A Model of Indivisible Consumption Categories

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Abstract

We offer a model that adapts the classical consumer choice model by assuming the indivisibility of consumption categories. Specifically, we assume that most consumption categories have a minimal expenditure threshold. This seems plausible with reference to real life (e.g., consumption related to car-ownership, or consumption related to buying a house). Our model provides a novel explanation for two puzzles: (1) why people often buy both insurance and lottery tickets, (2) why people often behave in a way consistent with narrow framing of their situation rather than acting out of global principles of maximization. We show that the complexity level of the consumer problem is non-monotonic and relative small for low wealth NP-hard for medium wealth and polynomial for high enough wealth. Next, we show that the model induce decreasing absolute risk aversion in wealth. we also show that investing in a lottery ticket (i.e., a product with high gain - small loss) can be worthwhile for those with low wealth and for a medium wealth the added value of the lottery decreases down to a negative value as wealth increases Finally, we show that for significant loss/gain the model induce **the fourfold pattern of risk attitude:** small probability inducing risk aversion for loss and risk loving for gain and high probability inducing risk loving for loss and risk aversion for gain.

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1 Introduction

Consumer choice is a central issue in the field of economics. Consumers seek a basket of goods that maximizes their utility under their budget constraints. The classical model of consumer choice assumes that all goods are divisible. Other models allow indivisibility of goods that induces a utility function that is not differentiated throughout its domain (see. Florig & Rivera, 2017; Friedman & Sákovics, 2015). Thaler (1985, 1990, 1999) has shown in a series of studies that consumers divide their budgets into consumption categories, wherein money assigned to one category is not fungible for use in another. Thaler found that the higher the budget, the more flexible the division into categories, suggesting that rich people tend to be more flexible or elastic when dividing their budgets among the different categories (Thaler, 1999, p.193).

We offer a model that incorporates the classical model with indivisibility using consumption categories. We assume that most consumption categories require a minimum consumption threshold that has a minimum cost like car ownership (car, fuel and compulsory insurance), housing (house, property tax, maintenance, etc.). Once the agent exceeds the threshold of minimal expenditure in any category, the utility from consumption in that category behaves just like the standard mode: continuous, increasing and concave.

The indivisible categories affecting an agent's risk preferences operate along four channels: (1) inducing risk aversion with respect to small changes, (2) inducing a preference for lotteries with a small probability to gain a large prize (positive skewness) , and that this preference decreases as a function of wealth, (3) for significant loss/gain the model induce the fourfold pattern of risk attitude: small probability inducing risk aversion for loss and risk loving for gain and high probability inducing risk loving for loss and risk aversion for gain (4) under some additional minor assumptions, absolute risk aversion decreases for any underlining utility function in each category.

The complexity of finding the basket of goods that maximize agents' payoff is non-monotonic in agents' wealth: hence, it is relative small for those of low-wealth, NP-hard for those of medium-wealth, and polynomial for high-enough wealth. This suggests that people with intermediate levels of wealth may depart more from utility maximization (due to complexity costs) than do either poor or rich people.

The paper is organized as follows: Section 2 presents the related literature and Section 3 presents our model with results on attitudes to risk and the complexity of the model. Section 4 contains the conclusion.

2 Related literature

2.1 The Concave-Convex Utility Function and Attitude to Risk in Wealth

One of the puzzles in the study of attitude to risk is the question of why individuals simultaneously buy both insurance and lottery tickets. Friedman and Savage (1948) propose that non-concave utility over wealth explains this behavior. Indivisible consumption categories provide micro-foundations for such non-concave utility over wealth and thus explain why agents repeatedly undertake skewed gambles. Markowitz (1952) addresses the non-concave utility induced by Friedman and Savage's indivisible categories because non-concavity occurs only at a point of change in the support (the categories support). This means that it creates a reference point that makes the non-concavities shift with wealth levels as support increases. Bailey et al. (1980) criticizes Friedman and Savage by showing that saving or borrowing can be preferable to gambling under several conditions. Hartley and Farrell (2002) show the implausibility of the conditions needed to avoid gambling under a non-concave utility function.

2.2 Risk Aversion in Wealth

Absolute risk aversion is one of the measures of risk aversion. Several studies show that relative risk aversion is almost constant (see Arrow, 1971; Gordon, Paradis, and Rorke, 1972, Kroll, Levy, and Rapoport, 1988 and Levy, 1994). Indivisible consumption categories induce a decrease in absolute risk aversion with respect to wealth that is implied by constant relative risk aversion.

2.3 Risk loving in Wealth

Studies show that low-earners spend a larger share of their income on legal gambling than do their more affluent counterparts and, in some studies, even a larger amount of money (Rintoul et al., 2014). This regressive incidence of gambling exists in different countries at different periods of time and in both micro- and macro-data. (see Clotfelter and Cook, 1991; Miyazaki et al., 1998; Beckcert and Lutter, 2009; Perez and Humphreys, 2013).

2.4 The Fourfold Pattern of Risk Attitude

Kahneman and Tversky (1979) present the fourfold pattern of risk attitude. Choices are between a sure thing px and a gamble yielding outcome x with a probability p and nothing otherwise, denoted (x, p). The fourfold pattern is that, as p increases from low to high, risk preferences in gains change from risk seeking to risk aversion, and those in losses change from risk aversion to risk seeking. For significant loss/gain that results in a support change a symmetric indivisible categories model induce the fourfold pattern. Markowitz (1952) proposed a triply inflected value function, which was convex over small gains but concave over larger ones, and concave over small losses but convex over larger ones. Scholten and Read (2014) show experiment data the show that the size of the amount change the risk preference and not only probability.

2.5 Categorization of the Consumer Problem

Thaler (1985, 1990, 1999) shows that individuals divide available consumption products into categories (health, entertainment, etc.) such that money belonging to one category is not fungible in another. According to Thaler, distribution of the total budget into different categories is used to achieve self-control over expenditures. In addition, categorization makes it easier to make rational decisions among different alternatives. To the best of my knowledge, there is no formal model for categorization.

2.6 Global Maximization or Sub-Optimal Decision-making and Narrow Framing?

Simon (1957) assumed that the human brain is limited in its computational ability and that each computation requires energy and time, which imposes a cost on the individual. Birnbaum (2008) argued that individuals' limited cognitive capacity limits identification of the best alternatives within a reasonable time frame. Classical models in economics assume that individuals have unlimited ability to analyze complex information, evaluate all possibilities, and make the optimal choice in each decision problem, but a large repository of literature demonstrates violations of the direct axioms of transitivity and independence, which implies that people often do not maximize their utility functions (see: Thaler, 1999; Kahneman & Tversky, 1974).

Benartzi and Thaler (1995) proposed that investors are loss-averse over fluctuations in the value of their financial wealth, which, since financial wealth is just one component of total wealth, constitutes narrow framing. Indivisible consumption categories can induce narrow framing as a function of wealth, where large medium range of wealth predicts sub-optimal decision making that can lead to narrow framing, but for individuals with low-income or high enough income the model's complexity is low and can lead to global-maximization as assumed by the classical model. Thaler found that the higher the budget, the more flexible the division into categories, suggesting that rich people tend to be more flexible when dividing the total budget among different categories (based on a series of interviews Thaler conducted in the early 1980s; see: Thaler, 1999 p.193). Jenny (2017) shows that the consumption of the poor who get cash or food vouchers as the same consumption, suggesting that poor agents are less sensitive to narrow-framing.

3. THEORY

Consider a consumer with a budget of B and K consumption categories (like housing, car ownership…) also consider that for each category there is consumption threshold that is represented by minimum consumption cost C_k Formally:

Define b_k as budget of category *k*.

An allocation of the budget B to the categories is defined as a vector b where $b \in R_+^K$ and

$$|b| = \sum_k b_k = B$$

The minimum consumption cost of the categories is defined as a vector c where $c \in R_+^K$

Define $f_k(b_k)$ as the utility of category k from consumption of b_k

$$f_k(b_k) = \begin{cases} u(b_k) & if\ b_k \geq c_k \\ 0 & if\ b_k < c_k. \end{cases}$$

We assume that u_k is a strictly increasing and strictly concave function where $u_k(0) = 0$. formally,

A1: Utility of a category is concave: $u`(b_k) > 0$ and $u``(b_k) < 0$.

A2: Limits of marginal utility: $for\ all\ k \in K\ \lim_{b_k^+ \to 0} u`(b_k) = \infty$ and $for\ all\ k \in K\ \lim_{b_k \to \infty} u`(b_k) = 0$

The utility of the consumer is the sum of the utilities of the categories:

$$U = \sum_1^K f_k(b_k)$$

Definition: A budget $b(B): \mathbb{R}_+ \to \mathbb{R}_+^K$ with $|b| = B$ will be called an **optimal allocation** if

$$U(b) = \max_{|\tilde{b}|=|b|} U(\tilde{b})$$

And b is minimal such budget with respect to the lexicographic order on $\mathbb{R}_+^K$.

We will definte the optimal allocation in a way that it has no removable discontinuity, namely, if $\lim_{B \to B_0} b(B) \neq b(B_0)$ under the lexicographic order tie-breaker defined above, we will let $b(B_0) = \lim_{B \to B_0} b(B)$.

Claim 1: For every category k there exists a budget B^k such that the optimal allocation b at budget B^k satisfies $b_k \geq c_k$ at the right neighborhood of B^k, and the optimal allocation b at a budget $B < B^k$ satisfies $b_k = 0$.

Proof: At every budget $B > c_k$, we can use category k for receiving utilization of $\frac{f_k(c_k)}{c_k} > 0$ by investment of c_k budget. Since we assume $f_i'(x) \rightarrow_{x\rightarrow\infty} 0$ for all i, then we must have $U'(b) \rightarrow_{x\rightarrow\infty} 0$. Therefore, there exist a B such that $U'(b) < \frac{f_k(c_k)}{c_k}$ for all $b > B$, and in particular an optimal allocation can be improved by the utilization obtained by the k-th category, for all $b > B + c_k$, namely the optimal allocation at $|b| = B + c_k$ satisfies $b_k > 0$. Denote by B^k the minimal such budget, and this finishes the argument.

Definition: We say that an entrance price consumer problem is a **generic** problem, if for every budget B, the set of optimal allocations of the budget

$$A = \underset{|b|=B}{\operatorname{argmax}}\, U(b)$$

Satisfies that the sets

$$A' = (U'(b))_{b\in A}$$

Satisfy that for every $b_1, b_2 \in \operatorname{argmax} A'$ we have $U^{(j)}(b_1) = U^{(j)}(b_2)$ for all $j \geq 2$.

Intuitively, we expect (up to some coincidence) that every significant change (jump) in the allocation to be affected by benefit caused due to direct change in utility or in the first derivative of the utility. Roughly speaking, assuming the model is stochastic and sampled uniformly from some continuous distribution, this generic behavior happens in probability 1.

Claim 2: In the generic entrance price consumer problem, for every (jump) discontinuity of the optimal allocation function $b(B)$ at a point B_0 (namely, $\lim_{B\to B_0^-} b(B) \neq \lim_{B\to B_0^+} b(B)$) we have either $U_+(B_0) > U_-(B_0)$ or $U'_+(B_0) > U'_-(B_0)$.

The proof of Claim 2 is follows from the definition of the genetic problem.

Claim 2 show that our model induce a concave-convex utility function that lead to risk aversion when the support is constant and a support chance can lead to risk loving because of the increase in the marginal utility. This result implied that the point of change in the support may induce behavior that is similar to a reference point in the prospect theory that is risk aversion for gain and risk loving for loss.

The complexity of our model:

The complexity of the consumer problem can induce narrow framing. We will show that our model induce NP-complete problem that is very hard to solve. We will also show that there is high-budget $B^H_{C,U}$ such that the complexity for $B > B^H_{C,U}$ is $o(n^c)$ and small-budget $B < B^L_{C,U}$ the complexity is o_1.

Define the consumer problem as **problem 1**. Formally:

Maximizing U
Such that:
$B \geq |b|$ and
$b_i \geq ci \; or \; b_i = 0$

Proposition 1: Problem 1 is NP-Complete.

The proof of Proposition 1 is presented in Appendix 1. In order to prove that Problem 1 is NP-hard, we describe a polynomial time-reduction from the Knapsack Problem, (which is known to be NP-Complete) to Problem 2. We prove that the support of

Problem 1 is the same as support for Problem 2 and, therefore, the solution of Problem 1 yields a solution to the Knapsack Problem.

From claim 1 we know there is a budget high enough such that all categories will have positive consumption and we define it as $B_{C,U}^{H}$ any higher budget will have positive consumption in all categories and in this higher regime the problem is a sum of continues and strictly concave functions that is $o(n^c)$ (see Bubeck, 2015).

$if\ max\{|c_i| \leq B\}$=A where A is a constant number of categories then the complexity is O(1).

<u>Utilization of money</u>:

Utilization of money simply define as the utility divided by the budget. We show that in a symmetric model from low levels of wealth utilization increases up to high levels of wealth. This suggests that individuals with low wealth suffer not only from low wealth but also from low utilization of their wealth.

Define level of utilization of money by the ratio: $Y = \frac{U(B)}{B}$

<u>Symmetric model:</u>

Assuming the K categories have the same utility function $\boldsymbol{f}(\boldsymbol{b})$ and the same consumption threshold cost (c).

Proposition 2: Let $n, m \in \mathbb{N}$ be two positive integers such that $n + m \leq K$ (total number of categories). Then,

$$Y(nc + x) < Y\big((n + m)c + x\big)$$

Proof: From symmetry

$$U(\ell c + x) = \ell u\left(c + \frac{x}{\ell}\right)$$

Therefore

$$Y(\ell c + x) = \frac{U(\ell c + x)}{\ell c + x} = \frac{u\left(c + \frac{x}{\ell}\right)}{c + \frac{x}{\ell}}$$

And this increases in ℓ since u is concave and $u(x) = 0$.

Symmetric model and category zero:

Assuming the K categories have the same utility function $\boldsymbol{f}(\boldsymbol{b})$ and the same consumption threshold cost (c) and a category with zero consumption threshold and utility function $u_0(b) < u(b) for\ all\ b > 0$.

Proposition 3: For all $0 < x < c$ and $n \in \mathbb{N}$, the marginal utilization for an incremental budget x is an increasing function of n, namely,

$$\frac{U\big((n+1)c + x\big) - U\big((n+1)c\big)}{x} \geq \frac{U(nc + x) - U(nc)}{x}$$

Moreover, there is a constant $0 < b_0 < c$ which will be defined later, such that there is equality for all $x \leq b_0$ and strict inequality for all $b_0 < x$.

The proof of Proposition 4 is presented in Appendix 2.

Theorem 1: For all $0 < x < c$ and $n \in \mathbb{N}$, the absolute risk aversion of U for an incremental budget x is an **decreasing** function of n, namely, for all increasing or constant absolute risk aversion and decreasing hyperbolic absolute risk aversion of utility for symmetric category assuming $\alpha u_0(b) = u(b)$.

$$-\frac{U''((n+1)c+x)}{U'((n+1)c+x)} \leq -\frac{U''(nc+x)}{U'(nc+x)}$$

The proof of Theorem 1 is presented in Appendix 3.

<u>Attitude toward high gain - small loss lotteries (lotto):</u>

It's well known that the lotto investment (very high gain and small loss) decreases with respect to wealth (see section 2). We prove that our model induces a similar result:

Definition: A lottery – You pay s, and with probability p you get $mc + s$, for a fixed $m \geq 1$. The expected change in budget is 0, namely

$$p(mc) + (1-p)(-s) = 0$$

$$p = \frac{s}{mc + s}$$

The expected change in utility is

$$\mathbb{E}[U] = pU(|b| + mc + s) + (1-p)U(|b| - s)$$
$$= \frac{s}{mc+s}U(|b| + mc) + \frac{mc}{mc+s}U(|b| - s)$$

Proposition 4: There exists $0 < \widetilde{b_0} < c$ such that for all $0 < x < \widetilde{b_0}$ we have

$$U(nc + x) > \mathbb{E}[U]$$

And for all $\widetilde{b_0} < x < c$ we have

$$\mathbb{E}[U] > U(nc + x)$$

We will estimate the new utilities.

$$U(|b|+mc)=U(|b|)+mu(c)$$

For all $x<b_0$ and $m+n<k$ the number of categories.

$$U(|b|-s)\sim U(|b|)-sU'(|b|)=U(|b|)-su_0'(x)$$

For s small enough.

Therefore,

$$\mathbb{E}[U]-U=\frac{s}{mc+s}\big(mu(c)\big)+\frac{mc}{mc+s}\big(-su_0'(x)\big)=\Big(\frac{ms}{mc+s}\Big)\big(u(c)-cu_0'(x)\big)$$

Therefore $\mathbb{E}[U]-U$ is positive if and only if

$$u(c)-cu_0'(x)>0$$

$$u_0'(x)>\frac{u(c)}{c}$$

Since u_0' is decreasing, and $\frac{u(c)}{c}>u'(c)$, there exists a unique $\widetilde{b_0}<b_0$ such that for all $0<x<\widetilde{b_0}$ we have $\mathbb{E}[U]<U(nc+x)$ and for all $\widetilde{b_0}<x<b_0$ we have $\mathbb{E}[U]>U(nc+x)$.

Note that for $b_0<x$ we obviously have $\mathbb{E}[U]-U>0$ since $U'(|b|)<\frac{u(c)}{c}$ and $U(|b|+mc)-U(|b|)>mu(c)$, and therefore

$$\begin{aligned}\mathbb{E}[U]-U&=\frac{s}{mc+s}\big(U(|b|+mc)-U(|b|)\big)+\frac{mc}{mc+s}\big(-sU'(|b|)\big)\\&=\frac{s}{mc+s}\Big(\big(U(|b|+mc)-U(|b|)-mcU'(|b|)\big)\Big)\\&>\frac{s}{mc+s}\left(mu(c)-mc\frac{u(c)}{c}\right)=\frac{ms}{mc+s}\big(u(c)-u(c)\big)=0\end{aligned}$$

Note that as long as $m+n<k$, $\mathbb{E}[U]-U$ **does not depend on n and only depends on x.**

When $m+n\geq k$, then $U(|b|+mc)-U(|b|)$ is **decreasing** in $|b|$ –

Let $b_1<b_2$ be budgets in the relevant domain.

$$\begin{aligned}&\big(U(|b_2|+mc)-U(|b_2|)\big)-\big(U(|b_1|+mc)-U(|b_1|)\big)\\&\quad=\big(U(|b_2|+mc)-U(|b_1|+mc)\big)-\big(U(|b_1|)-U(|b_2|)\big)\end{aligned}$$

Since U' is strictly decreasing for all $|b|>kc$ the above term is negative.

Therefore $\mathbb{E}[U] - U = \frac{s}{mc+s}\big(U(|b| + mc) - U(|b|)\big) + \frac{mc}{mc+s}\big(-sU'(|b|)\big)$ Is strictly decreasing in $|b|$, for $m + n \geq k$.

The Fourfold Pattern of Risk Attitude in the Symmetric Model:

Risk Aversion and Risk Loving for Loss

We will examine for which values of $p \in (0,1), a > 0$ we will prefer risking the money a in probability p, in contrast of losing pa of the budget. Namely, the condition for risk-loving is -

$$U(b) - U(b - pa) > p\big(U(b) - U(b - a)\big) + (1 - p)(0)$$

$$\frac{U(b) - U(b - pa)}{pa} > \frac{U(b) - U(b - a)}{a}$$

Similarly to the insurance case, the condition is exactly that

$$f_b(pa) > f_b(a)$$

We will examine this condition's dependency on the parameters a, p, x.

Proposition 5: (Classification of risk-loving budgets in the loss domains, for the full symmetric case)

Let $|b| = nc + x$, for $0 < x < c$, be the current budget, and assume $a < c + x$. We assume further that $u''' \geq 0$, namely u' is convex.

Under these assumptions, we are risk-loving if and only if

$$x < a \text{ and } p \in \left(\frac{x}{a}, 1\right)$$

Proof:

First we will prove a lemma about the behavior of the function $f_b(y)$.

Lemma: $f_b(y)$ satisfies –

1. Increasing for $y < x$ and decreasing for $y > x$
2. $f_b(y)$ has a jump discontinuity at $y = x$
3. $f_b(y) < u'(c)$ for all $0 < y < x$
4. $f_b(y) > u'(c)$ for all $x < y < c + x$

Proof of the lemma

Direct computation gives

$$f_b'(y) = \frac{U'(b-y)y - (U(b) - U(b-y))}{y^2} = \frac{1}{y}\left(U'(b-y) - \frac{U(b) - U(b-y)}{y}\right)$$

Since U' is decreasing, the above term is positive for $y < x$ – until the jump discontinuity.

For $y > x$, we are averaging increasing slopes, smaller than $u'(c)$, which are approaching $f_b(c) = \frac{U(b)-U(b-c)}{c} \geq \frac{u(c)}{c} > u'(c)$, therefore the function must decrease to it's final average.

Obviously $f_b'(y) < u'(c)$ for all $0 < y < x$, since we are averaging derivatives all smaller than the maximal derivative $u'(c)$. Afterwards, we include the jump discontinuity.

For $x < y < c$, since f_b is decreasing, and $f_b(c) = \frac{U(b+c)-U(b)}{c} \geq \frac{u(c)}{c}$, and therefore $f_b(y) \geq \frac{u(c)}{c} > u'(c)$, since u is concave.

For $y > c$, Note that

$$\frac{U(b) - U(b-y)}{y} \geq \frac{u(y)}{y}$$

Since we can allocate the extra y budget to a new category, obtaining utilization of $\frac{u(y)}{y}$, and optimal allocation can obviously only do better. Therefore we have

$$f_b(y) \geq \frac{u(y)}{y} = \frac{1}{y}\int_0^y u'(t)dt >_1 u'\left(\frac{y}{2}\right) \geq u'(c)$$

For all $y < 2c$, and (1) follows from a general claim about convex function, which uses the additional assumption $u'''(x) \geq 0$, this completes the proof of the lemma.

From the lemma, we can see that $f_b(pa) < f_b(a)$ for all $a < x$ and $p \in (0,1)$, from (1), therefore we have risk aversion for $a < x$. For $a > x$, If $p < \frac{x}{a}$, from (3) and (4) we have $f_b(pa) < u'(c) < f_b(a)$, and therefore we are risk averse. For $p > \frac{x}{a}$ we are risk loving from (1).

Now we will generalize the result inductively the the case $a > c + x$, using a similar lemma about the behavior of the accumulating slope average function f_b.

<u>Lemma</u>**:** $f_b(y)$ satisfies –

1. Increasing for $y < x$ and decreasing for $y > x$
2. $f_b(y)$ has a jump discontinuity at $y = nc + x$ for all $n \geq 0$
3. $f_b(y) < u'(c)$ for all $0 < y < x$

<u>Proof:</u>

1. For $y > x$, we will state that on each of the intervals $(nc + x, (n+1)c + x)$ the function has a final value $f_b\big((n+1)c + x\big) > u'(c)$, the function averages slopes smaller than $u'(c)$,

and therefore must decrease to it's final value.
The claim for $n = 0$ was proved in the previous lemma, and the rest follows by induction –

$$f_b\big((n+1)c+x\big) = \frac{U(b)-U(b-(n+1)c-x)}{(n+1)c+x}$$
$$= \frac{nc+x}{(n+1)c+x}\cdot\frac{U(b)-U(b-nc-x)}{nc+x} + \frac{c}{(n+1)c+x}$$
$$\cdot\frac{U(b-nc-x)-U(b-(n+1)c-x}{c} =$$

$$f_b\big((n+1)c+x\big) \geq_1 \frac{nc+x}{(n+1)c+x} f_b(nc+x) + \frac{c}{(n+1)c+x}\cdot\frac{u(c)}{c}$$

Where (1) follows from

$$U(b-nc-x) - U(b-(n+1)c-x) \geq u(c)$$

By the induction assumption, the first summand is $\geq \frac{u(c)}{c}$, and therefore the above quantity is also larger than $\frac{u(c)}{c}$ as a weighted average of such.

Therefore the function averages slopes smaller than each of the final values at the intervals $(nc+x, (n+1)c+x)$ and must be decreasing in those intervals. At the points $nc+x$ it obviously has a jump discontinuity.

2 and 3 follow from the previous lemma**.**

Therefore, we can infer that for all $a > 0$ there exists p_0 large enough, such that for all $p_0 < p < 1$ we are risk loving.

Let $a = mc + x + \alpha$, $\alpha < c$. From the previous lemma, the function f_b is decreasing in the interval $(mc+x, (m+1)c+x)$ and therefore $f_b(y) > f_b(a)$ for all $mc + x < y < a$. Therefore if $p > \frac{mc+x}{a}$ we have risk loving, namely $f_b(pa) > f_b(a)$.

Risk Aversion and Risk Loving for Gain

We will examine for which values of $p \in (0,1), a > 0$ we will prefer risking the money a in probability p, in contrast of losing pa of the budget. Namely, the condition for risk-loving is -

$$U(b + pa) - U(b) < p\big(U(b + a) - U(b)\big) + (1 - p)(0)$$

$$\frac{U(b + pa) - U(b)}{pa} < \frac{U(b + a) - U(b)}{a}$$

Define

$$g_b(y) = \frac{U(b + y) - U(b)}{y}$$

So the condition for risk loving is

$$g_b(pa) < g_b(a)$$

We will examine this condition's dependency on the parameters a, p, x.

Proposition 6: (Classification of risk-averse budgets in the gain domains, for the full symmetric case)

Let $|b| = nc + x$, for $0 < x < c$, be the current budget, and assume $a < 2c - x$. We assume further that $u''' \geq 0$, namely u' is convex.

Under these assumptions, we are risk-averse if and only if

$$0 < a < c - x \text{ and } p \in (0,1)$$

or

$$c - x < a < 2c - x \text{ and } p \in \left(\frac{c - x}{a}, 1\right)$$

Proof:

First we will prove a lemma about the behavior of the function $g_b(y)$.

<u>Lemma 1</u>: $g_b(y)$ satisfies –

4. Decreasing for $y < c - x$ and for $y > c - x$
5. $g_b(y)$ has a jump discontinuity at $y = c - x$
6. $g_b(y) < u'(c)$ for all $0 < y < c - x$
7. $g_b(y) > u'(c)$ for all $c - x < y < 2c - x$

<u>Proof of the lemma</u>

Direct computation gives

$$g_b'(y) = \frac{U'(b+y)y - (U(b+y) - U(b))}{y^2} = \frac{1}{y}\left(U'(b+y) - \frac{U(b+y) - U(b)}{y}\right)$$

Since U' is decreasing, the above term is always negative.

Obviously $g_b'(y) < u'(c)$ for all $0 < y < c - x$, since we are averaging derivatives all smaller than the maximal derivative $u'(c)$. Afterwards, we include the jump discontinuity.

For $c - x < y < c$, since g_b is decreasing, and $g_b(c) = \frac{U(b+c)-U(b)}{c} \geq \frac{u(c)}{c}$, and therefore $g_b(y) \geq \frac{u(c)}{c} > u'(c)$, since u is concave.

For $y > c$, Note that

$$\frac{U(b+y) - U(b)}{y} \geq \frac{u(y)}{y}$$

Since we can allocate the extra y budget to a new category, obtaining utilization of $\frac{u(y)}{y}$, and optimal allocation can obviously only do better. Therefore we have

$$g_b(y) \geq \frac{u(y)}{y} = \frac{1}{y}\int_0^y u'(t)dt >_1 u'\left(\frac{y}{2}\right) \geq u'(c)$$

For all $y < 2c$, and (1) follows from a general claim about convex function, which uses the additional assumption $u'''(x) \geq 0$, this completes the proof of the lemma.

From the lemma, we can see that $g_b(pa) > g_b(a)$ for all $a < c - x$ and $p \in (0,1)$, from (1), therefore we have risk aversion for $a < c - x$. For $a > c - x$, If $p < \frac{x}{a}$, from (3) and (4) we have $g_b(pa) < u'(c) < g_b(a)$, and therefore we are not risk-averse. For $p > \frac{x}{a}$ we are risk averse from (1).

Similarly to the loss case, one can obtain a similar result showing $g_b\big(nc + (c - x)\big) \geq \frac{u(c)}{c}$ and therefore g_b is decreasing in the intervals $\big(nc + (c - x), (n+1)c + (c - x)\big)$, since it averages slopes smaller than it's final value. Therefore, for p large enough, we will have $g_b(pa) > g_b(a)\left(1 > p > \frac{mc+(c-x)}{a}\right)$ – namely we are risk averse.

4 Discussion

Our paper presents novel explanations for the puzzle of attitude to risk where individuals simultaneously buy insurance and lottery tickets. We show that high gain - small loss lotteries (lotto) have positive added value for low-wealth individuals but the added value decreases as wealth increases down to a negative value. We show that for significant loss (that result in zero consumption of a consume category) small (resp., large) probability for loss can induce risk aversion (resp., risk loving) that is similar to finding in the loss domain (see Kahneman and Tversky (1992)). Our model predicts sub-optimal decision making in the medium range of wealth that can lead to narrow framing, but for individuals with low-income or high enough income the model's complexity is low and can lead to global-maximization as assumed by the classical model. We also refer to utilization of money, meaning the utility divided by the budget. We show that from low levels of wealth, utilization increases up to high levels of wealth. This suggests that individuals with low wealth suffer not only from low wealth but also from low utilization of their wealth.

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Appendix

Appendix 1: Proof of Proposition 1

Proof:

The Binary Knapsack Problem, Given a set of n items numbered from 1 up to n, each with a weight w_i and a value v_i and a maximum weight capacity W.

Let x_i be a Boolean variable stating whether we purchased item i.

Maximize $\sum_{i=1}^{n} v_i\, x_i$

Subject to $\sum_{i=1}^{n} w_i\, x_i \leq W\ and\ x_i \in \{0,1\}$.

The Binary Knapsack problem is known to be NP-Complete.

Claim:

The problem 1 is NP-Complete.

Proof:

In order to prove that a problem is NP-complete, two conditions must hold:

1. The problem belongs to the class NP.
2. The problem is NP-hard.

We show below that the consumer problem satisfies the above conditions.

Problem 1 belongs to NP, as it is possible to verify in polynomial time that a given solution is feasible.

In order to prove that problem 1 is NP-hard, we describe a polynomial time reduction from the Knapsack Problem, which is known to be NP-Complete, to problem 2. We will prove that the support of problem 1 is the same support of problem 2 and there for solving problem 1 give a solution to the Knapsack Problem.

Problem 2:

$maximize\ U(b) = \sum_k u(b_k)$ Where $u_k(b) = \begin{cases} 0 & if\ \ b < c_k \\ u_k(c_k) & if\ b \geq c_k \end{cases}$

Such that: $|b| \leq B$

We define an instance for problem 1 as follows:

$$U_k(x) = \begin{cases} 0 & if\ x < c_k \\ u_k(c_k) + \varepsilon(\frac{\sqrt{x - c_k}}{\sqrt{x - c_k} + 1}) & if\ x \geq c_k \end{cases}$$

Assuming that $\operatorname{argmax}_{|b|=B} \widetilde{U}(b)$ attains at $\tilde{b}_0$ and $\operatorname{argmax}_{|b|=B} U(b) = \{b_0, .., b_n\}$

Claim: For all $B \in \mathbb{R}$ there is $\varepsilon > 0$ such that for all b $|U(b \in b_i) - U(b \notin bi)| > \varepsilon * K$ then

$$supp\left(argmax_{|b|=B} \widetilde{U}(b)\right) \subseteq supp\left(argmax_{|b|=B} U(b)\right)$$

Proof: let assume that $supp\left(argmax_{|b|=B} U(b)\right) \notin supp\left(argmax_{|b|=B} \widetilde{U}(b)\right)$ then there is allocation $bj \notin bi$ that $\widetilde{U}(bj) > \widetilde{U}(bi \in b_i)$. By definition U(bi)>U(bj) and max($\widetilde{U}(bj) - U(bj)) < k * \varepsilon$ then the maximum $\widetilde{U}(b_j) - \widetilde{U}(bi) < k * \varepsilon - (U(b_i) - U(bj)$) but $\left(U(b_i) - U(bj)\right) > \varepsilon * K$ and there for$\widetilde{U}(b_j) - \widetilde{U}(bi) < 0$ contradiction.

Define: $u_k(c_k) = v_i\ and\ c_k = w_i$ and $|b| = W$ then the solution of problem 2 is the same solution of a Binary Knapsack problem with the same vector v, w and capacity of W.

The reduction can be computed in polynomial time.

Thus, we have proved that the consumer problem is NP-complete.

$if\ max\{|c_i| \leq B\}$=A where A is a constant number of categories then the complexity is $0_{(1).}$

There is a budget call B_H that in is optimal vector b where all $b_k > 0$ (from proposition 1). If $B \geq B_H$ then the complexity level is polynomial because the optimization problem is a sum of continues and strictly concave function that is $o(n^c)$ (see Bubeck, 2015)

Appendix 2: Proof of Proposition 3

Proof:

Step 1: the budget values where the support of U without category zero change are $B = nc$ where n is the number of categories in the support.

Proof: from $0 \leq B < c$ $U = \boldsymbol{f_0}(\boldsymbol{b})$ (no other option) in $B = c$ U= $\boldsymbol{f}(\boldsymbol{b})$ (because$\boldsymbol{f_0}(\boldsymbol{b}) < \boldsymbol{f}(\boldsymbol{b})$)

In $B = 2c$ U= $\mathbf{2} * \boldsymbol{f}(\boldsymbol{b})$ (because $\boldsymbol{f}(\mathbf{2}\boldsymbol{C}) - \boldsymbol{f}(\boldsymbol{C}) < \boldsymbol{f}(\boldsymbol{C}) - \boldsymbol{from\ concave}$)

We proof for n=2 now we proof in Induction for n and n+1:

Given $B = nc$ was a support change then when the budget increase in c there is a support change and there are n+1 categories with a budget of c. formally we claim:

$\sum\big(f(c+bi) - f(c)\big) < f(c)$ where $\sum bi = c$.

$f(c+bi) - f(c) < f`(c) * bi$ (Because f is strictly concave)

$\sum\big(f(c+bi) - f(c)\big) < f`(c) * \sum bi = f`(c) * c < f(c)$

(Because: $\frac{1}{c}\int_0^c f`(b)db = \frac{f(c)-f(0)}{c} > f`(c) = \frac{1}{c}\int_0^c f`(c)db$)

Step 2:

Now we prove that $U^{-}((n+1)c - U(nc) > U^{-}((n)c - U((n-1)c$ (the utility from extra c is increasing for all c where $B \leq kc$) (Moreover, one can similarly show that $U(nc + x) - U(nc)$ is increasing for all $0 < x < c$)

Define budget $\boldsymbol{b_0}$ such that $\boldsymbol{f`_0}(\boldsymbol{b_0}) = f`(c)$

Because $\boldsymbol{f`_0}(\boldsymbol{b}) < \boldsymbol{f`}(\boldsymbol{b})$ we have $f_0'(c) < f'(c) < f_0'(0) =_{A_3} \infty$ and from the intermediate value theorem we have $0 < b_0 < c$ such that $f_0'(b_0) = f'(c)$

For $x \leq b_0$ the optimal allocation of budget at point $|b| = nc + x$ is

$$(x, c, \cdots, c)$$

And therefore $U(nc + x) - U(nc) = u_0(x)$ which is independent of n and in particular increasing in n.

From now on we will assume $x > b_0$.

And define the optimal allocation of budget at point $|b| = nc + x$ to be

$$(b_0 + b^n, c + c^n, \cdots, c + c^n) \quad (2)$$

Since $x > b_0$ we have $b^n, c^n > 0$ and decreasing in n, since u, u_0 are concave (decreasing derivatives).

$\mathrm{U(nc +} \ \boldsymbol{b_0}) - \mathrm{U(nc)} = \boldsymbol{f_0}(\boldsymbol{b_0})$
$(\ \boldsymbol{b_0}\ go\ to\ category\ zero\ beuase\ the\ drivitise\ of\ zero\ is\ higher)$

$(\sum_1^{n+1} f(c + c^{n+1}) - f(c)) + \boldsymbol{f_0}(\boldsymbol{b_0} + b^{n+1}) - \boldsymbol{f_0}(\boldsymbol{b_0}) > \sum_1^n f(c + c^n) - f(c) + \boldsymbol{f_0}(\boldsymbol{b0} + b^n) - \boldsymbol{f_0}(\boldsymbol{b_0})\ (\mathbf{1})$

We have $kc^k + b^k = x - b_0$ for all k, therefore we invest the exact same amount of money into more categories, and the above inequality (1) obviously holds.

To go from the investment in RHS to LHS, we need to take $c^n - c^{n+1}$ and $b^n - b^{n+1}$ budget out of every category, and invest it in category $n + 1$. This money was invested with derivative **at most** $f'(c + c^{n+1})$. The money is re-invested in the interval $(c, c + c^{n+1})$ and therefore have derivative **at least** $f'(c + c^{n+1})$

$$f(c + c^n) - f(c + c^{n+1}) < f'(c + c^{n+1})(c^n - c^{n+1})$$

$$f(c + c^{n+1}) - f(c) > f'(c + c^{n+1})(c^{n+1} - c)$$

In the final computation, we will use a few preliminary identities -

$$c^{n+1} - c = n(c^n - c^{n+1})$$

$$f_i(c + c^n) - f_i(c + c^{n+1}) < f_i'(c + c^{n+1})(c^n - c^{n+1}) \leq f_{n+1}'(c + c^{n+1})(c^n - c^{n+1}) \quad (3)$$

$$f_{n+1}(c + c^{n+1}) - f_{n+1}(c) > f_{n+1}{}'(c + c^{n+1})(c^{n+1} - c) \quad (4)$$

$$c^{n+1} = \sum_{i=1}^{n} (c^n - c^{n+1}) \quad (5)$$

And therefore -

$$U(nc + x) =_2 f_0(b + b^n) + \sum_{i=1}^{n} f_i(c + c_i^n)$$

$$\sum_{i=1}^{n} f_i(c + c^n) - f_i(c) = \sum_{i=1}^{n} \big(f_i(c + c^{n+1}) - f_i(c)\big) + \big(f_i(c + c^n) - f_i(c + c^{n+1})\big)$$

$$<_3 \sum_{i=1}^{n} \big(f(c + c^{n+1}) - f(c) + f_{n+1}'(c + c^{n+1})(c^n - c^{n+1})\big) =_5$$

$$= \sum_{i=1}^{n} \big(f(c + c^{n+1}) - f(c)\big) + f_{n+1}'(c + c^{n+1})(c^{n+1}) <_4$$

$$< \sum_{i=1}^{n} \big(f_i(c + c^{n+1}) - f_i(c)\big) + f_{n+1}\big(c + c_i^{n+1}\big) - f_{n+1}(c) =$$

$$= \sum_{i=1}^{n+1} \big(f(c + c^{n+1}) - f_i(c)\big)$$

And therefore

$$U(nc+x) - U(nc) < U((n+1)c+x) - U\big((n+1)c\big)$$

Dividing by x gives the desired result –

$$\frac{U\big((n+1)c+x\big) - U\big((n+1)c\big)}{x} > \frac{U(nc+x) - U(nc)}{x} \blacksquare$$

Appendix 3: Proof of Theorem 1

Proof of theorem 3.2

Let u be a utility function satisfying the hyperbolic decreasing absolute risk aversion condition, namely,

$$-\frac{u''(x)}{u'(x)} = \frac{1}{ax+b}$$

For some constants $a, b \in \mathbb{R}$.

For all $0 < x < c$ and $n \in \mathbb{N}$, the absolute risk aversion of U for an incremental budget x is a decreasing function of n, namely,

$$-\frac{U''\big((n+1)c+x\big)}{U'\big((n+1)c+x\big)} \leq -\frac{U''(nc+x)}{U'(nc+x)}$$

Proof

First, we will give an explicit expression for $U(nc+x)$ -

$$U(nc+x) = \begin{cases} nu(c) + u_0(x) & for\ x < b_0 \\ nu\left(c + \frac{x - b_0 - g_n(x)}{n}\right) + u_0(b_0 + g_n(x)) & for\ \ x \geq b_0 \end{cases}$$

Where $g_n(x)$ is defined by the condition that the marginal utility of investing $b_0 + g_n(x)$ of the budget in u_0 and the rest of the budget uniformly in the symmetric categories is optimal. Namely,

$$u_0{}'(b_0 + g_n(x)) = u'\left(c + \frac{x - b_0 - g_n(x)}{n}\right)$$

which is well defined since all derivatives are decreasing.

Lemma 3.3

For all $n \in \mathbb{N}$ there exists $\beta_n > 0$ such that

$$g_n(x) = \beta_n(x - b_0)$$

For all $b_0 < x < c$.

Proof

Roughly speaking, the derivative $g_n{}'(x)$ depends on the ratio between u' and u'' in the points $c + \frac{x-b_0-g_n(x)}{n}$ and $b_0 + g_n(x)$ – How the first derivative changes in percentage. Therefore, one may get that assuming a condition about the quotient (which in our case, is the hyperbolic absolute risk aversion condition) gives a explicit formula for $g_n(x)$.

Formally, from the model assumptions we know $u_0 = \alpha u$ and therefore we obtain the relation

$$\frac{\left(u'\left(c + \frac{x - b_0 - g_n(x)}{n}\right)\right)}{u'\big(b_0 + g_n(x)\big)} = \alpha$$

Taking derivative with respect to x gives

$$\frac{\big(1 - g_n'(x)\big)}{n} \cdot u''\left(c + \frac{x - b_0 - g_n(x)}{n}\right) u'\big(b_0 + g_n(x)\big) - u'\left(c + \frac{x - b_0 - g_n(x)}{n}\right) g_n'(x) \cdot u''\big(b_0 + g_n(x)\big) = 0$$

$$\frac{\big(1 - g_n'(x)\big)}{n} \cdot u''\left(c + \frac{x - b_0 - g_n(x)}{n}\right) u'\big(b_0 + g_n(x)\big) = u'\left(c + \frac{x - b_0 - g_n(x)}{n}\right) g_n'(x) \cdot u''\big(b_0 + g_n(x)\big)$$

$$\frac{\big(1 - g_n'(x)\big)}{n} \frac{\left(u''\left(c + \frac{x - b_0 - g_n(x)}{n}\right)\right)}{u'\left(c + \frac{x - b_0 - g_n(x)}{n}\right)} = g_n'(x) \cdot \frac{u''\big(b_0 + g_n(x)\big)}{u'\big(b_0 + g_n(x)\big)}$$

Now we will substitute the hyperbolic absolute risk aversion condition and get

$$\frac{\big(1 - g_n'(x)\big)}{n} \cdot \frac{1}{a\left(c + \frac{x - b_0 - g_n(x)}{n}\right) + b} = g_n'(x) \cdot \frac{1}{a\big(b_0 + g_n(x)\big) + b}$$

$$\frac{1}{n}\frac{1}{a\left(c+\frac{x-b_0-g_n(x)}{n}\right)+b}$$

$$=\left(\frac{1}{a\big(b_0+g_n(x)\big)+b}+\frac{1}{n}\frac{1}{a\left(c+\frac{x-b_0-g_n(x)}{n}\right)+b}\right)g_n'(x)$$

This is a non-degenerate ordinary differential equation in the function $g_n(x)$, which by the existence and uniqueness theorem has a unique solution, with the initial condition $g_n(b_0) = 0$. We will guess the solution

$$g_n(x) = \beta_n(x - b_0)$$

and obtain the equation on β -

$$\frac{1-\beta_n}{n}\cdot\frac{1}{a\left(c+\frac{x-b_0-\beta_n x}{n}\right)+b}=\beta_n\cdot\frac{1}{a(b_0+\beta_n x)+b}$$

Rearranging gives a linear equation in x which can be solved by comparing coefficients.

$$\frac{1-\beta_n}{n}(a(b_0+\beta_n x)+b)=\beta_n\left(a\left(c+\frac{x-b_0-\beta_n x}{n}\right)+b\right)$$

$$a\frac{1-\beta_n}{n}\beta_n x+\frac{1-\beta_n}{n}(ab_0+b)=a\frac{1-\beta_n}{n}\beta_n x+\beta_n(ac-\frac{ab_0}{n}+ab)$$

$$\frac{1-\beta_n}{n}(ab_0+b)=\beta_n(ac-\frac{ab_0}{n}+b)$$

$$\frac{ab_0+b}{n}=\left(ac+\left(1+\frac{1}{n}\right)b\right)\beta_n$$

$$\beta_n=\frac{1}{n}\cdot\frac{ab_0+b}{ac+(1+1\backslash n)b}$$

And this finishes the proof of the lemma.

Back to the proof of proposition 3.2.

We will use the explicit expression for $U(nc+x)$ and $g_n(x)$ to obtain an explicit expression for the absolute risk aversion $-U''(nc+x)/U'(nc+x)$ and will show it is decreasing as a function of n.

$$U'(nc+x) = \begin{cases} u_0'(x) & for\ x < b_0 \\ \left(1-g_n'(x)\right)\cdot u'^{\left(c+\frac{x-b_0-g_n(x)}{n}\right)} + g_n'(x)\cdot u'_0\left(b_0+g_n(x)\right) = u'\left(c+\frac{x-b_0-g_n(x)}{n}\right) & x \geq b_0 \end{cases}$$

Therefore obviously $U'(nc+x)$ is increasing in n. The second derivative is -

$$U''(nc+x) = \begin{cases} u_0''(x) \\ \frac{1-g_n'(x)}{n}\cdot u''\left(c+\frac{x-b_0-g_n(x)}{n}\right) \end{cases}$$

Therefore, the absolute risk aversion is -

$$-\frac{U''(nc+x)}{U'(nc+x)} = \frac{1-g_n'(x)}{n}\cdot\frac{1}{a\left(c+\frac{x-b_0-g_n(x)}{n}\right)+b}$$
$$= \frac{1-g_n{}'(x)}{n(ac+b)+\left(x-b_0-g_n(x)\right)}$$

Both the enumerator and the denominator are increasing functions of n. To show that the whole expression is decreasing, we will compute separately the relative increment of each.

From Lemma 3.3 we know that

$$g_n(x) = \frac{1}{n}\cdot\frac{ab_0+b}{ac+(1+1\backslash n)b}(x-b_0)$$

And therefore

$$1-g_n'(x) = 1-\frac{1}{n}\cdot\frac{ab_0+b}{ac+(1+1\backslash n)b} = 1-\frac{ab_0+b}{nac+(n+1)b} = \frac{n(ac+b)+ab_0}{n(ac+b)+b}$$

And the relative increment is

$$\frac{1-g_{n+1}'(x)}{1-g_n'(x)} = \frac{(n+1)(ac+b)+ab_0}{(n+1)(ac+b)+b}\cdot\frac{n(ac+b)+b}{n(ac+b)+ab_0}$$
$$= \frac{(n+1)(ac+b)+ab_0}{n(ac+b)+ab_0}\cdot\frac{n(ac+b)+b}{(n+1)(ac+b)+b}$$
$$= \left(1+\frac{ac+b}{n(ac+b)+ab_0}\right)\left(1-\frac{ac+b}{(n+1)(ac+b)+b}\right)$$

Now, for the incremental change of the denominator –

$$n(ac+b)+\left(x-b_0-g_n(x)\right)$$
$$=n(ac+b)+\left(x-b_0-\frac{1}{n}\cdot\frac{ab_0+b}{ac+(1+1\backslash n)b}(x-b_0)\right)$$
$$=n(ac+b)+\left(\frac{n(ac+b)+ab_0}{n(ac+b)+b}\right)(x-b_0)$$

And the relative increment is

$$\frac{(n+1)(ac+b)+\left(x-b_0-g_{n+1}(x)\right)}{n(ac+b)+\left(x-b_0-g_n(x)\right)}$$
$$=\frac{(n+1)(ac+b)+\frac{(n+1)(ac+b)+ab_0}{(n+1)(ac+b)+b}(x-b_0)}{n(ac+b)+\left(\frac{n(ac+b)+ab_0}{n(ac+b)+b}\right)(x-b_0)}$$

The ratio between the second summand in the enumerator and denominator is exactly $\frac{1-g'_{n+1}(x)}{1-g'_n(x)}$ computed before, therefore to show that the whole term increments relatively faster than the previous term, it is enough to show that the first summand has relative increment which is higher then $\frac{1-g'_{n+1}(x)}{1-g'_n(x)}$. It's relative increment is just $\frac{n+1}{n}=1+\frac{1}{n}$.

Therefore all is left to show is that

$$\left(1+\frac{ac+b}{n(ac+b)+ab_0}\right)\left(1-\frac{ac+b}{(n+1)(ac+b)+b}\right)<1+\frac{1}{n}$$

The first multiplier is obviously less than $1+\frac{1}{n}$, and the second is less than 1, the result follows.

Let u be a utility function satisfying increasing or constant absolute risk aversion condition, namely,

$$\frac{d(-\frac{u''(x)}{u'(x)})}{d(x)}\geq 0$$

For all $0<x<c$ and $n\in\mathbb{N}$, the absolute risk aversion of U for an incremental budget x is a decreasing function of n, namely,

$$-\frac{U''\left((n+1)c+x\right)}{U'\left((n+1)c+x\right)}\leq-\frac{U''(nc+x)}{U'(nc+x)}$$

Proof

We know that:

$$-\frac{U''(nc+x)}{U'(nc+x)} = \frac{1-g_n'(x)}{n} \cdot -\frac{u''\left(c+\frac{x-b_0-g_n(x)}{n}\right)}{u'\left(c+\frac{x-b_0-g_n(x)}{n}\right)}$$

When n increase $-\frac{u''\left(c+\frac{x-b_0-g_n(x)}{n}\right)}{u'\left(c+\frac{x-b_0-g_n(x)}{n}\right)}$ is weakly decrease and we know that $\frac{1-g_n'(x)}{n}$ is decreasing in n, the result follows.