Speed limits

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1 Abstract

We consider a model in which a regulator determines the level of an activity—which is harmful to society—beyond which an agent will pay a fixed fine. The agent then chooses his level of activity accordingly. The agent has private information about the utility he gains from the activity, such that it is only publicly known that his utility is increasing in the level of the activity. We show that the regulator should adopt a random punishment scheme under certain condition in order to minimize harm. The results may help to solve the puzzle of why vague enforcement policies are implemented by competent governments even when tough enforcement is possible and costless.

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2 Introduction

It has been acknowledged that punishment is sometimes ineffective. An individual may derive a large benefit from some violation of the law and therefore will be willing to risk

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the punishment (Shavell, 1987). A similar idea appears in the Talmud, in which there is a principle that a decree should not be too difficult for the public to comply with. For example, the maximal driving speed to ensure safe driving is in all probability lower than the legal speed limit. However, if the speed limit were set too low, drivers might ignore it. Moreover, a low speed limit is often set, but informally the police allow drivers to exceed it somewhat without giving them a ticket. In another example, it is commonly argued that setting strict limits for toddlers is the best way to educate them. Nevertheless, it is often observed that parents, who can be assumed to want the best for their children, choose to be more flexible when disciplining the child. Obviously, disciplining a child is different from enforcing the law; nonetheless, there are some similarities. In this paper, we try to understand whether a vague or flexible enforcement policy, such as in the examples described, may indeed be optimal and if so why.

We consider a model in which an agent gains utility from engaging in a particular activity, but one that causes harm to the rest of society. The regulator chooses a level of the activity above which the agent will pay a fixed fine.[1] Now suppose that the regulator wishes to minimize harm. If he knows the agent's utility, then he imposes the fine at the lowest possible activity level at which the agent will prefer being at that level and not paying the fine to being at the maximal level possible and paying the fine. We assume that the agent's utility from engaging in the activity is private information. In particular, although it is commonly known that his utility is increasing in the activity level, it is not known whether he gains high or low utility from a given activity level. In these circumstances, if the regulator chooses an activity level that is too low, then the agent may choose the maximal activity level possible and will be willing to pay the fine. In this case, the harm caused to society is maximized. Alternatively, the regulator can choose a level sufficiently high such that with

[1] In the context of accidents, this activity level is sometimes referred to as the "level of negligence", or in other words the minimal effort level required by law to prevent an accident. If the agent's effort is below that level, he is responsible for damage caused in an accident and therefore required to pay a fine (see Shavell, 2009). Note that unlike in our model, here the agent is required to pay the fine only if an accident actually occurs.

certainty the agent prefers not to go beyond that level and therefore not to pay the fine to going beyond that level and paying the fine. In this case, the harm caused may still be substantial.

We show that if the probability that the agent is of a "low type" (i.e., gains a small amount of utility from a given activity level) and the maximal activity level are both sufficiently high, then—in order to minimize harm—the regulator should determine his policy in the following manner: Two activity levels are set. If the agent exceeds the first, then with a positive probability he incurs a fine, while if he exceeds the second, then he incurs the fine with certainty. In response, each type of agent stops exactly at one of the two levels: the low type stops at the low level and the high type stops at the high level, which is still lower than the highest possible level. This result suggests that it may be beneficial to implement a vague enforcement policy, even when it is costless to catch and punish an offender. For instance, consider the example of enforcing the speed limit. Although available technology enables a car to travel at extreme speeds, most drivers do not benefit much from driving at those speeds. The policy of enforcing the speed limit is often vague, even though it is possible to calibrate speed cameras to photograph cars that exceed any given speed level.

Note that we consider a regulator who chooses the probability that a fine, whose collection is assumed to be certain, is imposed when the agent exceeds a given activity level. Technically, this is the same as a regulator who chooses the probability of catching the agent at a given activity level, where the fine is always imposed if the agent is caught. In other words, a speed camera might photograph all drivers who exceed a given speed, but only some of them will receive a ticket. Alternatively, the camera might photograph only some of the drivers who are speeding and every one of them will receive a ticket. Therefore, the regulator's choice in our model can also be viewed as choosing the cumulative distribution of catching an offender who is always required to pay the fine if caught. Usually, and unlike in our model, a strategic choice of this type is assumed to be made by a regulator who has limited resources. This is further discussed in the review of the literature section.

Loosely speaking, there are two elements of deterrence in law enforcement: the probability of being caught (or the probability of being punished as in our model) and the size of the fine. By assuming that the fine is fixed, we can isolate the effect of the probability of enforcement. Furthermore, a fixed fine situation can capture scenarios in which the regulator is able to make only ad hoc decisions, such as deciding when exactly to collect the fine, as opposed to changing the size of the fine, which might require a cumbersome process of changing the legislation and is therefore less feasible. At a later stage, we will discuss other possible reasons why a fixed fine that is not always collected may be preferable to a graduated fine system.

The paper proceeds as follows. In the remainder of this section we review the literature. Section 3 describes the model. The results and an example that illustrates them presented in section 4. In section 5, we extend the model to consider social welfare and section 6 concludes.

Literature review

This paper is related to the economic literature on law enforcement that began with Gary Becker's (1968) seminal work and others that followed in his wake, including Stigler (1971), Polinsky and Shavell (1979) and others. In most of the literature, the problem examined is to achieve the "optimal" enforcement policy that maximizes welfare or minimizes crime level or both, where punishments (such as jail time) and enforcement efforts that raise the probability of catching an offender are both costly. In other words, if there were enough resources, it would be possible to construct a perfect enforcement system that eliminates or almost eliminates crime. In our model, the probability of being caught is set to be 1 and the size of the fine is fixed. However, since people's utility increases in the level of the activity, they may choose to break the law and pay the fine. This issue has been addressed by Shavell (1987) who considers a model in which sanctions are socially costly and endogenously determined, and each individual has a binary choice: either to commit a crime or not. Those individuals who gain high utility from committing the crime will choose to do so, regardless of whether or not they are punished, since the punishment is assumed to be bounded from above. He concludes that, unlike in the case of perfect information, under imperfect information on individuals' preferences, punishment will be imposed in the optimal enforcement system. In our model, the agent's choice variable is his activity level, which is continuous. Therefore our model can be viewed as complementary to Shavell (1987).[2]

Studies have also looked at the exploitation of imperfect information as part of enforcement policy (Polinsky and Shavell, 2000). An example would be the decision whether or not to reveal the location of speed cameras to the public (Calford and DeAngelo, 2020).

The idea that uncertainty can improve deterrence is not new. For example, uncertainty in enforcement policy has been found to improve deterrence when taking into account certain behavioral aspects of people's choices (Harel and Segal, 1999). A lab experiment conducted on speeding confirmed that uncertainty about the enforcement regime yields a significant reduction in violations (DeAngelo and Charness, 2012). However, in studies carried out so far, the main reason for not completely revealing the enforcement policy to the public has been based on the assumption that resources are limited and therefore it is impossible to enforce the law at all times and in all places. Thus, uncertainty as to where and how the law will be enforced may be more effective than providing full information.[3]

In contrast, it is costless to perfectly enforce the law in our model. In particular, it might be beneficial in our model to implement uncertain enforcement policies, but not because of a lack of resources, but rather because of constraints imposed by public preferences. We therefore complement the existing literature by offering a new explanation for why vague enforcement policies might be beneficial.

The closest model to ours is that of Craswell and Calfee (1986). In their setting, the

[2]See Polinsky and Shavell (2000) for a discussion of enforcement policy that takes into account the level of activity rather than just whether or not a crime has been committed.

[3]This approach appears in inspection games literature (see Avenhaus et al., 2002 for a survey).

magnitude of the fine and the probability that the fine is imposed both increase in the activity level, which is chosen by an individual with commonly known preferences. Unlike in our model, they assume that uncertainty is always present in law enforcement and therefore it is exogenously determined. They conclude that, depending on the shape of the distribution according to which the fine is imposed, uncertainty may increase or decrease the level of compliance. In particular, if there were no uncertainty in enforcement in their model, then social welfare would be maximized, since in that case the optimal level of compliance is achieved when a fine is imposed beyond a certain activity level. In our model, on the other hand, uncertainty is a strategic choice made by the regulator who also has the option of choosing a deterministic enforcement system. We derive conditions under which he chooses an uncertain law enforcement system rather than a deterministic one, with the main determinant being incomplete information regarding the agent types. Thus, although Craswell and Calfee (1986) indeed noted that, regardless of whether or not the regulator faces budget constraints, uncertainty in law enforcement may have a positive effect on compliance, we extend this point much further by showing that not only does uncertainty in law enforcement have positive effects, it may actually provide better results than any deterministic enforcement system.

3 The model

There are two players: an agent and a regulator. The agent chooses an activity level $x \in [0, \bar{x}]$, $\bar{x} < \infty$ from which he gains utility $\tilde{\alpha}u(x)$, where u is continuous and increasing and $\tilde{\alpha}$ is a random variable privately known to the agent. In particular, $\tilde{\alpha} = \alpha_L$ (*low type*) with probability p, and $\tilde{\alpha} = \alpha_H$ (*high type*) with probability of $1 - p$, where $\alpha_H > \alpha_L > 0$. The regulator's loss function is: $l(x)$, where l is a continuous and strictly increasing function of x. In the first stage of the game, the regulator chooses and publicly announces the distribution of a parameter $\hat{x}$. In the second stage, the agent chooses x. Then the value of $\hat{x}$ is realized and the agent pays a fine of $F > 0$ iff x is larger than the realization of $\hat{x}$. For $\hat{x} = \bar{x}$ the agent pays the fine iff $x = \bar{x}$. It is therefore assumed that the fine is only a punitive device and the revenue is distributed to the public after it is collected and does not provide any benefit to the regulator. Note that we also assume that the regulator's sole purpose is to minimize the loss from x and that he disregards the agent's utility from x. In section 5, it is shown that the model can easily be extended to a case in which the regulator maximizes social welfare and therefore takes into account both the harm caused by x and the agent's utility.

4 Equilibria

In this section, we prove our main result whereby, under some conditions, there exists an SPE in which the regulator randomizes over two particular activity levels, one of which is then chosen by the agent, depending on his type. In addition, it is shown that this strategy of the regulator, regardless of whether it is an equilibrium or not, is superior to any other type of randomization (or at least not worse). A numerical example is presented for illustration.

We assume that:

$$F < \alpha_L(u(\bar{x}) - u(0)), \tag{1}$$

This insures that the fine is not high enough that we obtain a corner solution, in which $\hat{x} = 0$.

As a benchmark, we first consider the case in which the agent's type is commonly known, namely, $p \in \{0, 1\}$ or $\tilde{\alpha} = \alpha$. Let $\hat{x}^*$ solve $\alpha u(\hat{x}^*) = \alpha u(\bar{x}) - F$. Note that by (1) and the continuity of u, there exists a unique $\hat{x}^* \in (0, \bar{x})$.

Lemma 1. Suppose that $p \in \{0, 1\}$. Then there exists a unique SPE in pure strategies, in which $\hat{x} = x = \hat{x}^*$. There is no other SPE in which $x < \hat{x}^*$ in any realization of x.

All proofs appear in the appendix.

Let $\hat{x}^* \equiv \hat{x}_L^*$ when $p = 1$ and $\hat{x}^* \equiv \hat{x}_H^*$ when $p = 0$.

Now consider the more general case, in which $\tilde{\alpha}$ is unknown to the regulator, namely $p \in (0, 1)$. For a given $\hat{x}$, denote by x_L the activity level chosen by a low type agent and by x_H the activity level chosen by a high type agent. The following Lemma presents the agent's best response to the regulator's strategies that we are interested in.

Lemma 2. a. Suppose that the regulator chooses some value $\hat{x}$ with certainty. If $\hat{x} \geq \hat{x}_H^*$, then $x = \hat{x}$. If $\hat{x} \in [\hat{x}_L^*, \hat{x}_H^*)$, then $x_L = \hat{x}$ and $x_H = \bar{x}$. If $\hat{x} < \hat{x}_L^*$, then $x = \bar{x}$.

b. $x_L \geq \hat{x}_L^*$ and $x_H \geq \hat{x}_H^*$.

c. Let $(q, x^{'}, x^{''})$ where $\hat{x}_L^* < x^{'} < \hat{x}_H^* < x^{''}$ and $0 < q < 1$, be a strategy chosen by the regulator, in which $\hat{x} = x^{'}$ with probability q and $\hat{x} = x^{''}$ with probability $1 - q$. There exists $(q, x^{'}, x^{''})$ such that $x_H = \hat{x}^{''}$ and $x_L = \hat{x}^{'}$.

Note that in Lemma 2c, we define a particular type of strategy that involves randomizing over exactly two activity levels. In the remainder of the analysis we show that under certain conditions, such a strategy satisfies an SPE.

Let $\hat{x}_d$ be the distribution of $\hat{x}$ chosen by the regulator (i.e., his strategy). In particular, $\hat{x}_d$ represents the probability that the agent is required to pay the fine when exceeding the activity level $\hat{x}$. In addition, let $l_{\hat{x}_d} = pEl(x_L) + (1-p)El(x_H)$ be the expected value of l at $\hat{x}_d$. If the regulator chooses $\hat{x}$ with certainty, then $l_{\hat{x}_d} = l_{\hat{x}}$

Definition 1. A particular triple $(q, x^{'}, x^{''})$ is *the regulator's best random strategy* if among all other triples that satisfy Lemma 2c there is no triple $(q^{'}, x_0^{'}, x_0^{''})$ such that $l_{(q^{'}, x_0^{'}, x_0^{''})} < l_{(q^{'}, x^{'}, x^{''})}$.

It is now straightforward to demonstrate the following lemma.

Lemma 3. There exists a triple: $(q, x^{'}, x^{''})$ that is the regulator's best random strategy.

Proposition 1 below shows that in order to find the distribution $\hat{x}_d$ that minimizes $l_{\hat{x}_d}$, among all non-deterministic distributions of $\hat{x}$, it is sufficient to consider only the strategy $(q, x^{'}, x^{''})$ that is the regulator's best random strategy. Proposition 2 then derives sufficient conditions under which this strategy is better than the best pure strategies. Theorem 1 establishes the main result.

Proposition 1. Let $\hat{x}_d = G$, where G is a non deterministic distribution. Suppose $(q, x^{'}, x^{''})$ is the regulator's best random strategy. Then there does not exist G such that $l_G < l_{(q,x^{'},x^{''})}$.

Proposition 2. Suppose that $l^{''}(x) \geq 0$ and $\frac{\partial u(x)}{\partial x} \to 0$ as $x \to \infty$. Then, there exists $(q, x^{'}, x^{''})$, $\bar{x}^*$ and $0 < p^* < 1$, such that for all $\bar{x} > \bar{x}^*$ and $p > p^*$, $(q, x^{'}, x^{''})$ is preferable for the regulator to choosing $\hat{x} = \hat{x}_L^*$ or $\hat{x} = \hat{x}_H^*$ with certainty.

Note that it is often claimed that the damage caused in a car accident increases exponentially with driving speed, which corresponds to the assumption in Proposition 2 that l is convex (or linear) in x.

Based on the analysis so far, we can now present the main result, which establishes the existence of an SPE in which the regulator randomizes over exactly two activity levels:

Theorem 1. Let $l^{''}(x) \geq 0$ and $\frac{\partial u(x)}{\partial x} \to 0$ as $x \to \infty$. If p and $\bar{x}$ are sufficiently large, then there exists an SPE, in which the regulator chooses his best random strategy $(q, x^{'}, x^{''})$, and $x_L = x^{'}$ and $x_H = x^{''}$.

The randomization policy proposed in Theorem 1 may correspond to some real-life situations, in which the enforcement policy is intentionally vague. However, in principle, the regulator can achieve a similar result by using different levels of fines: a small fine for exceeding some low level of activity and a larger fine for exceeding a higher level. As mentioned in the introduction, he may nevertheless prefer a single fine system and not always collect rather than applying a gradual fine system that is always enforced, since the latter system is generally less feasible. Furthermore, there may be other more "behaviorally related" reasons for his preference. For instance, there is evidence that a small fine may increase engagement in undesirable activity rather than prevent it, since the fine may be perceived by the public as a price paid for harm caused. (Gneezy and Rustichini, 2000).

To illustrate this result, it is perhaps useful to consider the following simple numerical example:

Example 1. Let $l(x) = u(x) = x, \alpha_L = 1, \alpha_H = 1.2, \bar{x} = 100, F = 50$ and $p = 0.7$.

Then:

$$\hat{x}_H^* = 70 \tag{2}$$

and

$$\hat{x}_L^* = 50 \tag{3}$$

Therefore:

$$l_{\hat{x}_H^*} = 70 \tag{4}$$

and

$$l_{\hat{x}_L^*} = 0.7 * \hat{x}_L^* + 0.3 * \bar{x} = 65 \tag{5}$$

The strategy: $(q, x^{'}, x^{''}) = (0.75, 52.5, 90)$, results in:

$$l_{(0.75,52.5,90)} = 0.7 * 52.5 + 0.3 * 90 = 63.5 \tag{6}$$

Note that this example only shows that there exists a mixed strategy of the form appearing in Lemma 2c which is preferred to the pure strategies. It may not be an equilibrium strategy, since there may be other mixed strategies of the same kind that are superior to it. Furthermore, for simplicity, we use a linear utility function in the example instead of a concave one as required by Proposition 2. And, indeed, the conditions in Proposition 2 are sufficient but not necessary.

5 Welfare

A key endeavor in the law enforcement literature is to find the enforcement policy that maximizes welfare. In particular, it is usually assumed that individuals benefit from their actions but those actions also cause harm to others and are therefore costly. A desirable enforcement policy therefore is based on the standard cost-benefit analysis. Note that in our model, the utility the agent gains from his action is not taken into account by the regulator. Nevertheless, in some sense, this assumption is made without a loss of generality, since usually the lowest activity level achievable is above the desirable activity level. Therefore, the regulator's purpose remains the same: minimizing the activity level as much as possible. This seems to be the case in various scenarios such as setting a speed limit to improve road safety or setting limits on travel distance during a lockdown. One simple way to change the model in order to capture this idea is as follows:

Let $\underline{x} \geq 0$ be the minimal activity level for which it is possible to impose a fine. An interpretation of $\underline{x}$ could be the formal speed limit mentioned in the law, such that a fine is not given for driving at a slower speed, or alternatively a rule of negligence in the tort laws (see footnote 1). For an interior solution, we also assume that $F < \alpha_L(u(\bar{x}) - u(\underline{x}))$ (note that this assumption is a generalization of (1)). Define the egalitarian expected level of welfare to be $w = p\alpha_L(Eu(x_L) - El(x_L)) + (1-p)\alpha_H(Eu(x_H) - El(x_H))$. Then,

Proposition 3. Suppose that the regulator maximizes w. If $u'' < 0 \leq l''$, $\alpha_H u'(\underline{x}) < l'(\underline{x})$ and $F < \alpha_L(u(\bar{x}) - u(\underline{x}))$, then the regulator chooses $\hat{x}_d$ which minimizes $El(x)$.

Note that the first two conditions on u and l in Proposition 3 imply that social welfare is maximized when $x_L = x_H = \underline{x}$ while the third condition implies that the regulator cannot force the agent to choose $x = \underline{x}$.

The proof is straightforward.

6 Conclusions

It is shown that a vague enforcement policy is desirable in some circumstances, even when cost of enforcement is not a constraint. A regulator who implements such a policy may therefore be both competent and "benevolent", since he understands that it is better not to impose a rule that the public is unable to comply with. On the contrary, a regulator who implements too strict an enforcement policy is either incompetent or not maximizing welfare, rather than efficiently enforcing the law.

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Appendix

Proof of Lemma 1. Let $\tilde{\alpha} = \alpha$. Suppose that the regulator chooses some $\hat{x} \in (0, \bar{x})$ with certainty. Since u is increasing, the agent either chooses $x = \hat{x}$ and therefore does not pay the fine or chooses $x = \bar{x}$ and does. Therefore, to minimize l, the regulator chooses the lowest value of $\hat{x}$ for which the agent chooses $x = \hat{x}$. This value solves $\alpha u(\hat{x}) = \alpha u(\bar{x}) - F$.

Assume to the contrary that $\hat{x}$ is a random variable which follows a distribution function $P(x)$ on $(0, \bar{x})$ that is chosen by the regulator. In particular, $P(x)$ represents the probability that the agent is required to pay the fine when choosing an activity level x. And suppose that the agent's best response to $P(x)$ is some $x < \hat{x}^*$. The expected utility of the agent is then $\alpha u(x) - P(x)F$. Since $\alpha u(\hat{x}^*) = \alpha u(\bar{x}) - F$ and $x < \hat{x}^*$,

$$\alpha u(x) < \alpha u(\bar{x}) - F < \alpha u(\bar{x}) - F + P(x)F,$$

or, by rearranging terms,

$$\alpha u(x) - P(x)F < \alpha u(\bar{x}) - F.$$

Thus, the agent prefers choosing $\bar{x}$ over x, a contradiction.

□

Proof of Lemma 2. By definition,

$$\alpha_L u(\hat{x}_L^*) = \alpha_L u(\bar{x}) - F \tag{7}$$

and

$$\alpha_H u(\hat{x}_H^*) = \alpha_H u(\bar{x}) - F. \tag{8}$$

Given that u is increasing in x, (7) and (8) imply Lemma 2a.

As in the case of Lemma 1, there is no equilibrium in which the realization of x_L is smaller than $\hat{x}_L^*$ and/ or the realization of x_H is smaller than $\hat{x}_H^*$.

Next, we prove the existence of $(q, x^{'}, x^{''})$. By (7), for every $\hat{x}_L^* < x^{'} < \hat{x}_H^*$ there exists $0 < q < 1$ such that $\alpha_L u(x^{'}) = \alpha_L u(\bar{x}) - qF$. Recall, that the agent pays the fine with probability q when choosing $x \in (x^{'}, x^{''}]$. Therefore, since u is increasing in x, for any given pair $(q, x^{'})$ that satisfies the above equality, $\alpha_L u(x^{'}) > \alpha_L u(x) - qF$ for all $x \in (x^{'}, x^{''}]$, and, therefore, the low type agent prefers $x^{'}$ over $x \in (x^{'}, x^{''}]$. Recall that when choosing $x > x''$, the agent pays the fine with certainty. Therefore, for the same pair $(q, x^{'})$ that satisfies the above equality, $\alpha_L u(x^{'}) > \alpha_L u(x) - F$ for all $x \in (x^{''}, \bar{x}]$, and, therefore, the low type agent prefers $x^{'}$ over $x \in (x^{''}, \bar{x}]$.

Note that by (8), for any $q > 0$ and $x^{'} < \hat{x}^*_H$, the high type agent prefers $\bar{x}$ over x for all $x \leq \hat{x}^*_H$.

Therefore, since the agent pays the fine with probability q when choosing $x = x^{''}$, the high type agent chooses $x = x^{''}$ if

$$\alpha_H u(x^{''}) - qF \geq \alpha_H u(\bar{x}) - F. \tag{9}$$

As mentioned above, by (8), (9) does not hold for $x^{''} = \hat{x}^*_H$ when $q > 0$. Obviously, for $x^{''} = \bar{x}$ it does. By the intermediate value theorem there exists $\hat{x}^*_H < x^{''} < \bar{x}$ for which the inequality in (9) holds. □

Proof of Lemma 3. Consider the set $S = \{(q, x^{'}, x^{''})\}$ to be a set of strategies that satisfy Lemma 2c and note that, by Lemma 2c, S is non-empty. By the proof of Lemma 2c, $q \in [a, 1]$ where $a > 0$, $x^{'} \in [\hat{x}^*_L, \hat{x}^*_H]$ and $x^{''} \in [b, \bar{x}]$ where $b > \hat{x}^*_H$. Since by the same proof, S is bounded and continuous by construction, $l_{(q,x^{'},x^{''})}$ has a minimum.[4]

□

Proof for Proposition 1. Assume in contrast that there is a distribution G such that $l_G < l_{(q,x^{'},x^{''})}$, where $(q, x^{'}, x^{''})$ is the regulator's best random strategy. In particular, let $x^{'}_0$ be the smallest realization of x_L and $x^{''}_0$ the smallest realization of x_H when the regulator chooses G. Let $q^{'} = G(x^{'}_0)$. Consider a strategy, in which the regulator chooses $\hat{x} = x^{'}_0$ with probability $q^{'}$ and $\hat{x} = x^{''}_0$ with probability $1 - q^{'}$. Then $x_L = x^{'}_0$ and $x_H = x^{''}_0$. By construction,

$$l_{(q^{'},x^{'}_0,x^{''}_0)} \leq l_G < l_{(q,x^{'},x^{''})},$$

a contradiction to $(q, x^{'}, x^{''})$ being the regulator's best random strategy.

[4]Note that although $(1, \hat{x}^*_L, \bar{x})$ is in S, it is not random and therefore does not strictly belong to the set of strategies that satisfy Lemma 2c. And, in principal, $(1, \hat{x}^*_L, \bar{x})$ can be a minimum point of S and therefore the regular's best random strategy. However, Proposition 2 will imply that this is not the case under the conditions appear in Theorem 1, in which we present our main result.

□

Proof of Proposition 2. For each $\bar{x}$ denote by $\hat{x}^*_H(\bar{x})$ the solution of (8), and by $\hat{x}^*_L(\bar{x})$ the solution of (7). Also, denote by $\hat{x}^{'}(\bar{x}, q)$ the solution of $\alpha_L u(x^{'}) = \alpha_L u(\bar{x}) - qF$, and by $\hat{x}^{''}(\bar{x}, q)$ a solution of (9). Recall that $\hat{x}^*_L(\bar{x}) < \hat{x}^{'}(\bar{x}, q) < \hat{x}^*_H(\bar{x}) < \hat{x}^{''}(\bar{x}, q)$.

By Lemma 2c

$$l_{(q, \hat{x}^{'}(\bar{x},q), \hat{x}^{''}(\bar{x},q))} = pl(\hat{x}^{'}(\bar{x}, q)) + (1-p)l(\hat{x}^{''}(\bar{x}, q)). \tag{10}$$

By Lemma 2a, if the regulator chooses $\hat{x}^*_H(\bar{x})$ with certainty, then $l_{\hat{x}^*_H(\bar{x})} = l(\hat{x}^*_H(\bar{x}))$. Since $\hat{x}^{'}(\bar{x}, q) < \hat{x}^*_H(\bar{x})$, by (10), and for sufficiently large $p < 1$, $l_{(q, \hat{x}^{'}(\bar{x},q), \hat{x}^{''}(\bar{x},q))} < l_{\hat{x}^*_H(\bar{x})}$.

If the regulator chooses $\hat{x}^*_L(\bar{x})$ with certainty, then $l_{\hat{x}^*_L(\bar{x})} = pl(\hat{x}^*_L(\bar{x})) + (1-p)l(\bar{x})$. Note that since $\frac{\partial u(x)}{\partial x} \to 0$ as $x \to \infty$, for each $\epsilon > 0$ there exists a sufficiently large $\bar{x}_0$ such that $|\hat{x}^*_H(\bar{x}_0) - \hat{x}^*_H(\bar{x}_1)| < \epsilon$, $|\hat{x}^*_L(\bar{x}_0) - \hat{x}^*_L(\bar{x}_1)| < \epsilon$, $|\hat{x}^{'}(\bar{x}_0, q) - \hat{x}^{'}(\bar{x}_1, q)| < \epsilon$ and $|\hat{x}^{''}(\bar{x}, q) - \hat{x}^{''}(\bar{x}_1, q)| < \epsilon$ for any $\bar{x}_1 > \bar{x}_0$. Therefore, by (10), for any given $p < 1$, there exists a sufficiently large $\bar{x}$ at which, $l_{\hat{x}^*_L(\bar{x})} > l_{(q, \hat{x}^{'}(\bar{x},q), \hat{x}^{''}(\bar{x},q))}$.

□

Proof of Theorem 1. Note that Lemma 2 and Propositions 1 and 2 imply that, under the conditions in Proposition 2, $(q, x^{'}, x^{''})$ that is the regulator's best random strategy, is better than any pure strategy and is not worse than any other mixed strategy. By Lemma 3, there exists $(q, x^{'}, x^{''})$ which is the regulator's best random strategy. □