments are made at period zero and are liquidated at period one. A security s is characterized by a pair (s_0, s_1), where s_0 is a real number representing the value (price) of the security at period zero, and s_1 is a random variable with finitely many positive values representing the value of the security at period one. If an agent with initial wealth w buys one unit of s, her wealth will be distributed as $w + s_1 - s_0$ and if she invests w in s, her wealth will be distributed as ws_1/s_0. Therefore $s_1 - s_0$ is the absolute return of s and s_1/s_0 is its relative return. Following Aumann and Serrano (2008) we assume that the absolute return of a security takes negative values with a positive probability and that its expectation is positive. Similarly, we assume that the relative return of a security takes values less than one with a positive probability and that its geometric mean is greater than one.[8]

We say that an agent with utility u and initial wealth w *A-accepts* security s if she benefits from buying one unit of it, i.e., if $Eu(w+s_1-s_0) > u(w)$. Otherwise she *A-rejects* it. Similarly, we say that the agent with initial wealth $w > 0$ *R-accepts* s if she benefits from investing w in s, i.e., if $Eu(ws_1/s_0) > u(w)$; otherwise she *R-rejects* it.[9]

We use the concepts of A-acceptance and R-acceptance to define two orders on agents representing their willingness to be exposed to absolute and relative risks respectively.

Definition 1.

[8] If the absolute return of a security is always positive or if the relative return of a security is always greater than one, its riskiness (absolute or relative, respectively) can be considered to be zero. On the other hand, if the mean of the absolute return of a security is negative then accepting securities with such i.i.d. absolute returns repeatedly will lead to bankruptcy with probability one; therefore its absolute riskiness can be considered to be infinity. For similar reasons, if the geometric mean of the relative return of a security is less than one, its relative riskiness can be considered to be infinity.

[9] The definitions of R-acceptance and R-rejection assume the investment of all initial wealth. This assumption is mostly reasonable for portfolios (rather than "securities"). We dispense with this assumption latter; see Section 2.3.

1. *Agent i is uniformly no less absolute-risk averse than agent j, written $i \succeq_A j$, if whenever i A-accepts a security at some wealth, j A-accepts that security at any wealth.*

2. *Agent i is uniformly no less relative-risk averse than agent j, written $i \succeq_R j$, if whenever i R-accepts a security at some (positive) wealth, j R-accepts that security at any (positive) wealth.*

The first part of Definition 1 is taken from Aumann and Serrano (2008) and the second part is our extension.

We call agent i uniformly more absolute-risk averse than j, denoted by $i \succ_A j$, if $i \succeq_A j$ but not $j \succeq_A i$. Similarly, we call agent i uniformly more relative-risk averse than j, denoted by $i \succ_R j$, if $i \succeq_R j$ but not $j \succeq_R i$. Note that both orders are partial orders.

Define an index as a positive real-valued function of securities. Given an index Q, security s is riskier than security r (according to Q) if $Q(s) > Q(r)$. If the value of the index depends only on absolute returns of securities, we call it an index of absolute returns; and if its value depends only on relative returns, we call it an index of relative returns.

We consider two axioms. The first part of each axiom is a simple translation of the corresponding Aumann-Serrano axiom, namely, from a gambles environment to a securities environment. Together, they characterize the index of absolute riskiness. By contrast, the second parts of the two axioms, which relate to relative returns, together characterizes a new index of relative riskiness.

Absolute Duality. If $i \succ_A j$, i A-accepts s at w, and if $Q(s) > Q(r)$, then j A-accepts r at w.
Relative Duality. If $i \succ_R j$, i R-accepts s at w, and if $Q(s) > Q(r)$, then j R-accepts r at w.

Duality asserts that if the more risk averse of two agents accepts the riskier of two assets, then a fortiori the less risk averse agent accepts the less risky asset.

Absolute Scaling. $Q(ts) = tQ(s)$ for all positive numbers t.
Relative Scaling. $Q(s^t) = tQ(s)$ for all positive numbers t.

The two versions of the scaling axiom, absolute and relative, embody the cardinal nature of riskiness; accepting two securities with the same returns doubles the risk. “Accepting” is interpreted as either A-accepting or R-accepting. More specifically, if the absolute return of a security is multiplied by two, then its absolute return is multiplied by two. Similarly, if the relative return of a security is raised to the power of two, its relative riskiness is multiplied by two.[10]

We define two indices: one is an index of absolute returns, A, and one is an index of relative returns, R. Given a security s, $A(s)$ and $R(s)$ are defined implicitly as follows:

$$Ee^{-(s_1-s_0)/A(s)} = 1. \tag{1}$$

$$\mathrm{E}\left(\frac{s_1}{s_0}\right)^{-1/R(s)} = 1. \tag{2}$$

While A is a translation of the index A^* to the securities environment, R is an altogether new index.[11] Note that Aumann and Serrano (2008) denote their index by “R.” But they do not distinguish between absolute and relative returns. We change their notation in accordance with the terms absolute and relative.

The following theorem asserts that the indices just defined satisfy the two axioms.

Theorem 1.

1. *For each security s, there is a unique positive number A that solves for* (1)*. A satisfies absolute duality and absolute scaling, and any index of absolute returns satisfying these two axioms is a positive multiple of A.*

2. *For each security s, there is a unique positive number R that solves for* (2)*. R satisfies relative duality and relative scaling, and any index of relative returns satisfying these two axioms is a positive multiple of R.*

[10]One can think of it as an investment in a security for two periods where the returns of the security are equal at both periods.

[11]A recent paper of Li (2013) proposes an index of riskiness of multiplicative gambles which is based on a similar principle.

Aumann and Serrano (2008) note in relation to A that duality and scaling are both essential: omitting either one of them results in admitting indices that are not positive multiples of A. But of the two properties, duality is the more central: together with certain weak conditions of continuity and monotonicity—but not scaling—it already implies that the index is ordinally equivalent to A. The same statement is true for our new index R.

It follows from (1) and (2) that

$$R(s) = A(\log s), \tag{3}$$

where $\log s$ is the security whose prices are: $(\log s)_0 = \log s_0$ and $(\log s)_1 = \log s_1$. Equation (3) does not imply that the orders induced by A and R are ordinally equivalent. In fact, there are many pairs of securities, s and r, for which $A(s) > A(r)$ but $R(s) < R(r)$.

The use of log return is quite commonly used in the financial literature to compare the riskiness of investments. In general, the main reason for using log returns instead of returns is to account for continuous compounding of interest or continuous growth of value. The present paper strengthens this idea by suggesting an axiomatic justification to use log returns when comparing the relative risks of different investments.

2.2 The Index A^*

Unlike the index A, the index A^* which characterized axiomatically by Aumann and Serrano (2008) is defined on additive gambles. An additive gamble is a random variable with real values—understood as dollar amounts—some of which are negative, and it has a positive expectation. By contrast, a multiplicative gamble is a random variable whose values are understood as relative returns, with an expectation greater than one and some values that are less than one. To make the distinction clearer, let us consider an agent with wealth w; if she accepts an additive gamble g, her wealth will be distributed as $w + g$. By contrast, if the agent invests w in a multiplicative gamble h, her wealth will be distributed as wh. Note that by definition, absolute returns of securities are additive gambles and relative returns of securities are multiplicative gambles.

Aumann and Serrano (2008) characterize their index by the duality and scaling (homogeneity) axioms. Our axioms of absolute duality and absolute scaling are a simple translation of their axioms to the environment of securities. To see the relation between A and the index A^*, note that the riskiness of an additive gamble g, denoted by $A^*(g)$, is defined implicitly by the equation

$$Ee^{-g/A^*(g)} = 1. \tag{4}$$

Obviously, it follows from (1) and (4) that for any security s,

$$A(s) \equiv A^*(s_1 - s_0).$$

Although Aumann and Serrano focus on additive gambles, at the end of their paper (Section 5.K) they address the problem of multiplicative gambles. Since they do not distinguish between absolute and relative riskiness, they apply the same definition to measure the riskiness of both additive and multiplicative gambles. Formally, the riskiness of a multiplicative gamble h is defined as

$$R^*(h) \equiv A^*(h - 1). \tag{5}$$

In our terms, the Aumann–Serrano riskiness of a multiplicative gamble is defined as the absolute riskiness of its net return. In principle, R^* can be applied to relative returns of securities, just as we apply A^* to absolute returns of securities, but such an application would induce a third order of riskiness on the set of securities which is different from the orders induced by A and R. In fact, it can be shown that for any security s,

$$R^*(s_1/s_0) = A^*(s_1 - s_0)/s_0.$$

Conceptually, then, R^* measures the absolute riskiness of a security per dollar.

2.3 $\mathbf{R}_\alpha$-acceptance

The axiom of relative duality that characterizes R uses the concept of R-acceptance. Recall that an agent with utility u and wealth w R-accepts a security s if and only if $Eu(ws_1/s_0) > u(w)$. This definition assumes that the invested capital is w, i.e., all the initial wealth. However, one could think of an alternative definition that considers an investment of only a fraction α of the wealth in a security. If the relative duality axiom referred to such a new definition of R-acceptance, it would characterize an index of relative riskiness different from R. To analyze it formally, note that if the initial wealth of an agent is w, investing αw in security s is equivalent to investing w in the security $s(\alpha)$ whose values at periods zero and one are defined as follows: $s(\alpha)_0 = s_0$ and $s(\alpha)_1 = s_0 + \alpha(s_1 - s_0)$. Investing either αw in s or w in $s(\alpha)$ causes the wealth to be distributed as $w + \alpha w(s_1/s_0 - 1)$. Given α, $0 < \alpha < 1$, we say that an agent with utility u and wealth w R$_\alpha$-accepts security s if she R-accepts $s(\alpha)$, i.e., if $Eu(ws(\alpha)_1/s(\alpha)_0) > u(w)$. Otherwise she R$_\alpha$-rejects it. Obviously, R-acceptance is a special case of R$_\alpha$-acceptance in which $\alpha = 1$. If we were to change the relative duality axiom to refer to R$_\alpha$-acceptance rather than to R-acceptance, we would get another index of relative riskiness, R_α, satisfying $R_\alpha(s) = R(s(\alpha))$ for any security s and $0 < \alpha < 1$.

While in general R_α and R are not ordinally equivalent, interestingly, when α goes to zero, the order induced by R_α becomes closer to the order induced by R^*. Formally,

$$\lim_{\alpha \to 0} \frac{R(s(\alpha))}{R(r(\alpha))} = \frac{R^*(s)}{R^*(r)} = \frac{A(s)/s_0}{A(r)/r_0}. \tag{6}$$

This implies that given two securities, s and r, if the absolute riskiness of s per dollar is higher than the absolute riskiness of r per dollar, then for α small enough, $s(\alpha)$ is relatively riskier than $r(\alpha)$. Put differently, when the capital at stake is very small, comparing either the relative riskiness of two investments or their absolute riskiness per dollar yield the same order of riskiness.[12] The proof of this statement is relegated to the Appendix.

3 Relation with Arrow-Pratt

3.1 Risk Aversion and Uniform Risk Aversion

The concepts of uniform absolute and relative risk aversion that underlie our treatment can be defined in terms of the well-known Arrow–Pratt coefficients of absolute and relative risk aversion.[13] Arrow (1965) and Pratt (1964) define

[12]It follows from the absolute scaling property that $A(s(\alpha)) = \alpha A(s)$. Hence, $A(s) > A(r)$ if and only if $A(s(\alpha)) > A(r(\alpha))$. Moreover, the right-hand side of Equation (6) equals $\frac{A(s(\alpha))/s_0}{A(r(\alpha))/r_0}$.

[13]Here, we draw on Aumann and Serrano's (2008) analysis of absolute riskiness and absolute risk aversion.

the coefficient of Absolute Risk Aversion (ARA) of an agent i with utility u_i and wealth w as

$$\rho_i(w) \equiv \rho(w, u_i) \equiv -u_i''(w)/u_i'(w),$$

and the coefficient of Relative Risk Aversion (RRA) as

$$\varrho_i(w) \equiv \varrho(w, u_i) \equiv w\rho(w, u_i).$$

The following lemma shows the relation between uniform risk aversion and Arrow–Pratt risk aversion.

Lemma 3.1.

1. *Agent i is no less uniformly absolute-risk averse than j if and only if $\rho_i(w_i) \geq \rho_j(w_j)$ for all w_i and w_j.*
2. *Agent i is no less uniformly relative-risk averse than j if and only if $\varrho_i(w_i) \geq \varrho_j(w_j)$ for all w_i and w_j.*

As Aumann and Serrano (2008) state, the Arrow–Pratt concept of absolute and relative risk aversion is a "local" concept in that it concerns i's attitude toward infinitesimally small risky assets at a specified wealth only.[14] In contrast, the concepts of uniform absolute-risk aversion and uniform relative-risk aversion are "global" in two senses: (1) they apply to risky assets of an arbitrary, finite size, which (2) may be taken at any wealth. However, these are only partial orders, whereas Arrow and Pratt define a numerical index (and hence a total order).

3.2 CARA and CRRA

An agent i is said to have Constant Absolute Risk Aversion (CARA) if her ARA is a constant α that does not depend on her wealth. In that case, i is called a CARA agent and her utility u a CARA utility, both with parameter α. There is an essentially unique CARA utility with parameter α, given by $u(w) = -e^{-\alpha w}$. An agent i is said to have Constant Relative Risk Aversion (CRRA) if the value of $\varrho_i(w)$ is constant for all w. CRRA expresses the idea that wealthier people are less risk averse. Here, wealth is assumed to be

[14] Schreiber (2012) takes the Arrow-Pratt approach by defining "local risks" based on the indices of riskiness described in the present paper.

positive. There is an essentially unique[15] CRRA utility with parameter α, given by

$$u_\alpha(x) = \begin{cases} \frac{(x^{1-\alpha}-1)}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1. \end{cases}$$

Though defined in terms of a local concept of risk aversion, CARA and CRRA can be defined in global terms as follows. An agent i has CARA if and only if, for any security s and any two wealth levels, i either A-accepts s at both levels or A-rejects s at both levels. Similarly, an agent i has CRRA if and only if, for any security s and any two wealth levels, i either R-accepts s at both levels or R-rejects s at both levels. This independence of wealth levels enables us to use the parameter of the CARA agents for ranking absolute riskiness of securities and to use the parameter of CRRA agents for ranking the relative riskiness of securities. Moreover, for each security s, there is precisely one "cutoff" value of the parameter, such that s is A-accepted by CARA agents with a smaller parameter and A-rejected by CARA agents with a larger parameter. Similarly, there is another "cutoff" value of the parameter, such that s is R-accepted by CRRA agents with a smaller parameter and R-rejected by CRRA agents with a larger parameter. These cutoff values are the reciprocal of the absolute riskiness and the relative riskiness (minus one). The following lemma formalizes this idea.

Lemma 3.2.

1. *The absolute riskiness $A(s)$ of a security s is the reciprocal of the number α such that any CARA agent with a parameter lower than α will A-accept the security and any CARA agent with a parameter (weakly) greater than α will A-reject the security.*

2. *The relative riskiness $R(s)$ of a security s is the reciprocal of the number $\gamma - 1$ such that any CRRA agent with a parameter lower than γ will R-accept the security and any CRRA agent with a parameter (weakly) greater than γ will R-reject the security.*

It follows from the lemma that for CARA agents, absolute riskiness is the only criterion for A-accepting or A-rejecting a security; and for CRRA agents, relative riskiness is the only criterion for R-accepting or R-rejecting

[15] Up to additive and positive multiplicative constants.

a security. For a more general family of utilities, there is no simple criterion based on A and R for accepting or rejecting securities. However, as the following lemma asserts, the riskier the security is, the less likely it is that agents will accept it.

Lemma 3.3.

1. *If* $\rho_i(x) < 1/A(s)$ *for all* x *between* $w+\min(s_1-s_0)$ *and* $w+\max(s_1-s_0)$, *then* i *A-accepts* s *at* w*; and*
   if $\rho_i(x) > 1/A(s)$ *for all such* x, *then* i *A-rejects* s *at* w.

2. *If* $\varrho_i(x) < 1/R(s)+1$ *for all* x *between* $w\cdot\min(s_1/s_0)$ *and* $w\cdot\max(s_1/s_0)$, *then* i *R-accepts* r *at* w*; and*
   if $\varrho_i(x) > 1/R(s) + 1$ *for all such* x, *then* i *R-rejects* s *at* w.

To sum up, the higher the absolute riskiness of a security is, the more likely it is that decision makers will A-reject it; and the higher the relative riskiness of a security is, the more likely it is that decision makers will R-reject it.

4 Properties

Some of the properties of this section were studied by Aumann and Serrano (2008) in relation to the absolute riskiness of additive gambles. Here, we relate these properties to absolute and relative riskiness of securities, together with a few other relevant properties not studied by Aumann and Serrano.

1. Monotonicity with Respect to Stochastic Dominance.

Stochastic dominance is one of the fundamental concepts in the theory of decision making under risk. As Aumann and Serrano (2008) write, "the most uncontroversial, widely accepted notions of riskiness are provided by the concepts of *stochastic dominance*." (Hadar and Russell 1969, Hanoch and Levy 1969, Rothschild and Stiglitz 1970). To explain these notions, let $\widetilde{w}_1$ and $\widetilde{w}_2$ be two discrete random variables representing distributions over wealth levels. We say that $\widetilde{w}_1$ First-Order Dominates (FOD) $\widetilde{w}_2$ if $\widetilde{w}_1 \geq \widetilde{w}_2$ for sure and $\widetilde{w}_1 > \widetilde{w}_2$ with positive probability. We say that $\widetilde{w}_1$ Second-Order Dominates (SOD) $\widetilde{w}_2$ if $\widetilde{w}_2$ may be obtained from $\widetilde{w}_1$ by "mean-preserving spreads," i.e., by replacing some of $\widetilde{w}_1$'s values with random variables whose mean is that value. We say that $\widetilde{w}_1$ stochastically dominates $\widetilde{w}_2$ if there is a random variable distributed like $\widetilde{w}_1$ that dominates $\widetilde{w}_2$ (in the above sense).

Since the definition of stochastic dominance can be applied to both absolute and relative returns, it suggests two orders of stochastic dominance on securities. More specifically, we say that security s *absolutely stochastically dominates* security r if the absolute return of s stochastically dominates the absolute return of r, i.e., if $s_1 - s_0$ stochastically dominates $r_1 - r_0$. Similarly, we say that security s *relatively stochastically dominates* security r if s_1/s_0 stochastically dominates r_1/r_0.

As the following lemma asserts, A and R are compatible with stochastic dominance.

Lemma 4.1.

1. *If s absolutely stochastically dominates r, then $A(r) > A(s)$.*
2. *If s relatively stochastically dominates r, then $R(r) > R(s)$.*

It is important to note that the two orders of stochastic dominance are not ordinally equivalent. To see that, let s be a security with $s_0 = 1$ and $s_1 = [1.2, 0.5; 0.9, 0.5]$, and let r be a security with $r_0 = 2$ and $r_1 = [2.5, 0.5; 1.8, 0.5]$. It is easy to see that r_1/r_0 stochastically dominates s_1/s_0 but $r_1 - r_0$ does not stochastically dominate $s_1 - s_0$.[16]

2. The Risk-Free Alternative.

By definition, A and R depend only on the current and the future prices of the security. A more general definition of riskiness would take into account the risk-free alternative available for investors in the economy. Let $r_f \geq 1$ be the risk-free (gross) return available for investors, such that investing w in the risk-free asset yields wr_f at the next period. To fit the definitions to the new situation, we redefine the concepts of A-acceptance and R-acceptance. Since the present value of r_1 is r_1/r_f, we say that an agent A-accepts a security s if $Eu(w + s_t/r_f - s_0) > u(w)$ and R-accepts the security if $Eu(ws_t/(s_0 r_f)) >$

[16] Though the two orders are not equivalent, they are not contradictory either. Stochastic dominance is only a partial order since the members of many pairs of securities do not stochastically dominates each other. A direct implication is that if s absolutely stochastically dominates r, then r does not relatively stochastically dominate s, and vice versa.

$u(w)$. Obviously, the concepts A-acceptance and R-acceptance studied in Section 2.1 are only special cases of the new definitions in which $r_f = 1$.

Indices of riskiness that take into account the risk-free interest rate can be defined as follows:

$$A_f(s, r_f) \equiv A(s^{r_f}), \tag{7}$$

and

$$R_f(s, r_f) \equiv R(s^{r_f}), \tag{8}$$

where s^{r_f} is the security whose values are $s_0^{r_f} = s_0$ and $s_1^{r_f} = s_1/r_f$.[17] Obviously, if we substitute the new definitions of A-acceptance and R-acceptance into the duality axiom, Theorem 1 will be true for A_f and R_f instead of A and R.

Since, for a security s, the values of s_1 are assumed to be positive, the absolute return $s_1 - s_0$ stochastically dominates $s_1/r_f - s_0$ and the relative return s_1/s_0 stochastically dominates $s_1/(r_f s_0)$. Hence, it follows from Lemma 4.1 that any security becomes riskier when the risk-free interest rate is higher. Indeed, every agent who A-accepts (R-accepts) a security at a certain risk free interest rate will A-accept (R-accept) it if the risk-free interest rate is lower. It is interesting to note that the order induced by A_f (R_f) for two different rates of r_f are not ordinally equivalent. If the risk-free interest rate increases, it makes any risky asset even riskier—but not at the same rate.

We proceed now to study the properties of A and R; that is, we assume that $r_f = 1$.

3. Investing Only a Fraction of Wealth.

Recall that $s(\alpha)$ denotes the security whose prices are: $s(\alpha)_0 = s_0$ and $s(\alpha)_1 = s_0 + \alpha(s_1 - s_0)$. Investing w in $s(\alpha)$ is equivalent to investing only $w\alpha$ in s as in both cases, the wealth at period 1 is distributed as $w + \alpha w(s_1/s_0 - 1)$. The following lemma asserts that if $\alpha < 1$, the investment of only αw in security s is less risky than investing w in s.

Lemma 4.2. *$R(s(\alpha)) < R(s)$ for $0 < \alpha < 1$.*

[17] In order for the right-hand sides of Equations (7) and (8) to be well defined, we have to assume that the absolute and relative returns of s^{r_f} satisfy the limitations that we imposed on returns in Section (2.1).

Intuitively, investing only a fraction of wealth exposes the investor to smaller risks.

4. Portfolio Diversification.

The widely accepted idea that diversified portfolios are preferable to non-diversified portfolios was emphasized in the pioneering work of Markowitz (1952). In our context, we will say that a measure of riskiness satisfies the property of "superiority of diversification" if a diversified portfolio consisting of two securities with different returns is less risky than the riskier security. An important implication of this property is that a diversified portfolio consisting of securities whose returns are i.i.d. is less risky than each one of the securities. Indeed, A and R satisfy this property.

Formally, let h and k be two securities and let $a_\alpha(h,k)$ be a security whose absolute return is a weighted average of the absolute returns of h and k, i.e., $a_\alpha(h,k)$'s absolute return is $\alpha(h_1 - h_0) + (1-\alpha)(k_1 - k_0)$, where $0 < \alpha < 1$. If the absolute returns of h and k are not equal,[18] we have

$$A(a_\alpha(h,k)) < \max(A(h), A(k)). \tag{9}$$

Similarly, let $r_\alpha(h,k)$ be the security whose absolute return is $\alpha(h_1/h_0) + (1-\alpha)(k_1/k_0)$, where $0 < \alpha < 1$. If their relative returns are not equal, we have

$$R(r_\alpha(h,k)) < \max(R(h), R(k)). \tag{10}$$

5. Normal and Log-Normal Distributions.

By our earlier definition, the value of a security at period 1 has only finitely many values and so its distribution cannot be normal or log-normal. We therefore redefine the price of a security at period 1 as any random variable for which A and R are well defined.[19]

Aumann and Serrano (2008) show that if an additive gamble g is normally distributed with variation σ and expectation μ, then $A^*(g) = \sigma^2/(2\mu)$.

[18] "Not equal" means that at least on one event they take different values; identical distributions are not necessarily equal.

[19] The indices A and R are not well defined for every continuous random variable; see Schulze (2010) who studies for which probability distributions the index A^* is well defined.

Hence, if the absolute return of security s is normally distributed with variation σ and expectation μ, then $A(s) = \sigma^2/(2\mu)$. So it follows from Equation (3) that if s_1/s_0 has a log-normal distribution with parameters μ and σ, then $R(s) = \sigma^2/2\mu$, where σ^2 is the variance of $\log s_1/s_0$ and μ is the expectation of $\log s_1/s_0$.

6. Translation Invariance

Let x be a positive number. We denote by $s+x$ the security whose prices are $(s+x)_0 = s_0 + x$ and $(s+x)_1 = s_1 + x$. Similarly, we denote by xs the security whose prices are $(xs)_0 = xs_0$ and $(xs)_1 = xs_1$. It is easy to verify the following observations:

1. $A(s+\delta) = A(s)$ for every $\delta > 0$.
2. $R(\lambda s) = R(s)$ for every $\lambda > 0$.

Adding δ to the price of the security does not affect its absolute local riskiness since the buying price and the selling price offset each other. Multiplying a security by λ does not affect its relative return and, therefore, does not affect its relative riskiness.

7. Continuity.

We say that an index of absolute returns Q is continuous if $Q(s^n) \to Q(s)$ whenever the absolute returns of s^n are uniformly bounded and converge to the absolute return of s in probability. Similarly, we say that an index of relative returns Q is continuous if $Q(s^n) \to Q(s)$ whenever the relative returns of s^n are uniformly bounded and converge to the relative return of s in probability. By these definitions, the riskiness indices A and R are continuous. In words, when the returns of two securities are likely to be close, their riskiness levels are close. The proof for the continuity of A appears in Aumann and Serrano (2008). Since the log function is continuous, it follows from (3) that R is also continuous.

8. Diluted Securities.

Let s be a security and let p be a number strictly between zero and one. Define the "diluted" security s^p as the security whose price at period 1, s_1^p, is s_1 with probability p and s_0 with probability $1-p$, and whose price at period zero, s_0^p equals s_0. Then, $A(s^p) = A(s)$ and $R(s^p) = R(s)$. Though at first this may sound counterintuitive, any expected utility maximizer A-accepts s^p

if and only if she A-accepts s, and any expected utility maximizer R-accepts s^p if and only if she R-accepts s.

The proof of the claim in relation to absolute riskiness appears in Aumann and Serrano (2008), and the claim in relation to relative riskiness follows from the first claim and Equation (3).

9. Repeated Investments.

Investing repeatedly in a security with i.i.d. returns is just as risky as investing in the security for only one period. Formally, let s and s' be two securities with i.i.d. absolute returns. Then

$$A(s + s') = A(s) = A(s'),$$

where $s + s'$ is the security whose prices at periods zero and one are the sum of the prices of s and s' at these periods. Similarly, if the relative returns of s and s' are i.i.d., then

$$R(ss') = R(s) = R(s'),$$

where ss' is the security whose prices at periods zero and one are the multiplication of the prices of s and s' at these periods. It follows that the riskiness of the securities do not depend on the investment time horizon, when securities returns are assumed to be i.i.d. over time. This contrasts with expectation and variance, which are correlated with the investment time horizon. The proof of the statement on absolute returns (and some additional results on dependent investments, which can be easily translated to relative terms) appears in Aumann and Serrano (2008). The statement on relative returns follows from the first statement and Equation (3).

5 Alternative Characterizations

In Section 2, adopting the approach of Aumann and Serrano (2008), we characterized the indices of absolute and relative riskiness by duality and scaling. As we saw there, although A and R are based on two axioms, it is only the duality axiom that determines the orders of riskiness. In this section we show that, in addition to duality, there are two alternative characterizations that induce the same orders of riskiness that are induced by A and R.

5.1 Probability for Bankruptcy

Meilijson (2009) suggests an interpretation of the index A^* of riskiness connecting the concept of riskiness to the probability of great losses as a result of investing repeatedly in assets with i.i.d. absolute returns. In this section we extend his idea to relative returns.

Given a security s, we define the infinite random process s^a as follows. For any $t = 1, 2,$ ..., the t random absolute return of the process, defined as $s^a_t - s^a_{t-1}$ (when s^a_{t-1} is given), is identically independently distributed (i.i.d.) as $s_1 - s_0$. It turns out that for two securities s and h, if $A(s) > A(h)$, then there is a real value B such that the probability that s^a will go below x at some t is greater than the probability that h^a will go below x at some t, for all $x < B$. Formally,

Theorem 2. *$A(s) > A(h)$ if and only if there exists a real number B such that for all $x < B$,*

$$P(\exists t \text{ s.t. } h^a_t < x) < P(\exists t \text{ s.t. } s^a_t < x). \tag{11}$$

Similarly, given a security s, we define the infinite random process s^r as follows. For any $t = 1, 2,$ ..., the t random relative return of the process, defined as s^r_t / s^r_{t-1} (when s^r_{t-1} is given), is identically independently distributed (i.i.d.) as s_1/s_0. Here again, for any two securities s and h, if $R(s) > R(h)$, then there is a real value B such that the probability that s^r will go below x at some t is greater than the probability that h^r will be below x at some t, for all $x < B$. Formally,

Theorem 3. *$R(s) > R(h)$ if and only if there exists a real number B such that for all $x < B$,*

$$P(\exists t \text{ s.t. } h^r_t < x) < P(\exists t \text{ s.t. } s^r_t < x). \tag{12}$$

Theorem 2 is proved by Meilijson (2009). Theorem 3 follows from Theorem 2 and Equation (3).

Obviously, these theorems suggest an alternative characterization of the indices A and R: s is absolutely riskier than h if and only if Equation (11) holds, and s is relatively riskier than h if and only if Equation (12) holds. Hence, A and R are the only indices that satisfy Equations (11) and (12) and the scaling axiom.

5.2 Wealth Uniform Dominance

Hart (2011) defines an order of riskiness on additive gambles, called *wealth uniform dominance*, and shows its equivalency to the order induced by the Aumann–Serrano index. We take Hart's approach and define two variants of his wealth uniform dominance order, except that we refer to securities rather than gambles, and show their equivalency to the orders induced by A and R. In his analysis, Hart (2011) relates to a specific set of utilities, U^*, which he calls "regular utilities." The properties of utilities in U^* that are relevant to our discussion are that (1) their absolute risk aversion decreases (weakly) with wealth (DARA) and (2) their relative risk aversion increases (weakly) with wealth (IRRA).

Hart's wealth uniform dominance order is defined as follows.

A gamble g wealth uniformly dominates a gamble h whenever:

$$\text{if } g \text{ is rejected by u at all } w > 0$$
$$\text{then } h \text{ is rejected by u at all } w > 0,$$

for every utility $u \in U^*$.

Similarly, we define two orders of wealth-uniform dominance for securities.

Definition 2.

1. *A security s absolutely wealth uniformly dominates a security r, denoted $s \geq_{AW} r$, whenever:*

   $$\textit{if } s \textit{ is A-rejected by } u \textit{ at all } w > 0$$
   $$\textit{then } r \textit{ is A-rejected by } u \textit{ at all } w > 0,$$

   for every utility $u \in U^*$.

2. *A security s relatively wealth uniformly dominates a security r, denoted $s \geq_{RW} r$, whenever:*

   $$\textit{if } s \textit{ is R-rejected by } u \textit{ at all } w > 0$$
   $$\textit{then } r \textit{ is R-rejected by } u \textit{ at all } w > 0,$$

for every utility $u \in U^*$.

We have the following theorem.

Theorem 4.

1. *For any two securities s and r, $s \geq_{AW} r$ if and only if $A(s) \leq A(r)$.*

2. *For any two securities s and r, $s \geq_{RW} r$ if and only if $R(s) \leq R(r)$.*

Obviously, the orders $\geq_{AW}$ and $\geq_{RW}$ can substitute the duality axiom, and, together with the scaling axiom, they characterize A and R in a unique way.

This approach may seem more intuitive than our first approach: s is riskier than r if any decision maker who rejects s also rejects r. Rejecting here is interpreted in the sense of wealth uniformly. However, riskiness according to this approach is relevant to a limited set of utilities U^*.

6 Conclusions

This paper advances the idea that risks arising from investments have two aspects, namely, absolute and relative. The paper characterizes indices of absolute and relative riskiness by the Aumann-Serrano principle of duality between risk and risk aversion. The index of absolute riskiness is simply the index A^* applied to absolute returns, and the index of relative riskiness, which is a new index, is a function of relative returns. Both indices reflect the idea that "risk is what risk averters hate" (Machina and Rothschild 2008).

Since the indices differ in their properties, they are each better suited to different situations. Nevertheless, the index of relative riskiness has two important advantages that make it, in our opinion, more relevant in many cases. First, it is more likely that agents care about the relative return rather than the absolute return of an investment. In stock markets, for example, it is quite common to believe that only relative returns should affect investment decisions. The price of a single security is almost irrelevant. Second, it is reasonable to assume that investors have utilities that are close to CRRA utilities. For such investors, as follows from Lemmas 3.2 and 3.3, the index of relative riskiness is much more relevant.

Finally, it is noteworthy that although the two indices are not ordinally equivalent, there are many pairs of securities on which the indices do agree.

It would be interesting to characterize this set of securities for which absolute and relative riskiness agree. It is to be hoped that further study will clarify this connection.

Appendix

A Proofs

Since A is basically the index A^* applied to absolute returns, many of the statements in relation to A and their proofs appear already in Aumann and Serrano (2008). Here we mostly focus on the proofs of statements in relation to R.

In this section, investors i and j have utility functions u_i and u_j and Arrow–Pratt coefficients ϱ_i and ϱ_j of relative risk aversion. Since utilities may be modified by additive and positive multiplicative constants, we assume throughout that

$$u_i(1) = u_j(1) = 0 \text{ and } u_i'(1) = u_j'(1) = 1. \tag{13}$$

Lemma A.1. *For some $\delta > 1$, suppose that $\varrho_i(w) > \varrho_j(w)$ at each w with $1/\delta < w < \delta$. Then $u_i(w) < u_j(w)$ whenever $1/\delta < w < \delta$ and $w \neq 1$.*

Proof. Let y be a number, $1/\delta < y < \delta$. If $y > 1$, then, by equation (13),

$$\log u_i'(y) = \log u_i'(y) - \log u_i'(1) = \int_1^y [\log u_i'(z)]'dz = \int_1^y \frac{u_i''(z)}{u'(z)}dz$$
$$= \int_1^y -(\varrho_i(z)/z)dz < \int_1^y -(\varrho_j(z)/z)dz = \log u_j'(y).$$

If, on the other hand, $y < 1$, the reasoning is similar but the inequality is reversed, because then $\int_1^y = -\int_y^1$. Thus when $y > 1$, $\log u_i'(y) < \log u_j'(y)$ and also $u_i'(y) < u_j'(y)$, and when $y < 1$ $\log u_i'(y) > \log u_j'(y)$ and also $u_i'(y) > u_j'(y)$. So if $w > 1$, then, by (13),

$$u_i(w) = \int_1^w u_i'(y)dy < \int_1^w u_j'(y)dy = u_j(w);$$

and if $w < 1$, then

$$u_i(w) = -\int_w^1 u_i'(y)dy < -\int_w^1 u_j'(y)dy = u_j(w).$$

□

Corollary 5. *If $\varrho_i(w) \leq \varrho_j(w)$ for all $w > 0$, then $u_i(w) \geq u_j(w)$ for all $w > 0$.*

Lemma A.2. *For any security s, its relative riskiness $R(s)$ is well defined.*

Proof. For a given security s, we denote by $\hat{s}$ the relative return of s, $\hat{s} = s_1/s_0$. We define the function f_s as follows:

$$f_s(\beta) \equiv \mathrm{E}\,\hat{s}^\beta = \Sigma p_i \hat{s}_i^\beta, \tag{14}$$

where β is a real number. The first and second derivatives of f_s are

$$f_s'(\beta) = \Sigma p_i \hat{s}_i^\beta \log \hat{s}_i, \tag{15}$$
$$f_s''(\beta) = \Sigma p_i \hat{s}_i^\beta (\log \hat{s}_i)^2. \tag{16}$$

Since by definition at least one of the values of $\hat{s}$ is greater than one and at least one of the values is less than one,

$$\lim_{\beta \to \pm\infty} f_s(\beta) = \infty. \tag{17}$$

In addition, since f_s'' is positive for all β, f_s' increases with β, which implies that f_s has a single minimum point. It follows from (14) that $f_s(0) = 1$. If $f_s'(0) \neq 0$, there should be another value of β, for which $f_s(\beta) = 1$. Based on this insight, we define β^* as follows:

1. If $f_s'(0) > 0$, then there is only one additional value of β, $\beta = \beta^*$, in which $f_s(\beta^*) = 1$ and $\beta^* < 0$.
2. If $f_s'(0) < 0$, then there is only one additional value of β, $\beta = \beta^*$, in which $f_s(\beta^*) = 1$ and $\beta^* > 0$.
3. If $f_s'(0) = 0$, then there is no other value of β, $\beta \neq 0$, in which $f_s(\beta) = 1$. In this case we set $\beta^* = 0$.

Since we assumed that the weighted geometric mean of the relative return of securities is greater than one, $f_s'(0) = \Sigma p_i \log \hat{s}_i > 0$, and the first case, in which $\beta^* < 0$, is satisfied. Defining $R(s) = -1/\beta^*$ shows the existence of $R(s)$ and also that $R(s) > 0$. This completes the proof. □

Lemma A.3. *For any two securities s and r,*

$$R(r) > R(s) \Leftrightarrow f_s(-1/R(r)) < 1.$$

Proof. We use the definition of f_s of the previous proof. Since $f_s'(0) > 0$, $\beta^* = -1/R(s) < 0$, and the minimum point of f_s is between $-1/R(s)$ and 0 (scenario 1 in the proof of (A.2)). This, together with the continuity of f_s, implies that for any β, $-1/R(s) < \beta < 0$, $f_s(\beta) < 1$. Since $-1/R(s) < -1/R(r) < 0$, $f_s(-1/R(r)) < 1$. □

Lemma A.4. *For any utility function u_α and value of $\delta > 1$ there is a security $s = s(\alpha, \delta)$, such that $u_\alpha(\hat{s}) = 0$ and $\forall i$, $1/\delta < \hat{s}_i < \delta$, where $\hat{s} = s_1/s_0$ and $\hat{s}_i s$ are the values that $\hat{s}$ takes.*

Proof. Let $f(\epsilon)$ be defined as $f(\epsilon) = \epsilon u_\alpha(\sqrt{1/\delta}) + (1-\epsilon)u_\alpha(\sqrt{\delta})$. It is easy to see that if $\epsilon = 0$, then $f(\epsilon) > 0$, and if $\epsilon = 1$, then $f(\epsilon) < 0$. Since f is continuous in ϵ, $f(\epsilon^*) = 0$ for some ϵ^* between zero and one. The desired security is the one whose relative return takes the value $\sqrt{1/\delta}$ with probability ϵ^* and the value $\sqrt{\delta}$ with probability $1 - \epsilon^*$. □

Lemma A.5. *If $\varrho_i(w_i) > \varrho_j(w_j)$, then there is a security s that j R-accepts at w_j and i R-rejects at w_i.*

Proof. Without loss of generality, $w_i = w_j = 1$, and so $\varrho_i(1) > \varrho_j(1)$.[20] Let ϱ be a number between $\varrho_i(w)$ and $\varrho_j(w)$, $\varrho_i(w) > \varrho > \varrho_j(w)$. Since u_i and u_j are twice continuously differentiable, it follows that there is a number $h > 1$ such that $\varrho_i(w) > \varrho > \varrho_j(w)$ at each w with $1/h < w < h$. By Lemma A.4, there is a security $s(\varrho, h)$ such that u_ϱ is indifferent between R-accepting or R-rejecting it. Therefore, by Lemma A.1,

$$u_i(w) < u_\varrho(w) < u_j(w) \text{ whenever } 1/\delta < w < \delta \text{ and } w \neq 1 \qquad (18)$$

implies that $u_i(\hat{s}(\varrho, h)) < 0 < u_j(\hat{s}(\varrho, h))$, where $\hat{s} = s_1/s_0$. Hence i R-rejects the security but j R-accepts it. □

[20] For arbitrary w_i and w_j, define $u_i^*(x) = [u_i(xw_i) - u_i(w_i)]/(w_i u_i'(w_i))$ and u_j^* similarly, and apply the current reasoning to u_i^* and u_j^*. u_i^* and u_j^* R-accept or R-reject securities at $x = 1$, just as u_i and u_j R-accept or R-reject securities at w_i and w_j, respectively. In addition, $u_i^*(1) = u_j^*(1) = 0$ and $u_i^{*\prime}(1) = u_j^{*\prime}(1) = 1$.

Proof of Lemma 3.1. The proof of the first part of the lemma appears in Aumann and Serrano (2008). Here we prove the second part. We have to show that $\varrho_i(w) \geq \varrho_j(w)$ for all wealth levels w if and only if i is no less uniformly relative-risk averse than j.
"If": Assume that there are w_i and w_j with $\varrho_i(w_i) < \varrho_j(w_j)$. By Lemma $A.5$, there is a security that i R-accepts at w_i and j R-rejects at w_j, thereby contradicting i being less uniformly relative-risk averse than j.

"Only if": Assuming that $\varrho_i(w_i) \geq \varrho_j(w_j)$ for all wealth levels w_i and w_j, we must show that for both wealth levels, w_i and w_j, and security s, if i R-accepts s at w_i, then j R-accepts s at w_j. Without loss of generality, $w_i = w_j = 1$, and so we must show that

if i R-accepts s at 1, then j R-accepts s at 1.

From Corollary 5 (with i and j reversed), we conclude that $u_j(w_j) \geq u_i(w_i)$ for each w_i and w_j, and so $\mathrm{E}\, u_j(\hat{s}) \geq \mathrm{E}\, u_i(\hat{s})$, where $\hat{s} = s_1/s_0$. That yields the above claim. □

Lemma A.6. *An agent i has a CRRA utility if and only if for any security s and any two wealth levels, i either R-accepts s at both levels or R-rejects s at both levels.*

Proof. We denote by $\hat{s}$ the relative return of s, i.e., $\hat{s} = s_1/s_0$. Recall that all CRRA utility functions have the form

$$u_\alpha(x) = \begin{cases} \frac{(x^{1-\alpha}-1)}{1-\alpha} & \text{if } \alpha \neq 1 \\ \log(x) & \text{if } \alpha = 1 \end{cases} \tag{19}$$

for $\alpha > 0$.

"Only if": Let $u_\alpha(x)$ be a CRRA utility with parameter α. u_α R-accepts s at w if and only if $\mathrm{E}\, u_\alpha(w\hat{s}) > u_\alpha(w)$, that is, if and only if $\mathrm{E}\, u_\alpha(\hat{s}) > u_\alpha(1)$.

"If": It follows from Lemma A.5; just take $j = i$. □

Proof of Theorem 1. The first part of the theorem appears in Aumann and Serrano (2008). Here we prove the second part.

For $\alpha > 0$, let $u_\alpha(x)$ be the CRRA utility function with parameter α. The functions u_α satisfy (13), and so by Lemma $A.1$ (with δ arbitrarily large) their graphs are nested; that is,

$$\text{if } \alpha > \beta, \text{ then } u_\alpha(x) < u_\beta(x) \text{ for all } x > 0, x \neq 1. \tag{20}$$

The existence of $R(s)$ is proved in Lemma $A.2$.

To see that R satisfies the duality axiom, let i, j, r, h, and w be as in the hypothesis of that axiom; without loss of generality, $w = 1$. Set $\gamma \equiv 1 + 1/R(s)$, $\eta \equiv 1 + 1/R(h)$, $\alpha_i = \inf \varrho_i$ and $\alpha_j = \sup \varrho_j$. For a given security s, we denote by $\hat{s} = s_1/s_0$ the relative return of s. Thus

$$\mathrm{E}\, u_\gamma(\hat{s}) = 0 \text{ and } \mathrm{E}\, u_\eta(\hat{h}) = 0. \tag{21}$$

By hypothesis, $R(s) > R(h)$, so $\eta > \gamma$. By Corollary 5,

$$u_i(x) \leq u_{\alpha_i}(x) \text{ and } u_{\alpha_j}(x) \leq u_j(x) \text{ for all x.} \tag{22}$$

Now assume $\mathrm{E}\, u_i(\hat{s}) > 0$; we must prove that $\mathrm{E}\, u_j(\hat{h}) > 0$. From $\mathrm{E}\, u_i(\hat{s}) > 0$ and (22), it follows that $\mathrm{E}\, u_{\alpha_i}(\hat{s}) > 0$. So by (21), $E_\gamma(\hat{s}) = 0 < \mathrm{E}\, u_{\alpha_i}(\hat{s})$. So by (20), $\gamma > \alpha_i$. By Lemma 3.1 $\alpha_i \geq \alpha_j$ so $\eta > \gamma$ yields $\alpha_j < \eta$. Since (21), (20) and (22) yield $0 < \mathrm{E}\, u_\eta(\hat{h}) < \mathrm{E}\, u_{\alpha_j}(\hat{h}) < \mathrm{E}\, u_j(\hat{h})$, it follows that R satisfies the duality axiom.

That R satisfies the scaling axiom is immediate, and so, indeed, R satisfies the two relative axioms.

In the opposite direction, let Q be an index that satisfies the relative axioms. We first show that

$$\text{Q is ordinally equivalent to R.} \tag{23}$$

If this is not true, then there must exist s and r that are ordered differently by Q and R. This means either that the respective orderings are reversed, that is,

$$Q(s) > Q(r) \text{ and } R(s) < R(r), \tag{24}$$

or that the equality holds for exactly one of the two indices, that is,

$$Q(s) > Q(r) \text{ and } R(s) = R(r) \tag{25}$$

or

$$Q(s) = Q(r) \text{ and } R(s) > R(r). \tag{26}$$

If either (25) or (26) holds, then by the scaling axiom, replacing s by s^δ for sufficiently small $\delta > 1$ leads to reversed inequalities. So without loss of generality we may assume (24).

Now let $\gamma \equiv 1 + 1/R(s)$ and $\eta \equiv 1 + 1/R(r)$; then (21) holds. By (24), $\gamma > \eta$. Choose μ and ν so that $\gamma > \mu > \nu > \eta$. Then $u_\gamma(x) < u_\mu(x) < u_\nu(x) < u_\eta(x)$ for all $x \neq 0$. So by (21) $\mathrm{E}\, u_\mu(\hat{s}) > \mathrm{E}\, u_\gamma(\hat{s}) = 0$ and $\mathrm{E}\, u_\nu(\hat{r}) < \mathrm{E}\, u_\eta(\hat{r}) = 0$. So if i and j have utility functions u_μ and u_ν, respectively, then i R-accepts s and j R-rejects r. But from $\mu > \nu$ and Lemma (3.1), it follows that $i \succ j$, contradicting the duality axiom for Q. So (23) is proved.

To see that Q is a positive multiple of R, let s^* be an arbitrary but fixed security and set $\lambda \equiv Q(s^*)/R(s^*)$. If s is any security and $t \equiv Q(s)/Q(s^*)$, then $Q((s^*)^t) = tQ(s^*) = Q(s)$, and so $tR(s^*) = R((s^*)^t) = R(s)$ by the ordinal equivalence between Q and S, and $R(s)/R(s^*) = t = Q(s)/Q(s^*)$, and $Q(s)/R(s) = Q(s^*)/R(s^*) = \lambda$, and $Q(s) = \lambda R(s)$. This completes the proof of Theorem A. □

Needless to say, both duality and scaling are essential to Theorem 1. Thus the mean log $E \log s$ satisfies scaling but violates duality, while the index $[R(s)]$, where $[x]$ denotes the integer part of x, satisfies duality but violates scaling. Neither $E \log s$ nor $[R(s)]$ is even ordinally equivalent to R.

Proof of (6) ***in Section 2.3.*** It is enough to show that

$$\lim_{\alpha \to 0} R(s(\alpha))/\alpha = A(s)/s_0. \tag{27}$$

Indeed, following Equation (3), $R(s) = A^*(\log(s_1/s_0))$; hence

$$R(s(\alpha))/\alpha = A^*(\log(1 + \alpha(s_1/s_0 - 1)))/\alpha,$$

which equals $A^*(\log(1+\alpha(s_1/s_0-1))/\alpha)$ (by scaling). Since A^* is continuous, the limit of this expression as α goes to zero equals $A^*(s_1/s_0 - 1) = A(s)/s_0$. □

Proof of Lemma 3.2. For the proof of the first part of the lemma, see Aumann and Serrano (2008). Here we prove only the second part.

An agent with a CRRA utility with parameter γ R-accepts security s if and only if

$$f_s(1 - \gamma) > 1,$$

where $f_s(\beta)$ is the function defined in (14). Since for all $\beta < \beta^*$, $f_s(\beta) > 1$ and for all $\beta^* < \beta < 0$, $f_s(\beta) < 1$ (by the proof of Lemma A.2), every CRRA agent with a parameter greater than $1 - \beta^*$ R-rejects s and every CRRA agent with a parameter lower than $1 - \beta^*$ R-accepts s. □

Proof of Lemma 3.3. For the proof of the first part of the lemma, see Aumann and Serrano (2008). Here we prove only the second part. Let u_i be i's utility and assume that $\varrho_i(x) < 1/R(s) + 1$ for all x between $w \min \hat{s}$ and $w \max \hat{s}$, where $\hat{s} = s_1/s_0$. Define a utility u_j as follows: when x is between $w \min \hat{s}$ and $w \max \hat{s}$, define $u_j(x) \equiv u_i(x)$; when $x \leq w \min \hat{s}$, define $u_j(x)$ to equal a CRRA utility with parameter $\varrho_i(w \min \hat{s})$ and $u_j(w \min \hat{s}) = u_i(w \min \hat{s})$ and $u_j'(w \min \hat{s}) = u_i'(w \min \hat{s})$; when $x \geq w \max \hat{s}$, define $u_j(x)$ to equal a CRRA utility with parameter $\varrho_i(w \max \hat{s})$ and $u_j(w \max \hat{s}) = u_i(w \max \hat{s})$ and $u_j'(w \max \hat{s}) = u_i'(w \max \hat{s})$. Let u_k be a CRRA utility with parameter $[1/R(s) + 1] - \epsilon$. Then

$$\min_x \varrho_k(x) > \max_x \varrho_j(x)$$

for positive ϵ sufficiently small. By Lemma 3.2, a CRRA person with parameter $[1/R(s) + 1]$ is indifferent between R-accepting and R-rejecting s. Therefore, k, who is less risk averse, R-accepts s, and so j also R-accepts s. But between the minimum and maximum of $w\hat{s}$, the utilities of i and j are the same. So i R-accepts s at w. □

Proof of Lemma 4.1. The first part of the lemma is proved in Aumann and Serrano (2008). Here we prove only the second part.

For $\gamma \geq 0$, set $f(\gamma) = E\hat{s}^{1-\gamma}/(1-\gamma)$ and $f_*(\gamma) = E\hat{s}_*^{1-\gamma}/(1-\gamma)$. Let s and s_* be two securities whose relative returns are $\hat{s}$ and $\hat{s}_*$, respectively. If s first-order relatively dominates s_*, then $f(\gamma) < f_*(\gamma)$ whenever $\gamma > 1$. It follows that the unique positive root of $f_* = 1$ is less than that of $f = 1$, an so $R(s_*) > R(s)$.

If s second-order relatively dominates s_*, then, $f(\gamma) < f_*(\gamma)$, too, because of the strict convexity of $x^{1-\gamma}/(1-\gamma)$ as a function of x for all $x > 0$. The remainder of the proof is as before. □

Proof of Lemma 4.2. Let $\hat{s} = [x_1, p_1; x_2, p_2; ...; x_n, p_n]$ be the relative return of a security s. For convenience, we denote $\epsilon = \hat{s} - 1$, where $\epsilon_i = x_i - 1$. We have to show that for any $0 < \alpha < 1$, $R(s) > R(s(\alpha))$. That is the result of the following lemma (whose claim is a bit stronger).

Lemma A.7. *Every agent who R-accepts s would R-accept $s(\alpha)$ for all* $0 < \alpha < 1$.

Proof. By definition, for every concave function u and two different numbers x and y,

$$u(\alpha x + (1-\alpha)y) > \alpha u(x) + (1-\alpha)u(y).$$

Submitting $x = w + w\epsilon$ and $y = w$, we get

$$u(w + \alpha w\epsilon) > \alpha u(w + w\epsilon) + (1 - \alpha)u(w),$$

and so,

$$Eu(w + \alpha w\epsilon) > \alpha Eu(w + w\epsilon) + (1 - \alpha)u(w).$$

If an agent with utility u and wealth w R-accepts s, then $Eu(w+w\epsilon) > u(w)$ implies that $Eu(w + \alpha w\epsilon) > u(w)$, which means that the agent R-accepts $s(\alpha)$. In terms of Hart (2011), $s(\alpha)$ *acceptance dominates* s. □

Now, if anyone who R-accepts s also R-accepts k (but not vice versa) then R(s)>R(k); otherwise there would have been a CRRA agent who would R-accept s but R-reject k. □

Proof of the Portfolio Diversification Property. The proof of Equation (9) follows from the subadditivity of A (see Equation 5.8.2 in Aumann and Serrano 2008), which implies that

$$A(a_\alpha(h, k)) \leq \alpha A(h) + (1 - \alpha)A(k),$$

and equality obtains if and only if the absolute return of h is a positive multiple of the absolute return of k. If $A(h) \neq A(k)$ then $\alpha A(h) + (1 - \alpha)A(k) < \max(A(h), A(k))$, and it follows that unless the returns of k and h are equal, $A(a_\alpha(h, k)) < \max(A(h), A(k))$.

To prove Equation (10), let $\hat{h} = h_1/h_0$ and $\hat{k} = k_1/k_0$ be the relative returns of h and k, respectively, and assume that $\hat{h}$ and $\hat{k}$ are not equal. Without loss of generality assume that $R(h) \geq R(k)$. From the convexity of the function $f(x) = x^{(-1/R(h))}$ we get

$$E(\alpha\hat{h} + (1 - \alpha)\hat{k})^{-1/R(h)} < \alpha E\hat{h}^{-1/R(h)} + (1 - \alpha)E\hat{k}^{-1/R(h)} \leq 1,$$

where the last inequality follows from Lemma A.3. Hence, it follows from the same lemma and from the convexity of $f(x)$ that unless the returns of h and k are equal, $R(r_\alpha(h, k)) < R(h)$.

Note that while it is shown that A is convex, it follows from the proof that R is quasiconvex. Many of our tests further indicate that R is also convex, but at this stage we do not have formal proof of this. Therefore, the convexity of R remains a conjecture. □

Proof of Theorem 4. Here we prove only the second part of the theorem; for the proof of the first part see Hart (2011).

Let r and k be two securities such that $R(k) > R(r)$. If agent i R-rejects r at any wealth level, then it follows from Lemma 3.3 that for any $w_0 > 0$ there is $w'_0 \in (w_0 \min r, w_0 \max r)$ for which $\varrho_i(w'_0) \geq 1/R(r) + 1$. Since IRRA is assumed, it follows that for all $w > 0$, $\varrho_i(w) > 1/R(r) + 1$. That implies that for all $w > 0$ $\varrho_i(w) > 1/R(k) + 1$. So it follows from 3.3 that k is R-rejected at all w.

The opposite direction is proved as follows. Assume that r wealth uniformly dominates k but that $R(r) > R(k)$. Let $x = (1/(-1 + R(r)) + 1/(-1 + R(k)))/2$. According to Lemma 3.2, a CRRA agent with parameter x R-rejects r but R-accepts k at any $w > 0$, a contradiction. □

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