All probabilities are equal, but some probabilities are more equal than others*

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Abstract

There are several procedures for selecting people at random. Modern and ancient stories as well as some experiments suggest that individuals may not view all such lotteries as "fair." In this paper, we compare alternative procedures and show conditions under which some procedures are preferred to others. These procedures give all individuals an equal chance of being selected, but have different structures. We analyze these procedures as multi-stage lotteries. In line with previous literature, our analysis is based on the observation that multi-stage lotteries are not considered indifferent to their probabilistic one-stage representations.

Keywords: Fair lotteries, two-stage lotteries, reduction axiom, PORU

JEL # D63, D91

1 Introduction

Since Diamond (1967), it became clear that social lotteries can and should be used to enhance fairness and equality in the allocation of indivisible goods.

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Unlike Harsanyi (1955,1975), who concentrated on the ex-post perspective, social lotteries can create a certain degree of ex-ante equality, at least until the outcome of the lottery is revealed. Such procedures were axiomatized both at the social level (see, e.g., Epstein and Segal (1992)) and at the individual level (Karni and Safra (2002)). The literature is however silent regarding the actual randomization mechanism society should use. Our aim in this paper is to highlight a neglected aspect of the different procedures — their attractiveness and fairness as viewed by those who participate in them. Our main point is that if the aim of social lotteries is to eliminate complaints about favoritism, then these lotteries must be deemed fair not only by the social planner, but more importantly, by those who benefit or suffer from their consequences. And since fairness is often identified with equality, society should not use procedures that yield what subjects may believe to be a differential treatment. It may also be that participants, rightly or wrongly, believe that some procedures are more vulnerable to potential cheating than others. Moreover, suppose that participants are not indifferent between procedures just because they have preferences for the way uncertainty is resolved. Pareto concerns then dictate that social planners should pay attention to the mechanisms they use and not just to the probabilities they produce. This is clearly demonstrated by the evolvement of the draft lottery during the Vietnam-war era, discussed in section 5 below. Such arguments suggest that social planners should take into consideration individual perception and preferences over randomization mechanisms, even if the planners themselves concern themselves only with the implied probability distribution.

Suppose we want to use a random procedure in order to select one out of n people to receive a certain indivisible good. One can think of two natural such procedures. Either put slips with the n people's names in a box and pick one slip at random, or put one green and $n-1$ red balls in an urn, and then ask each to pick a ball, where the one who picked green is selected. Both methods seem fair and are statistically equivalent, as they give each person the same $\frac{1}{n}$ probability of being selected.

But the two may not seem identical to the n participants. To begin with, in the first procedure there is only one acting person (which may or may not be one of the n people), while in the second procedure, each of the n individuals takes an active part.[1] More importantly, the two procedures

[1]See e.g. Eliaz and Rubinstein (2014, P4), where more than 40% of the participants

represent totally different lotteries. The first is a simple lottery, while for almost each of the n individuals the second procedure offers a compound lottery which is likely to be appreciated differently by participants according to the order of the draws.

Formally, we analyze two basic procedures. 1. Pre-ordered draws, where $n-k$ balls of one color (red) and k balls of another color (green) are put into an urn and, in their turn, subjects pick a random ball without replacement. The selected persons are those who picked the green balls. 2. Names lottery, where the n names are put in an urn and an organizer randomly selects one or some of them. We also discuss variants of these procedures: 3. Pre-ordered draws when a winning ball is added, and if too many people are selected, the procedure starts over again. 4. Names lottery, where names are repeatedly picked up (with replacement), and the first person whose name appears in two successive draws is selected. We compare these procedures and show how their desirability changes with the size of the group and the identity of each participant. Most, but not all of our results assume that the number of chosen people is either one or all but one.

Most of the procedures we discuss involve multi-stage lotteries and our analysis depends on the evaluation of such lotteries. The literature describes two ways in which multi-stage lotteries can be transformed into simple, one-stage lotteries. The first is using the reduction axiom, according to which each possible outcome is listed with its compound probability, obtained by multiplying the probabilities along the path to that outcome. Alternatively, each simple lottery is replaced by its certainty equivalent and, when applied recursively, the value of the multi-stage lottery is computed (see Kreps and Porteus (1978) and Segal (1990)). Expected utility is the only theory under which decision makers are indifferent between these two simplifications. Several theoretical models use this distinction between the two methods to explain phenomena like ambiguity aversion (Segal (1987), Klibanoff, Marinacci, and Mukerji (2005)) or to analyze variants of the housing allocation (one-sided matching) problem (Dillenberger and Segal (2021)). Experiments tend to support the second approach (see Starmer (2000), Halevy (2007), Abdellaoui, Klibanoff, and Placido (2015), Harrison, Martinez-Correa, and Swarthout (2015), and Masatlioglu, Orhun, and Raymond (2017)). As we are interested in people's subjective evaluation of social lotteries, we follow

were not indifferent to the question of who is going to flip a coin between a pair of twins — their mother or a doctor.

this approach.

2 Preliminaries

Let $\mathcal{T} = \{\mathfrak{g}, \mathfrak{b}\}$ be a set of two outcomes ("good" and "bad") and let $\mathcal{A} = [a, b] \subset \Re$ be a set of monetary payoffs, large enough so that each person in society has (personal) g, b such that he is indifferent between $\mathfrak{g}$ and g and between $\mathfrak{b}$ and b. This indifference is preserved even when outcomes are uncertain — he is also indifferent between $(\mathfrak{g}, p; \mathfrak{b}, 1-p)$ and $(g, p; b, 1-p)$.

Let $\succeq$ be a preference relation over lotteries. Given a two-stage lottery $L = (X_1, q_1; \ldots; X_\ell, q_\ell)$ where $X_1, \ldots, X_\ell$ are simple lotteries, we assume that the lottery L is evaluated by decision makers using recursive utility. That is, L is transformed to the simple lottery $C(L) = (\text{CE}(X_1), q_1; \ldots; \text{CE}(X_\ell), q_\ell)$, where $\text{CE}(X)$ is the certainty equivalent of X, satisfying $(\text{CE}(X), 1) \sim X$ (see Segal (1990)). Denote by $R(L)$ the one-stage lottery obtained from L by multiplying the probabilities. Dillenberger (2010) suggested the following assumption:

Definition 1. The preference relation $\succeq$ satisfies PORU (preferences for one-shot resolution of uncertainty) if for every $L = (X_1, q_1; \ldots; X_\ell, q_\ell)$, $R(L) \succeq C(L)$.

In particular, PORU implies the following weaker requirement, which is the one we use throughout (note that $\succeq$ is now used to compare a simple lottery with a compound lottery):

Definition 2. The preference relation $\succeq$ satisfies weak-PORU if for all x, y

$$(x, pq;\, y, 1-pq) \succeq ((x, p;\, y, 1-p), q;\, y, 1-q) \tag{1}$$

Dillenberger (2010) proved that under some standard assumptions, PORU holds if and only if preferences also satisfy negative certainty independence: "If the sure outcome x is not enough to compensate the decision maker for the risky prospect X, then mixing it with any other lottery, thus eliminating its certainty appeal, will not result in the mixture of x being more attractive than the corresponding mixture of X" (p. 1980). Bernasconi and Bernhofe (2020) found that inexperienced decision makers behave according to eq. (1) for $x > y \geqslant 0$, although these preferences are less significant for low values of pq. For more support of preferences for one-shot resolution of uncertainty see references in the Introduction.

For some of our results we use the rank dependent (RD) model (Quiggin (1982)). According to this model, for $x \geqslant y$

$$\mathrm{RD}(x,p;y,1-p) = u(x)f(p) + u(y)[1-f(p)] \tag{2}$$

where f is continuous and strictly increasing, $u(0)=0$, $f(0)=0$, and $f(1)=1$.[2] For $X_1 \succeq X_2$, the value of the two-stage lottery $(X_1,q;X_2,1-q)$ is

$$\begin{aligned}&\mathrm{RD}(u^{-1}(\mathrm{RD}(X_1)),q;\ u^{-1}(\mathrm{RD}(X_2)),1-q) = \\ &\mathrm{RD}(X_1)f(q) + \mathrm{RD}(X_2)[1-f(1-q)]\end{aligned} \tag{3}$$

Let $g(p) = 1-f(1-p)$. Observe that $g(0)=0, g(1)=1$, and g is concave iff f is convex. We can rewrite eq. (2) as

$$u(y)g(1-p) + u(x)[1-g(1-p)] \tag{4}$$

When using the RD model, we will assume that the utility from being selected for a good outcome is 1, and the utility from being selected for a bad outcome is 0.

In the RD model, risk aversion (in the sense of rejection of mean preserving spreads) requires that f is convex (and g concave, see Chew, Karni, and Safra (1987)). The elasticity of a function $h(p)$ is given by $\eta^h(p) = \frac{ph'(p)}{h(p)}$. Increasing elasticity of f is linked to the common ratio effect[3] and to the recursive model of ambiguity (Segal (1987,1987a)). The following two lemmas discuss the connection between weak-PORU and properties of f and g.

Lemma 1. The RD model satisfies weak-PORU whenever the elasticity of f is increasing and correspondingly, the elasticity of g is decreasing.[4]

[2] Most of our results require only two different outcomes, so apply to the larger set of biseparable utilities (Ghirardato and Marinacci (2001)), including, for example, Gul's (1991) disappointment aversion theory. However, claims 9 and 10 below need more than two outcomes, therefore we use the RD model throughout. For $x_1 > \ldots > x_n$, the RD value of the lottery $X = (x_1,p_1;\ldots;x_n,p_n)$ is $u(x_1)f(p_1) + \sum_{i=2}^n u(x_i)[f(\sum_{j=1}^i p_j) - f(\sum_{j=1}^{i-1} p_j)]$.

[3] For example, $(1M,1) \succ (5M,0.8;\ 0,0.2)$ while $(5M,0.04;\ 0,0.96) \succ (1M,0.05;0,0.95)$. See Allais (1953), MacCrimmon and Larsson (1979), Starmer (2000) and further references there.

[4] These two assumptions do not contradict each other, see Segal (1987, p. 185).

Lemma 2. The four combinations of convexity–concavity and increasing–decreasing elasticity of h are possible. However, if the first non-zero derivative of h at 0 is finite, then increasing elasticity implies convexity, although convexity does not imply increasing elasticity.

In the sequel, we analyze changes in individuals' welfare due to changes in the basic parameters, for example, the number n of individuals from whom we select. These are discrete variables, but we consider them as continuous so that we can differentiate with respect to them. The results are of course meaningful only for the integer values of these variables.

3 Pre-ordered draws

Consider the following procedure. Two types of balls, green and red, are put in an urn. Agents are pre-ordered and one after the other they draw from the urn with no replacement. Those who drew green are selected for a good outcome and those who drew red receive a bad outcome. Just before his turn arrives, each person is told how many green and red balls were already drawn. Denote by $P(n,k)$ the procedure in which the urn initially contains k green and $n-k$ red balls.

Consider $P^i(n,1)$, the lottery faced by individual i in procedure $P(n,1)$ *before* the first ball is drawn. With probability $\frac{i-1}{n}$, someone draws the green ball before i's turn arrives, i is not selected, and gets 0. (Recall that he'll find out whether this happened just before his turn arrives). With the remaining probability, $\frac{n-i+1}{n}$, the green ball will not be drawn before his turn and then he is facing a lottery in which with probability $\frac{1}{n-i+1}$ he draws the green ball, is selected, and gets 1 and with probability $\frac{n-i}{n-i+1}$ he draws a red ball, is not selected, and gets 0. In other words, individual i is facing the two-stage lottery

$$P^i(n,1) = \left(\left(1, \tfrac{1}{n-i+1}; 0, \tfrac{n-i}{n-i+1}\right), \tfrac{n-i+1}{n}; \, 0, \tfrac{i-1}{n}\right) \tag{5}$$

Likewise, in procedure $P(n,n-1)$ person i wins for sure if the red ball was picked by one of the first $i-1$ participants. The probability of this event is $\frac{i-1}{n}$. If they all picked a green ball, then he has probability $\frac{1}{n-i+1}$ to pick the losing ball. Ex-ante, he is facing the two stage lottery

$$P^i(n,n-1) = \left(\left(0, \tfrac{1}{n-i+1}; 1, \tfrac{n-i}{n-i+1}\right), \tfrac{n-i+1}{n}; \, 1, \tfrac{i-1}{n}\right) \tag{6}$$

Procedures $P^i(n, 1)$ and $P^i(n, n-1)$ are depicted in Figure 1.

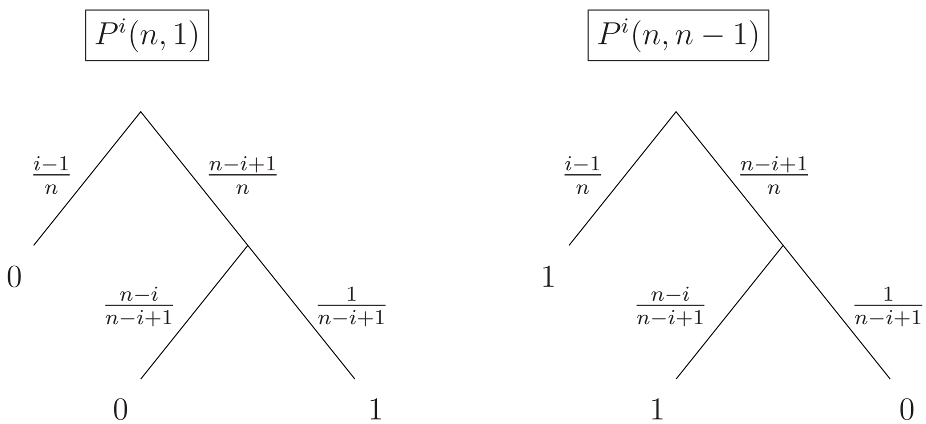

Figure 1: Procedures $P^i(n, 1)$ and $P^i(n, n-1)$ for person i

In this section we deal with some issues concerning the pre-ordered procedure and two of its possible variants. First, we show that the position in the procedure is not a matter of indifference. Moreover, preferences for such positions depend on the way information about other participants is revealed (section 3.1). We then analyze preferences for the number of individuals taking part in the procedure (section 3.2), and compare changes in the group size to a possible manipulation of the urn, where a winning ball is added and if necessary, the procedure is repeated (section 3.3).

3.1 Does the order matter?

When $i = 1$ and when $i = n$, the two-stage lottery of eq. (5) becomes $(1, \frac{1}{n}; 0, \frac{n-1}{n})$. The first person is obviously facing a simple lottery — either he draws green (the probability of this event is $\frac{1}{n}$), or he does not. Before person 1 draws, the last person to draw knows that when his turn arrives the last remaining ball will be either green or red, and ex ante, the probabilities of these events are $\frac{1}{n}$ and $\frac{n-1}{n}$. All other agents face a real two-stage lottery with the reduced form $(1, \frac{1}{n}; 0, \frac{n-1}{n})$. Likewise, for $i = 1$ and $i = n$, the two-stage lottery of eq. (6) becomes $(1, \frac{n-1}{n}; 0, \frac{1}{n})$ while all other agents face a real two-stage lottery with this reduced form. We thus get our first result:

Claim 1. Assume weak-PORU. In the pre-ordered procedure P, if only one person is selected to receive a different outcome from everyone else, then it is best to be either the first or the last to draw.[5]

In $P^i(n, j)$ person i may get one of several simple lotteries. If PORU (and not just weak-PORU) is assumed, then the claim holds for any number of selected individuals.

Claim 1 does not depend on the specific model decision makers use for the evaluation of lotteries (but it depends of course on the analysis of this procedure as a two-stage lottery). More detailed results can be obtained for specific models like rank-dependent. Using eqs. (3) and (2), the value of lottery (5) is

$$\mathrm{RD}(P^i(n,1)) = f\left(\tfrac{1}{n-i+1}\right) f\left(\tfrac{n-i+1}{n}\right) \tag{7}$$

Similarly, by eqs. (3) and (4), the value of lottery (6) is

$$\begin{aligned} \mathrm{RD}(P^i(n,n-1)) &= g\left(\tfrac{n-i+1}{n}\right)\left[1 - g\left(\tfrac{1}{n-i+1}\right)\right] + \left[1 - g\left(\tfrac{n-i+1}{n}\right)\right] \\ &= 1 - g\left(\tfrac{n-i+1}{n}\right) g\left(\tfrac{1}{n-i+1}\right) \end{aligned} \tag{8}$$

The position in the queue changes the structure of the two-stage lottery in which each person participates and obviously, the values of $\mathrm{RD}(P^i(n,1))$ and $\mathrm{RD}(P^i(n,n-1))$ change with i.

Claim 2. Let $i^* = n + 1 - \sqrt{n}$. If the elasticity of f is increasing, then $\mathrm{RD}(P^i(n,1))$ is decreasing in i until i^* and increasing thereafter. The same holds for $\mathrm{RD}(P^i(n,n-1))$ if the elasticity of g is decreasing.

As mentioned above, increasing elasticity of f is associated with the common ratio effect, hence so is the behavior described by the first part of Claim 2.

Claim 2 suggests a possible tool social planners can use to improve social welfare. If individuals have different, yet known, preferences over lotteries, then different orders of the draws in the P procedure create by themselves different distributions of utilities. We do not explore this possibility as we

[5] This is exactly the sentiment expressed by Mrs. Montgomery and Mr. Belmont in Graham Greene's *Dr. Fischer of Geneva, or the Bomb Party.*

only assume general properties of preferences, rather than specific functional forms.

Our analysis and results are quite sensitive to the exact protocol used for the social lottery. We assumed so far that people know in advance that they'll find out just before their turn arrives how many balls of each color were already drawn beforehand. But it is easy to outline alternative scenarios. Consider for example procedure $Q(n,k)$, which is the same as $P(n,k)$, only that everyone is watching the outcome of each draw right away. We call this procedure pre-ordered with full information. In $Q(n,1)$, person i will participate in an i-stage lottery, where in each of the first $i-1$ stages either (i) the person whose turn arrives picks the green ball, the procedure terminates, and person i receives 0, or (ii) all the first $i-1$ people picked red, and in his turn, person i has $\frac{1}{n-i+1}$ chance of drawing the green ball. Figure 2 depicts the lotteries $Q^i(n,1)$ and $Q^i(n,n-1)$ faced by person i.

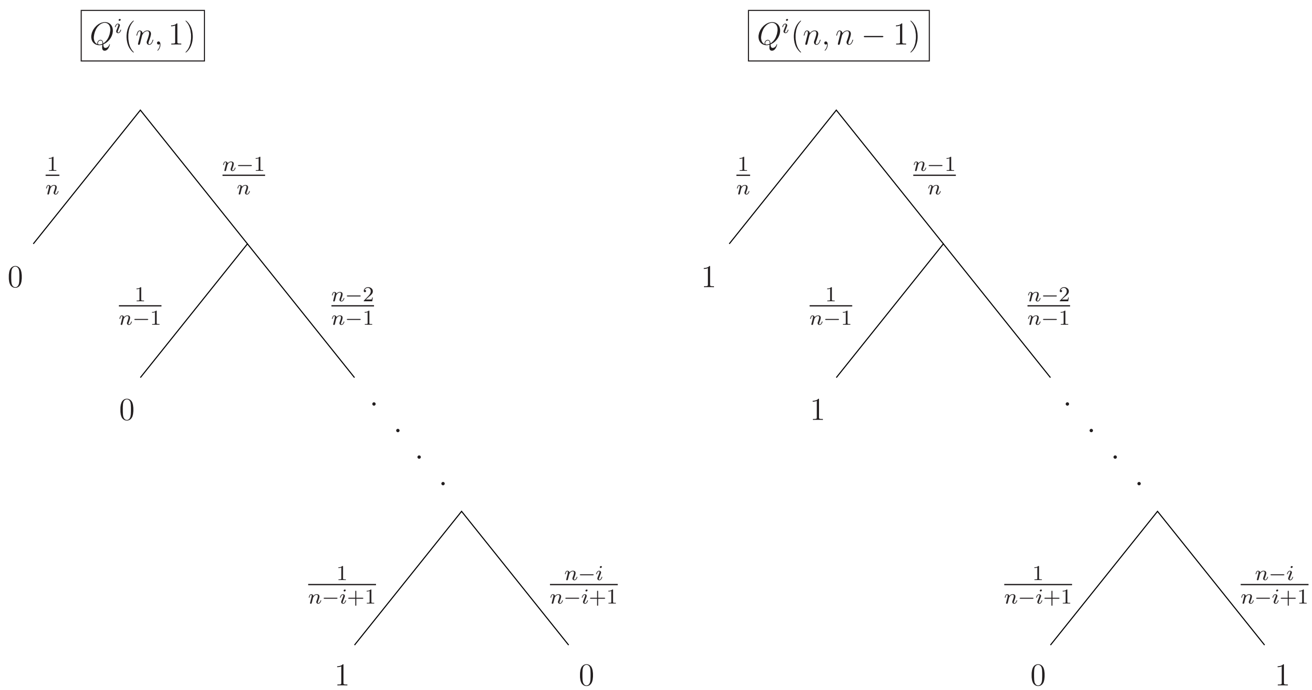

Figure 2: Procedures $Q^i(n,1)$ and $Q^i(n,n-1)$ for person i

Claim 3. Assuming weak-PORU, the values of the pre-ordered with full information procedures $Q^i(n,1)$ and $Q^i(n,n-1)$ are decreasing with i.

Claim 2 and 3 show how sensitive subjects may be to the exact protocols. Although both P and Q describe pre-ordered procedures, observing all stages of the lottery entirely changes its desirability to those who are at the end of the line to draw.

Remark 1. Claims 1 and 3 (as well as Claim 8 below assume weak-PORU. Experimental evidence suggests however that although some people judge fewer stages to be more fair, not all agree (see the cited experiment by Eliaz and Rubinstein (2014)). If instead of weak-PORU the opposite is assumed, that is, if for all x, y, $((x,p;\, y, 1-p), q;\, y, 1-q) \succeq (x, pq;\, y, 1-pq)$, then the three claims using weak-PORU are reversed. For example, Claim 1 will say that it is *worst* to be first or last and Claim 3 will state "increasing with i."

3.2 Is more always better?

When $n-1$ out of n people are going to be selected for a good outcome, it seems almost obvious that each of them would like the number n to be as high as possible, since the ex-ante probability of being selected, $\frac{n-1}{n}$, is increasing with n. But as we have seen in the previous section, the structure and evaluation of the multi stage lotteries faced by the n individuals may cause them to have preferences over lotteries with the same reduced probabilities. In fact, some people may even prefer a lower value of n.

Consider person #900 out of 1,000 in procedure $P(n,n-1)$. By the time his turn arrives, it is very likely that the red ball was already picked by someone else, in which case he knows for sure that he is going to get the good outcome. However, if he is #900 out of 10,000, then it is very likely that all 899 people before him picked green balls. As he will have to participate in a lottery in which picking the red ball is still possible, he may prefer the former case to the latter. We do not claim that this is likely to happen, only that it *may* happen. However, as the next claim shows, first-order stochastic dominance implies that this will never happen to candidates who succeed the added person.

Claim 4. Let $I = \{1, i_0, \ldots, n\}$. Suppose an individual is added at position i_0 without changing the order of the rest, pushing each person in $\{i_0, \ldots, n\}$

one stage down. Then using the $P(n+1,1)$ procedure in order to allocate a desirable outcome, every individual $i \in I$ in the original list becomes strictly worse off and using $P(n+1,n)$, every individual $i \in I$ becomes strictly better off.

To analyze the change in the welfare of those who precede that added person, we restrict attention to the RD model. We say that a function h satisfies condition $(*)$ for individual i if

$$\frac{i-1}{n} \leqslant \frac{\eta^h\left(\frac{1}{n-i+1}\right)}{\eta^h\left(\frac{n-i+1}{n}\right)} \tag{9}$$

Claim 5. Consider the allocation of desirable goods. Suppose an individual is added at position i_0 without changing the order of the rest and let person $1 < i < i_0$ be an RD maximizer. If f satisfies condition $(*)$, then in the $P(n+1,1)$ procedure person i becomes worse off and if g satisfies condition $(*)$, then in $P(n+1,n)$ person i becomes better off.

The LHS of eq. (9) is always less than 1. The RHS is greater than 1 whenever the elasticity at $\frac{1}{n-i+1}$ is greater than the elasticity at $\frac{n-i+1}{n}$. As $\frac{1}{n-i+1} \geqslant \frac{n-i+1}{n}$ iff $i \geqslant i^* = n+1-\sqrt{n}$, this observation leads to the following corollary:

Corollary 1. Suppose an individual is added at position $i_0 > i^*$ without changing the order of the rest. If the elasticity of f is increasing, then for $i^* \leqslant i < i_0$, $\mathrm{RD}(P^i(n,1))$ is decreasing in n. If the elasticity of g is decreasing, then for $i \leqslant i^*$, $\mathrm{RD}(P^i(n,n-1))$ is increasing in n.

Clearly condition $(*)$ for individual i is satisfied for $h(p) = p^\alpha$ (observe that the LHS of eq. (9) is less than 1 and the elasticity of $p^\alpha \equiv \alpha$). However, it does not necessarily hold for all functions f, even if they are convex. For example, fix n and let

$$f(p) = \begin{cases} \frac{n-1}{n^2}p & p \leqslant \frac{1}{n-1} \\ \\ \frac{(n-1)(n^2-1)}{(n-2)n^2}p + 1 - \frac{(n-1)(n^2-1)}{(n-2)n^2} & p > \frac{1}{n-1} \end{cases}$$

The function f is convex, it does not satisfy condition $(*)$ for individual 2, and, indeed, individual 2 prefers $P^2(n+1,1)$ to $P^2(n,1)$ when another person is added after him.

3.3 Adding winning balls

Consider the following $W(n, n-1)$ procedure for selecting $n-1$ out of n people for a good outcome. Put n green and one red ball in an urn and let the n candidates draw balls (with no replacement) according to a pre-arranged order.[6] If someone picked the red ball, then the other $n-1$ people are selected. If not, repeat this procedure using the same order. Like procedure P, participants are informed about the outcome of the previous stages just before their turn arrives. They also find out if the procedure needs to be repeated before it starts all over again. We show in this section that although this procedure may not be attractive to all, such preferences are not universal as some individuals will prefer it to $P(n, n-1)$.

Person i is facing a (potentially repeated) three-stage lottery. In the first stage, individuals before him draw balls. The probability that one of them has drawn the red ball is $\frac{i-1}{n+1}$, in which case the procedure terminates and person i is selected. If not, move to the second stage which is person i's turn. In this stage, his probability of drawing a red ball is $\frac{1}{n-i+2}$, in which case he gets 0 and the procedure is over. Otherwise, move to the third stage, where the probability of person i being selected is the probability that one of the last $n-i$ people draws the red ball, that is, $\frac{n-i}{n-i+1}$. If this does not happen, then the procedure is repeated. Person i is thus facing the following multi-stage lottery $W^i(n, n-1)$:[7]

$$\left(1, \tfrac{i-1}{n+1}; \left(\left(1, \tfrac{n-i}{n-i+1}; W^i(n, n-1), \tfrac{1}{n-i+1}\right), \tfrac{n-i+1}{n-i+2}; 0, \tfrac{1}{n-i+2}\right), \tfrac{n-i+2}{n+1}\right) \tag{10}$$

We show next that there is a connection between preferences for adding balls and preferences for adding individuals as discussed in section 3.2. In the unintuitive case in which a person in the pre-ordered procedure prefers not to add a candidate when all but one are selected for a good outcomes, he will also prefer not to add an extra ball. This result does not depend on the functional form used to evaluate lotteries. On the other hand, if $P^i(n, n-1)$ is improving when a person is added at the end of the order, then it is possible

[6] Such a procedure is suggested in *Talmud Yerushalmi*, Sanhedrin 1:7 as a possible solution to unfairness of pre-ordered draws.

[7] The lottery $W^i(n, n-1)$ is an infinite-stage lottery, as the procedure may need to repeat itself again and again, albeit with shrinking probability. Since the value of the lottery must be between the values of its most extreme possible outcomes, the value of lottery (10) is well defined.

(though not guaranteed) that person i will like to add a winning ball to the urn, even if it means that with some positive probability he'll have to go through the whole procedure again.

Claim 6. If $P^i(n, n-1) \succeq P^i(n+1, n)$, that is, if when participating in the pre-ordered procedure $P^i(n, n-1)$, person i prefers not to add another participant to the end of the order, then he also prefers $P^i(n, n-1)$ to $W^i(n, n-1)$. But if $P^i(n+1, n) \succ P^i(n, n-1)$, then any preferences between $W^i(n, n-1)$ are $P^i(n, n-1)$ possible.

The general analysis of $W^i(n, 1)$, where one person is to be selected for a good outcome but two green balls (and $n-1$ red balls) are put in the urn is quite complicated. We offer here analysis of the preferences of the first person to draw, for the case in which his preferences are RD with $f(p) = p^\alpha$. This function has constant elasticity α, and indeed, as in this case $f(pq) = f(p)f(q)$, these preferences are neutral with respect to weak-PORU.

Claim 7. Suppose that the preferences of person 1 are RD with $f(p) = p^\alpha$. Then $W^1(n, 1) \succeq P^1(n, 1)$ iff $\alpha \leqslant 1$.

4 Names lotteries

In this section we analyze a family of procedures that can be labeled "names lottery." In such procedures the names of the candidates are put in an urn, and an impartial observer draws the desired number of names out of it. Depending on the exact procedure, the analysis of such lotteries too depends on decision makers' attitudes towards multi-stage lotteries.

4.1 One appearance

The simplest case of names lottery is $N(n, 1)$, where the observer draws one name from the urn and the individual whose name has been drawn is selected to receive the desired outcome.[8] Observe that all individuals are facing the same lottery $\left(1, \frac{1}{n}; 0, \frac{n-1}{n}\right)$.

[8] Such is the lottery conducted by Nestor to determine which of the nine Greek heroes will duel Hector (*Iliad* VII, 171–182).

Similarly, we can define the procedure $N(n,k)$ where the n names are placed in an urn and the k names drawn by an impartial observer are selected for the desired outcome. Unlike $N(n,1)$, this is a multi-stage lottery, where with probability $\frac{1}{n}$, each person is selected in the first round. If not selected, with probability $\frac{1}{n-1}$ he is selected in the second round, and so on until k names have been drawn.[9] Under procedure $N(n,k)$ each individual is thus facing the lottery

$$\left(1, \tfrac{1}{n}; \left(1, \tfrac{1}{n-1}; \left(\ldots \left(1, \tfrac{1}{n-k+1}; 0, \tfrac{n-k}{n-k+1}\right) \ldots\right), \tfrac{n-2}{n-1}\right), \tfrac{n-1}{n}\right) \tag{11}$$

These procedures are depicted in Figure 3.

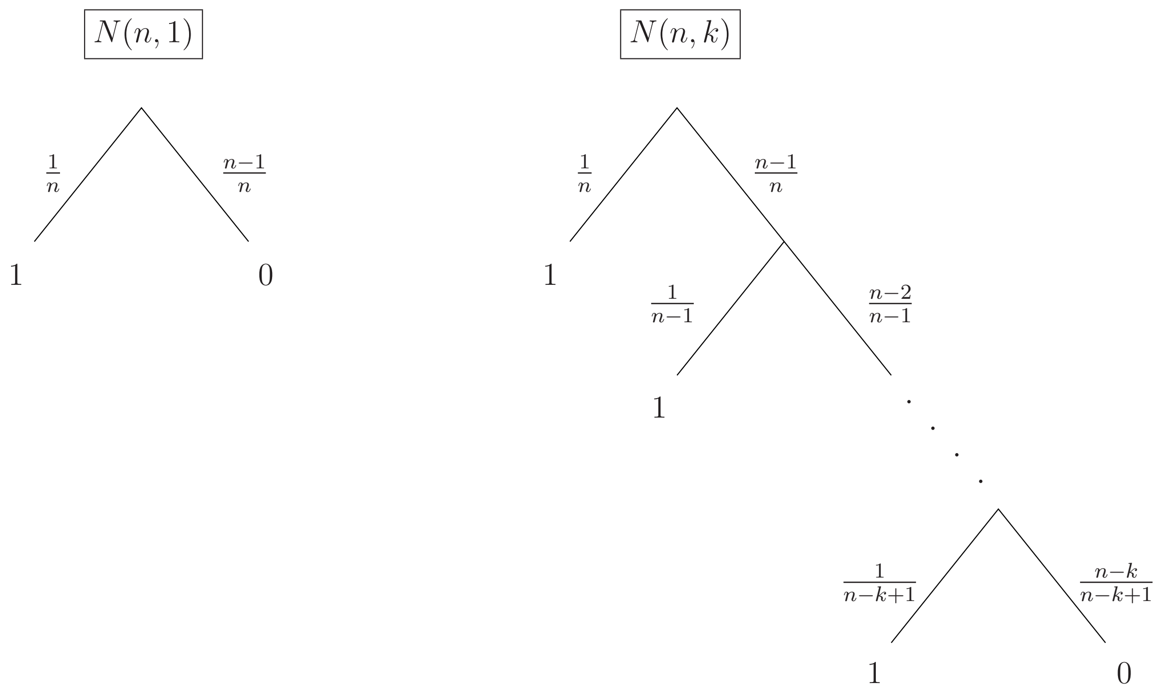

Figure 3: Procedures $N(n,1)$ and $N(n,k)$

We first compare the pre-ordered $P(n,k)$ and the names $N(n,k)$ procedures when exactly one person is going to get a different outcome from the

[9]Another possibility is to prepare $\binom{n}{k}$ slips with all combinations of k people and to pick one at random. This may become cumbersome even for relatively small numbers like $n = 30$ and $k = 10$ where more than 30 million slips are needed. We do not discuss this procedure here.

rest, that is, $k = 1$ or $k = n - 1$. This analysis requires only the weak-PORU assumption. Comparing Figures 1 and 3, it is clear that the names procedure is preferred to the pre-ordered one when one person is selected. Not surprisingly, these preferences are reversed when $n - 1$ persons are selected, as the pre-ordered procedure still induces only two stages of uncertainty, while the names procedure requires $n - 1$ stages.

Claim 8. Assuming weak-PORU, person $1 < i < n$ prefers $N(n, 1)$ to $P^i(n, 1)$ (with indifference for $i = 1, n$) and for all i, $P^i(n, n-1)$ is preferred to $N(n, n-1)$.

The reason $N(n, 1)$ is preferred to $P^i(n, 1)$ for all but the first and last persons to draw is that the former is a single-shot lottery, while the latter requires two steps. But this advantage disappears once two or more persons are selected. While $P^i(n, k)$ remains a two-stage lottery — the first stage determines the composition of the urn when person i's turn arrives and the second stage is the actual lottery played by that person — procedure $N(n, k)$ requires k stages. As we show next, this makes a difference already for the case $k = 2$, that is, when two people are selected.

Consider $P^i(n, 2)$ where $2 < i < n-2$. Initially the urn contains two green balls, but when person i's turn arrives, the number of the green balls still in the urn may be zero, one, or two. He'll therefore face, respectively, the lotteries $(0, 1)$ (that is, zero for sure), $(1, \frac{1}{n-i+1}; 0, \frac{n-i}{n-i+1})$, or $(1, \frac{2}{n-i+1}; 0, \frac{n-i-1}{n-i+1})$. The probability of having no green balls left is the probability that two of the first $i-1$ participants picked green, that is, $\frac{(i-1)(i-2)}{n(n-1)}$. Likewise, the probabilities the one or two of the green balls are left are $\frac{2(i-1)(n-i+1)}{n(n-1)}$ and $\frac{(n-i+1)(n-i)}{n(n-1)}$ (see Figure 4). Obviously, weak-PORU cannot be applied to $P^i(n, 2)$, but the RD model enables comparisons of such lotteries. The next claim shows that while the RD model does not imply that when society needs to choose two members everyone will prefer the names lottery to the pre-ordered one, there are conditions under which the the pre-ordered lottery will be chosen.

Claim 9. Consider an RD decision maker with $f(p) = p^\alpha$, $\alpha > 1$. Then for all i and n, $P^i(n, 2)$ is preferred to $N(n, 2)$. However, this result does not hold for all convex functions f.

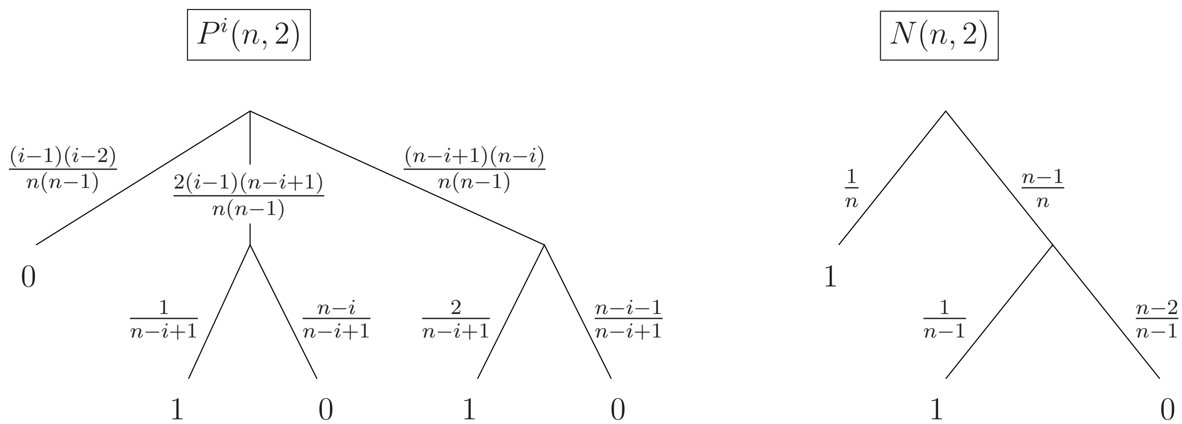

Figure 4: Procedures $P^i(n,2)$ and $N(n,2)$

4.2 Two in a row

In this section we discuss a variant of the names lottery procedure where one of n people is selected. The names of the candidates are written on n balls and are put in an urn from which they are repeatedly drawn (with replacement). The first name to appear twice in a row is selected. Denote this procedure $T(n)$.

Consider this procedure from the point of view of person i. In the first draw, there are two possibilities. A: his name appears, B: the name of somebody else appears. The probabilities of these events are $\frac{1}{n}$ and $\frac{n-1}{n}$, respectively. Outcome A leads to a lottery, where with probability $\frac{1}{n}$ his name is drawn again and he receives the good. With probability $\frac{n-1}{n}$ another name appears, and therefore he receives B. Outcome B leads to another lottery, where with probability $\frac{1}{n}$ his name appears and he receives A, with probability $\frac{1}{n}$ the name that was drawn in the last round appears again, leaving person i with zero (as the other person wins), and with probability $\frac{n-2}{n}$ another name appears, and person i receives outcome B again. See Figure 5.

This procedure terminates with probability 1, and since all participants are in a symmetric position, each has probability $\frac{1}{n}$ to be selected. If preferences satisfy PORU, then obviously $N(n,1) \succ T(n)$. Observe however that

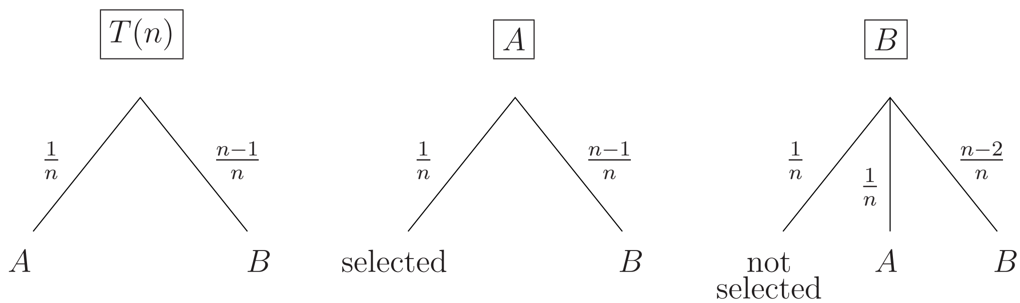

Figure 5: Procedures $T(n)$

since B permits three different outcomes, weak-PORU cannot be applied to $T(n)$. We show next that within the RD model, a different condition than weak-PORU determines preferences over $N(n, 1)$ and $T(n)$.

Claim 10. An RD decision maker with a convex probability transformation function f prefers $N^j(n, 1)$ to $T^j(n)$ for $j \in \{\text{win,lose}\}$, while concave f leads to the opposite preferences.[10]

5 Draft Lotteries

To recruit more soldiers during the Vietnam war, the US Army used a draft lottery. Slips bearing the numbers 1–366 (representing the days of the year) were put in capsules, mixed, and drawn at random to determine the order of the date-of-births (dob) by which new recruits will be drafted. We call this method Procedure A. The lottery took place on December 1, 1969, and the first to be drawn was 258 (September 14), the second 115 (April 24), the last one being 160 (June 8).[11] Evidently, the capsules were not properly mixed (see Fienberg (1971)) and the army looked for an alternative method.

Since it is physically hard to properly mix 366 capsules, a simple solution would be to divide the year into six 61-days segments and then have two lotteries: One will determine the order of the six segments, while the other

[10] Claims 7 and 9 assumed more restrictive conditions. Recall that p^α is convex for $\alpha > 1$ and concave when $\alpha < 1$.

[11] A second lottery took place on the same day to determine the order of potential recruits in each date (Starr (1997, §14)), but as this stage is necessary for all methods discussed in this section we'll ignore it.

will dictate the order of the 61 days (presumably, shuffling 61 capsules is a lot more likely to yield a true mix than doing it with 366 capsule). Call it Procedure B. Suppose the segments are randomly ordered 4–2–5–6–1–3, and the first of the 61 numbers to be drawn is 15. Then the first date to be drafted will be the 198th day of the year (which is the 15th day of the 4th segment), that is, July 16, the second date will be 76 (the 15th day of the 2nd segment), that is, March 16, and so on. This method was not used. Instead, the following Procedure C was employed.

Two drums, each with 366 capsules[12] containing the numbers 1–366, were prepared for the lottery held on 1 July 1970, one representing day of the year, the other the ranks in the draft.[13] Pairs of capsules were drawn simultaneously from the two drums, matching dates of birth with the number of the draft.[14] Surprisingly, this method is quite ancient. According to the biblical story,[15] once they conquered the land of Canaan, the Israeli tribes were to allocate it by a lottery. The execution of this command is described in *Joshua* 18:1–10 with no particulars regarding the actual lottery used. Rabbinical sources[16] describe it as a double lottery with two urns, one containing slips with the names of the tribes, the other slips of the territories. Two young priests stood by them and simultaneously picked pairs of slips from the two urns, thus matching tribes with their territories.[17] A similar method was used in England until 1826 — a pictorial reproduction of this method can be found in Raven (2016, Fig. 5.1).

The three draft lotteries A–C depicted above produce the same probability distribution over matching dates of birth and the order of the draft — each dob has probability $1/366$ of being number i to be drafted, $i = 1, \ldots, 366$. But they have different structures. Lottery A is a 366-stage lottery, where at

[12]In fact there were only 365 capsules, but for the consistency of our analysis we modified it to deal with leap years.

[13]The capsules were put in the drums in a random order, fully described in Rosenblatt and Filliben (1971).

[14]This mechanism effectively randomizes the order at which subjects are playing the "real" lottery. Preferences for such randomizations are expressed by Mr. Belmont in Graham Greene's *Dr. Fischer of Geneva, or the Bomb Party.*

[15]*Numbers* 26:55–56 and 33:54.

[16]*Talmud Yerushalmi*, Yoma 4:1 (41b) and *Talmud Bavli*, Bava Batra 122a.

[17]We found no hint that this source was known to those who designed the draft lottery, nor that it was noted in the literature since. Fienberg (1971), in his discussion of the Vietnam draft lotteries, mentions the biblical lottery, but not the rabbinical extended story of the double barrels.

stage i each person whose dob has not yet been selected is facing the lottery "draft" with probability $\frac{1}{367-i}$ and "continue to stage $i+1$" with probability $\frac{366-i}{367-i}$. Lottery B is a 62-stage lottery, where the first stage determines the order of the six groups, and the remaining 61 steps are similar to the stages of procedure A. Lottery C is also a 366-stage lottery. At stage i each person whose dob is still in the urn is facing a lottery where with probability $1/(367-i)^2$ his dob is selected and matched with each of the $367-i$ ranking numbers still in the urn, and with probability $\frac{366-i}{367-i}$ he continues to the next stage. Since these lotteries involve more than two outcomes they are not covered by the claims of our paper. But the fact that society is not indifferent between them (procedure A was considered in retrospect inferior to C and C was probably better than B) supports our main claim: The fact that two social lotteries induce the same distribution over the allocation of the social outcomes does not mean that participants are indifferent between them, and consequentially, nor should society be indifferent to the choice of the random mechanism.

6 Concluding remarks

A fundamental requirement of fairness is equal treatment of equals. When random selection from equals is needed, this principal requires giving each candidate an equal chance (see Taurek (1977) and Broome (1984,1998)). But having a mechanism which is considered fair by the social planner may not be enough. Members of society, too, should consider it fair and that it treats all of them in an equal way, at least to the extent that no one will believe that someone else was treated better than him.[18] The fact that all probabilities are equal does not imply that all individuals will consider the mechanisms creating them as such and some mechanisms will be considered to create more equal treatments than others. This argument is supported by the experimental results of Eliaz and Rubinstein (2014). For example, they asked subjects the following question.

> One prize is to be awarded to one person from among 20 candidates. Compare the fairness (from the point of view of the

[18] We ignore here people preferences for equality (see e.g. Karni and Safra (2002)) and consider only their selfish preferences not to feel discriminated.

candidates) of the following procedures for selecting who will get the prize.

(A) A computer program repeatedly draws a name at random, and the prize is awarded to the first person whose name is drawn twice.

(B) A computer program draws one of the names at random and that person is awarded the prize.

Almost half of their subjects were not indifferent between (A) and (B).[19] The findings of Eliaz and Rubinstein are indeed not conclusive — 21% preferred (A) to (B), 23% preferred (B) to (A), while 56% were indifferent.

The argument that having preferences over probabilistically-equivalent procedures is irrational and that violations of probability theory may expose decision makers to Dutch books (see de Finetti (1937) and Yaari (1985)) is irrelevant. There is no point in using a "fair" mechanism unless it is deemed fair by those who should bear its consequences. And if adding a green ball to the urn, or having a names lottery rather than sequential draws, will make people feel that the procedure is more fair, then so be it.

It is clear from our discussion that there is no one method that is obviously better than all other methods. For example, names lotteries are better than the pre-ordered procedure when one person is to be selected, but not necessarily when society wants to select two (or more) people. Some people like, and some do not like possible manipulations of the procedure (e.g., by adding winning balls). This is in agreement with the scant experimental evidence and with the anecdotal literary evidence we mentioned. Nor do we claim to exhaust the list of possible procedures. For example, one can think of combinations of mechanisms where one method is used to create a short list out of which the same or another mechanism will select some people. Our main point is however established — all mechanisms are not the same.

[19] Unlike the lotteries of Figures 1 and 3, the compound lottery obtained from procedure (A) is not a simple sequence of win/lose probabilities and therefore weak-PORU cannot be applied. Also, although this procedure terminates after at most $n+1$ draws, the obtained multi-stage lottery lacks the elegant recursive form of the "two in a row" procedure (see section 4.2).

Appendix: Proofs

Proof of Lemma 1: By Segal (1987, Lemma 4.1), if the elasticity of h is increasing, then $h(p)h(q) \leqslant h(pq)$ and if it is decreasing, then $h(p)h(q) \geqslant h(pq)$.[20]

Assume $u(y) = 0$. For $x > y$, $u(x) > 0$ and the RD value of $(x, pq;\, y, 1-pq)$ is $u(x)f(pq)$ while the value of $((x, p;\, y, 1-p), q;\, y, 1-q)$ is $u(x)f(p)f(q)$. For $x < y$, $u(x) < 0$ and the corresponding values of the above lotteries are $u(x)g(pq)$ and $u(x)g(p)g(q)$. ■

Proof of Lemma 2: $h_1(p) = \frac{p+p^2}{2}$ is convex with increasing elasticity, $h_2(p) = \frac{p+\sqrt{p}}{2}$ is concave with increasing elasticity, $h_3(p) = 2p^3$ for $p \leqslant \frac{1}{2}$ and $h_3(p) = \frac{3}{2}p - \frac{1}{2}$ for $p > \frac{1}{2}$ is convex with decreasing elasticity, and $h_4(p) = \sin(\frac{\pi p}{2})$ is concave with decreasing elasticity.

Suppose that the elasticity of h is increasing. We show first that under the lemma's assumptions, this implies that $\eta_h \geqslant 1$. Since h is increasing, its first non-zero derivative at 0 must be positive. Let $f^{(n+1)}(0) = d > 0 = f^{(n)}(0) = \ldots = f'(0)$, where $d < \infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{p\to 0} \frac{ph'(p)}{h(0)} &= 1 + \lim_{p\to 0} \frac{ph''(p)}{h'(p)} = \ldots = n + \lim_{p\to 0} \frac{ph^{(n+1)}(p)}{h^{(n)}(p)} \\ &= n + d \cdot \lim_{p\to 0} \frac{p}{h^{(n)}(p)} = n + 1 \geqslant 1 \end{aligned} \tag{12}$$

Then

$$\begin{aligned} \eta_h' \geqslant 0 \quad &\Longleftrightarrow \quad h'h + ph''h - p[h']^2 \geqslant 0 \\ &\Longleftrightarrow \quad h'h[1 - \tfrac{ph'}{h}] \geqslant -ph''h \end{aligned}$$

By eq. (12), the LHS of the last inequality is negative, hence $h'' > 0$.

The first non-zero derivative at zero of the above function h_3 is finite, the function is convex, yet its elasticity is decreasing. ■

Proof of Claim 2: We prove the claim for f. Differentiate (7) with respect

[20] The statement of Lemma 4.1 in (1987) is wrong (it suggests an iff result). The proof there only proves the above statement. Observe that we do not make assumptions regarding the convexity or concavity of h.

to i to obtain

$$\frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right) - \frac{1}{n} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right) \leqslant 0 \iff$$

$$\frac{\frac{1}{n-i+1} f'\left(\frac{1}{n-i+1}\right)}{f\left(\frac{1}{n-i+1}\right)} \leqslant \frac{\frac{n-i+1}{n} f'\left(\frac{n-i+1}{n}\right)}{f\left(\frac{n-i+1}{n}\right)}$$

Clearly $\frac{1}{n-i+1} \leqslant \frac{n-i+1}{n}$ iff $i \leqslant i^*$. Assuming increasing elasticity of f, the last inequality therefore holds iff $i \leqslant i^*$. The proof for g is similar. ■

Proof of Claim 3: For $i \leqslant n-1$, the first $i-1$ stages of $Q^i(n,1)$ and $Q^{i+1}(n,1)$ are the same. The last two stages of $Q^{i+1}(n,1)$ are

$$((1, \tfrac{1}{n-i}; 0, \tfrac{n-i-1}{n-i}), \tfrac{n-i}{n-i+1}; 0, \tfrac{1}{n-i+1})$$

By weak-PORU it is inferior to $(1, \frac{1}{n-i+1}; 0, \frac{n-i}{n-i+1})$, which is the last stage of $Q^i(n,1)$. The proof of $Q(n, n-1)$ is similar. ■

Proof of Claim 4: We prove the claim for $P(n+1,1)$. For all $i \geqslant i_0$, person i in $P(n,1)$ becomes person $i+1$ in $P(n+1,1)$. Using eq. (5), it follows by first-order stochastic dominance that for person $i \geqslant i_0$

$$\begin{aligned} P^i(n,1) &= \left(\left(1, \tfrac{1}{n-i+1}; 0, \tfrac{n-i}{n-i+1}\right), \tfrac{n-i+1}{n};\, 0, \tfrac{i-1}{n}\right) \\ &\succ \left(\left(1, \tfrac{1}{n-i+1}; 0, \tfrac{n-i}{n-i+1}\right), \tfrac{n-i+1}{n+1};\, 0, \tfrac{i}{n+1}\right) = P^{i+1}(n+1,1) \end{aligned}$$

Also, $P^1(n,1) = (1, \frac{1}{n}; 0, \frac{n-1}{n}) \succ (1, \frac{1}{n+1}; 0, \frac{n}{n+1}) = P^1(n+1,1)$. The proof for $P(n+1,n)$ is similar. ■

Proof of Claim 5: Let $1 < i < i_0$ and recall that the RD value of $P^i(n,1)$ is given by eq. (7). Its derivative with respect to n is

$$-\frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right) + \frac{i-1}{n^2} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right)$$

which is non-positive iff

$$\frac{i-1}{n^2} f'\left(\frac{n-i+1}{n}\right) f\left(\frac{1}{n-i+1}\right) \leqslant \frac{1}{(n-i+1)^2} f\left(\frac{n-i+1}{n}\right) f'\left(\frac{1}{n-i+1}\right) \iff$$

$$\frac{i-1}{n} \times \frac{\frac{n-i+1}{n} f'\left(\frac{n-i+1}{n}\right)}{f\left(\frac{n-i+1}{n}\right)} \leqslant \frac{\frac{1}{n-i+1} f'\left(\frac{1}{n-i+1}\right)}{f\left(\frac{1}{n-i+1}\right)} \iff$$

$$\frac{i-1}{n} \leqslant \frac{\eta^f\left(\frac{1}{n-i+1}\right)}{\eta^f\left(\frac{n-i+1}{n}\right)}$$

The proof for procedure $P(n+1,n)$ is similar, starting with eq. (8). ■

Proof of Claim 6: Suppose that $P^i(n,n-1) \succeq P^i(n+1,n)$. Observe that the first $i-1$ steps of the $P^i(n+1,n)$ and $W^i(n,n-1)$ procedures are equivalent (see Figure 6 for a graphical comparison of the two procedures.) With probability $\frac{i-1}{n+1}$ one of the first $i-1$ participants will pick the red ball and the procedure will terminate with person i receiving the desired outcome. In the second stage, in both procedures, person i draws a red ball with probability $\frac{1}{n-i+2}$, he wins 0, and the procedure is over. The difference between the two procedures arises when person i draws a green ball. In the P procedure, he is selected and the procedure is over for him. In the W procedure, he is selected when someone after him draws the red ball, otherwise the procedure must be repeated. Since the continuation value of the W procedure is less than the value of being selected, it follows that $P^i(n+1,n) \succeq W^i(n,n-1)$. Therefore $P^i(n,n-1) \succeq W^i(n,n-1)$.

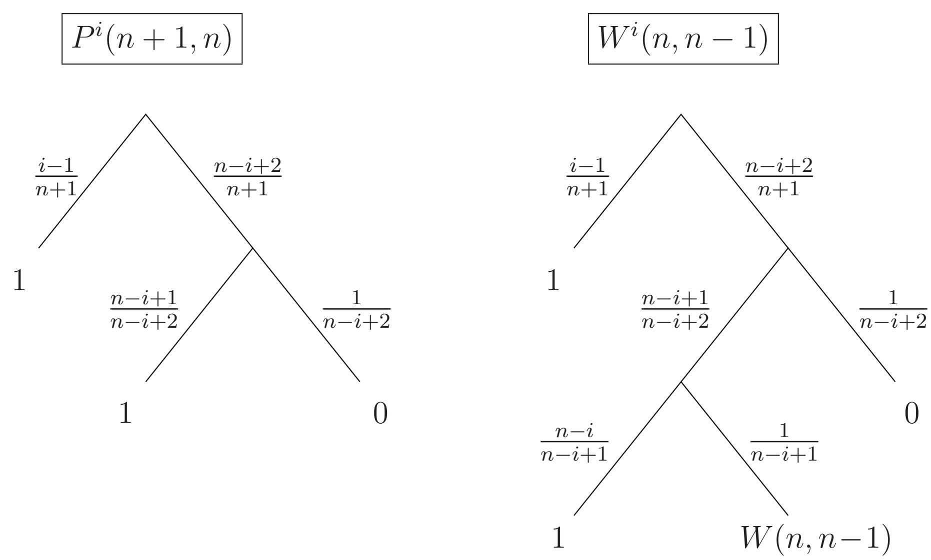

Figure 6: Procedures $P^i(n+1,n)$ and $W^i(n,n-1)$ for person i

Suppose now that $P^i(n+1,n) \succ P^i(n,n-1)$. We show that it is possible to have both relations between $W^i(n,n-1)$ and $P^i(n,n-1)$.

1. $W^i(n,n-1) \succ P^i(n,n-1)$: Consider RD preferences. let $i^* = n+1-\sqrt{n}$, $s = f\left(\frac{n-i}{n-i+1}\right) \leqslant \frac{n-i}{n-i+1}$, $r = f\left(\frac{i-1}{n}\right) = \max\{0, s + \left(\frac{i^2-2(n+1)i+(n^2+n+1)}{n}\right)(s -$

$1)\}$, and let f be the piecewise linear function

$$f(p) = \begin{cases} \frac{nrp}{i-1} & p \leqslant \frac{i-1}{n} \\ \frac{n(1-r)p}{n-i+1} + \frac{nr-i+1}{n-i+1} & p > \frac{i-1}{n} \end{cases}$$

The function f is convex, increasing in n, and is such that individual $i \leqslant i^*$ prefers $W^i(n, n-1)$ to $P^i(n, n-1)$. By the first part of the claim it follows that $P^i(n+1, n) \succ P^i(n, n-1)$.

2. $P^i(n, n-1) \succ W^i(n, n-1)$: Consider again RD preferences with $f(p) = \frac{p^2+p}{2}$. For $n = 5$ and $i = 2$ we get $\mathrm{RD}(P^2(6,5)) = 0.7472 > \mathrm{RD}(P^2(5,4)) = 0.6975 > \mathrm{RD}(W^2(5,4)) = 0.6745$. ■

Proof of Claim 7: In his turn, person 1 picks a green ball with probability $\frac{2}{n+1}$ and red with probability $\frac{n-1}{n+1}$. In the former case, he wins 1 if no one else picks a green ball (the probability of this event is $\frac{1}{n}$), or the procedure starts over again. In the latter case he receives 0 if only one green ball is picked by the rest (the probability of this event is $\frac{2}{n}$), otherwise the procedure is repeated (see Figure 7).

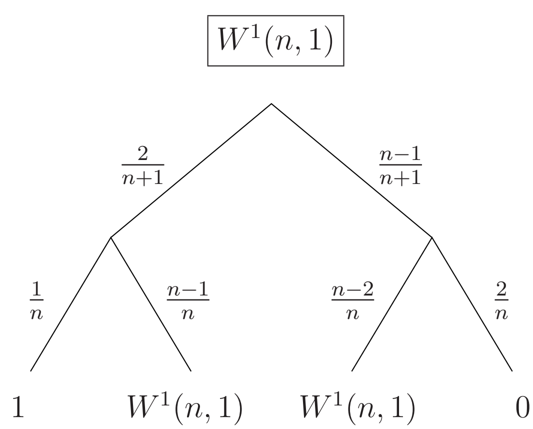

Figure 7: Procedure $W^1(n,1)$ for person 1

By first-order stochastic dominance the left branch is better than the right one. Denote $s = \mathrm{RD}(W^1(n,1))$, hence

$$\begin{aligned} s &= \left(f(\tfrac{1}{n}) + s[1-f(\tfrac{1}{n})]\right) f(\tfrac{2}{n+1}) + s f(\tfrac{n-2}{n})[1-f(\tfrac{2}{n+1})] \Longrightarrow \\ s &= \frac{f(\frac{2}{n+1}) f(\frac{1}{n})}{1-f(\frac{2}{n+1}) + f(\frac{2}{n+1})f(\frac{1}{n}) - f(\frac{n-2}{n}) + f(\frac{2}{n+1})f(\frac{n-2}{n})} \end{aligned}$$

Recall that $\mathrm{RD}(P^1(n,1)) = f(\frac{1}{n})$ and substitute $f(p) = p^\alpha$ to obtain that $W^1(n,1) \succeq P^1(n,1)$ iff

$$\begin{aligned} &\left(\tfrac{2}{n+1}\right)^\alpha \geqslant 1 - \left(\tfrac{2}{n+1}\right)^\alpha + \left(\tfrac{2}{n^2+n}\right)^\alpha - \left(\tfrac{n-2}{n}\right)^\alpha + \left(\tfrac{2n-4}{n^2+n}\right)^\alpha \iff \\ &2^\alpha(n^\alpha - 1) \geqslant [n^\alpha - (n-2)^\alpha][(n+1)^\alpha - 2^\alpha] \end{aligned} \tag{13}$$

For $\alpha < 1$ the function n^α is concave, therefore for $n = 2$, $2^\alpha = n^\alpha - (n-2)^\alpha$ and $n^\alpha - 1 > (n+1)^\alpha - 2^\alpha$. For $\alpha < 1$, the derivatives of $2^\alpha - [n^\alpha - (n-2)^\alpha]$ and $[n^\alpha - 1] - [(n+1)^\alpha - 2^\alpha]$ wrt n are positive, and inequality (13) is satisfied. The proof for the case $\alpha > 1$ is similar. ■

Proof of Claim 8: Since $N(n,1) = P^1(n,1) = P^n(n,1) \succeq P^i(n,1)$, the first claim follows immediately by Claim 1.

Procedure $P^i(n, n-1)$ yields person i the lottery $\left(\left(0, \frac{1}{n-i+1}; 1, \frac{n-i}{n-i+1}\right), \frac{n-i+1}{n}; 1, \frac{i-1}{n}\right)$ (see eq. (6)). By weak-PORU,

$$\begin{aligned} &\left(\left(0, \tfrac{1}{n-i+1}; 1, \tfrac{n-i}{n-i+1}\right), \tfrac{n-i+1}{n}; 1, \tfrac{i-1}{n}\right) = \\ &\left(\left(0, \prod_{j=i}^{n-1} \frac{n-j}{n-j+1}; 1, 1 - \prod_{j=i}^{n-1} \frac{n-j}{n-j+1}\right), \prod_{j=1}^{i-1} \frac{n-j}{n-j+1}; 1, 1 - \prod_{j=1}^{i-1} \frac{n-j}{n-j+1}\right) \succeq \\ &\left(1, \tfrac{1}{n}; \left(1, \tfrac{1}{n-1}; \left(\ldots \left(1, \tfrac{1}{2}; 0, \tfrac{1}{2}\right) \ldots\right), \tfrac{n-2}{n-1}\right), \tfrac{n-1}{n}\right) \end{aligned}$$

Which is the lottery obtained from procedure $N(n, n-1)$. ■

Proof of Claim 9: Let $A = \mathrm{RD}(N(n,2))$ and $B = \mathrm{RD}(P^i(n,2))$. Then

$$A = f\left(\tfrac{1}{n}\right) + f\left(\tfrac{1}{n-1}\right)\left[1 - f\left(\tfrac{1}{n}\right)\right]$$

and by fnt. 2, B equals

$$f\left(\tfrac{2}{n-i+1}\right) f\left(\tfrac{(n-i+1)(n-i)}{n(n-1)}\right) + f\left(\tfrac{1}{n-i+1}\right)\left[f\left(1 - \tfrac{(i-1)(i-2)}{n(n-1)}\right) - f\left(\tfrac{(n-i+1)(n-i)}{n(n-1)}\right)\right]$$

For $f(p) = p^\alpha$ we get that $B - A$ equals

$$\frac{(2^\alpha-1)(n-i)^\alpha}{n^\alpha(n-1)^\alpha} + \frac{1}{(n-i+1)^\alpha}\left[1 - \frac{(i-1)(i-2)}{n(n-1)}\right]^\alpha - \frac{1}{n^\alpha} - \frac{1}{(n-1)^\alpha} + \frac{1}{n^\alpha(n-1)^\alpha} > 0 \iff$$
$$(2^\alpha - 1)(n-i)^\alpha + (n+i-2)^\alpha - (n-1)^\alpha - n^\alpha + 1 > 0$$

(Observe that $(n-i+1)(n+i-2) = n(n-1) - (i-1)(i-2)$). The last inequality holds since $\alpha > 1$, $2^\alpha - 1 > 1$, $1 < n-i < n-1 < n < n+i-2$, and p^α is a convex function.

To show that the claim does not hold for all convex functions f, let

$$f(p) = \begin{cases} 0.1p & p \leqslant 0.75 \\ 3.7p - 2.7 & p > 0.75 \end{cases}$$

For $n = 10$ and $i = 5$ we get $A = 0.021 > B = 0.009$. ■

Proof of Claim 10: The analysis depends on whether the selected person receives a desired or an undesired outcome.

(i) The selected person wins a desired outcome. By FOSD, $A \succ B$. By eqs. (3) and (2), $\mathrm{RD}(T) = f(\frac{1}{n})u(A) + [1 - f(\frac{1}{n})]u(B)$, $u(A) = f(\frac{1}{n}) + [1 - f(\frac{1}{n})]u(B)$, and by fnt. 2

$$\begin{aligned} u(B) &= f(\tfrac{1}{n})u(A) + [f(\tfrac{n-1}{n}) - f(\tfrac{1}{n})]u(B) \\ &= [f(\tfrac{1}{n})]^2 + [f(\tfrac{n-1}{n}) - [f(\tfrac{1}{n})]^2]u(B) \Longrightarrow \\ u(B) &= \frac{[f(\frac{1}{n})]^2}{1 - f(\frac{n-1}{n}) + [f(\frac{1}{n})]^2} \end{aligned}$$

Substituting into the values of A and T we get

$$\begin{aligned} u(A) &= f(\tfrac{1}{n}) + \frac{[1 - f(\frac{1}{n})] \times [f(\frac{1}{n})]^2}{1 - f(\frac{n-1}{n}) + [f(\frac{1}{n})]^2} \\ \mathrm{RD}(T) &= [f(\tfrac{1}{n})]^2 + \frac{[f(\frac{1}{n}) + 1] \times [1 - f(\frac{1}{n})] \times [f(\frac{1}{n})]^2}{1 - f(\frac{n-1}{n}) + [f(\frac{1}{n})]^2} \\ &= \frac{[f(\frac{1}{n})]^2[2 - f(\frac{n-1}{n})]}{1 - f(\frac{n-1}{n}) + [f(\frac{1}{n})]^2} \end{aligned}$$

Recall that $\mathrm{RD}(N(n,1)) = f(\frac{1}{n})$, hence $\mathrm{RD}(T) \leqslant \mathrm{RD}(N(n,1))$ iff

$$\begin{aligned}
&\frac{f(\frac{1}{n})[2 - f(\frac{n-1}{n})]}{1 - f(\frac{n-1}{n}) + [f(\frac{1}{n})]^2} \leqslant 1 \iff \\
&f(\tfrac{1}{n})[1 - f(\tfrac{1}{n})] + f(\tfrac{1}{n})[1 - f(\tfrac{n-1}{n})] \leqslant 1 - f(\tfrac{n-1}{n}) \iff \\
&f(\tfrac{1}{n}) + f(\tfrac{n-1}{n}) \leqslant 1
\end{aligned}$$

which is the case whenever f is convex.

(ii) The selected person wins an undesired outcome. By FOSD, $B \succ A$. By eqs. (3) and (4), $\mathrm{RD}(T) = f(\frac{n-1}{n})u(B) + [1 - f(\frac{n-1}{n})]u(A)$, $u(A) = f(\frac{n-1}{n})u(B)$, and

$$\begin{aligned}
u(B) &= f(\tfrac{1}{n}) + [f(\tfrac{n-1}{n}) - f(\tfrac{1}{n})]u(B) + [1 - f(\tfrac{n-1}{n})]u(A) \\
&= f(\tfrac{1}{n}) + [f(\tfrac{n-1}{n}) - f(\tfrac{1}{n})]u(B) + [1 - f(\tfrac{n-1}{n})]f(\tfrac{n-1}{n})u(B) \Longrightarrow \\
u(B) &= \frac{f(\frac{1}{n})}{1 - 2f(\frac{n-1}{n}) + f(\frac{1}{n}) + [f(\frac{n-1}{n})]^2}
\end{aligned}$$

Substituting into the values of A and T we get

$$\begin{aligned}
u(A) &= \frac{f(\frac{1}{n})f(\frac{n-1}{n})}{1 - 2f(\frac{n-1}{n}) + f(\frac{1}{n}) + [f(\frac{n-1}{n})]^2} \\
\mathrm{RD}(T) &= \frac{f(\frac{1}{n})f(\frac{n-1}{n}) + f(\frac{1}{n})f(\frac{n-1}{n})[1 - f(\frac{n-1}{n})]}{1 - 2f(\frac{n-1}{n}) + f(\frac{1}{n}) + [f(\frac{n-1}{n})]^2} \\
&= \frac{f(\frac{1}{n})f(\frac{n-1}{n})[2 - f(\frac{n-1}{n})]}{1 - 2f(\frac{n-1}{n}) + f(\frac{1}{n}) + [f(\frac{n-1}{n})]^2}
\end{aligned}$$

Recall that now $\mathrm{RD}(N(n,1)) = f(\frac{n-1}{n})$ (the selected person wins the undesired outcome), hence $\mathrm{RD}(T) \leqslant \mathrm{RD}(N(n,1))$ iff

$$\begin{aligned}
&\frac{f(\frac{1}{n})[2 - f(\frac{n-1}{n})]}{1 - 2f(\frac{n-1}{n}) + f(\frac{1}{n}) + [f(\frac{n-1}{n})]^2} \leqslant 1 \iff \\
&2f(\tfrac{1}{n}) - f(\tfrac{1}{n})f(\tfrac{n-1}{n}) \leqslant 1 - 2f(\tfrac{n-1}{n}) + f(\tfrac{1}{n}) + [f(\tfrac{n-1}{n})]^2 \iff \\
&[f(\tfrac{1}{n}) + f(\tfrac{n-1}{n})][1 - f(\tfrac{n-1}{n})] \leqslant 1 - f(\tfrac{n-1}{n})
\end{aligned}$$

Which, again, is the case whenever f is convex. ■

References

[1] Abdellaoui, M., P. Klibanoff, and L. Placido, 2015. "Experiments on compound risk in relation to simple risk and to ambiguity," *Management Science*, 61:1306–1322. https://doi.org/10.1287/mnsc.2014.1953

[2] Allais, M., 1953. "Le comportement de l'homme rationnel devant le risque: Critique des postulates et axiomes de l'ecole Americaine," *Econometrica* 21:503–546. https://doi.org/1907921

[3] Bernasconi, M. and J. Bernhofe, 2020. "Catch me if you can: Testing the reduction of compound lotteries axiom in a tax compliance experiment," *Journal of Behavioral and Experimental Economics*, 84:1–14. https://doi.org/10.1016/j.socec.2019.101479

[4] Broome, J., 1984. "Selecting people randomly," *Ethics*, 95:38-55. https://doi.org/10.1086/292596

[5] Broome, J., 1998, "Kamm on fairness," *Philosophy and Phenomenological Research*, 58:955-961. https://doi.org/2653739

[6] Chew, S.H., E. Karni, and Z. Safra, 1987. "Risk aversion in the theory of expected utility with rank-dependent probabilities," *Journal of Economic Theory* 42:370–381. ttps://doi.org/10.1016/0022-0531(87)90093-7

[7] de Finetti, B. 1937. "La Prevision: Ses Lois Logiques, ses Sources Subjectives," *Annales del'Institut Henri Poincaré*, 7:1–68. Translated into English by Kyburg, H. E. Jr., "Foresight: Its Logical Laws, its Subjective Sources," in Kyburg, H. E. Jr. and Smokler, H. E., eds. (1964), Studies in Subjective Probability, Wiley: New York.

[8] Diamond, P.A., 1967. Cardinal welfare, individualistic ethics, and interpersonal comparison of utility. *Journal of Political Economy*, 75:765–766.

[9] Dillenberger, D., 2010. "Preferences for one-shot resolution of uncertainty and Allais-type behavior," *Econometrica* 78:1973–2004. https://doi.org/10.3982/ECTA8219

[10] Dillenberger, D. and U. Segal, 2021. "Allocation mechanisms without reduction," *American Economic Review: Insights*, 3:455–69. 10.1257/aeri.20200728

[11] Eliaz, K. and A. Rubinstein, 2014. "On the fairness of random procedures," *Economics Letters*, 123:168–170. https://doi.org/10.1016/j.econlet.2014.01.042

[12] Epstein, L.G. and U. Segal, 1992. "Quadratic social welfare functions," *Journal of Political Economy*, 100:691–712. https://doi.org/10.1086/261836

[13] Fienberg, S.E., 1971. "Randomization and social affairs: The 1970 draft lottery," *Science* 171:255–161. 10.1126/science.171.3968.255

[14] Ghirardato, P. and M. Marinacci, 2001. "Risk, ambiguity, and the separation of utility and beliefs," *Mathematics of Operations Research*, 26:864–890. https://doi.org/10.1287/moor.26.4.864.10002

[15] Gul, F., 1991. "A theory of disappointment aversion," *Econometrica* 59:667–686. https://doi.org/2938223

[16] Halevy, Y., 2007. "Ellsberg revisited: An experimental study," *Econometrica*, 75:503–536. https://doi.org/10.1111/j.1468-0262.2006.00755.x

[17] Harsanyi, J.C., 1955. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of Political Economy*, 63:309–321. https://doi.org/10.1086/257678

[18] Harsanyi, J.C., 1975. "Nonlinear social welfare function," *Theory & Decision* 6:311–332. https://doi.org/10.1007/BF00136200

[19] Harrison, G.W., J. Martínez-Correa, and J.T. Swarthout, 2015. "Reduction of compound lotteries with objective probabilities: Theory and evidence," *Journal of Economic Behavior & Organization*, 119:32–55. https://doi.org/10.1016/j.jebo.2015.07.012

[20] Karni, E. and Z. Safra, 2002: "Individual sense of justice: A utility representation," *Econometrica* 70:263–284. https://doi.org/10.1111/1468-0262.00275

[21] Klibanoff, P., M. Marinacci, and S. Mukerji, 2005. "A smooth model of decision making under ambiguity," *Econometrica*, 73:1849–1892. https://doi.org/10.1111/j.1468-0262.2005.00640.x

[22] Kreps, D.M. and E.L. Porteus, 1978. “Temporal resolution of uncertainty and dynamic choice theory,” *Econometrica* 46:185–200. https://doi.org/1913656

[23] MacCrimmon, K.R., and S. Larsson, 1979. “Utility Theory: Axioms Versus ‘Paradoxes’,” in M. Allais and O. Hagen, Eds.: *Expected Utility Hypotheses and the Allais Paradox*. Dordrecht: D. Reidel. https://doi.org/10.1007/978-94-015-7629-1_15

[24] Masatlioglu, Y., Y. Orhun, and C. Raymond, 2017. “Skewness and intrinsic preferences for information,” Mimeo. http://dx.doi.org/10.2139/ssrn.3232350

[25] Quiggin, J., 1982. “A theory of anticipated utility,” *Journal of Economic Behavior & Organization* 3:323–343. https://doi.org/10.1016/0167-2681(82)90008-7

[26] Raven, 2016. “Debating the lottery in Britain c. 1750–1830,” in M. Zollinger, ed.: *Random Riches*. New York: Routledge.

[27] Rosenblatt, J.R. and J.J. Filliben, 1971. “Randomization and the draft lottery,” *Science* 171:306–308. 10.1126/science.171.3968.306

[28] Segal, U., 1987. “The Ellsberg paradox and risk aversion: an anticipated utility approach,” *International Economic Review* 28:175–202. https://doi.org/2526866

[29] Segal, U., 1987a. “Some remarks on Quiggin’s anticipated utility,” *Journal of Economic Behavior & Organization* 8:145–154. https://doi.org/10.1016/0167-2681(87)90027-8

[30] Segal, U., 1990. “Two stage lotteries without the reduction axiom,” *Econometrica* 58:349–377. https://doi.org/2938207

[31] Starmer, C., 2000. “Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk,” *Journal of Economic Literature* 38:332–382. 10.1257/jel.38.2.332

[32] Starr, N., 1997. “Nonrandom risk: The 1970 draft lottery,” *Journal of Statistics Education* 5:2.

[33] Taurek, J., 1977. “Should the numbers count?,” *Philosophy & Public Affairs* 6:293–316.

[34] Yaari, M.E., 1985. “On the role of ‘dutch books’ in the theory of choice under risk,” Nancy Schwartz Memorial Lectures.