When (not) to Persuade Consumers: Persuasive and Demarkting Information Designs

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Abstract

We analyze the boundaries of Bayesian persuasion for firms that provide information to consumers about uncertain product match, and can engineer the information structure to be persuasive. Monopolists that cannot alter the prices they charge can utilize persuasion to increase their profits, as usual. However, when prices are endogenous or when competition is rampant, firms opt to lower their persuasive claims and instead choose to truthfully reveal the product's match. In cases where competition is strengthened even further, firms will choose to demarket their products in order to soften competition. A platform that designs the information displayed for sellers on the platform faces similar incentives, but will never choose to persuade consumers with endogenous prices as it will lower the total revenue it can receive.

Keywords: Bayesian persuasion, information design, platform design, demarketing.

1 Introduction

Persuasion is often used by marketers to try and convince customers to purchase products. One way that firms can persuade is by designing the information environment to influence the decision making of potential consumers. For example, Amazon & Airbnb list product attributes in a descending order based on the attribute's average rating when presenting the product; car dealers often encourage consumers to take a fully-loaded car for a test drive so that consumers have a better experience with the car, and pharmaceutical companies can design a drug commercial to highlight the strengths and weaknesses of the drug.

From the firm's perspective, while a persuasive information environment can help influence the consumers in the aggregate, the firm has no control on each individual consumer's response to the persuasive information. For example, firms can choose to present the top rated product feature first hoping consumers would care, but the firm does not know whether an individual consumer would value these specific features. Similarly creating a drug commercial that focuses on one aspect of the drug does not guarantee unanimous acceptance among consumers. This implies that the firm cannot manipulate the response of each individual consumer by targeting them with personalized information. We also expect that consumers, who anticipate the firm's persuasion incentives will rationally discount the level of persuasion. As a result, the firm faces an information design trade-off between displaying more persuasive information that might increase consumer demand even when products do not necessarily match consumer tastes, and risk losing demand due to consumers' rational discounting of information. In fact, it might be possible that firms will be better off by not persuading consumers, or even choosing a demarketing strategy that would claim the firm's product are not as good as they are.

How, then, should a firm respond to consumers' rational expectations when deciding the level of persuasion? In this paper we analyze the information design decisions of firms when they sell products with uncertain match utility to consumers and do not have private information that allows them to target signals individually. We ask whether it is always profitable for firms to try and persuade consumers to buy their products, and if it is, then what level of ex ante persuasion they should choose. Importantly, we look at two moderating factors of persuasion which are pricing power and competition. We ask whether these factors encourage or discourage persuasion, and also how they affect the profitability and pricing levels in the market. Additionally, we also analyze the information design incentives of shopping platforms that design the information presented by sellers to consumers while letting sellers set their own prices.

When providing more information to consumers, firms need to decide how revealing this information is about the true match utility of the product. The most revealing information design would be to give the product to the consumers and let them try it for long enough. The least revealing is to not provide any information or to provide homogeneous information, i.e., to say that the product will always be a great match, even though it sometimes might not be. The latter approach might seem beneficial at first, but we expect rational consumers to not be persuaded by such an uninformative signal. Within these two extremes, the literature on Bayesian persuasion, starting with Kamenica and Gentzkow (2011), has shown that there are credible information designs that can simultaneously be not fully revealing as well as increase the benefits to the firm beyond the fully revealing design. This result hinges on the assumption that increasing demand through persuasion is always beneficial for the firm, but as we show, this is not always the case. Persuasion often comes at the expense of lowering consumer surplus and willingness to pay, which has potential effects on the profit a firm can achieve as well as its ability to respond to competition.

This trade-off, the fundamental aspect of our research, implies that persuasion may not contribute to firms' success. Our findings show that pricing and competition incentives dramatically alter the incentives to persuade consumers. In some cases, we even find that sellers and platforms would like to "demarket" a product in equilibrium (Miklós-Thal and Zhang 2013, Kim and Shin 2016, Harbaugh and To 2020), instead of persuading consumers to purchase.

We start by analyzing a monopolist who sells a product of uncertain quality (or match) to consumers in Section 2. We compare the optimal information design when prices are exogenous vs endogenous, and find that persuasion only arises in equilibrium when prices are exogenous. With endogenous prices, increasing persuasion lowers the willingness to pay enough to offset the increase in demand, making persuasion unattractive from a profit point of view. The monopolist is better off selecting a fully revealing information design that allows it to maximize the price it can charge consumers. In that sense, pricing and persuasion are strategic substitutes: a monopolist may find it not profitable to use persuasion techniques given that consumers' Bayesian updating can mitigate the power of persuasion by dampening the firm's pricing power.

Next, in Section 3 we turn to study the impact of competition on the incentives to persuade. We extend the model to a duopoly where the firms compete by choosing their information designs. We again compare the case of exogenous prices to the case of firms that set endogenous prices after selecting their information designs. With exogenous prices, we see a reversal of the monopoly case. A monopoly with exogenous prices can use persuasion to increase its profits, but under competition, the incentives to persuade are weakened by the competitive effect, because persuading consumers to buy a product that might not match lowers their expected utility a-priori, and drives them to buy from the competitor.

When prices are endogenized, we find three interesting results. First, there is no equilibrium in pure strategies in the pricing stage – because consumers now receive two signals (one for each product), the market is effectively segmented into three segments, and the competition over them is similar to that in Varian (1980) and Narasimhan (1988), yielding mixed strategies. Second, we find that in the subgame perfect equilibrium only one firm elects to provide additional information to consumers, while the other opts to not provide any signal. The reason for this asymmetry is that it allows firms to soften their competition in the pricing game. Third, for the firm that does provide signals to consumers, there are cases where it will choose to persuade customers, there are cases where it will be fully revealing, but there are also cases where it will demarket, and try to convince consumers that the product is not a good match when it actually is. Hence, in contrast to the case of exogenous prices, when prices are endogenized we find a full range of possible levels of persuasion depending on the a priori match probability of the product. The intuition is that when there is a low probability of a match, firms do not compete too strongly with one another, and persuasion can benefit the firm that provides information. As the a priori probability of a match increases, competition intensifies, and fully revealing the match of the product, or demarketing, deescalates the price competition. One interesting aspect of this result is that firms might find it more profitable to reduce the efficiency of information they provide to consumers because of strategic concerns about competitor reactions.

Lastly, we extend the model to include a platform's decision on information design when hosting two competing sellers in Section 4. Interestingly, we find that the platform will choose either full revelation or demarketing. In some cases, the information design will be asymmetric, where the platform will choose to provide information only for one seller and not for both, even though the sellers are symmetric a priori. The intuition behind these results is that the platform receives a fixed share of the revenue from the sellers. As such, it would like to maximize sales on the platform, without enticing the sellers to compete too much on prices. When the utility of a non-matching product is low enough, and the chances of a mismatch are high enough, the platform is better off with full revelation, since the chances of direct competition (e.g., the consumer considering both products to match at the same time) are low. When the a priori chance of a match increases, direct competition among the sellers is more likely, and in that case the platform would like to demarket one of the sellers in order to soften the price competition.

The results of our analysis shed light on the limits to Bayesian persuasion, which looks at cases where firms cannot select credible signal realizations (unlike, e.g., Kihlstrom and Riordan 1984, Gerstner 1985, Milgrom and Roberts 1986, Erdem et al. 2008, Mayzlin and Shin 2011). We believe this scenario is quite realistic for many firms, and especially for those competing on a platform. Unlike much of the prior Bayesian persuasion research we find that firms do not always benefit from persuading customers, and are better off designing their information in a way that fully helps consumers to ascertain the match utility of products. In the case of monopoly this is in contrast to the findings of Iyer and Zhong (2020) and Jerath and Ren (2020), because of the effect of endogenous pricing. Our findings add to the literature on competition in information provision that finds that the effect of competition on information design is ambiguous in general (Gentzkow and Kamenica 2016, Board and Lu 2018). We also contribute to the stream of literature on platform design and the interaction between firms and consumers, where it is shown that platforms, as well as firms, are often better off withholding information from consumers, in order to soften competition and increase market revenues (Vellodi 2018, Romanyuk and Smolin 2019). Finally, we contribute to the growing literature in marketing on Bayesian persuasion and information design (Gardete and Hunter 2020, Subramanian and Zhang 2021).

To summarize, we show that using persuasive approaches can weaken a firm's pricing power and competitive position, and for platforms creates increased competition among firms that might lower the overall revenue of the platform. We identify cases where firms and platform might not want to persuade consumers and even choose a demarketing strategy, in which they claim that their product is not as good as it is. The results indicate that the power of persuasion may be limited in realistic settings for competing firms and platforms.

2 Model

A mass one of consumers each have demand for one product, which provides ex-ante uncertain utility u. The product can be good match (G) with probability α or bad match (B) with probability $1-\alpha$. The product, if purchased at a price m, provides the realized utility $u-m$ where $u \in \{u_G, u_B\}$ and $u_G = 1$ and $u_B < u_G$. We assume thet ex-ante, $\mathbb{E}[u] - m = \alpha u_G + (1-\alpha)u_B - m < 0$. In other words, the ex-ante expected utility from purchasing the product is negative, and after purchasing the product, its match is revealed to each customer. We also assume that $m < u_G$, which means that there are potential gains to trade with more information about the product.

A monopolist producer can provide information about the utility of the product before purchase in the form of a signal $s \in \{g, b\}$. We denote the expected utility of consuming the product after observing the signal s as $u(s)$, i.e., $u(s) = E[u|s]$. Given the signal and the price, the consumer will purchase the product if $u(s) - m > 0$.

We will call a signal fully revealing if after observing its realization the consumer can infer the true value of u exactly. That is, if $Pr(G|s = g) = 1$ and $Pr(B|s = b) = 1$ the signal is fully revealing.[1] Using Bayes rule, this is equivalent to stating that $Pr(s = g|G) = 1$ and $Pr(s = b|B) = 1$. Consistent with the Amazon example, the monopolist cannot pick which signal s the consumer observes. Rather, it can design the information environment for all consumers to persuade them to buy the product by setting the revelation properties of the signal, i.e., the monopolist can commit to values p_G and p_B such that $p_G = Pr(s = g|G)$ and $p_B = Pr(s = g|B)$, as described in Table 1.

Table 1: Firm's Information design.

		State	
		G	B
Realized Signal	g	p_G	p_B
	b	$1 - p_G$	$1 - p_B$

We further assume that there is a cost to design a non-revealing signal of $\frac{(1-p_G)^2}{2} + \frac{p_B^2}{2}$, which means that designing a perfectly revealing signal is effortless for the information designer. This assumption allows us to emphasize cases where persuasion is desired even when there are strong incentives to be fully revealing. It also assists in guaranteeing the existence of pure strategy equilibria in information design in most cases. This cost can be seen as a persuasion cost, since inducing consumers to believe a state which is different from the true state is costly. We will use the term persuasion if in equilibrium firms claim the product is better than it actually is (i.e., set $p_B > 0$); we will say that the strategy is to demarket if in equilibrium firms claim that a product is not as good as it is, i.e., they set $p_G < 1$ which means that good products can yield a bad signal.

2.1 Monopoly with Exogenous Prices

We begin our analysis with the case of exogenous prices m. The timing of the game is as following:

1. The monopolist decides whether to provide additional information, and if so, they pick p_G and p_B which are observed by the consumers.

2. If additional information was provided, consumers observe a signal realization s and purchase the product if $u(s) - m > 0$.

[1] To simplify notation, we shorten conditioning on the state in which $u = u_G$ to conditioning on G.

The firm's profit is:

$$m\mathbb{E}_s[\mathrm{I}(u(s) - m > 0)] - \frac{(1-p_G)^2}{2} - \frac{p_B^2}{2} \tag{1}$$

where $\mathrm{I}(\cdot)$ is the indicator function.

We focus on equilibria which are obedient (Bergemann and Morris 2019), where a consumer will follow the signal they observe if the firm provides one. Specifically, these are cases where consumers will purchase the product if they observe $s = g$ and not purchase if $s = b$. Other equilibria (for example, where the consumers ignore any signal provided to them, or where the consumers buy regardless of the signal) might exist, but are quite intuitive and do not provide any insight on firm strategies. Hence, in our analysis, we only analyze the cases where obedient equilibria exist.

After observing a signal, the consumers update their beliefs about the utility of the product. Using Bayes' rule, the updated expected utility from consuming when observing $s = g$ is:

$$u(g) = \frac{Pr(G|g)E[u(g)|G] + Pr(B|g)E[u(b)|B]}{Pr(g)} \tag{2}$$

$$= \frac{\alpha p_G u_G + (1-\alpha)p_B u_B}{\alpha p_G + (1-\alpha)p_B} \tag{3}$$

And similarly $u(b) = \frac{\alpha(1-p_G)u_G + (1-\alpha)(1-p_B)u_B}{\alpha(1-p_G) + (1-\alpha)(1-p_B)}$.

In an obedient equilibrium a consumer buys only after receiving the signal g, i.e., if the following incentive compatibility constraints hold:

$$u(g) - m \geq 0 \tag{4}$$

$$u(b) - m < 0 \tag{5}$$

Maximizing the firm's profit in (1) subject to the incentive compatibility constraints yields the following:

Proposition 1. *When* $\mathbb{E}[u] - m < 0$*, there is a unique obedient equilibrium in which the firm will set* $p_G^* = 1$ *and* $p_B^* = \min\left(\frac{\alpha(u_G - m)}{(1-\alpha)(m-u_B)}, m(1-\alpha)\right)$*. Specifically,* $0 < p_B^* < 1$.

Proof. All proofs appear in the Appendix. □

Proposition 1 echoes previous results in Kamenica and Gentzkow (2011) and Bergemann and Morris (2019) about the conditions for Bayesian persuasion to be an equilibrium strategy. When

α is large, the consumer will a priori have a high expected value for the product and persuasion cannot increase profits since the consumer will consume the product anyway. When α is smaller and u_B is smaller than m, the firm always has an incentive to reveal the true state of the product if it is good, and persuade consumers in a bad state by setting p_B to be between zero and one. Setting $p_G^* = 1$ is optimal because it both increases revenue and lowers costs, and does not make the consumer trust the signal less. Setting p_B too high, however, will lower the expected utility from buying and will cause the consumer to not buy even when receiving a good signal. Hence, there is an maximum positive level of p_B that the monopolist will set, which is lower than 1.

This result is not surprising by itself as it has been shown previously in the literature. We use it as a benchmark to demonstrate how pricing and competition incentives dramatically alter the incentives to persuade.

2.2 Monopoly with Endogenous Pricing

Now, assume that the monopolist can select the price of its products after it sets the information design. Unlike the case with exogenous prices, the firm faces a trade-off between increasing the consumer's expected utility of a product by providing more revealing information, and persuading the consumer to buy the product. Increasing the expected utility allows the firm to charge higher prices and extract more rent, at the risk of lowered demand, while increasing persuasion increases demand, but lowers the willingness to pay of the consumer.

The timing of the game matches that of the previous section, but in step 1 the firm also chooses the price m for its products. Because information is symmetric and the firm does not have a priori private information about the product's utility, it cannot use prices to signal the utility of its products.

The firm's profit is the same as in Equation (1), the only difference being that m is an endogenous choice variable. The consumer's incentive compatibility constraints in an obedient equilibrium are also the same as with exogenous prices. This means that a consumer purchases the product after observing $s = g$ if:

$$u(g) = Pr(G|g)u_G + Pr(B|g)u_B - m = \frac{\alpha p_G u_G + (1-\alpha)p_B u_B}{\alpha p_G + (1-\alpha)p_B} - m \geq 0 \tag{6}$$

and when they observe a signal b, they will not purchase with $u(b) - m < 0$.

Because the price m is endogenous, the firm can extract the maximum revenue by charging a price m equal to $u(g)$. If we plug-in the equilibrium price into the expression for the firm's profit, the expression for profit becomes:

$$\alpha p_G u_G + (1-\alpha) p_B u_B - \frac{(1-p_G)^2}{2} - \frac{p_B^2}{2} \tag{7}$$

Compared to the case of exogenous prices, the ability to pick prices removes the conditioning of the profit on the purchase probability for the customer, as prices will be endogenously selected to induce buying, and in equilibrium all consumers will purchase the product. Hence, the information design p_G and p_B now solely control the willingness to pay of the customer. Solving for optimal information design, we find the following:

Proposition 2. *The unique obedient equilibrium is fully revealing with $p_G^* = 1$ and $p_B^* = 0$, and occurs only when $u_B < 0$. When $u_B > 0$ the firm does not want to provide information.*

Proposition 2 provides new insight about the effects of pricing on persuasion decisions. In this framework pricing serves as a strategic substitute to persuasion within a certain range of values for u_B. Moreover, there is no case where persuasion is viable, unlike the typical case analyzed in the literature. This means that the pricing and persuasion trade-off may often be strong enough to dissuade the monopolist from persuading consumers to buy products of low quality. Potentially, this implies that consumers should not worry as much from being "falsely" persuaded when the seller is a monopolist, because the pricing power of the monopolist makes persuasion a less valuable strategy.

3 Duopoly

We now proceed to analyze the persuasion incentives under duopolistic competition. As before, we first analyze persuasion with exogenous prices and then extend our analysis to incorporate pricing decisions.

3.1 Duopoly with Exogenous Prices

When firms cannot select prices, the revenue per unit m is fixed and independent from the firms' decisions. Because we assumed that without additional information and with exogenous prices,

$\mathbb{E}[u] < m$, firms have to provide information to generate any sales. This allows us to focus on the case where two firms, indexed by 1 and 2, provide a signal s_1 and s_2, respectively. The signals of the firms segment consumers into four segments with sizes detailed in Table 2.[2]

Table 2: Consumer Segmentation by Signals

Signals	Size
g_1, g_2	$(\alpha p_{G1} + (1-\alpha)p_{B1})\,(\alpha p_{G2} + (1-\alpha)p_{B2})$
g_1, b_2	$(\alpha p_{G1} + (1-\alpha)p_{B1})\,(\alpha(1-p_{G2}) + (1-\alpha)(1-p_{B2}))$
b_1, g_2	$(\alpha(1-p_{G1}) + (1-\alpha)(1-p_{B1}))\,(\alpha p_{G2} + (1-\alpha)p_{B2})$
b_1, b_2	$(\alpha(1-p_{G1}) + (1-\alpha)(1-p_{B1}))\,(\alpha(1-p_{G2}) + (1-\alpha)(1-p_{B2}))$

In an obedient equilibrium, a consumer will consume product j only if $u(s_j = g) > u(s_{-j})$ and $u(s_j = g) \geq m$, while we assume that if the expected utilities are tied, the consumer will choose either product with equal probability. Firm j's profit when the other firm uses information design p_{G-j} and p_{B-j} as its strategy is therefore:

$$\begin{aligned}\pi_j(p_{Gj}, p_{Bj}) = {} & m/2\,(\alpha p_{Gj} + (1-\alpha)p_{Bj})\,(\alpha p_{G-j} + (1-\alpha)p_{B-j}) \\ & + m\,(\alpha p_{Gj} + (1-\alpha)p_{Bj})\,(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})) - \frac{(1-p_{Gj})^2}{2} - \frac{p_{Bj}^2}{2} \quad (8)\end{aligned}$$

The first term is firm j's profit when both firms generate a signal g, while the second is the profit when the competitor generates a signal b. There is no profit when firm j generates signal b as consumers will not buy after seeing b in an obedient equilibrium.

Firm j's profit is increasing in p_{Gj} and decreasing in p_{Bj} for any value of p_{G-j} and p_{B-j}, yielding the following result:

Proposition 3. *In an obedient equilibrium when duopoly firms compete without pricing decisions, both firms set $p_G^* = 1$ and $p_B^* = 0$ regardless of the value of u_B.*

Proposition 3 shows a strikingly different result compared to the monopoly case in Proposition 1. In an obedient equilibrium, the competition removes the incentive of firms to persuade, even when pricing is not involved. The intuition is that trying to persuade consumers to consume in a bad state by setting a positive p_B gives the competitor the option to be more revealing, which will lower the competitor's costs as well as increase the competitor's demand. Hence, the best response

[2] We shorten the notation $s_j = g$ to g_j.

to a competitor's actions is always to be more revealing. One implication we may draw from this proposition is that in competitive markets where firms do not have pricing power, consumers may experience less persuasion.

3.2 Duopoly with Pricing

After showing that pricing incentives for a monopoly and competition incentives for a duopoly lower the incentives to persuade consumers, we now turn to analyze the effect of the interaction between pricing and competition on the persuasion decisions of firms. We again focus on obedient equilibria where consumers do not buy the product if they received a bad signal, but will buy it if they received a good signal and the price is low enough. A requirement for obedient equilibria to exist is that u_B is low enough such that it is not profitable for firms to sell a product to consumers by charging a very low price. The full game with endogenous information design has two stages – an information design stage followed by pricing stage. To aid the analysis, we first study pricing in a subgame where duopoly firms have chosen a fully revealing information design in the design stage. We then generalize the result by endogenizing the information design in the first stage. The following lemma describes the pricing equilibrium when both firms have chosen a fully revealing information design:

Lemma 1. *When firms fully reveal their product quality, i.e., set* $p_{Gj} = 1$ *and* $p_{Bj} = 0$*, if* $u_B < \frac{\alpha(1-\alpha)}{1-\alpha+\alpha^2}$*, there exists a mixed strategy pricing equilibrium such that each firm will randomize its prices between* $[(1-\alpha)u_G, u_G]$*, with CDF* $F(p) = \frac{p-(1-\alpha)u_G}{\alpha p}$*.*

The pricing equilibrium described in Lemma 1 is similar in spirit to that of Varian (1980) and Narasimhan (1988) where consumers are segmented into a loyal segment of each store and a switchers segment that chooses among the two stores. In our case the loyals are those who receive only one good signal, and the switchers are those who receive two good signals. Therefore, the game has no pure strategy equilibrium in prices. Consequently each firm in equilibrium randomizes its prices in a range $[(1-\alpha)u_G, u_G]$. The upper bound u_G is the reservation price of consumers, $u_j(g) = u_G$, yielding profit $\alpha(1-\alpha)u_G$.

The non-existence of pure strategy pricing equilibria will also hold when the information design decision is endogenoized in the design stage, which we now turn to analyze. Similarly to the previous cases, in the design stage, firm j decides whether to send out information (p_{Gj} and p_{Bj})

or not to send any information and let consumers use their prior beliefs. If the firms picks $p_{Gj} = Pr(s_j = g|v_j = u_G)$ and $p_{Bj} = Pr(s_j = g|v_j = u_B)$, such a design will have a cost of $\frac{(1-p_{Gj})^2}{2} + \frac{p_{Bj}^2}{2}$. That means that trying to persuade consumers by deviating from fully revealing the match of the product is more costly. The updated expected utilities of the consumers after observing a signal are $u_j(g) = \frac{\alpha p_{Gj} u_G + (1-\alpha) p_{Bj} u_B}{\alpha p_{Gj} + (1-\alpha) p_{Bj}}$ and $u_j(b) = \frac{\alpha(1-p_{Gj}) u_G + (1-\alpha)(1-p_{Bj}) u_B}{\alpha(1-p_{Gj}) + (1-\alpha)(1-p_{Bj})}$. If a firm chooses to not send any signal, it will experience no cost and consumers will use their prior beliefs. After observing the choices of information design, firms select the price to charge p_j in the pricing stage.

The firm choices of information design in the first stage define three types of pricing subgames: (1) both firms do not provide additional information, (2) both firms choose an information design (p_{Bj}, p_{Gj}), and (3) one firm chooses an information design (p_{Bj}, p_{Gj}) while the other does not provide any signal. Our solution concept is that of a subgame perfect equilibrium.

3.2.1 Subgame 1: Both firms do not provide a signal

Because consumers do not update their beliefs in this subgame, they will have symmetric willingness to pay for the products from both firms. After observing the price, the consumers will buy from the firm with the lowest price, leading to Bertrand duopoly competition. In equilibrium both firms will set prices to zero and make zero profit.

3.2.2 Subgame 2: Both firms provide a signal

Similarly to the case analyzed in Section 3.1 the signals segment consumers into the four segments described in Table 2.

In obedient equilibria where consumers do not purchase after receiving a bad signal, the profit of firm j setting price p_j when the competitor sets price p_{-j} will be:

$$\begin{aligned}\pi_j &= (\alpha p_{Gj} + (1-\alpha) p_{Bj}) \left(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})\right) p_j \\ &\quad + (\alpha p_{Gj} + (1-\alpha) p_{Bj}) \left(\alpha p_{G-j} + (1-\alpha) p_{B-j}\right) \mathbb{I}[p_j < p_{-j}] - \frac{(1-p_{Gj})^2}{2} - \frac{p_{Bj}^2}{2} \qquad (9)\end{aligned}$$

In any such equilibrium firms will mix their pricing following the same reasoning as in Lemma

1. The highest price they can charge will be $u_j(g)$, and the expected equilibrium profit will be:

$$\mathbb{E}[\pi_j] = (\alpha p_{Gj} + (1-\alpha)p_{Bj})\left(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})\right)u_j(g) - \frac{(1-p_{Gj})^2}{2} - \frac{p_{Bj}^2}{2} \quad (10)$$

Plugging in $u_j(g) = \frac{\alpha p_{Gj} u_G + (1-\alpha)p_{Bj} u_B}{\alpha p_{Gj} + (1-\alpha)p_{Bj}}$ and using $u_G = 1$ we can prove the following:

Lemma 2. *In an obedient Nash equilibrium of this subgame, both firms will set $p^*_{Gj} = 1$.*

Lemma 2 shows that in equilibrium both firms will design their signal to fully reveal the match of the product if it is good by setting $p_{Gj} = 1$. The reason is that the profit of firm j increases in p_{Gj} for any value of α and p_{Bj}. Using the result that $p^*_{Gj} = 1$, we solve for the full Nash equilibrium and prove the following:

Lemma 3. *In the subgame where both firms provide additional information:*

- *When $u_B \leq 0$, firms will set $p^*_{Bj} = 0$ such that all firms will be fully revealing and not engage in any persuasion.*
- *When $0 < u_B < \bar{u}_B$ there will be persuasion where firms set $p^*_{Bj} = \frac{u_B(1-\alpha)^2}{1+u_B(1-\alpha)^2} > 0$ and $p_{Gj} = 1$.*[3]

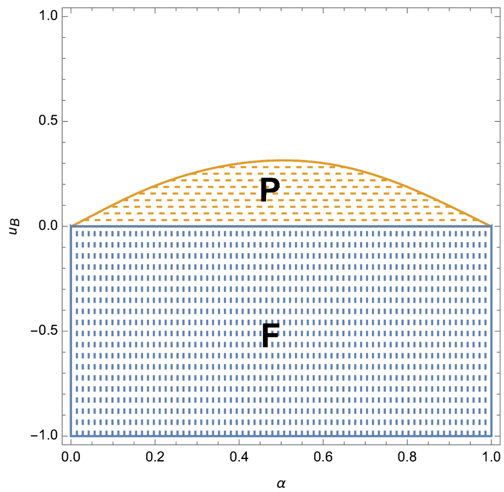
Lemma 3 shows that when both firms provide signals, they will engage in persuasion with price competition, but only when $\bar{u}_B > u_B > 0$. This is sharply different from the monopoly case with pricing, or the duopoly case without pricing, where persuasion is not possible. As we have shown, the ability to set a price creates a trade-off for players between persuading that increases demand but lowers the willingness to pay. Under competition, however, the pricing side of the trade-off is weakened. Increasing the expected utility provided to consumers does not allow to charge higher prices necessarily, because the competitors will respond with lower prices. Hence, increasing persuasion when $\bar{u}_B > u_B > 0$ increases the demand for the firm, as more consumers receive a good signal, but because competition already maintained a lower price, it will not have as much effect on prices. When $u_B < 0$, in contrast, increasing p_B would indeed increase demand, but also will intensify competition. As a result, persuasion is only possible when $u_B > 0$.

Figure 1 illustrates the the range of parameters where equilibrium exists in the subgame in which both firms provide information signals. When $u_B < 0$ (Region F), the equilibrium information

[3]The expression for $\bar{u}_B$ appears in the proof in the Appendix.

design is $p^*_{Gj} = 1$ and $p^*_{Bj} = 0$. When $u_B \geq 0$ (Region P), the equilibrium information design is $p^*_{Gj} = 1$ and $1 > p^*_{Bj} > 0$.

Figure 1: Parameter ranges for equilibria in Subgame 2



Region P: Both firms choose persuasion ($p^*_{Gj} = 1$ and $0 < p^*_{Bj} < 1$).
Region F: Both firms choose full revelation ($p^*_{Gj} = 1$ and $p^*_{Bj} = 0$).

3.2.3 Subgame 3: Only one firm provides additional information

In this subgame, only one firm provides a signal and sets p_B and p_G in the design stage, while the other remains quiet. For this result to be possible in equilibrium, it is required that $\mathbb{E}[u] \geq 0$. Otherwise, the firm that does not provide a signal will have zero demand, and a subgame perfect equilibrium will not include this information design structure. Without loss of generality, we assume that firm 2 which we call passive, chooses to be quiet, while firm 1, which we call active, sets p_G and p_B and provides information. The next lemma describes the resulting equilibrium price distributions when the firms compete in the pricing stage:

Lemma 4. *When one firm is active, sets p_G and p_B and provides information, while the other firm is passive, and does not provide additional information, there is a mixed strategy pricing equilibrium such that the passive firm will randomize prices between $[\Pr(b) * \mathbb{E}[u], \mathbb{E}[u]]$ and the active firm will randomize prices between $[u(g) - \Pr(g)\mathbb{E}[u], u(g)]$.*

Lemma 4 shows that in the asymmetric equilibrium both firms will mix their pricing within two separate price ranges. The active firm will charge higher prices on average and will sell to consumers who receive a good signal. The passive firm will charge lower prices on average and will sell to consumers who receive a bad signal. Because the firms provide different levels of information to consumers and sell to consumers of different segments with different information, the active firm can charge up to $u(g)$, while the passive firm can charge up to $\mathbb{E}[u]$. In order to make sure that the passive firm does not poach too many consumers from the active firm, the lowest price that the active firm charges is equal to the highest surplus the consumers can receive if they switch to buying from the passive firm. Overall, the mixing strategies are similar to the benchmark analyzed in Lemma 1, with asymmetry in prices due to the different level of information provided to consumers. The proof of Lemma 4 in the Appendix fully characterizes the price distributions of firms 1 and 2.

Next, we use backward induction to find firm 1's information design policy in the first stage. The profit maximization problem for firm 1 is:

$$\begin{aligned}\max_{p_{G1},p_{B1}} \pi_1(p_{G1},p_{B1}) &= \Pr(g) * (u(g) - \Pr(g)\mathbb{E}[u]) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \\ &= ((1-\alpha)p_{B1} + \alpha p_{G1}) \left(\frac{(1-\alpha)u_B p_{B1} + \alpha u_G p_{G1}}{(1-\alpha)p_{B1} + \alpha p_{G1}} - ((1-\alpha)p_{B1} + \alpha p_{G1})\left((1-\alpha)u_B + \alpha u_G\right) \right) \\ &\quad - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \\ s.t. &\quad 0 \le p_{B1} \le 1, 0 \le p_{G1} \le 1\end{aligned} \tag{11}$$

The analysis of this constrained maximization problem yields the following result:

Lemma 5. *When $\mathbb{E}[u] \ge 0$ (which is required for an asymmetric equilibrium), the active firm will select:*

- *Persuasion ($p_B > 0, p_G = 1$) for low values of α and high values of u_B.*
- *Full revelation ($p_B = 0, p_G = 1$) for intermediate values of α.*
- *Demarketing ($p_B = 0, p_G < 1$) for high values of α.*

The surprising result in Lemma 5 is that the active firm may choose to persuade ($p_B > 0$) when α is low, but also to demarket ($p_G < 1$) when α is high, such that it sometimes sends a bad signal when the product is actually a match and yields utility u_G.

The intuition can be gleaned from Lemma 4, and from figure 2(b) that illustrates the ranges of prices and average prices in equilibrium for different levels of α. When α is very low, the active firm does not face strong competition from the passive firm, and both firms can increase their prices with α without too much competition. As α increases even more, the active firm now faces a passive firm that provides enough expected utility to the customers who receive a negative signal to pose stronger competition. This requires the active firm to both provide information to the customer and reveal the true match of the product, and also to lower prices in order to keep demand at a desired level. Finally, for high values of α, the competition is even stronger, the passive firm receives lower demand, which causes it to lower prices somewhat and compete more aggressively. If the active firm responds by lowering p_G, it gives more demand to the passive firm and softens price competition, which allows the active firm to raise its prices. Hence, when there is a high chance that the products in the market will match consumer preferences, an active firm might choose to demarket its products. For example, a few brands have made it harder to recognize that their products are part of the brand's family, which may cause a consumer to confuse the product with an inferior brand.

Figure 2 illustrates the range of parameters where an equilibrium exists in the subgame with one passive and one active firm. Region P is where the active firm persuades ($p_B > 0$). Region F is where it chooses full revelation ($p_B = 0$) and Region D is where it demarkets ($p_G < 1$). Figure 2(b) shows the dynamics of price competition between the two firms when α increases. As we can see, while firm 2's average price always increases in α, firm 1's average price will first increase but then decrease. The intuition is that an increase in α causes the range of prices the firms use to overlap, which results in stronger competition. The only firm that can respond through information is the active firm, and as a result, it would gradually transition from persuasion to full revelation and eventually to demarketing, to avoid head-to-head competition.

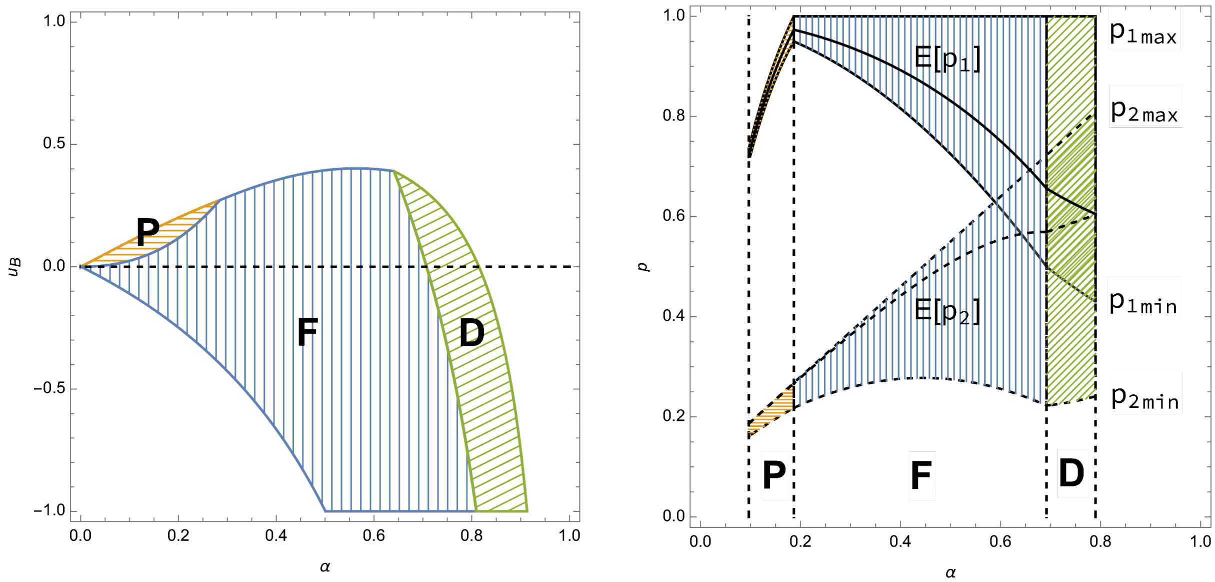
3.2.4 Equilibrium Selection

In subgame 1, the profits of both firms are zero. Hence we can focus on the equilibria in subgames 2 and 3 only. A firms' strategy in the first stage is to choose to be either passive (not provide information) or active (provide information) and set values for p_G and p_B. We will call the outcome in subgame 2 an active-active equilibrium (aa for short) while it will be called a passive-active (pa)

Figure 2: Equilibria and Price Distributions in Subgame 3

(a) Parameter ranges for equilibria

(b) Price distributions when $u_B = 0.1$



(Left) Figure (a) shows the range where equilibria exist and the corresponding strategy of the active firm in each range. The active firm selects: (i) Persuasion (Region P), (ii) Full revelation (Region F), or (iii) Demarketing (Region D). (Right) Figure (b) shows the maximum, minimum, and expected prices of both firms when $u_B = 0.1$. Solid black lines indicate firm 1 (active) and black dashed lines indicate firm 2 (passive).

in subgame 3.

In the standard definition of an obedient equilibrium, consumers buy only after observing a good signal. However, in a subgame with a passive strategy, consumers do not observe a signal from the passive firm. We extend the definition of an obedient equilibrium to account for this case, by assuming that consumers will buy from the firm that provides them the highest surplus. Another issue with endogenous pricing is that firms might want to charge a price lower than $u(b)$ and sell to consumers after they observe a bad signal. We restrict firms to not having a profitable strategy of pricing below $u(b)$, which maintains the spirit of an obedient equilibrium.

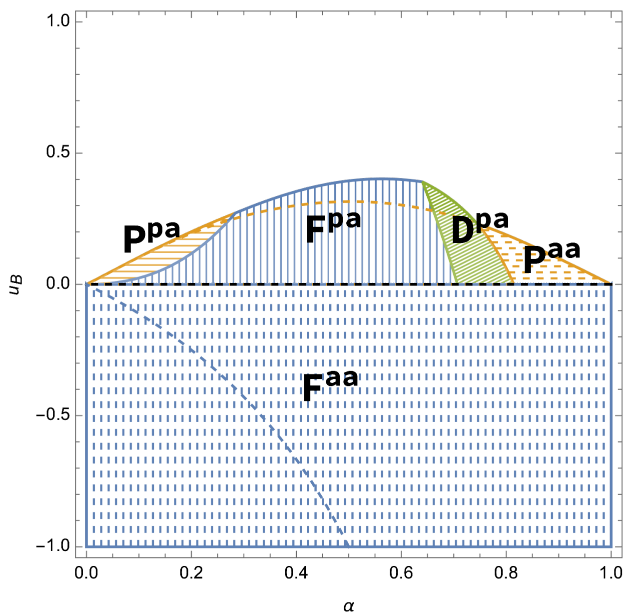
Using backwards induction, we arrive at the following subgame perfect equilibrium (SPE):

Proposition 4. *When $u_B < 0$, the active-active equilibrium is the subgame perfect obedient equilibrium, and both firms choose full revelation. When $u_B > 0$, for low values of α the passive-active equilibrium is the SPE, while for high values of α it is the active-active. Specifically, the passive-active equilibrium shifts from persuasion, through full revelation ending with demarketing as α*

increases, while the active-active equilibrium will be with persuasion.

Figure 3 illustrates these results. The colored area contains the range of parameters where a subgame perfect equilibrium in pure strategies exists. The boundary of the areas in color is determined by the region where profitable deviations do not exist in the second stage. When $u_B < 0$, region F^{aa} has an active-active equilibrium with full revelation. When $u_B > 0$, we observe the shift from persuasion to full revelation to demarketing through regions P^{pa}, F^{pa} and D^{pa} as α increases. Finally, when α is large, the passive firm has a profitable deviation and will not remain passive in the *pa* subgame, and hence the only remaining equilibrium is the *aa* equilibrium with persuasion in region P^{aa}.

Figure 3: Equilibrium Selection



The intuition is that when $u_B < 0$, a passive firm will not make any profit, and hence a passive-active equilibrium is not feasible for negative u_B. For $u_B > 0$, we see a more nuanced result. When α is small, having one firm as passive and one firm as active softens the competition between the firms, as they split the market such that the active firm sells to those who receive a good signal, while the passive can sell to those who received a bad signal from the active firm. In that sense, a passive-active equilibrium enlarges the market. If both firms were active, they could only sell when they provide a good signal, which makes them compete more fiercely and lowers their profit. When α is large enough, by switching to being active, the passive firm can now have a high probability of finding consumers who receive a good signal from it. The effect of having a larger market through providing good signals, despite the increased competition, is stronger than the possibility of selling to those who only received a bad signal from the active firm.

To summarize, in a duopoly with information design and price competition, there are two possible types of equilibria. A passive-active equilibrium is expected to arise where one firm provides no additional information, while another will choose among persuasion, full revelation, or demarketing based on how probable good matches are. Essentially, for the active firm, information design creates a tradeoff between demand enhancement and price competition. On one hand, persuasion increases sales. On the other hand, the passive firm who is facing a smaller demand is forced to price more aggressively in order to compete for market share. When α is small, the chance for a good match is small so the demand enhancement effect dominates for the active firm, while as α gets larger, the price competition effect slowly becomes more substantial. In this case, the active firm will then turn to full revelation or even demarketing in order to avoid head to head price competition. The active-active case will arise when $u_B < 0$ or when α is extremely large, because the passive firm can deviate to being active and gain a higher profit.

4 Persuasion within a Platform

We now analyze a shopping platform that intermediates the sales of products between the two competing sellers and buyers. Examples that match this scenario are Amazon.com and eBay.com that design what the product description or listing environment looks like for sellers, as well as AirBnB.com that designs what a host's listing appears like. The platform cannot dictate the price each seller is going to charge, but can change the information environment to influence consumers, which will in turn affect the platform's revenues. We explore whether the platform has an incentive to intervene in the persuasion process by deciding what information will be presented to customers and how. For example, Amazon presents the highest rated features for each product, and the order of the features and which features to emphasize is made by Amazon depending on the product's ratings. For one product Amazon might present "ease of use" and "modern design" as the highly ranked features, while for a competing product it might present "sturdy" and "lightweight" as the prominent features. The platform also decides how many images and what length of description to present for each listing, and even the order of the listings to show to consumers.

We have two goals in the analysis. First, we investigate the optimal information design for the platform. Second, we analyze the level of persuasion the platform chooses. Specifically, we ask (1) Does the platform generate bias away from full revelation, i.e., selects $(p_{Gj}, p_{Bj}) \neq (1, 0)$, and (2) does the platform persuade more or less compared to the Duopoly case in Section 3.

4.1 Platform with Exogenous Prices

Initially, we assume that the revenue per unit m is exogenous, and the platform can decide on p_{Gj} and p_{Bj} for each firm. We assume that the platform's profit is a fixed share of the total sales of both firms, which is standard practice in online platforms.

The following result summarizes the information design decision of the platform when prices are exogenous:

Proposition 5. *When $\mathbb{E}[u] - m < 0$, there is a unique obedient equilibrium in which the platform will set $p^*_{Gj} = 1$ and $p^*_{Bj} = \min\left(\frac{\alpha(u_G - m)}{(1-\alpha)(m-u_B)}, \frac{(1-\alpha)^2 m}{(1-\alpha)^2 m+1}\right)$. Specifically, $0 < p^*_{Bj} < 1$.*

When prices are exogenous, the two sellers cannot respond by changing prices if demand is low, and the platform faces a decision similar to the case of a monopoly without pricing in Proposition 1. In this case, the assumption that $\mathbb{E}[u] - m < 0$ implies that α is small. Consequently, the same intuition from the monopoly case applies. Setting $p^*_G = 1$ is optimal because it both increases revenue and lowers costs, and does not make the consumer trust the signal less. Setting p_B too high, however, will lead the consumer to not take the signal as credible and will cause the consumer to not buy even when receiving a good signal. Comparing to the case of duopoly, we find that the platform will persuade more, so that total demand and total revenue increases. This is possible because in the platform case, the sellers cannot respond by being more truthful towards consumers and steal demand from their competitors. By centralizing information design at the platform level, the information competition is softened, which allows the platform (and sellers) to be less truthful and persuade more.

4.2 Platform with Pricing

Similarly to the case of duopoly competition, when prices are endogenous the game has two stages. In the first stage, the platform can select to set p_{Gj} and p_{Bj} for each seller, or decide to not provide any information for any of the sellers. After observing the information design environment, firms then decide the optimal prices p_j. Again, we assume the platform obtains profit based on a fixed share of the total sales from both firms. We solve for the platform's optimal persuasion strategy by using backward induction. We consider two types of scenarios for the platform: A design where the platform provides information for both sellers (which we call symmetric), and an asymmetric one, where the platform only provides information for one of the sellers.

The symmetric design applies to a case where the platform designs the information environment (such as the rating and reviews system, or the product page), but does not strategically intervene in how information is presented to customers once it is created. The asymmetric design allows the platform to also favor one seller over the other (for example, by adding more information to their listings, or by featuring the results), and this will result in one seller having more information than the other. One example, mentioned in the introduction, is a platform that decides to emphasize different aspects of a listing for each seller, depending on the review results of each seller's product.

4.2.1 Symmetric design: the platform provides information for both sellers

When we focus on obedient equilibria where consumers only buy products after they receive a good signal, similarly to the previous results, the equilibrium prices in the second stage will be in mixed strategies given an information design. Therefore, plugging in $u_j(g) = \frac{\alpha p_{Gj} u_G + (1-\alpha) p_{Bj} u_B}{\alpha p_{Gj} + (1-\alpha) p_{Bj}}$ the platform's optimization problem will be:

$$\max_{p_{Gj}, p_{Bj}} \pi_P = \pi_j + \pi_{-j} =$$
$$\sum_{j=1}^{2} \left\{ (\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j}))\,(\alpha p_{Gj} u_G + (1-\alpha) p_{Bj} u_B) - \frac{(1-p_{Gj})^2}{2} - \frac{p_{Bj}^2}{2} \right\} \quad (12)$$

$$\text{s.t. } p_{Bj} \leq 1, -p_{Bj} \leq 0, p_{Gj} \leq 1, -p_{Gj} \leq 0$$

We consider cases where u_B could be positive or negative. An obedient equilibrium requires that $u_j(g) \geq 0$, or otherwise consumers will not buy with a good signal. Using the expression for $u_j(g) = \frac{\alpha u_G p_{Gj} + (1-\alpha) u_B p_{Bj}}{\alpha p_{Gj} + (1-\alpha) p_{Bj}}$, the condition implies

$$\alpha u_G p_{Gj} + (1-\alpha) u_B p_{Bj} \geq 0 \quad (13)$$

which translates to the following condition on the possible values of u_B:

$$u_B \geq \begin{cases} -\frac{\alpha p_{Gj}}{(1-\alpha)p_{Bj}}, \text{if } p_{Bj} \in (0,1] \\ -\infty, \text{if } p_{Bj} = 0 \end{cases} \tag{14}$$

The analysis of the solution to the platform's optimization problem (12) along with condition (14) yields the following results:

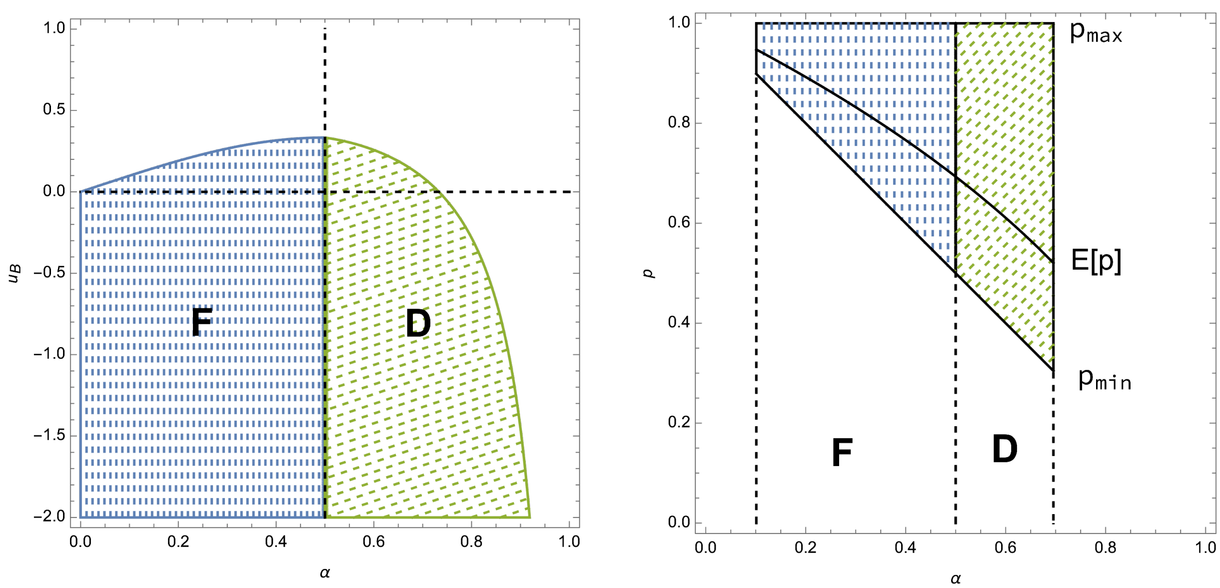
Proposition 6. *In the obeident equilibrium the platform will choose to provide the same information design for both sellers. When* $\alpha \in [0, 0.5]$, *the platform will choose full revelation. When* $\alpha \in (0.5, 1]$, *the platform will choose to demarket and set* $p_{Gj} < 1, p_{Bj} = 0$.

Figure 4 illustrates the range of parameters where equilibrium exists. Region F indicates full revelation while Region D indicates demarketing. The upper curve is the boundary that support an equilibrium in the second pricing stage.

Figure 4: Equilibria and Price Distributions under a Symmetric Platform Design

(a) Equilibria

(b) Price distributions when $u_B = 0.1$



The left figure presents the range of parameters α and u_B where equilibria exist and their type (Full revelation or Demarketing), while the right figure shows the maximum, minimum and expected prices of both firms when $u_B = 0.1$.

The platform's choice to demarket is interesting. The intuition is that the platform wants to avoid head to head competition between the two firms. By constraining both firms' ability to be truthful when the signal is good, it is essentially softening the price war, which has little effect on demand, but increases total sales.

4.2.2 Asymmetric design: the platform provides information only for one firm

We perform the analysis in two parts: (i) when $\mathbb{E}[u] \geq 0$ and (ii) when $\mathbb{E}[u] < 0$. When $\mathbb{E}[u] \geq 0$, we know from Lemma 4 that the profits of the two firms are $\Pr(g) * (u(g) - \Pr(g)\mathbb{E}[u])$ and $\Pr(b)\mathbb{E}[u]$, respectively. In this case, the platform's optimization problem becomes:

$$\begin{aligned}
\max_{p_{G1}, p_{B1}} \pi_P =& \pi_j + \pi_{-j} \\
=& \Pr(b)\mathbb{E}[u] + \Pr(g) * (u(g) - \Pr(g)\mathbb{E}[u]) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \\
=& ((1-\alpha)p_{B1} + \alpha p_{G1}) \left(\frac{(1-\alpha)u_B p_{B1} + \alpha p_{G1}}{(1-\alpha)p_{B1} + \alpha p_{G1}} - (\alpha + (1-\alpha)u_B)((1-\alpha)p_{B1} + \alpha p_{G1}) \right) \\
&+ (\alpha + (1-\alpha)u_B)((1-\alpha)(1-p_{B1}) + \alpha(1-p_{G1})) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \\
&s.t. \quad 0 \leq p_{B1} \leq 1, 0 \leq p_{G1} \leq 1
\end{aligned} \tag{15}$$

Next, we consider the case where $\mathbb{E}[u] < 0$. Without loss of generality, we assume that the platform does not provide information for firm 2, while it sets p_{G1} and p_{B1} for firm 1. In the second stage, firm 2 will price at zero, earn zero profit, and no consumer will buy from firm 2 since $\mathbb{E}[u] < 0$. In the obedient equilibrium, the consumer would buy from firm 1 if $u(g) - p_1 \geq 0$. As a result, firm 1 can extract the maximum revenue by charging a price p_1 equal to $u(g)$. The platform's optimization problem now becomes:

$$\begin{aligned}
\max_{p_{G1}, p_{B1}} \pi_P =& \pi_j + \pi_{-j} \\
=& 0 + \Pr(g) * u(g) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \\
=& ((1-\alpha)p_{B1} + \alpha p_{G1}) \left(\frac{(1-\alpha)u_B p_{B1} + \alpha p_{G1}}{(1-\alpha)p_{B1} + \alpha p_{G1}} \right) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \\
&s.t. \quad 0 \leq p_{B1} \leq 1, 0 \leq p_{G1} \leq 1
\end{aligned} \tag{16}$$

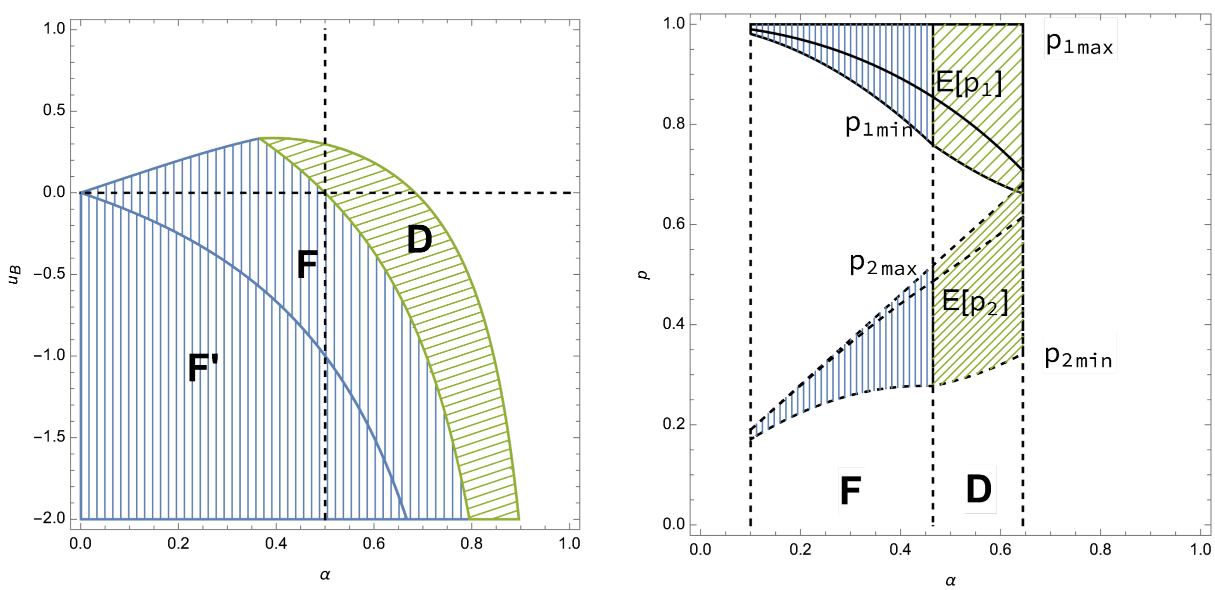
The platform will choose full revelation and set $p_{B1}^* = 0$ and $p_{G1}^* = 1$ since when $\mathbb{E}[u] < 0$, the platform's profit is monotonically increasing in p_{G1} and decreasing in p_{B1}. Combining the results of the analysis from these two cases, we find the following:

Proposition 7. *When the platform sets an asymmetric information design, if* $\mathbb{E}[u] > 0$*, it will select full revelation when* α *is small, and demarket with* $p_{G1} < 1$ *and* $p_{B1} = 0$ *when* α *is large. If* $\mathbb{E}[u] < 0$*, it will select full revelation.*

Figure 5 illustrates the results. Region F and F' are where the platform will pick full revelation, while Region D indicates where demarketing will be used. The upper curve is the condition required to support an equilibrium in the second stage. The intuition behind the results is similar to the case with symemtric information design — when α is high, the platform would like the firms to avoid price competition. By using demarketing, the platform restricts the ability of the stronger firm to compete over customers by lowering its prices and increasing consumer surplus.

Figure 5: Equilibria and Price Distributions under an Asymmetric Platform Design

(a) Equilibria (b) Price distributions when $u_B = 0.1$



The left figure shows the range of parameters and corresponding strategy that the platform chooses for the seller that provides information. The platform selects: (i) Full revelation (Region F and F'), or (ii) Demarketing (Region D). In Region F, $\mathbb{E}[u] > 0$ and both sellers are active in the market. In Region F', $\mathbb{E}[u] < 0$ and firm 2 shuts down. The right figure shows the maximum, minimum, and expected prices of both firms when $u_B = 0.1$. The solid lines characterize prices for firm 1 which provides information and the dashed lines characterize prices for firm 2.

4.2.3 The optimal information design of the platform

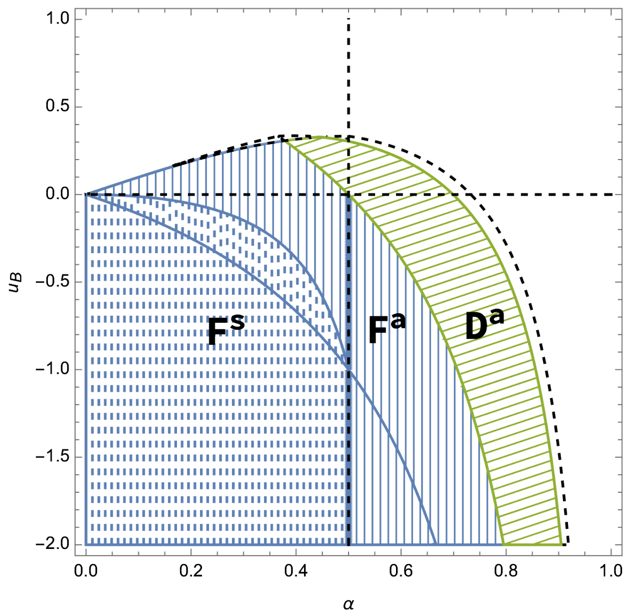
Having analyzed the two possible strategies of the platform, we now shift to finding the equilibrium choice. We first rule out the case where the platform provides no information for both firms. If the platform prevents any information transmission, the profit of both firms and then the platform is zero (due to convergence to Bertrand competition), which is the least preferred by the platform. Hence, we will focus on the cases with obedient equilibria with information transmission. The next proposition fully characterizes the equilibrium choice of the platform.

Proposition 8. *The optimal information design for the platform is:*

- *For any $u_B < 0$, there are four cutoff points $\alpha_1(u_b) < \alpha_2(u_b) < \alpha_3(u_b) < \alpha_4(u_b)$ and the platform selects:*
  - *Full revelation ($p_B = 0, p_G = 1$) with symmetric design when $\alpha < \alpha_1$, or asymmetric design when $\alpha_1 \leq \alpha < \alpha_2$.*
  - *Demarketing ($p_B = 0, p_G < 1$) with asymmetric design when $\alpha_2 \leq \alpha < \alpha_3$, or with symmetric design when $\alpha_3 \leq \alpha < \alpha_4$.*
- *For any $u_B > 0$, there are three cutoff points $\alpha_2(u_b) < \alpha_3(u_b) < \alpha_4(u_b)$ and the platform selects:*
  - *Full revelation ($p_B = 0, p_G = 1$) with asymmetric design when $\alpha < \alpha_2$.*
  - *Demarketing ($p_B = 0, p_G < 1$) with asymmetric design when $\alpha_2 \leq \alpha < \alpha_3$, or with symmetric design when $\alpha_3 \leq \alpha < \alpha_4$.*

Figure 6 illustrates these results ("F" represents Full Revelation, "D" represents Demarketing; "s" represents a symmetric design and "a" stands for asymmetric.). In Region $\mathrm{F^s}$ ($\mathrm{F^a}$), the platform chooses full revelation for both (one of the) firms. In Region $\mathrm{D^s}$ ($\mathrm{F^a}$), the platform chooses demarketing for both (one of the) firms.

Figure 6: Platform Equilibrium Selection



The results not only echo our previous findings, but also add new insights about the impact of the platform that chooses the information design. Similarly to before, when α increases, the platform moves away from full revelation to demarketing, in order to deescalate price competition between the two firms. The additional insight is that centralizing the information design at the platform level allows it to select the equilibrium that the sellers will play in the pricing game. This allows the platform to make sure the sellers do not compete too strongly, even in parameter ranges where they would have deviated under duopoly competition. Consequently, the platform is able to make a profit even when the information design is symmetric, a case which is not possible with duopoly competition. In other words, the ability to not persuade customers benefits a platform more than it benefits duopolists.

5 Conclusion

It is often true that providing more information to customers about a product can help consumers make better decisions and make a marketer more successful. This fact, however, raises a dilemma for marketers – how much information should they provide, and how truthful and revealing should this information be about the product?

Many marketers are often tempted to use information as a persuasion mechanism, assuming that they are benefiting the firm or the platform in the process. The recent literature on Bayesian persuasion has provided a formal argument underpinning this decision – the results show that firms can increase their demand by credibly persuading consumers to purchase products even if the products do not match the consumers' preferences.

Our analysis has revealed that when applied to more nuanced marketing scenarios, persuasion is not often the best approach, and truthfulness (or even humility executed through demarketing) are often preferred. The results show that when pricing and competition incentives are added to the mix, firms who choose to persuade consumers often lose profits because persuaded consumers are either willing to pay less (monopoly), or switch more easily to competitors (duopoly). These results also apply to a platform that picks an information design for sellers on the platform, since persuasion increases competition and lowers the platform's revenue. The ability to change information designs allows the firms to appear more differentiated for consumers, which counteracts the incentive to persuade. Often this will result in an asymmetric equilibrium where one firm provides information and the other does not. In extreme cases, when competing products are expected to match consumer tastes with high a priori probability, firms can further differentiate by demarketing, or "hiding" information about good products, because this approach increases perceived differentiation by consumers. The result is decreased price competition even further beyond full revelation.

The results predict that persuasion will be less common with competition or when firms have pricing power. We also help explain the choice of platforms and online sellers in how they disclose product information, and how they design search experiences Dukes and Liu (2016). Similarly to previous findings, we show that information design can be used strategically to alter the level of competition in a market or on a platform. Firms should realize that they might not want to feature the same attributes as their competitors, while platforms might create implicit differentiation by featuring different attributes even for very similar products.

The results also have implications for how platforms and regulators should respond to persuasive information which might have negative consequences (e.g., fake reviews, misleading native advertising, influencer marketing). For example, when considering the level of enforcement against fake reviews (He et al. 2020) platforms might realize that allowing persuasive fake reviews might lower their revenues. Similarly, regulators have mandated that social media influencers disclose financial relationships with the brands to discourage misleading consumers. Our model predicts that the level of disclosure should affect the persuasiveness of the messages used by influencers, but will also affect prices in this market.

We limited our analysis to the fundamental cases of monopoly, duopoly and a platform to maintain parsimony, but there are additional avenues to explore the limits of persuasion. We did not consider the implications of heterogeneity in information design, e.g., consumers with different tastes. The analysis of heterogeneity provides a fruitful avenue for future research, but remains an open question due to the technical complexity that it raises, even in simple models. Another promising avenue for future research is targeted information design where firms can personalize the information provided to customers based on their tastes.

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A Proofs

Proof of Proposition 1. When $\alpha < \frac{m-u_B}{u_G-u_B}$, the consumer will not consume without additional information since $\mathbb{E}[u] < 0$. Hence, a sufficient condition for an obedient equilibrium to exist is an information design where $u(g) \geq 0$. First, we notice the condition on α only holds if $u_B < m$. Moreover, the revenue of the firm increases in p_G and the cost decreases in p_G, hence it is optimal to set $p_G^* = 1$. As the expected utility of the consumer $u(g)$ increases in p_G and $u(b)$ decreases in p_G, setting $p_G = 1$ will also not violate the IC constraints.

The solution to the unconstrained maximization problem of the monopolist when $p_G = 1$ is $p_B = m(1-\alpha)$, while the incentive compatibility constraints can be written as:

$$p_B \leq \frac{\alpha(u_G - m)}{(1-\alpha)(m-u_B)} \tag{17}$$

$$(1-\alpha)(1-p_B)(u_B - m) \leq 0 \tag{18}$$

The second condition holds for any $u_B < m$, and hence in equilibrium, only the first constraint might be binding. The firm will increase p_B as much as possible as it increases its revenue, as long as the cost is not too high.

That means it will set $p_B^* = \min\left(m(1-\alpha), \frac{\alpha(u_G-m)}{(1-\alpha)(m-u_B)}\right)$ which maximizes p_B subject to making sure the consumers still consume the product. This equilibrium is unique because of the concavity of the profit function with respect to p_B. As a final remark, notice that for every value of $u_G > m$, $u_B < m$ and $\alpha < \frac{m-u_B}{u_G-u_B}$, $0 < p_B^* < 1$. □

Proof of Proposition 2. First, we notice that the revenue increases in p_G and the cost decreases in p_G. Hence it is optimal to always set $p_G^* = 1$.

When $u_B > 0$, the profit from setting $1 > p_B > 0$ is $(1-\alpha)p_B u_B - \frac{p_B^2}{2}$, but without sending a signal, the profit is $(1-\alpha)u_B$ which is larger because $1 \geq p_B \geq 0$. Hence, the firm will not want to send a signal when $u_B > 0$, and an obedient equilibrium does not exist.

When $u_B < 0$, the revenue of the firm decreases in p_B as well as the cost. Hence the firm would like to set p_B as low as possible, implying that $p_B^* = 0$. The profit in this case is higher than the expected profit without a signal. □

Proof of Proposition 3. We prove this by contradiction. We first assume there exists a equilibrium where duopoly firms set $p_B > 0$, then we show that at least one firm will deviate from this equilibrium.

If in equilibrium both firms set $p_B > 0$, then we know that changing p_B will affect two segments of consumers: the segment of consumers who obtain good signals from both firms, and the segment of consumers who obtain a good signal from one firm. In a symmetric equilibrium where both firm choose $p_B^* > 0$, $u_j(g) = Pr(G_j|g_j)u_G + Pr(B_j|g_j)u_B = \frac{\alpha p_{Gj}u_G + (1-\alpha)p_{Bj}u_B}{\alpha p_{Gj} + (1-\alpha)p_{Bj}} < \frac{p_{Gj}u_G}{p_{Gj}} = u_G$, because $\frac{\partial u_j(g)}{\partial p_B} = \frac{(1-\alpha)\alpha p_G(u_B - u_G)}{(\alpha p_G + (1-\alpha)p_B)^2} < 0$. Hence, deviating by decreasing p_B will decrease costs, and will attract more customers from the first segment, thus increasing demand. This contradicts the assumed equilibrium. Given that in any equilibrium $p_B^* = 0$, the optimal decision for both firms is then to set $p_G^* = 1$. □

Proof of Lemma 1. Since only the consumer who obtains a good signal will buy, all firms will set prices above $u_j(b)$. Then the profit of a firm setting price p_j when the competitor sets price p_{-j} is:

$$\pi_j(p_j, p_{-j}) = \left(\alpha(1-\alpha) + \alpha^2 \mathbb{I}[p_j < p_{-j}]\right) p_j \tag{19}$$

where $\mathbb{I}[\cdot]$ is the indicator function.

Because in a mixed strategy the firm should be indifferent between setting different prices, p_{min} will satisfy:

$$\pi_j(p_{min}, p_{-j}) = \left(\alpha^2 + \alpha(1-\alpha)\right) p_{min} = \alpha(1-\alpha)u_j(g) \tag{20}$$

resulting in $p_{min} = (1-\alpha)u_G$. The lowest price should be higher than u_B (otherwise some consumers may buy with a bad signal), which means we need to restrict $u_j(b) < (1-\alpha)u_j(g)$, which implies $u_B < (1-\alpha)u_G$.

Finally, if we denote by $F(\cdot)$ the cdf of the equilibrium prices in $[p_{min}, p_{max}]$, then we can solve for $F(\cdot)$ in:

$$\pi_j(p) = \left(\alpha^2(1-F(p)) + \alpha(1-\alpha)\right) p = \alpha(1-\alpha)u_j(g) \tag{21}$$

resulting in the equilibrium mixed strategy price distribution $F(p) = \frac{p-(1-\alpha)u_j(g)}{\alpha p}$.

A final step in the analysis is to verify that no firm has a profitable deviation. In our case, a firm could charge a price lower than $u_j(b) = u_B$, sell products to consumers who receive a bad signal, and potentially make higher profit.

If a firm sets a price lower than u_B, when the competition mixes their pricing according to $F(p)$, its expected profit will be:

$$\pi_j(p) = \left(\alpha^2 + \alpha(1-\alpha) + (1-\alpha)^2 + (1-\alpha)\alpha(1 - F(u_{-j}(g) - u_j(b) + p))\right) p \tag{22}$$

The second derivative of the profit with respect to price is $\frac{2(1-\alpha)^2 u_j(g)(u_j(b)-u_j(g))}{(p-u_j(b)+u_j(g))^3}$, which is negative for any $p \leq u_j(b)$. Hence the function is concave. The first derivative at $p = u_j(b)$ is $1 - \alpha(1-\alpha) - \frac{(1-\alpha)^2 u_j(b)}{u_j(g)} > 1 - \alpha(1-\alpha) - (1-\alpha)^2 > 0$.

As a result, a deviating firm can maximize its profit by setting the price to $u_j(b)$, yielding a profit

$$(\alpha^2 + \alpha(1-\alpha) + (1-\alpha)^2)u_j(b)$$

If this deviation profit is lower than the equilibrium profit of $\alpha(1-\alpha)u_j(g)$, no firm will want to deviate.

Hence, we need that $u_j(b) = u_B < \min\left((1-\alpha)u_j(g), \frac{\alpha(1-\alpha)}{1-\alpha+\alpha^2}u_j(g)\right) = \frac{\alpha(1-\alpha)}{1-\alpha+\alpha^2}u_j(g)$ when $u_j(g) = u_G = 1$ and $0 < \alpha < 1$ to maintain an equilibrium where consumers do not buy with a bad signal. □

Proof of Lemma 2. Calculating $\frac{\partial \pi_j}{\partial p_{Gj}} = \alpha\left((1-\alpha)(1-p_{B-j}) + \alpha(1-p_{G-j})\right) + 1 - p_{Gj} \geq 0$, $\forall p_{Gj}, p_{G-j}, p_{Bj}, p_{B-j} \in [0,1]$. And we also have $\frac{\partial^2 \pi_j}{\partial p_{Gj}^2} = -1$, which implies π_j is concave in p_{Gj}. So, it is maximized at the corner when $p_{Gj} = 1$. □

Proof of Lemma 3. Lemma 2 shows that both firms will set $p_G^* = 1$ in any obedient equilibrium. When $u_B \leq 0$, Firm's j's payoff has the FOC

$$\frac{\partial \pi_j}{\partial p_{Bj}}|_{p_{Gj}=1, p_{G-j}=1} = u_B(1-\alpha)^2(1-p_{B-j}) - p_{Bj} \leq 0, \forall p_{Bj}, p_{B-j} \in [0,1]$$

implying that they will set $p_{Bj}^* = 0$ and $p_{Gj}^* = 1$. Since $u_B < 0$, the condition $u_B < \frac{\alpha(1-\alpha)}{\alpha^2-\alpha+1} u_G$ always holds and ensures that no firm will deviate in the second stage to lower pricing given p_{Gj} and p_{Bj}.

When $u_B > 0$ we have $\left.\frac{\partial \pi_j}{\partial p_{Bj}}\right|_{p_{Gj}=1, p_{G-j}=1} = u_B(1-\alpha)^2(1-p_{B-j}) - p_{Bj}$. Both firms will not set $p_{Bj} = 1$, because $\left.\frac{\partial \pi_j}{\partial p_{Bj}}\right|_{p_{Gj}=1, p_{G-j}=1, p_{Bj}=1} = u_B(1-\alpha)^2(1-p_{B-j}) - 1 < u_B - 1 < 0$. Given that $p_{B-j} \neq 1$, the fact that $\left.\frac{\partial \pi_j}{\partial p_{Bj}}\right|_{p_{Gj}=1, p_{G-j}=1, p_{Bj}=0, p_{B-j}\neq 1} = u_B(1-\alpha)^2(1-p_{B-j}) > 0$ suggests that both firms will set $p_{Bj} \in (0,1)$ because π_j is concave in p_{Bj}. Solving the FOCs

$$\left.\frac{\partial \pi_j}{\partial p_{Bj}}\right|_{p_{Gj}=1, p_{G-j}=1} = u_B(1-\alpha)^2(1-p_{B-j}) - p_{Bj} = 0$$

we find the solution $p_{B1}^* = p_{B2}^* = \frac{u_B(1-\alpha)^2}{1+u_B(1-\alpha)^2}$. The concavity of π_j w.r.t. to p_{Bj} and the positive solutions for $0 < u_B < 1$ imply that this is a unique candidate for a Nash equilibrium. For it to be a Nash equilibrium, the lowest price charged by any firm in the second stage needs to be higher than $u_j(b)$, and no firm should have a profitable deviation. In a mixed strategy equilibrium, the indifference between setting any price leads to the lowest price p_{min} satisfying:

$$\begin{aligned}&[(\alpha p_{Gj} + (1-\alpha)p_{Bj})(\alpha p_{G-j} + (1-\alpha)p_{B-j}) + (\alpha p_{Gj} + (1-\alpha)p_{Bj})(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j}))]\, p_{min} \\ &= (\alpha p_{Gj} + (1-\alpha)p_{Bj})(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j}))\, u_j(g)\end{aligned}$$

resulting in $p_{min} = \frac{(\alpha-1)\left(-\alpha+(\alpha-1)^3 u_B^2 - \alpha(\alpha-1)^2 u_B\right)}{((\alpha-1)^2 u_B+1)(\alpha+(\alpha-1)^2 u_B)}$. The constraint that the lowest price is higher than $u_j(b)$, which deters consumers from buying with a bad signal, yields

$$u_B < \frac{4(\alpha-3)\alpha + \frac{4\sqrt[3]{-2}\alpha^3(3-2\alpha)}{\sqrt[3]{\mathcal{M}}} + 2^{2/3}\left(1-i\sqrt{3}\right)\sqrt[3]{\mathcal{M}}}{12(\alpha-1)^2}$$

where

$$\begin{aligned}\mathcal{M} &= \alpha(\alpha((\alpha-3)\alpha(\alpha(7\alpha-15)+36)+81)-27) \\ &+ 3(1-\alpha)\alpha\sqrt{9\alpha^8 - 54\alpha^7 + 177\alpha^6 - 420\alpha^5 + 702\alpha^4 - 810\alpha^3 + 648\alpha^2 - 324\alpha + 81}\end{aligned}$$

By denote $F(\cdot)$ a the cdf of the equilibrium prices in $[p_{min}, p_{max}]$, we can solve for $F(\cdot)$ in:

$$\begin{aligned}\pi_j(p) =& \left(\alpha p_{Gj} + (1-\alpha)p_{Bj}\right)\left(\alpha p_{G-j} + (1-\alpha)p_{B-j}\right)(1-F(p))p \\ &+ \left(\alpha p_{Gj} + (1-\alpha)p_{Bj}\right)\left(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})\right)p \\ =& \left(\alpha p_{Gj} + (1-\alpha)p_{Bj}\right)\left(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})\right)u_j(g)\end{aligned}$$

resulting in the equilibrium price distribution $F(p) = \frac{(\alpha-1)^2 p u_B + (\alpha-1)u_j(g)+p}{p(\alpha+(\alpha-1)^2 u_B)}$ within the range $[\frac{\alpha+(\alpha-1)^3\left(-u_B^2\right)+\alpha(\alpha-1)^2 u_B}{\alpha+(\alpha-1)^2 u_B}, u_j(g)]$. If a firm sets a price lower than $u_j(b)$, when the competition mixes their pricing according to $F(p)$, the firm's expected profit will be:

$$\begin{aligned}\pi_j(p) = & \, p\left((1-\alpha)\left(1-p_{Bj}\right) + \alpha\left(1-p_{Gj}\right)\right)\left((1-\alpha)p_{B-j} + \alpha p_{G-j}\right)\left(1 - F(u_{-j}(g) - u_j(b) + p)\right) \\ &+ p\left((1-\alpha)\left(1-p_{Bj}\right) + \alpha\left(1-p_{Gj}\right)\right)\left((1-\alpha)\left(1-p_{B-j}\right) + \alpha\left(1-p_{G-j}\right)\right) \\ &+ p\left((1-\alpha)p_{Bj} + \alpha p_{Gj}\right)\left((1-\alpha)\left(1-p_{B-j}\right) + \alpha\left(1-p_{G-j}\right)\right) \\ &+ p\left((1-\alpha)p_{Bj} + \alpha p_{Gj}\right)\left((1-\alpha)p_{B-j} + \alpha p_{G-j}\right)\end{aligned}$$

The function is concave with respect to any price $p \leq u_j(b)$. The payoff is increasing when $p = u_j(b)$ for $\alpha \in [0,1]$ and $0 < u_B < 1$. As a result the maximum deviation profit will be achieved when setting $p = u_j(b)$, yielding a profit $\frac{u_B\left(\alpha^2-\alpha+(\alpha-1)^4 u_B^2+(\alpha+1)(\alpha-1)^2 u_B+1\right)}{((\alpha-1)^2 u_B+1)^2}$. The condition

$$u_B < \bar{u}_B = \frac{2(\alpha-3)\alpha - \frac{2\sqrt[3]{2}(\alpha(\alpha(\alpha(2\alpha-3)+3)-6)+3)}{\sqrt[3]{\mathcal{N}}} + 2^{2/3}\sqrt[3]{\mathcal{N}}}{6(\alpha-1)^2}$$

where

$$\begin{aligned}\mathcal{N} =& -7\alpha^6 + 36\alpha^5 - 90\alpha^4 + 153\alpha^3 - 144\alpha^2 + 54\alpha \\ &+ 3\sqrt{3}(1-\alpha)\sqrt{3\alpha^{10} - 18\alpha^9 + 69\alpha^8 - 194\alpha^7 + 383\alpha^6 - 556\alpha^5 + 588\alpha^4 - 388\alpha^3 + 132\alpha^2 - 16\alpha + 4}\end{aligned}$$

ensures that the deviation profit is lower than the equilibrium profit to maintain an equilibrium. □

Proof of Lemma 4. Assume it is not profitable for firm 1 to lower the price and sell products to consumers who receive bad signal from it. So this constraint ensures that firm 1 only focuses on consumers who receive good signals. After using backward induction to find the optimal information design policy in the first stage, we will check that firm 1 has not have the incentive to deviate to a price lower than $u(b)$.

We first suppose both firms are using mixed strategies and neither firm has a mass point. For firm 2, it is mixing between $[p_{2\min}, \mathbb{E}[u]]$. When firm 2 charges $\mathbb{E}[u]$, it could only sell products to consumers receiving bad signals from firm 1. The profit is $\Pr(b) * \mathbb{E}[u]$. When firm 2 charges $p_{2\min}$, it could obtain the whole market. So $p_{2\min} = \Pr(b) * \mathbb{E}[u]$. For the prices between $\Pr(b) * \mathbb{E}[u]$ and

$\mathbb{E}[u]$, the indifferent condition for firm 2 is

$$p_2 * [\Pr(b) + \Pr(g)(1 - F_1(u(g) - \mathbb{E}[u] + p_2))] = \Pr(b) * \mathbb{E}[u], \qquad \text{for } \Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u] \quad (23)$$

So, the profit for firm 2 is $\Pr(b) * \mathbb{E}[u]$. And firm 2 is mixing between $[\Pr(b) * \mathbb{E}[u], \mathbb{E}[u]]$. Notice that in order for this equilibrium to hold, we must have $\mathbb{E}[u] \geq 0$. Otherwise, the profit for firm 2 is negative.

For firm 1, first notice that firm 1 will not charge a prices lower than $u(g) - \Pr(g)\mathbb{E}[u]$. This is because we have the constraint above to ensure that firm 1 will only focuses on consumers receiving good signal from it. Firm 2 is mixing between $[\Pr(b) * \mathbb{E}[u], \mathbb{E}[u]]$, and the maximum surplus that firm 2 gives to the consumers is $\mathbb{E}[u] - \Pr(b) * \mathbb{E}[u] = \Pr(g) * \mathbb{E}[u]$. So for firm 1, when charging $u(g) - (\mathbb{E}[u] - \Pr(b)\mathbb{E}[u]) = u(g) - \Pr(g)\mathbb{E}[u]$, it gives consumer $\Pr(g) * \mathbb{E}[u]$ surplus. Thus, in this case, firm 1 could obtain all consumers receiving good signals from it(since firm 2 has no mass point at the lowest price $\Pr(b)\mathbb{E}[u]$). The profit for firm 1 is $\Pr(g) * (u(g) - \Pr(g)\mathbb{E}[u])$ at this point. Then, for the prices higher than $u(g) - \Pr(g)\mathbb{E}[u]$, firm 1 is facing a trade-off between setting a higher price and losing proportion of $\Pr(g)$, the indifferent condition for firm 1 is

$$p_1 * \Pr(g) * (1 - F_2(p_1 - (u(g) - \mathbb{E}[u]))) - \frac{(1 - p_{G1})^2}{2} - \frac{p_{B1}^2}{2} = \Pr(g) * (u(g) - \Pr(g)\mathbb{E}[u]) - \frac{(1 - p_{G1})^2}{2} - \frac{p_{B1}^2}{2} \quad (24)$$
$$\text{for } u(g) - \Pr(g)\mathbb{E}[u] \leq p_1 \leq u(g)$$

So, the profit for firm 1 is $\Pr(g) * (u(g) - \Pr(g)\mathbb{E}[u]) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2}$. And firm 1 is mixing between $[u(g) - \Pr(g)\mathbb{E}[u], u(g)]$

Solving equation (23), we could get

$$F_1(p_1) = \begin{cases} 0, & \text{for } p_1 \leq p_{2\min} + (u(g) - \mathbb{E}[u]) = u(g) - \Pr(g)\mathbb{E}[u] \\ 1 + \frac{\Pr(b)}{\Pr(g)} - \frac{\Pr(b)\mathbb{E}[u]}{\Pr(g)(p_1 - (u(g) - \mathbb{E}[u]))}, & \text{for } u(g) - \Pr(g)\mathbb{E}[u] \leq p_1 \leq u(g) \\ 1, & \text{for } p_1 \geq u(g) \end{cases} \quad (25)$$

Plugging in $p_1 = u(g) - \Pr(g)\mathbb{E}[u]$, $F_1(u(g) - \Pr(g)\mathbb{E}[u]) = 1 + \frac{\Pr(b)}{\Pr(g)} - \frac{\Pr(b)\mathbb{E}[u]}{\Pr(g)(u(g) - \Pr(g)\mathbb{E}[u] - (u(g) - \mathbb{E}[u]))} = 1 + \frac{\Pr(b)}{\Pr(g)} - \frac{\Pr(b)\mathbb{E}[u]}{\Pr(g)\Pr(b)\mathbb{E}[u]} = 0$. Plugging in $p_1 = u(g)$, $F_1(u(g)) = 1 + \frac{\Pr(b)}{\Pr(g)} - \frac{\Pr(b)\mathbb{E}[u]}{\Pr(g)(u(g) - (u(g) - \mathbb{E}[u]))} = 1$. This suggests that firm 1 has no mass point.

The expectation of firm 1's price $\mathbb{E}(p_1)$ is

$$\mathbb{E}(p_1) = \int_{u(g) - \Pr(g)\mathbb{E}[u]}^{u(g)} pdf(p_1) * p_1 \, \mathrm{d}p_1 = \int_{u(g) - \Pr(g)\mathbb{E}[u]}^{u(g)} \frac{\Pr(b)\mathbb{E}[u]}{\Pr(g)(p_1 - (u(g) - \mathbb{E}[u]))^2} * p_1 \, \mathrm{d}p_1 \quad (26)$$

Solving equation (24), what we get is

$$F_2(p_2) = \begin{cases} 0, & \text{for } p_2 \leq \Pr(b) * \mathbb{E}[u] \\ 1 - \frac{u(g) - \Pr(g)\mathbb{E}[u]}{(p_2 + (u(g) - \mathbb{E}[u]))}, & \text{for } \Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u] \\ 1, & \text{for } p_2 \geq \mathbb{E}[u] \end{cases} \tag{27}$$

Notice that when plugging in $p_2 = \mathbb{E}[u]$, $F_2(\mathbb{E}[u]) = 1 - \frac{u(g) - \Pr(g)\mathbb{E}[u]}{(\mathbb{E}[u] + (u(g) - \mathbb{E}[u]))} = \frac{\Pr(g)\mathbb{E}[u]}{u(g)} < 1$. This implies that firm 2 has a mass point at the highest price $\mathbb{E}[u]$ with a probability $1 - \frac{\Pr(g)\mathbb{E}[u]}{u(g)}$. Since 2 has a mass point at the highest price $\mathbb{E}[u]$ and firm 1 has no mass point, equation (24) remains unchanged.

The expectation of firm 2's price $\mathbb{E}(p_2)$ is

$$\begin{aligned} \mathbb{E}(p_2) &= \int_{\Pr(b)\mathbb{E}[u]}^{\mathbb{E}[u]} pdf(p_2) * p_2 \, \mathrm{d}p_2 + \mathbb{E}[u] * \left(1 - \frac{\Pr(g)\mathbb{E}[u]}{u(g)}\right) \\ &= \int_{\Pr(b)\mathbb{E}[u]}^{\mathbb{E}[u]} \frac{u(g) - \Pr(g)\mathbb{E}[u]}{(p_2 + (u(g) - \mathbb{E}[u]))^2} * p_2 \, \mathrm{d}p_2 + \mathbb{E}[u] * \left(1 - \frac{\Pr(g)\mathbb{E}[u]}{u(g)}\right) \end{aligned} \tag{28}$$

□

Proof of Lemma 5. Plugging in $u_G = 1$, the Lagrangian of firm 1's maximization problem becomes

$$\begin{aligned} \mathcal{L}(p_{G1}, p_{B1}) = &((1-\alpha)p_{B1} + \alpha p_{G1}) \left(\frac{(1-\alpha)p_{B1}u_B + \alpha p_{G1}}{(1-\alpha)p_{B1} + \alpha p_{G1}} - ((1-\alpha)p_{B1} + \alpha p_{G1})((1-\alpha)u_B + \alpha) \right) \\ &- \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2} - \lambda_1(p_{G1} - 1) - \lambda_2(p_{B1} - 1) - \lambda_3(-p_{G1}) - \lambda_4(-p_{B1}) \\ &s.t. p_{Bj} \leq 1, -p_{Bj} \leq 0, p_{Gj} \leq 1, -p_{Gj} \leq 0 \end{aligned} \tag{29}$$

The Kuhn-Tucker conditions for this maximization problem is

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha - 2(\alpha-1)\alpha p_{B1}((\alpha-1)u_B - \alpha) + p_{G1}\left(-2\alpha^3 + 2(\alpha-1)\alpha^2 u_B - 1\right) + 1 - \lambda_1 + \lambda_3 = 0 \tag{30}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p_{B1}} &= p_{B1}\left(-2\alpha(\alpha-1)^2 + 2(\alpha-1)^3 u_B - 1\right) - (\alpha-1)(2\alpha p_{G1}((\alpha-1)u_B - \alpha) + u_B) - \lambda_2 + \lambda_4 = 0 \\ &\lambda_i \geq 0, p_{G1} \leq 1, p_{B1} \leq 1, -p_{G1} \leq 0, -p_{B1} \leq 0, \text{ and,} \\ &\lambda_1(p_{G1} - 1) = 0, \lambda_2(p_{B1} - 1) = 0, \lambda_3(-p_{G1}) = 0, \lambda_4(-p_{B1}) = 0 \end{aligned} \tag{31}$$

- If $p_{G1}^* = 0$

  Then, $\lambda_1(p_{G1} - 1) = 0$. So $\lambda_1 = 0$

(30) of the Kuhn-Tucker conditions now becomes

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = -2(\alpha-1)\alpha p_{B1}\left((\alpha-1)u_B-\alpha\right)+\alpha+1+\lambda_3 = 0 \tag{32}$$

$$\lambda_3 \geq 0 \tag{33}$$

$\lambda_3 = 2(\alpha-1)\alpha p_{B1}\left((\alpha-1)u_B-\alpha\right)-\alpha-1 < 0$ when $\alpha \in [0,1]$, which contradicts $\lambda_3 \geq 0$.

So, $p^*_{G1} = 0$ could not be the solution.

- If $p^*_{B1} = 1$

  Then, $\lambda_4(-p_{B1}) = 0$. So $\lambda_4 = 0$

  (31) of the Kuhn-Tucker conditions now becomes

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = -2\alpha^3+4\alpha^2-2\alpha-(\alpha-1)\left(2\alpha p_{G1}\left((\alpha-1)u_B-\alpha\right)+u_B\right)+2(\alpha-1)^3u_B-1-\lambda_2 = 0 \tag{34}$$

$$\lambda_2 \geq 0 \tag{35}$$

  $\lambda_2 = -2\alpha^3+4\alpha^2-2\alpha-(\alpha-1)\left(2\alpha p_{G1}\left((\alpha-1)u_B-\alpha\right)+u_B\right)+2(\alpha-1)^3u_B-1 < 0, \alpha \in [0,1]$ contradicting $\lambda_2 \geq 0$.

  So, $p^*_{B1} = 1$ could not be the solution.

- If $p^*_{G1} \in (0,1), p^*_{B1} \in (0,1)$

  Then, $\lambda_i = 0, i = 1,2,3,4$

  (30) and (31) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha-2(\alpha-1)\alpha p_{B1}\left((\alpha-1)u_B-\alpha\right)+p_{G1}\left(-2\alpha^3+2(\alpha-1)\alpha^2u_B-1\right)+1 = 0 \tag{36}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = p_{B1}\left(-2\alpha^3+4\alpha^2-2\alpha+2(\alpha-1)^3u_B-1\right)-(\alpha-1)\left(2\alpha p_{G1}\left((\alpha-1)u_B-\alpha\right)+u_B\right) = 0 \tag{37}$$

$$p^*_{G1} \in (0,1), p^*_{B1} \in (0,1) \tag{38}$$

  Solving the equations (36) and (37), we could then find the solution candidate:

$$p_{G1} = \frac{-2\alpha^4+2\alpha^3+2\alpha^2-3\alpha+2(\alpha-1)^2\left(2\alpha^2-1\right)u_B-2(\alpha-1)^3\alpha u_B^2-1}{-4\alpha^3+4\alpha^2-2\alpha+\left(4\alpha^3-8\alpha^2+6\alpha-2\right)u_B-1}$$

$$p_{B1} = -\frac{(\alpha-1)\left(2(\alpha+1)\alpha^2+\left(-4\alpha^3+2\alpha-1\right)u_B+2(\alpha-1)\alpha^2u_B^2\right)}{-4\alpha^3+4\alpha^2-2\alpha+\left(4\alpha^3-8\alpha^2+6\alpha-2\right)u_B-1}$$

$p^*_{G1} \in (0,1), p^*_{B1} \in (0,1)$ cannot hold at the same time.

So, $p^*_{G1} \in (0,1), p^*_{B1} \in (0,1)$ could not be the solution.

- If $p^*_{G1} = 1, p^*_{B1} \in (0,1)$

  Then, $\lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

  (30) and (31) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha\left(-2\alpha^2 - 2(\alpha-1)p_{B1}\left((\alpha-1)u_B - \alpha\right) + 2(\alpha-1)\alpha u_B + 1\right) - \lambda_1 = 0 \tag{39}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = p_{B1}\left(-2\alpha^3 + 4\alpha^2 - 2\alpha + 2(\alpha-1)^3 u_B - 1\right) - (\alpha-1)\left(2\alpha\left((\alpha-1)u_B - \alpha\right) + u_B\right) = 0 \tag{40}$$

  Solving the (40), we could then get the solution candidate $p_{B1} = \frac{(\alpha-1)\left(\left(2\alpha^2-2\alpha+1\right)u_B - 2\alpha^2\right)}{-2\alpha^3+4\alpha^2-2\alpha+2(\alpha-1)^3 u_B - 1}$ and $p_{G1} = 1$

  - $p_{B1} \in (0,1)$ and $\mathbb{E}[u] \geq 0$ yields the condition that $u_B > \frac{2\alpha^2}{2\alpha^2-2\alpha+1}$ for $\alpha \in [0, \frac{1}{2})$
  - $\lambda_1 = \alpha\left(-2\alpha^2 - 2(\alpha-1)p_{B1}\left((\alpha-1)u_B - \alpha\right) + 2(\alpha-1)\alpha u_B + 1\right)$
    $= -\frac{\alpha\left(2\alpha^3-6\alpha^2+2\alpha-2\left(2\alpha^3-6\alpha^2+5\alpha-1\right)u_B + 2(\alpha-1)^3 u_B^2 + 1\right)}{-2\alpha^3+4\alpha^2-2\alpha+2(\alpha-1)^3 u_B - 1} \geq 0, \Rightarrow u_B < \frac{2\alpha^2+\sqrt{4\alpha^2-6\alpha+3}-4\alpha+1}{2(\alpha-1)^2}$
    for $\alpha \in [0.5, 0.741348]$
  - If firm 1 sets a price lower than $u(b)$, when firm 2 uses mixed pricing strategies according to $F_2(p) = 1 - \frac{u(g)-\Pr(g)\mathbb{E}[u]}{(p+(u(g)-\mathbb{E}[u]))}$ for $\Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u]$, its expected profit will be:

$$\pi_1(p_1) = p_1 * [\Pr(g) + \Pr(b)(1 - F_2(p_1 + (\mathbb{E}[u] - u(b)))), \text{ for } p_1 \leq u(b) \tag{41}$$

    The second derivative of the profit with respect to price is negative for any $p \leq u(b)$. Hence the function is concave. The first derivative at $p = u(b)$ is positive when $\alpha \in [0,1], u_B < 1$. As a result, a deviating firm can maximize its profit by setting the price to $u(b)$. The deviating profit should be lower than the equilibrium profit, yielding a constraint root of a 5-th order polynomial

  So, $p^*_{G1} = 1, p^*_{B1} = \frac{(\alpha-1)\left(\left(2\alpha^2-2\alpha+1\right)u_B - 2\alpha^2\right)}{-2\alpha^3+4\alpha^2-2\alpha+2(\alpha-1)^3 u_B - 1} > 0$ could be a possible solution.

- If $p^*_{G1} = 1, p^*_{B1} = 0$

  Then, $\lambda_2 = 0, \lambda_3 = 0$

(30) and (31) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = -2\alpha^3 + \alpha + 2(\alpha-1)\alpha^2 u_B - \lambda_1 = 0 \tag{42}$$
$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = -(\alpha-1)\left(2\alpha\left((\alpha-1)u_B - \alpha\right) + u_B\right) + \lambda_4 = 0 \tag{43}$$
$$\lambda_1 \geq 0, \lambda_4 \geq 0 \tag{44}$$

- $\mathbb{E}[u] \geq 0$ must hold, i.e. $\alpha + (1-\alpha)u_B \geq 0, \Rightarrow u_B \geq -\frac{\alpha}{(1-\alpha)}$.
- $\lambda_1 = -2\alpha^3 + \alpha + 2(\alpha-1)\alpha^2 u_B \geq 0, \Rightarrow u_B < \frac{2\alpha^2-1}{2(\alpha-1)\alpha}$, for $\alpha \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$
- $\lambda_4 = (\alpha-1)\left(2\alpha\left((\alpha-1)u_B - \alpha\right) + u_B\right) \geq 0, \Rightarrow u_B < \frac{2\alpha^2}{2\alpha^2-2\alpha+1}, \alpha \in [0, \frac{1}{2}]$
- For firm 1, the lowest price should be higher than $u(b)$, otherwise, consumer might buy from firm 1 when receiving bad signal. Thus, we should have constraint $p_1\min - u(b) = u(g) - \Pr(g)\mathbb{E}[u] - u(b) = -\alpha\left(\alpha + (1-\alpha)u_B\right) - u_B + 1 > 0, \Rightarrow u_B < \frac{\alpha^2-1}{\alpha^2-\alpha-1}$
- If firm 1 sets a price lower than $u(b)$, when firm 2 uses mixed pricing strategies according to $F_2(p) = 1 - \frac{u(g)-\Pr(g)\mathbb{E}[u]}{(p+(u(g)-\mathbb{E}[u]))}$ for $\Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u]$, its expected profit will be:

  $$\pi_1(p_1) = p_1 * [\Pr(g) + \Pr(b)(1 - F_2(p_1 + (\mathbb{E}[u] - u(b)))), \text{ for } p_1 \leq u(b) \tag{45}$$

  The second derivative of the profit with respect to price is negative for any $p \leq u(b)$. Hence the function is concave. The first derivative at $p = u(b)$ is positive when $\alpha \in [0,1], u_B < 1$. As a result, a deviating firm can maximize its profit by setting the price to $u_j(b)$. The deviating profit should be lower than the equilibrium profit, yielding a constraint $u_B < -\frac{\sqrt{4\alpha^6-8\alpha^5+8\alpha^3-4\alpha^2+1}-1}{2(\alpha-1)^2\alpha}$
- The intersection of above constraint is $-\frac{\alpha}{(1-\alpha)} \leq u_B < \frac{2\alpha^2}{2\alpha^2-2\alpha+1}$ for $\alpha \in [0, 0.28398]$; $-\frac{\alpha}{(1-\alpha)} \leq u_B < \frac{1-\sqrt{4\alpha^6-8\alpha^5+8\alpha^3-4\alpha^2+1}}{2(\alpha^3-2\alpha^2+\alpha)}$ for $\alpha \in [0.28398, 0.640388]$; $-\frac{\alpha}{(1-\alpha)} \leq u_B < \frac{2\alpha^2-1}{2(\alpha-1)\alpha}$ for $\alpha \in [0.640388, \frac{1}{\sqrt{2}}]$

So, $p^*_{G1} = 1, p^*_{B1} = 0$ could be a possible solution.

- If $p^*_{B1} = 0, p^*_{G1} \in (0,1)$

  Then , $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$

  (30) and (31) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha + p_{G1}\left(-2\alpha^3 + 2(\alpha-1)\alpha^2 u_B - 1\right) + 1 = 0 \tag{46}$$
$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = -(\alpha-1)\left(2\alpha p_{G1}\left((\alpha-1)u_B - \alpha\right) + u_B\right) + \lambda_4 = 0 \tag{47}$$
$$\lambda_4 \geq 0 \tag{48}$$

Solving the (46), we could then get the solution candidate $p_{G1} = -\frac{\alpha+1}{-2\alpha^3+2(\alpha-1)\alpha^2 u_B-1}$ and $p_{B1} = 0$

- $p_{G1} \in (0,1)$ and $\mathbb{E}[u] \geq 0$ yields the condition that $u_B > \frac{2\alpha^2-1}{2(\alpha-1)\alpha}$ for $\alpha \in [\frac{1}{2}, \frac{1}{\sqrt{2}}]$
- $\lambda_4 = (\alpha-1)\left(2\alpha p_{G1}\left((\alpha-1)u_B-\alpha\right)+u_B\right) = (\alpha-1)\left(u_B - \frac{2\alpha(\alpha+1)((\alpha-1)u_B-\alpha)}{-2\alpha^3+2(\alpha-1)\alpha^2 u_B-1}\right) \geq 0$, yielding a constraint that $u_B < -\frac{-4\alpha^3+\sqrt{8\alpha^3+4\alpha^2-4\alpha+1}+2\alpha-1}{4(\alpha-1)\alpha^2}$ for $\alpha \in [0, \frac{1}{2}]$
- If firm 1 sets a price lower than $u(b)$, when firm 2 uses mixed pricing strategies according to $F_2(p) = 1 - \frac{u(g)-\Pr(g)\mathbb{E}[u]}{(p+(u(g)-\mathbb{E}[u]))}$ for $\Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u]$, its expected profit will be:

$$\pi_1(p_1) = p_1 * [\Pr(g) + \Pr(b)(1 - F_2(p_1 + (\mathbb{E}[u] - u(b)))), \text{ for } p_1 \leq u(b) \tag{49}$$

  The second derivative of the profit with respect to price is negative for any $p \leq u(b)$. Hence the function is concave. The first derivative at $p = u(b)$ is positive when $\alpha \in [0,1], u_B < 1$. As a result, a deviating firm can maximize its profit by setting the price to $u(b)$. The deviating profit should be lower than the equilibrium profit, yielding a constraint root of a 4-th order polynomial

□

Proof of Proposition 4. We first consider the case where $E[u] \leq 0$

Table 3: Payoffs when $E[u] \leq 0$

	passive	active
passive	0,0	$0, \pi_2^M((\emptyset,0),((p_{G2}^M,p_{B2}^M),u_2(g)))$
active	$\pi_1^M(((p_{G1}^M,p_{B1}^M),u_1(g)),(\emptyset,0)),0$	$\pi_1^D(((p_G^D,p_B^D),F_1(p)),((p_G^D,p_B^D),F_2(p)))$, $\pi_2^D(((p_G^D,p_B^D),F_1(p)),((p_G^D,p_B^D),F_2(p)))$

(passive, active) can not be the subgame perfect equilibrium because for firm 1, it can always profitably deviate it by setting $p_{G1} = 1$, $p_{B1} = 0$ and $p = \epsilon$. This yields a non-negative profit.

Suppose (active, active) is the SPE, then we need to check that firm 1 (same for firm 2 because of symmetry) does not have the incentive to deviate.

$\pi_1^D(((p_G^D,p_B^D),F_1(p)),((p_G^D,p_B^D),F_2(p))) > \pi_1^D(((p'_{G1},p'_{B1}),F'_1(p)),((p_G^D,p_B^D),F'_2(p)))$

RHS $\pi_1^D(((p'_{G1},p'_{B1}),F'_1(p)),((p_G^D,p_B^D),F'_2(p)))$

$= (\alpha p'_{G1} + (1-\alpha)p'_{B1})\left(\alpha\left(1-p_G^D\right)+(1-\alpha)\left(1-p_B^D\right)\right)u_1(g)' - \frac{\left(1-p'_{G1}\right)^2}{2} - \frac{p'^2_{B1}}{2}$

and $\pi_1^D(((p_G^D,p_B^D),F_1(p)),((p_G^D,p_B^D),F_2(p))) > \pi_1((\emptyset,F'_1(p)),((p_G^D,p_B^D),F'_2(p))) = 0$

When $E[u] > 0$

Table 4: Payoffs when $E[u] > 0$

	passive	active
passive	0,0	$\pi_1^D((\emptyset, F_1(p)),((p_{G2}^a, p_{B2}^a), F_2(p)))$, $\pi_2^D((\emptyset, F_1(p)),((p_{G2}^a, p_{B2}^a), F_2(p)))$
active	$\pi_1^D(((p_{G1}^a, p_{B1}^a), F_1(p)),(\emptyset, F_2(p)))$, $\pi_2^D(((p_{G1}^a, p_{B1}^a), F_1(p)),(\emptyset, F_2(p)))$	$\pi_1^D(((p_G^D, p_B^D), F_1(p)),((p_G^D, p_B^D), F_2(p)))$, $\pi_2^D(((p_G^D, p_B^D), F_1(p)),((p_G^D, p_B^D), F_2(p)))$

Suppose (active, active) is the SPE, then we need to check that firm 1 (same for firm 2 because of symmetry) does not have the incentive to deviate.

$\pi_1^D(((p_G^D, p_B^D), F_1(p)),((p_G^D, p_B^D), F_2(p))) > \pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_G^D, p_B^D), F'_2(p)))$

RHS $\pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_G^D, p_B^D), F'_2(p)))$

$= (\alpha p'_{G1} + (1-\alpha)p'_{B1}) \left(\alpha\left(1-p_G^D\right) + (1-\alpha)\left(1-p_B^D\right)\right) u_1(g)' - \frac{\left(1-p'_{G1}\right)^2}{2} - \frac{p'^2_{B1}}{2}$

and $\pi_1^D(((p_G^D, p_B^D), F_1(p)),((p_G^D, p_B^D), F_2(p))) > \pi_1^D((\emptyset, F'_1(p)),((p_G^D, p_B^D), F'_2(p)))$

RHS $\pi_1^D((\emptyset, F'_1(p)),((p_G^D, p_B^D), F'_2(p))) = \left(\alpha\left(1-p_G^D\right) + (1-\alpha)\left(1-p_B^D\right)\right) E[u]$

Suppose (passive, active) is the SPE, then we need to check that firms do not have the incentive to deviate.

$\pi_1^D((\emptyset, F_1(p)),((p_{G2}^a, p_{B2}^a), F_2(p))) > \pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_{G2}^a, p_{B2}^a), F'_2(p)))$

RHS $\pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_{G2}^a, p_{B2}^a), F'_2(p)))$

$= (\alpha p'_{G1} + (1-\alpha)p'_{B1}) (\alpha(1-p_{G2}^a) + (1-\alpha)(1-p_{B2}^a)) u_1(g)' - \frac{\left(1-p'_{G1}\right)^2}{2} - \frac{p'^2_{B1}}{2}$

and $\pi_2^D((\emptyset, F_1(p)),((p_{G2}^a, p_{B2}^a), F_2(p))) > \pi_2^D((\emptyset, F'_1(p)),((p'_{G2}, p'_{B2}), F'_2(p)))$

We first consider the case where $E[u] > 0$ and $u_B < 0$.

(active, active) is SPE

The profit in (active, active) equilibrium is $\pi_1^D(((p_G^D, p_B^D), F_1(p)),((p_G^D, p_B^D), F_2(p))) = \alpha(1-\alpha)$

If the firm deviates by not providing any signals, it would get a profit of $\pi_1^D((\emptyset, F'_1(p)),((p_G^D, p_B^D), F'_2(p))) = \left(\alpha\left(1-p_G^D\right) + (1-\alpha)\left(1-p_B^D\right)\right) E[u] = (1-\alpha)(\alpha + (1-\alpha)u_B)$

When $u_B < 0$, $\pi_1^D(((p_G^D, p_B^D), F_1(p)),((p_G^D, p_B^D), F_2(p))) > \pi_1^D((\emptyset, F'_1(p)),((p_G^D, p_B^D), F'_2(p)))$. Thus, firms do not have the incentive to deviate to not provide signals.

(passive, active) is not SPE

The profit in (passive, active) equilibrium is $\pi_1^D((\emptyset, F_1(p)),((p_{G2}^a, p_{B2}^a), F_2(p)))$
$= (\alpha(1-p_{G2}^a) + (1-\alpha)(1-p_{B2}^a))(\alpha + (1-\alpha)u_B)$

If the firm deviates by providing signals, the profit is $\pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_{G2}^a, p_{B2}^a), F'_2(p)))$
$= (\alpha p'_{G1} + (1-\alpha)p'_{B1}) (\alpha(1-p_{G2}^a) + (1-\alpha)(1-p_{B2}^a)) u_1(g)' - \frac{\left(1-p'_{G1}\right)^2}{2} - \frac{p'^2_{B1}}{2}$

The deviating profit is monotonically increasing in p'_{G1}, $\frac{\partial \pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_{G2}^a, p_{B2}^a), F'_2(p)))}{\partial p'_{G1}} = \alpha((1-\alpha)(1-p_{B2}^a) + \alpha(1-p_{G2}^a)) - p'_{G1} + 1 > 0$. So the deviating firm will always choose $p'_{G1} = 1$. When $u_B < 0$, the deviating profit is monotonically decreasing in p'_{B1}, $\frac{\partial \pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_{G2}^a, p_{B2}^a), F'_2(p)))}{\partial p'_{B1}} = u_B(1-\alpha)((1-\alpha)(1-p_{B2}^a) + \alpha(1-p_{G2}^a)) - p'_{B1} < 0$. So the deviating firm will choose $p'_{G1} = 1, p'_{B1} = 0$. And the corresponding profit is $\max \pi_1^D(((p'_{G1}, p'_{B1}), F'_1(p)),((p_{G2}^a, p_{B2}^a), F'_2(p))) =$

$\alpha\left(\alpha\left(1-p_{G2}^{a}\right)+\left(1-\alpha\right)\left(1-p_{B2}^{a}\right)\right)$. The deviating profit is higher than that in (passive, active) equilibrium, i.e. $\pi_1^D(((p'_{G1},p'_{B1}),F'_1(p)),((p_{G2}^a,p_{B2}^a),F'_2(p))) > \pi_1^D((\emptyset,F_1(p)),((p_{G2}^a,p_{B2}^a),F_2(p)))$. Hence, (passive, active) is not SPE.

Next, we consider the case where $u_B > 0$.

(passive, active) is SPE

The profit in (passive, active) equilibrium is $\pi_1^D((\emptyset,F_1(p)),((p_{G2}^a,p_{B2}^a),F_2(p)))$
$=\left(\alpha\left(1-p_{G2}^{a}\right)+\left(1-\alpha\right)\left(1-p_{B2}^{a}\right)\right)\left(\alpha+(1-\alpha)u_B\right)$

Recall that $\frac{\partial\pi_1^D(((p'_{G1},p'_{B1}),F'_1(p)),((p_{G2}^a,p_{B2}^a),F'_2(p)))}{\partial p'_{B1}} = u_B(1-\alpha)\left(\left(1-\alpha\right)\left(1-p_{B2}^{a}\right)+\alpha\left(1-p_{G2}^{a}\right)\right)-p'_{B1}$. When $u_B>0$, the deviating firm will choose $p'_{B1}=u_B(1-\alpha)\left(\left(1-\alpha\right)\left(1-p_{B2}^{a}\right)+\alpha\left(1-p_{G2}^{a}\right)\right)$

$\pi_1^D(((p'_{G1},p'_{B1}),F'_1(p)),((p_{G2}^a,p_{B2}^a),F'_2(p)))-\pi_1^D((\emptyset,F_1(p)),((p_{G2}^a,p_{B2}^a),F_2(p)))$
$=-\frac{1}{2}(1-\alpha)u_B\left(\left(1-\alpha\right)\left(1-p_{B2}^{a}\right)+\alpha\left(1-p_{G2}^{a}\right)\right)\left(2-(1-\alpha)u_B\left(\left(1-\alpha\right)\left(1-p_{B2}^{a}\right)+\alpha\left(1-p_{G2}^{a}\right)\right)\right)<0$. The deviating profit is lower than that in (passive, active) equilibrium. So, (passive, active) is the SPE.

(active, active) is not SPE

$\pi_1^D(((p_G^D,p_B^D),F_1(p)),((p_G^D,p_B^D),F_2(p)))-\pi_1^D((\emptyset,F'_1(p)),((p_G^D,p_B^D),F'_2(p)))$
$=\left(\alpha p_G^D+(1-\alpha)p_B^D u_B\right)\left(\alpha\left(1-p_G^D\right)+\left(1-\alpha\right)\left(1-p_B^D\right)\right)-\frac{\left(1-p_G^D\right)^2}{2}-\frac{\left(p_B^D\right)^2}{2}-\left(\alpha\left(1-p_G^D\right)+\left(1-\alpha\right)\left(1-p_B^D\right)\right)E[u]$
$=-\frac{(1-\alpha)^2u_B\left((1-\alpha)^2u_B+2\right)}{2((1-\alpha)^2u_B+1)^2}<0$. So, firms will have the incentive to deviate by not sending any signals. (active, active) is not SPE.

□

Proof of Proposition 5. When $\mathbb{E}[u]>0$, the consumer will consume without additional information and the profit for the platform is m. So when $\mathbb{E}[u]>0$, the platform will choose not sending any signals.

When $\mathbb{E}[u]<0$, the maximization problem for the platform is

$$\max_{p_{Bj},p_{Gj}} \pi_P = \pi_j+\pi_{-j} \tag{50}$$
$$= m\left(1-\left(\left(1-\alpha\right)\left(1-p_{B1}\right)+\alpha\left(1-p_{G1}\right)\right)\left(\left(1-\alpha\right)\left(1-p_{B2}\right)+\alpha\left(1-p_{G2}\right)\right)\right) \tag{51}$$
$$-\frac{p_{B1}^2}{2}-\frac{p_{B2}^2}{2}-\frac{1}{2}\left(1-p_{G1}\right)^2-\frac{1}{2}\left(1-p_{G2}\right)^2$$
$$s.t. p_{Bj}\le 1, -p_{Bj}\le 0, p_{Gj}\le 1, -p_{Gj}\le 0$$

The consumer will follow the signal they receive if the following incentive compatibility constraints hold:

$$u_j(g)-m\ge 0 \tag{52}$$
$$u_j(b)-m\le 0 \tag{53}$$

First notice that $\frac{\partial\pi_P}{\partial p_{G1}}=\alpha m\left(\left(1-\alpha\right)\left(1-p_{B2}\right)+\alpha\left(1-p_{G2}\right)\right)+1-p_{G1}\ge 0, \forall p_{Gj}\in[0,1], p_{Bj}\in$

$[0,1]$. The revenue of the firm increases in p_{Gj} and the cost decreases in p_{Gj}, hence it is optimal to set $p^*_{Gj} = 1$. As the expected utility of the consumer $u_j(g)$ increases in p_G and $u_j(b)$ decreases in p_{Gj}, setting $p^*_{Gj} = 1$ will also not violate the IC constraints.

$\frac{\partial \pi_P}{\partial p_{B1}}|_{p_{Gj}=1, p_{B1}=p_{B2}} = -p_{B1} + (1-\alpha)m\left((1-\alpha)(1-p_{B2}) + \alpha(1-p_{G2})\right) = (1-\alpha)^2 m - p_{B1}\left((1-\alpha)^2 m + 1\right)$. When $p_{Bj} = \frac{(1-\alpha)^2 m}{(1-\alpha)^2 m+1} < 0$, $\frac{\partial \pi_P}{\partial p_{B1}} > 0$; when $p_{Bj} = \frac{(1-\alpha)^2 m}{(1-\alpha)^2 m+1} > 0$, $\frac{\partial \pi_P}{\partial p_{B1}} < 0$.

The solution to the unconstrained maximization problem of the monopolist when $p_G = 1$ is $p_B = \frac{(1-\alpha)^2 m}{(1-\alpha)^2 m+1}$, while the incentive compatibility constraints can be written as:

$$p_{B1} \leq \frac{\alpha(u_G - m)}{(1-\alpha)(m-u_B)} \tag{54}$$

$$(1-\alpha)(1-p_B)(u_B - m) \leq 0 \tag{55}$$

So, it is optimal for the platform to set $p_{Bj} = \min\left(\frac{(1-\alpha)^2 m}{(1-\alpha)^2 m+1}, \frac{\alpha(u_G-m)}{(1-\alpha)(m-u_B)}\right) < 1$ □

Proof of Proposition 6. The Lagrangian is

$$\begin{aligned}\mathcal{L}(p_{G1}, p_{B1}) = & \left((1-\alpha)(1-p_{B2}) + \alpha(1-p_{G2})\right)\left((1-\alpha)u_B p_{B1} + \alpha u_G p_{G1}\right) - \frac{p_{B1}^2}{2} - \frac{1}{2}(1-p_{G1})^2 \\ & + \left((1-\alpha)(1-p_{B1}) + \alpha(1-p_{G1})\right)\left((1-\alpha)u_B p_{B2} + \alpha u_G p_{G2}\right) - \frac{p_{B2}^2}{2} - \frac{1}{2}(1-p_{G2})^2 \\ & - \lambda_1(p_{G1}-1) - \lambda_2(p_{G2}-1) - \lambda_3(p_{B1}-1) - \lambda_4(p_{B2}-1) \\ & - \lambda_5(-p_{G1}) - \lambda_6(-p_{G2}) - \lambda_7(-p_{B1}) - \lambda_8(-p_{B2}) \\ & s.t. p_{Bj} \leq 1, -p_{Bj} \leq 0, p_{Gj} \leq 1, -p_{Gj} \leq 0\end{aligned} \tag{56}$$

The Kuhn-Tucker conditions for this maximization problem is

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha + (\alpha-1)\alpha(u_B+1)p_{B2} - p_{G1} - 2\alpha^2 p_{G2} + 1 - \lambda_1 + \lambda_5 = 0 \tag{57}$$

$$\frac{\partial \mathcal{L}}{\partial p_{G2}} = \alpha + (\alpha-1)\alpha(u_B+1)p_{B1} - p_{G2} - 2\alpha^2 p_{G1} + 1 - \lambda_2 + \lambda_6 = 0 \tag{58}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = (\alpha-1)\left(u_B(-2(\alpha-1)p_{B2} - 1) + \alpha(u_B+1)p_{G2}\right) - p_{B1} - \lambda_3 + \lambda_7 = 0 \tag{59}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B2}} = (\alpha-1)\left(u_B(-2(\alpha-1)p_{B1} - 1) + \alpha(u_B+1)p_{G1}\right) - p_{B2} - \lambda_4 + \lambda_8 = 0 \tag{60}$$

$$\begin{aligned}& \lambda_i \geq 0, p_{G1} \leq 1, p_{G2} \leq 1, p_{B1} \leq 1, p_{B2} \leq 1, -p_{G1} \leq 0, -p_{G2} \leq 0, -p_{B1} \leq 0, -p_{B2} \leq 0 \text{ and,} \\ & \lambda_1(p_{G1}-1) = 0, \lambda_2(p_{G2}-1) = 0, \lambda_3(p_{B1}-1) = 0, \lambda_4(p_{B2}-1) = 0, \\ & \lambda_5(-p_{G1}) = 0, \lambda_6(-p_{G2}) = 0, \lambda_7(-p_{B1}) = 0, \lambda_8(-p_{B2}) = 0\end{aligned}$$

We then try to find the KKT solutions of the problem

- If $p^*_{Gj} = 0$

Then, $\lambda_1(p_{G1}-1)=0, \lambda_2(p_{G2}-1)=0$. So $\lambda_1=0$, and $\lambda_2=0$

(57) of the Kuhn-Tucker conditions now becomes

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = (\alpha + (\alpha-1)\alpha\,(u_B+1)\,p_{B1}+1) + \lambda_5 = 0 \tag{61}$$

$$\lambda_5 \geq 0 \tag{62}$$

$\lambda_5 < 0, \forall \alpha \in [0,1]$, which contradicts $\lambda_5 \geq 0$.

So, $p^*_{Gj}=0$ could not be the solution.

- If $p^*_{Bj}=1$

  Then, $\lambda_7(-p_{B1})=0, \lambda_8(-p_{B2})=0$. So $\lambda_7=0$ and $\lambda_8=0$

  (59) of the Kuhn-Tucker conditions now becomes

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = u_B((\alpha-1)\,(-2\alpha+\alpha p_{G2}+1)) + (\alpha-1)\alpha p_{G2} - 1 - \lambda_3 = 0 \tag{63}$$

$$\lambda_3 \geq 0 \tag{64}$$

$\lambda_3 = u_B((\alpha-1)\,(-2\alpha+\alpha p_{G2}+1)) + (\alpha-1)\alpha p_{G2} - 1 < 0, \forall \alpha \in [0,1]$, which contradicts $\lambda_3 \geq 0$.

So, $p^*_{Bj}=1$ could not be the solution.

- If $p^*_{Gj} \in (0,1), p^*_{Bj} \in (0,1)$

  Then, $\lambda_i = 0, i = 1,2,\ldots,8$

  (57) and (59) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha + (\alpha-1)\alpha\,(u_B+1)\,p_{B1} - p_{G1} - 2\alpha^2 p_{G1} + 1 = 0 \tag{65}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = (\alpha-1)\,(u_B\,(-2(\alpha-1)p_{B1}-1) + \alpha\,(u_B+1)\,p_{G1}) - p_{B1} = 0 \tag{66}$$

$$p^*_{Gj} \in (0,1), p^*_{Bj} \in (0,1) \tag{67}$$

Solving the equations (65) and (66), we could then find the solution candidate:

$$p_{Gj} = \frac{-\alpha + (\alpha-1)^2 u_B(\alpha(u_B-1)-2)-1}{\alpha^2((\alpha-2)\alpha-1) + \alpha^2(\alpha-1)^2 u_B^2 - 2\,(\alpha^2+1)\,(\alpha-1)^2 u_B - 1}$$

$$p_{Bj} = \frac{(\alpha-1)(-\alpha(\alpha+1) + (\alpha-1)\alpha u_B + u_B)}{\alpha^2((\alpha-2)\alpha-1) + \alpha^2(\alpha-1)^2 u_B^2 - 2\,(\alpha^2+1)\,(\alpha-1)^2 u_B - 1}$$

$p^*_{Gj} \in (0,1), p^*_{Bj} \in (0,1)$ cannot hold at the same time.

So, $p_{Gj}^{*} \in (0,1), p_{Bj}^{*} \in (0,1)$ could not be the solution.

- If $p_{Gj}^{*} = 1, p_{Bj}^{*} \in (0,1)$

  Then, $\lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = 0, \lambda_7 = 0, \lambda_8 = 0$

  (57) and (59) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha\left(-2\alpha + (\alpha-1)\left(u_B+1\right)p_{B2} + 1\right) - \lambda_1 = 0 \tag{68}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = 0, \Rightarrow p_{Bj} = \frac{(\alpha-1)\left(\alpha + (\alpha-1)u_B\right)}{2(\alpha-1)^2 u_B + 1} \tag{69}$$

$$\lambda_1 \geq 0, p_{Bj} \in (0,1) \tag{70}$$

  First $p_{Bj} = \frac{(\alpha-1)(\alpha+(\alpha-1)u_B)}{2(\alpha-1)^2 u_B+1} \in (0,1)$ yields the constraints that $u_B > \frac{\alpha}{1-\alpha} \vee u_B < \frac{\alpha^2-\alpha-1}{(\alpha-1)^2}$

  Remember that by (14), $\alpha u_G p_{Gj} + (1-\alpha)u_B p_{Bj} \geq 0, \Rightarrow \alpha + (1-\alpha)u_B \frac{(\alpha-1)(\alpha+(\alpha-1)u_B)}{2(\alpha-1)^2 u_B+1} \geq 0$, yields the constraint that $u_B \geq -\frac{1}{2(\alpha-1)^2}$

  The intersection of above two conditions is $u_B > \frac{\alpha}{1-\alpha}$

  If a firm sets a price lower than $u_j(b)$, when the competition mixes their pricing according to $F(p)$, its expected profit will be:

$$\begin{aligned}\pi_1(p) = {} & p\left((1-\alpha)\left(1-p_{B1}\right) + \alpha\left(1-p_{G1}\right)\right)\left((1-\alpha)p_{B2} + \alpha p_{G2}\right)\left(1 - F(u_2(g) - u_1(b) + p)\right) \\ & + p\left((1-\alpha)\left(1-p_{B1}\right) + \alpha\left(1-p_{G1}\right)\right)\left((1-\alpha)\left(1-p_{B2}\right) + \alpha\left(1-p_{G2}\right)\right) \\ & + p\left((1-\alpha)p_{B1} + \alpha p_{G1}\right)\left((1-\alpha)\left(1-p_{B2}\right) + \alpha\left(1-p_{G2}\right)\right) \\ & + p\left((1-\alpha)p_{B1} + \alpha p_{G1}\right)\left((1-\alpha)p_{B2} + \alpha p_{G2}\right)\end{aligned}$$

  A firm can deviate from the equilibrium by setting the price to $u_j(b)$. The deviating profit should be smaller than the expected profit in the equilibrium However, when $u_B > \frac{\alpha}{1-\alpha}$, this does not hold. A firm could profitably deviate from the equilibrium and set price $u_i(b)$. Therefore, there is no value of u_B that can support such an equilibrium.

  So, $p_{Gj}^{*} = 1, p_{Bj}^{*} \in (0,1)$ could not be the solution.

- If $p_{Gj}^{*} = 1, p_{Bj}^{*} = 0$

  Then, $\lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = 0$

(57) and (59) of The Kuhn-Tucker conditions become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \left(\alpha - 2\alpha^2\right) - \lambda_1 = 0 \tag{71}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = (\alpha - 1)\left(\alpha\left(u_B + 1\right) - u_B\right) + \lambda_7 = 0 \tag{72}$$

$$\lambda_1 \geq 0, \lambda_7 \geq 0 \tag{73}$$

- $\lambda_1 = 2\left(\alpha - 2\alpha^2\right) \geq 0, \Rightarrow \alpha \leq \frac{1}{2}$
- $\lambda_7 = 2(1-\alpha)\left(\alpha\left(u_B + 1\right) - u_B\right) \geq 0, \Rightarrow u_B \leq \frac{\alpha}{1-\alpha}$
- Using the condition to support an equilibrium in the second stage, we need the condition that $u_j(b) < \frac{\alpha(1-\alpha)}{\alpha^2-\alpha+1}u_j(g)$. $\Rightarrow u_B < \frac{\alpha(1-\alpha)}{\alpha^2-\alpha+1}$
- Notice that $\frac{(1-\alpha)\alpha}{\alpha^2-\alpha+1} - \frac{\alpha}{1-\alpha} = \frac{\alpha^2}{(\alpha-1)(\alpha^2-\alpha+1)} < 0$, which means $u_B < \frac{\alpha(1-\alpha)}{\alpha^2-\alpha+1}$ is the more restrictive constraint.

So, when $\alpha \in [0, \frac{1}{2}]$ and $u_B < \frac{\alpha(1-\alpha)}{\alpha^2-\alpha+1}$ $p^*_{Gj} = 1, p^*_{Bj} = 0$ is a KKT solution.

- If $p^*_{Bj} = 0, p^*_{Gj} \in (0,1)$

  Then , $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0, \lambda_5 = 0, \lambda_6 = 0$

  (57) and (59) of the Kuhn-Tucker conditions now becomes

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = 0, \Rightarrow p_{Gj} = \frac{\alpha + 1}{2\alpha^2 + 1} \tag{74}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = (\alpha - 1)\left(\alpha\left(u_B + 1\right)p_{G2} - u_B\right) + \lambda_7 = 0 \tag{75}$$

$$\lambda_7 \geq 0, p^*_{Gj} \in (0,1) \tag{76}$$

  - $p^*_{Gj} \in (0,1), \Rightarrow \alpha \in (0.5, 1]$
  - $\lambda_7 = (1-\alpha)\left(\alpha\left(u_B+1\right)p_{G2} - u_B\right) = (1-\alpha)\left(\frac{\alpha(\alpha+1)+\left(-\alpha^2+\alpha-1\right)u_B}{2\alpha^2+1}\right) \geq 0, \Rightarrow u_B < \frac{\alpha(\alpha+1)}{\alpha^2-\alpha+1}$. Notice that $\frac{\alpha(\alpha+1)}{\alpha^2-\alpha+1} > 1$ when $\alpha \in (0.5, 1]$, so as long as $u_B < 1$, $\lambda_7 \geq 0$
  - Because in a mixed strategy the firm should be indifferent between setting different prices, p_{min} will satisfy:

$$\begin{aligned}
&\left[\left(\alpha p_{Gj} + (1-\alpha)p_{Bj}\right)\left(\alpha p_{G-j} + (1-\alpha)p_{B-j}\right)\right]p_{min} \\
&+ \left[\left(\alpha p_{Gj} + (1-\alpha)p_{Bj}\right)\left(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})\right)\right]p_{min} \\
= &\left(\alpha p_{Gj} + (1-\alpha)p_{Bj}\right)\left(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j})\right)u_j(g)
\end{aligned}$$

    resulting in $p_{min} = \frac{\alpha^2-\alpha+1}{2\alpha^2+1}$. The lowest price should be higher than $u_j(b)$ (otherwise some consumers may buy with a bad signal), which means we need to restrict $u_j(b) < \frac{\alpha^2-\alpha+1}{2\alpha^2+1}$, which implies $u_B < \frac{4\alpha^5-3\alpha^4+4\alpha^3-4\alpha^2+2\alpha-1}{(\alpha-1)(2\alpha^2+1)^2}$.

If we denote by $F(\cdot)$ the cdf of the equilibrium prices in $[p_{min}, p_{max}]$, then we can solve for $F(\cdot)$ in:

$$\begin{aligned}\pi_j(p) = &(\alpha p_{Gj} + (1-\alpha)p_{Bj})(\alpha p_{G-j} + (1-\alpha)p_{B-j})(1-F(p))p \\ &+ (\alpha p_{Gj} + (1-\alpha)p_{Bj})(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j}))\,p \\ = &(\alpha p_{Gj} + (1-\alpha)p_{Bj})(\alpha(1-p_{G-j}) + (1-\alpha)(1-p_{B-j}))\,u_j(g)\end{aligned}$$

resulting in the equilibrium mixed strategy price distribution $F(p) = \frac{(2\alpha^2+1)\left((-\alpha^2+\alpha-1)u_j(g)+p\right)}{\alpha(\alpha+1)p}$.

Thus, each firm will randomize its prices between $[\frac{\alpha^2(2\alpha-1)-(2\alpha^3-2\alpha^2+\alpha-1)u_B}{\alpha^2-\alpha+1}, 1]$, with CDF $F(p) = \frac{(2\alpha^2+1)\left((-\alpha^2+\alpha-1)u_j(g)+p\right)}{\alpha(\alpha+1)p}$

If a firm sets a price lower than $u_j(b)$, when the competition mixes their pricing according to $F(p)$, its expected profit will be:

$$\begin{aligned}\pi_1(p) = &\,p\left((1-\alpha)(1-p_{B1}) + \alpha(1-p_{G1})\right)\left((1-\alpha)p_{B2} + \alpha p_{G2}\right)(1 - F(u_2(g) - u_1(b) + p)) \\ &+ p\left((1-\alpha)(1-p_{B1}) + \alpha(1-p_{G1})\right)\left((1-\alpha)(1-p_{B2}) + \alpha(1-p_{G2})\right) \\ &+ p\left((1-\alpha)p_{B1} + \alpha p_{G1}\right)\left((1-\alpha)(1-p_{B2}) + \alpha(1-p_{G2})\right) \\ &+ p\left((1-\alpha)p_{B1} + \alpha p_{G1}\right)\left((1-\alpha)p_{B2} + \alpha p_{G2}\right)\end{aligned}$$

The second derivative of the profit with respect to price is negative for any $p \leq u_j(b)$. Hence the function is concave. The first derivative at $p = u_j(b)$ is positive when $\alpha \in (0.5, 1], u_B < 1$. As a result, a deviating firm can maximize its profit by setting the price to $u_j(b)$, yielding a profit $-\frac{(3\alpha^4+4\alpha^2-\alpha+1)\left(-2\alpha^3+\alpha^2+(2\alpha^3-2\alpha^2+\alpha-1)u_B\right)}{(\alpha^2-\alpha+1)(2\alpha^2+1)^2}$. This deviation profit should be lower than the equilibrium profit, which yields the constraint $u_B < \frac{\alpha\left(6\alpha^6-4\alpha^5+9\alpha^4-7\alpha^3+2\alpha^2-1\right)}{(\alpha-1)(2\alpha^2+1)(3\alpha^4+4\alpha^2-\alpha+1)}$

– The intersection of above constraints is $u_B < \frac{\alpha\left(6\alpha^6-4\alpha^5+9\alpha^4-7\alpha^3+2\alpha^2-1\right)}{(\alpha-1)(2\alpha^2+1)(3\alpha^4+4\alpha^2-\alpha+1)}$

So, when $\alpha \in (\frac{1}{2}, 1]$ and $u_B < \frac{\alpha\left(6\alpha^6-4\alpha^5+9\alpha^4-7\alpha^3+2\alpha^2-1\right)}{(\alpha-1)(2\alpha^2+1)(3\alpha^4+4\alpha^2-\alpha+1)}$ $p^*_{Gj} = \frac{\alpha+1}{2\alpha^2+1}, p^*_{Bj} = 0$ is a KKT solution.

In the platform optimization problem, there is only one KKT solution when $\alpha \in [0, \frac{1}{2}]$ or $\alpha \in (\frac{1}{2}, 1]$. Since Kuhn-Tucker conditions are necessary, the single KKT solution must be the maximum. (Otherwise, it contradicts with the Weierstrass Theorem.) □

Proof of Proposition 7. First, $\mathbb{E}[u] \geq 0$ should hold. Otherwise, no consumer would buy from firm 2. $\alpha + (1-\alpha)u_B \geq 0, \Rightarrow u_B \geq -\frac{\alpha}{1-\alpha}$

$$\mathcal{L}(p_{G1}, p_{B1}) = ((1-\alpha)p_{B1} + \alpha p_{G1}) \left(\frac{(1-\alpha)p_{B1}u_B + \alpha p_{G1}}{(1-\alpha)p_{B1} + \alpha p_{G1}} - ((1-\alpha)p_{B1} + \alpha p_{G1})\left((1-\alpha)u_B + \alpha\right) \right)$$
$$+ (\alpha + (1-\alpha)u_B)\left((1-\alpha)(1-p_{B1}) + \alpha(1-p_{G1})\right) - \frac{(1-p_{G1})^2}{2} - \frac{p_{B1}^2}{2}$$
$$- \lambda_1(p_{G1}-1) - \lambda_2(p_{B1}-1) - \lambda_3(-p_{G1}) - \lambda_4(-p_{B1}) \quad (77)$$

The Kuhn-Tucker conditions for this maximization problem is

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha - 2\alpha\left((\alpha-1)u_B - \alpha\right)\left((\alpha-1)p_{B1} - \alpha p_{G1}\right) + \alpha\left((\alpha-1)u_B - \alpha\right) - p_{G1} + 1 - \lambda_1 + \lambda_3 = 0 \quad (78)$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = p_{B1}\left(-2\alpha^3 + 4\alpha^2 - 2\alpha + 2(\alpha-1)^3u_B - 1\right) - (\alpha-1)\alpha\left(2p_{G1}\left((\alpha-1)u_B - \alpha\right) + u_B - 1\right) - \lambda_2 + \lambda_4 = 0$$
$$\lambda_i \geq 0, p_{G1} \leq 1, p_{B1} \leq 1, -p_{G1} \leq 0, -p_{B1} \leq 0, \text{ and,} \quad (79)$$
$$\lambda_1(p_{G1}-1) = 0, \lambda_2(p_{B1}-1) = 0, \lambda_3(-p_{G1}) = 0, \lambda_4(-p_{B1}) = 0$$

- $p_{G1}^* = 0$

  Equation (78) of the Kuhn-Tucker conditions now becomes

  $$\frac{\partial \mathcal{L}}{\partial p_{G1}} = -\alpha^2 + \alpha - 2(\alpha-1)\alpha p_{B1}\left((\alpha-1)u_B - \alpha\right) + (\alpha-1)\alpha u_B + 1 + \lambda_3 = 0 \quad (80)$$
  $$\lambda_3 \geq 0 \quad (81)$$

  $\lambda_3 = \alpha^2 - \alpha + 2(\alpha-1)\alpha p_{B1}\left((\alpha-1)u_B - \alpha\right) - (\alpha-1)\alpha u_B - 1 < 0$, which contradicts $\lambda_3 \geq 0$.

  So, $p_{G1}^* = 0$ could not be the solution.

- $p_{B1}^* = 1$

  Equation (79) of the Kuhn-Tucker conditions now becomes

  $$\frac{\partial \mathcal{L}}{\partial p_{B1}} = -2\alpha^3 + 4\alpha^2 - 2\alpha - \alpha(\alpha-1)\left(2p_{G1}\left((\alpha-1)u_B - \alpha\right) + u_B - 1\right) + 2(\alpha-1)^3u_B - 1 - \lambda_2 = 0 \quad (82)$$
  $$\lambda_2 \geq 0 \quad (83)$$

  $\lambda_2 = -2\alpha^3 + 4\alpha^2 - 2\alpha - \alpha(\alpha-1)\left(2p_{G1}\left((\alpha-1)u_B - \alpha\right) + u_B - 1\right) + 2(\alpha-1)^3u_B - 1 < 0$, which contradicts $\lambda_2 \geq 0$.

  So, $p_{B1}^* = 1$ could not be the solution.

- If $p^*_{G1} \in (0,1), p^*_{B1} \in (0,1)$

  Then, $\lambda_i = 0, i = 1,2,3,4$

  Equation (78) and (79) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha - 2\alpha\left((\alpha-1)u_B - \alpha\right)\left((\alpha-1)p_{B1} - \alpha p_{G1}\right) + \alpha\left((\alpha-1)u_B - \alpha\right) - p_{G1} + 1 = 0 \tag{84}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = p_{B1}\left(-2\alpha^3 + 4\alpha^2 - 2\alpha + 2(\alpha-1)^3 u_B - 1\right) - (\alpha-1)\alpha\left(2p_{G1}\left((\alpha-1)u_B - \alpha\right) + u_B - 1\right) = 0 \tag{85}$$

$$p^*_{G1} \in (0,1), p^*_{B1} \in (0,1) \tag{86}$$

  Solving the equations, we could then find the solution candidate:

$$p_{G1} = \frac{-2\alpha^4 + 2\alpha^3 + 3\alpha^2 - 3\alpha + \left(4\alpha^4 - 8\alpha^3 + \alpha^2 + 5\alpha - 2\right)u_B - 2(\alpha-1)^3\alpha u_B^2 - 1}{-4\alpha^3 + 4\alpha^2 - 2\alpha + \left(4\alpha^3 - 8\alpha^2 + 6\alpha - 2\right)u_B - 1}$$

$$p_{B1} = \frac{(1-\alpha)\alpha\left(2\alpha^2 + 2\alpha + \left(1 - 4\alpha^2\right)u_B + 2(\alpha-1)\alpha u_B^2 + 1\right)}{-4\alpha^3 + 4\alpha^2 - 2\alpha + \left(4\alpha^3 - 8\alpha^2 + 6\alpha - 2\right)u_B - 1}$$

  $p_{B1} > 0$ does not hold when $u_B \geq -\frac{\alpha}{1-\alpha}$

  So, $p^*_{G1} \in (0,1), p^*_{B1} \in (0,1)$ could not be the solution.

- $p^*_{G1} = 1, p^*_{B1} \in (0,1)$

  Then, $\lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

  Equation (78) and (79) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha\left(-\alpha - 2\left((\alpha-1)u_B - \alpha\right)\left((\alpha-1)p_{B1} - \alpha\right) + (\alpha-1)u_B + 1\right) - \lambda_1 = 0 \tag{87}$$

$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = p_{B1}\left(-2\alpha^3 + 4\alpha^2 - 2\alpha + 2(\alpha-1)^3 u_B - 1\right) - (\alpha-1)\alpha\left(-2\alpha + (2\alpha-1)u_B - 1\right) = 0 \tag{88}$$

  Solving the equation, we could then get the solution candidate $p_{B1} = \frac{(\alpha-1)\alpha(-2\alpha+(2\alpha-1)u_B-1)}{-2\alpha^3+4\alpha^2-2\alpha+2(\alpha-1)^3u_B-1}$ and $p_{G1} = 1$. $p_{B1} \in (0,1)$ is not possible when $u_B \geq -\frac{\alpha}{1-\alpha}$

  So, $p^*_{G1} = 1, p^*_{B1} \in (0,1)$ could not be the solution.

- If $p^*_{G1} = 1, p^*_{B1} = 0$

  Then, $\lambda_2 = 0, \lambda_3 = 0$

Equation (78) and (79) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = \alpha\left(-2\alpha^2 - \alpha + \left(2\alpha^2 - \alpha - 1\right)u_B + 1\right) - \lambda_1 = 0 \tag{89}$$
$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = -(\alpha - 1)\alpha\left(-2\alpha + (2\alpha - 1)u_B - 1\right) + \lambda_4 = 0 \tag{90}$$
$$\lambda_1 \geq 0, \lambda_4 \geq 0 \tag{91}$$

- $\lambda_1 = \alpha\left(-2\alpha^2 - \alpha + \left(2\alpha^2 - \alpha - 1\right)u_B + 1\right) \geq 0, \Rightarrow u_B \leq \frac{-2\alpha^2-\alpha+1}{-2\alpha^2+\alpha+1}$
- $\lambda_4 = (\alpha - 1)\alpha\left(-2\alpha + (2\alpha - 1)u_B - 1\right) \geq 0$ always holds when $u_B \geq -\frac{\alpha}{1-\alpha}$
- If firm 1 sets a price lower than $u(b)$, when firm 2 uses mixed pricing strategies according to $F_2(p) = 1 - \frac{u(g)-\Pr(g)\mathbb{E}[u]}{(p+(u(g)-\mathbb{E}[u]))}$ for $\Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u]$, its expected profit will be:

$$\pi_1(p_1) = p_1 * [\Pr(g) + \Pr(b)(1 - F_2(p_1 + (\mathbb{E}[u] - u(b))))], \text{ for } p_1 \leq u(b) \tag{92}$$

  The second derivative of the profit with respect to price is is negative for any $p \leq u(b)$. Hence the function is concave. The first derivative at $p = u(b)$ is positive when $\alpha \in [0, 1], u_B < 1$. As a result, a deviating firm can maximize its profit by setting the price to $u_j(b)$. The deviating profit should be lower than the equilibrium profit, yielding a constraint $u_B < -\frac{\sqrt{4\alpha^6-8\alpha^5+8\alpha^3-4\alpha^2+1}-1}{2(\alpha-1)^2\alpha}$
- The intersection of above constraint is $u_B < -\frac{\sqrt{4\alpha^6-8\alpha^5+8\alpha^3-4\alpha^2+1}-1}{2(\alpha-1)^2\alpha}$ for $\alpha \in [0, 0.366025]$; $u_B < \frac{-2\alpha^2-\alpha+1}{-2\alpha^2+\alpha+1}$ for $\alpha \in [0.366025, 1]$

So, $p^*_{G1} = 1, p^*_{B1} = 0$ could be a possible solution.

- If $p^*_{B1} = 0, p^*_{G1} \in (0, 1)$

  Then , $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$

  Equation (78) and (79) of the Kuhn-Tucker conditions now become

$$\frac{\partial \mathcal{L}}{\partial p_{G1}} = -\alpha^2 + \alpha + p_{G1}\left(-2\alpha^3 + 2(\alpha - 1)\alpha^2 u_B - 1\right) + (\alpha - 1)\alpha u_B + 1 = 0 \tag{93}$$
$$\frac{\partial \mathcal{L}}{\partial p_{B1}} = -(\alpha - 1)\alpha\left(2p_{G1}\left((\alpha - 1)u_B - \alpha\right) + u_B - 1\right) + \lambda_4 = 0 \tag{94}$$
$$\lambda_4 \geq 0 \tag{95}$$

Solving the (46), we could then get the solution candidate $p_{G1} = -\frac{-\alpha^2+\alpha+(\alpha-1)\alpha u_B+1}{-2\alpha^3+2(\alpha-1)\alpha^2 u_B-1}$ and $p_{B1} = 0$

- $p_{G1} \in (0, 1) \Rightarrow u_B > \frac{-2\alpha^2-\alpha+1}{-2\alpha^2+\alpha+1}$

– $\lambda_4 = \frac{(\alpha-1)\alpha\left(2\alpha^2+2\alpha+\left(1-4\alpha^2\right)u_B+2(\alpha-1)\alpha u_B^2+1\right)}{-2\alpha^3+2(\alpha-1)\alpha^2 u_B-1} \geq 0$ always holds when $u_B \geq -\frac{\alpha}{1-\alpha}$

– If firm 1 sets a price lower than $u(b)$, when firm 2 uses mixed pricing strategies according to $F_2(p) = 1 - \frac{u(g)-\Pr(g)\mathbb{E}[u]}{(p+(u(g)-\mathbb{E}[u]))}$ for $\Pr(b) * \mathbb{E}[u] \leq p_2 \leq \mathbb{E}[u]$, its expected profit will be:

$$\pi_1(p_1) = p_1 * [\Pr(g) + \Pr(b)(1 - F_2(p_1 + (\mathbb{E}[u] - u(b)))), \text{ for } p_1 \leq u(b) \tag{96}$$

The second derivative of the profit with respect to price is negative for any $p \leq u(b)$ and $u_B > \frac{2\alpha^2+\alpha-1}{2\alpha^2-\alpha-1}$. Hence the function is concave. The first derivative at $p = u(b)$ is positive when $u_B > \frac{2\alpha^2+\alpha-1}{2\alpha^2-\alpha-1}$. As a result, a deviating firm can maximize its profit by setting the price to $u(b)$. The deviating profit should be lower than the equilibrium profit, yielding a constraint root of a 4-th order polynomial

□

Proof of Proposition 8. If the platform chooses p_{Gj} and p_{Bj} for both firms

If $\alpha \in [0, 0.5]$ and $u_B < \frac{\alpha(1-\alpha)}{\alpha^2-\alpha+1}$(condition to support an equilibrium in the second stage), then the optimal information design policy for the platform is $p_{Gj} = 1, p_{Bj} = 0$ and the profit for the platform is $\pi_p|_{p_{Gj}=1,p_{Bj}=0} = 2(1-\alpha)\alpha$,

If $\alpha \in (0.5, 1]$ and $u_B < \frac{\alpha\left(6\alpha^6-4\alpha^5+9\alpha^4-7\alpha^3+2\alpha^2-1\right)}{(\alpha-1)(2\alpha^2+1)(3\alpha^4+4\alpha^2-\alpha+1)}$(condition to support an equilibrium in the second stage), then $p_{Gj} = \frac{\alpha+1}{2\alpha^2+1} < 1, p_{Bj} = 0$ and the profit for the platform is $\pi_p|_{p_{Gj}=\frac{\alpha+1}{2\alpha^2+1},p_{Bj}=0} = \frac{(2-\alpha)\alpha}{2\alpha^2+1}$,

If the platform chooses p_{Gj} and p_{Bj} for only one firm and let the other firm do no send any

When $u_B > \frac{-2\alpha^2-\alpha+1}{-2\alpha^2+\alpha+1}$, $p_{G1} = -\frac{-\alpha^2+\alpha+(\alpha-1)\alpha u_B+1}{-2\alpha^3+2(\alpha-1)\alpha^2 u_B-1}$ and $p_{B1} = 0$

$\pi_p^{*as} = \frac{\alpha\left(-5\alpha^3+4\alpha^2+\alpha-4\right)-5(\alpha-1)^2\alpha^2 u_B^2+2\left(5\alpha^4-7\alpha^3+\alpha^2+2\alpha-1\right)u_B}{-4\alpha^3+4(\alpha-1)\alpha^2 u_B-2} > 2(1-\alpha)\alpha$ when $\alpha \in [0, 0.5]$

$\pi_p^{*as} = \frac{\alpha\left(-5\alpha^3+4\alpha^2+\alpha-4\right)-5(\alpha-1)^2\alpha^2 u_B^2+2\left(5\alpha^4-7\alpha^3+\alpha^2+2\alpha-1\right)u_B}{-4\alpha^3+4(\alpha-1)\alpha^2 u_B-2} > \frac{(2-\alpha)\alpha}{2\alpha^2+1}$ when $\alpha \in (0.5, 1]$

When $-\frac{\alpha}{1-\alpha} \leq u_B \leq \frac{-2\alpha^2-\alpha+1}{-2\alpha^2+\alpha+1}$

$\pi_p^{*as} = (1-\alpha)(\alpha + (1-\alpha)u_B) + \alpha(1 - \alpha(\alpha + (1-\alpha)u_B)) > 2(1-\alpha)\alpha$ when $\alpha \in [0, 0.5)$ and $u_B > \frac{\alpha^2}{\alpha^2+\alpha-1}$

$\pi_p^{*as} = (1-\alpha)(\alpha + (1-\alpha)u_B) + \alpha(1 - \alpha(\alpha + (1-\alpha)u_B)) > \frac{(2-\alpha)\alpha}{2\alpha^2+1}$ when $\alpha \in (0.5, 1]$

When $u_B < -\frac{\alpha}{1-\alpha}$

$\pi_p^{*as} = \alpha < 2\alpha(1-\alpha)$ when $\alpha \in [0, 0.5)$ and $u_B < -\frac{\alpha}{1-\alpha}$

$\pi_p^{*as} = \alpha > \frac{(2-\alpha)\alpha}{2\alpha^2+1}$ when $\alpha \in (0.5, 1]$ and $u_B < -\frac{\alpha}{1-\alpha}$

□