CAKE CUTTING - FAIR AND SQUARE

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Abstract The classic fair cake-cutting problem [Steinhaus, 1948] is extended by introducing geometric constraints on the allocated pieces. Specifically, agents may demand to get their share as a square or a rectangle with a bounded length/width ratio. This is a plausible constraint in realistic cake-cutting applications, notably in urban and agricultural economics where the "cake" is land. Geometric constraints greatly affect the classic results of the fair division theory. The existence of a proportional division, giving each agent 1/n of his total cake value, is no longer guaranteed. We prove that it is impossible to guarantee each agent more than 1/(2n-1) of his total value. Moreover, we provide procedures implementing partially proportional division, giving each agent 1/(An-B) of his total value, where A and B are constants depending on the shape of the cake and its pieces. Fairness and social welfare implications of these procedures are analyzed in various scenarios.

1. INTRODUCTION AND RELATED WORK

Consider a group of 1000 people arriving at a newly discovered island. They decide to settle in the island and one of their first tasks is to divide the land among them. However, the island is heterogeneous - parts of it are covered with fruit trees, other parts are covered with grass, while other parts are barren. Additionally, the settlers have different preferences regarding the different parts of the island. Some of them think that the beach is the most valuable part, feeling that it should be divided to 1000 small parcels enabling each person access to the sea, while others consider the inner rain forests to be the most valuable part. How can the island be divided among the settlers such that each of them gets a land-plot that he or she considers a fair share?

The answer to the question can be based on the application of the solution to the classic problem of **fair cake-cutting**. In this problem, an infinitely divisible set (the "cake") has to be allocated among n agents with different valuation functions over the subsets of this set. The goal is to divide the cake into n disjoint subsets allocating each agent a single subset such that the allocation

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is *fair*. The earliest fairness criterion studied in the context of cake-cutting is *proportionality* - each of the n agents should get a piece which they value as at least $1/n$ of the entire cake. Steinhaus [1948] proved that a proportional cake-cutting always exists, by describing an elegant division procedure developed by Banach and Knaster. This result was later improved and extended in many ways, for example, by reducing the number of queries each agent has to answer during the division process [e.g. Even and Paz, 1984, Webb, 1997], by giving different proportions to agents with different rights [e.g. Berliant et al., 1992, McAvaney et al., 1992, Robertson and Webb, 1997], by satisfying stronger fairness criteria such as envy-freeness [e.g. Brams and Taylor, 1996, Barbanel and Brams, 2004] or egalitarian-equivalence [e.g. LiCalzi and Nicolò, 2009], by optimizing various efficiency and social welfare criteria in addition to fairness [e.g. Moulin, 2004, Cohler et al., 2011, Hüsseinov and Sagara, 2013], and by developing strategy-proof procedures that induce the agents to reveal their true preferences [e.g. Nicolò and Yu, 2008, Chen et al., 2013]. This is an active research topic, with new results published every several months.

Alas, when the existing cake-cutting procedures are used to divide the island among the settlers, it turns out that the resulting land-plots are practically unusable, since most of the procedures disregard the **geometric shape** of the allocated pieces of land. Many procedures assume that the cake is a general set, and allow each piece to be any subset of the cake [e.g. Brams and Taylor, 1995, Robertson and Webb, 1998, Chambers, 2005]. This means that each "piece" may actually consist of a large number of tiny "crumbs" (Figure 1/left). This is not a problem when dividing an actual dessert cake, but it raises a serious difficulty when dividing land, as a large collection of tiny pieces of land has no practical use.

On the other extreme, many division procedures assume a very restricted cake - the cake is the one-dimensional interval [0,1], and each piece is a one-dimensional sub-interval [e.g. Su, 1999, Nicolò and Yu, 2008, Aumann and Dombb, 2010, Chen et al., 2013]. When such a procedure is applied to a two-dimensional island, each agent gets a long strip of land that runs from one side of the island to the opposite side (Figure 1/middle)[2] and these strips may be too narrow to use.[3]

Dall'Aglio and Maccheroni [2009] acknowledge the importance of having nicely-shaped pieces in resolving land disputes. They prove that, if the cake is a simplex in any number of dimensions, then there exists an envy-free and proportional division of the cake into polytopes. However, this proof is existential - no procedures are given.

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[2]As Douglas R. Woodall elucidates: "the cake is simply a compact interval which without loss of generality I shall take to be [0,1]. If you find this thought unappetizing, by all means think of a three-dimensional cake. Each point P of division of my cake will then define a plane of division of your cake: namely, the plane through P orthogonal to [0,1]." [Woodall, 1980]

[3]Several papers assume that the cake is a circle [Thomson, 2007, Brams et al., 2008, Barbanel et al., 2009], but still a one-dimensional circle. The pieces are arcs which, if projected on a two-dimensional circle, result in thin wedge-like land-plots.

Relatively few papers explicitly relate to a **two-dimensional cake**. Two of them discuss the problem of dividing a disputed territory between several bordering countries, with the constraint that each country should get a piece that is adjacent to its border: Hill [1983] proved that such a division exists, and Beck [1987] complemented this proof with a procedure for constructing such a division. Although the adjacency constraint is important and useful in some cases, it is not relevant to the general land division scenario. Iyer and Huhns [2009] describe a procedure that asks each of the n agents to draw n disjoint rectangles on the map of the two-dimensional cake. These rectangles are supposed to represent the "desired areas" of the agent. The procedure tries to give each agent one of his n desired areas. However, it does not succeed unless each rectangle proposed by an individual intersects at most one other rectangle drawn by any other agent. If even a single rectangle of Alice intersects two rectangles of George, then the procedure fails and no agent gets any piece.

In this paper, we look for a fair division of a two-dimensional cake, where each agent gets a single (contiguous) piece that satisfies some **explicit geometric constraints**. A formal general definition of the constraint appears in Section 2. It covers the specific requirement that *each piece must be a fat rectangle* - its length/width ratio is bounded by a pre-specified constant. If that constant is 1, then our constraint becomes: *each piece must be a square* (hence the phrase "fair-and-square" in the title of this paper).[4] This constraint is natural in many practical scenarios, for example, when dividing an urban land for building houses, or when dividing an agricultural land for cultivation. Consider, for example, an urban land-plot of 900 square meters. This land-plot may have a high value if it is shaped as a 30-by-30 square or a 45-by-20 rectangle, but it may be useless if it is a 1-by-900 strip or 900 separated 1-by-1 squares.

In addition to land division, the problem of two-dimensional cake-cutting may be relevant for fair division of advertisement areas in newspapers or websites. The importance of geometric constraints there is obvious. As a third possible application, consider several radio companies that compete over frequencies. To increase the effectiveness of their service, instead of dividing only the frequencies, the companies can also divide the times of the day. For example, a particular radio frequency band in the 1.0-1.2 GHZ range may be allotted from 7am to 9pm for broadcasting traffic information, while it may be used for downloading data onto wireless devices during off-peak hours [Iyer and Huhns, 2009]. This is a two-dimensional division problem of the frequencies and the broadcasting time. Here, too, a piece that is too narrow in either dimension may be unusable. A fourth possible application is related to *redistricting* - partitioning a state to a fixed number of districts with a fixed number of citizens in each district, for the purpose of elections to the house of representatives. The partitioning may have a substantial effect on the final outcome of the elections. As the parties usually have different preferences regarding the desirable partition, it is natural to look

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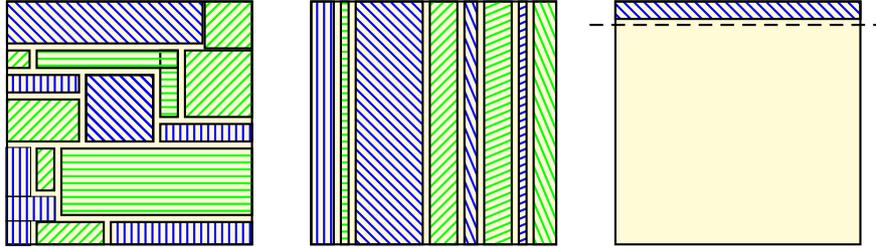[4]Alternative geometric constraints are discussed in subsection 5.4.

Figure 1: Problems with common cake-cutting procedures.
Left: *Many cake-cutting procedures allocate to each agent an arbitrarily large set of arbitrarily small disconnected pieces ("crumbs"). Such a set of land-plots is unusable.*
Middle: *Many cake-cutting procedures assume that the cake is the 1-dimensional interval [0,1], and thus allocate to each agent a long and narrow strip of land, which might be unusable.*
Right: *The divide-and-choose procedure might force the chooser to choose between a valuable strip that is too narrow to be useful and a wide strip that has no value.*

for fair division solutions.[5] The law sets geometric constraints on the districts, but the constraints are ill-defined.[6]

Obviously, the existing procedures for fair cake-cutting cannot guarantee that the pieces are squares or fat rectangles. Even the simple divide-and-choose procedure for two agents might produce long and narrow strips that are unusable. For example, suppose that the chooser values only the northern shore of the island, e.g., because he plans to build a hotel near the sea (See Figure 1/right). If the divider decides to draw the division line just to the south of the shore, then the chooser has two bad options: either choose the northern shore, which is near the sea but too narrow for meaningful building, or choose the southern part, which is sufficiently wide but has no value because it has no access to the sea. [7]

It is therefore natural to ask the following two questions: (a) Does there always exist a proportional division such that each piece is a fat rectangle? (b) If such a division exists, what procedure can implement it?

In Section 3, our first main result gives a *negative* answer to the former question: in some cases, there may be no proportional division with fat pieces! This is in sharp contrast to the unconstrained cake-cutting problem, where a proportional division always exists [Steinhaus, 1948, Dubins and Spanier, 1961]. This impossibility result also quantifies the loss of proportionality due to the fatness

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[5]Landau et al. [2009] developed a "fair redistricting" procedure in which two parties divide the state between them and each party redistricts its share of the state.

[6]See: http://www.redistrictingthenation.com/glossary.aspx#compactness

[7]Recently, several mathematicians have studied some geometric problems related to fair division of squares, rectangles and other polygons [e.g. Christ et al., 2011, Dumitrescu and Tóth, 2012, Karasev et al., 2013]. However, these problems do not take into account agents with subjective valuation functions.

Notation	Meaning
C	A land-cake - a polygon that is to be divided fairly.
n	Number of agents participating in a land-cake division.
P_i	A land-plot - a part of the land-cake that is given to agent $i \in \{1, ..., n\}$
V_i	The valuation function of agent i - a function over land-plots.
L	A specific length/width ratio of a rectangle. By convention, $L \geq 1$.
L-ratio rectangle	A rectangle with length/width ratio of exactly L.
R	An upper bound on length/width ratios of rectangles Since all realistic lands can be approximated by polygons, in this paper the focus is on polygonal cakes.($R \geq 1$).
R-fat rectangle	A rectangle with length/width ratio of at most R.
$Prop(C, n, R)$	The value that can be guaranteed to each of n agents, when dividing the land-cake C to R-fat rectangular pieces.

TABLE I

NOTATION USED THROUGHOUT THE PAPER

constraint, by calculating an upper bound on the value that can be guaranteed to each agent when the pieces must be fat rectangles. In Section 4, we present a *positive* result that provides some comfort by describing a procedure for cutting a cake such that each agent gets a fat rectangle with a guaranteed lower bound on its value. Several possible extensions to the model are discussed in Section 5, in particular, allowing each agent to hold a subjective geometric constraint, allowing each agent to get two disjoint land-plots and allowing each agent to require a rectangle with an exact length/width ratio. We also discuss a couple of alternative geometric constraints. An alternative problem formulation, that relates our work to the literature on non-additive valuations, is presented in Section 6. A table summarizing our results and some interesting open questions are presented in the concluding Section 7.

2. THE MODEL

We study a 2-dimensional cake that represents land that has to be divided among n agents. All realistic lands can be approximated by polygons; in this paper the focus is therefore on cakes that are polygons. To distinguish between the whole polygon and its pieces, we refer to the former as a *land-cake* and to the latter as *land-plots*.

DEFINITION 1   Given an integer $n \geq 1$ and a land-cake C, an **n-allocation** of C is an n-tuple of disjoint land-plots, $(P_1, ... P_n)$, such that, for every i, $P_i \subseteq C$.
[8]

Each agent is assumed to have a subjective valuation function over land-plots that satisfies the following standard assumptions:

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[8]Note that the definition does not require that the entire land-cake is divided, since in reality it often happens that some land is left undivided. In other words, we make a *free disposal* assumption.

- Each valuation function is a *measure*, i.e., non-negative (the value of each land-plot is at least 0) and additive (the value of a whole is equal to the sum of the values of its parts).[9]
- Each valuation function is *absolutely continuous with respect to area*, i.e., the value of each land-plot with an area of 0 is 0. This entails that it is *infinitely divisible* in all directions [see Hill and Morrison, 2009], i.e., if the value of a land-plot is V, it is possible to divide it horizontally, vertically and diagonally, such that one of the pieces has a value of pV, for every $p \in (0, 1)$.

The subjective valuation functions encapsulate the different preferences of the agents, i.e., some of the agents may prefer a land-plot near the sea while others may prefer a land-plot with a fertile soil, etc. Our version of the cake-cutting problem is defined below.

DEFINITION 2   A **cake-cutting instance** is a tuple:

$$(C, \ n, \ (V_1, ..., V_n), \ (UP_{1,...,}UP_n), \ FA)$$

where:
- C is the land-cake to divide;
- $n \geq 1$ is an integer - the number of agents;
- $V_1 ... V_n$ are infinitely divisible measures - the valuation functions;
- $UP_1...UP_n$ (Usable Plots) are sets of land-plots in C.
- FA (Fair Allocations) is a set of n-tuples of values.

A **solution** to a cake-cutting instance is an n-allocation $(P_1, ... P_n)$, such that:
- a. For every i, $P_i \in UP_i$.
- b. $(V_1(P_1), ... V_n(P_n)) \in FA$.

The UP_i (Usable Plots) sets represent the geometric constraints and the FA (Fair Allocations) set represents the fairness condition that the allocation should satisfy. Note that in the general case each agent may have different geometric constraints, although usually we will assume that the geometric constraints of all agents are identical or have a similar nature.

What kind of geometric constraints make sense in the context of land division? One important constraint is *connectedness*: each land-plot should be a single connected piece of land.[10] A second constraint is that each land-plot should be an *axis-parallel rectangle* - a rectangle with sides parallel to the x and y axes. This constraint is natural in agricultural and urban planning scenarios. The two constraints are important but insufficient because they allow the land-plots to be arbitrarily narrow and thus unusable.

Two additional constraints that come to mind are *length constraint* (each land-plot should be a rectangle at least M meters long in each edge, where M is a

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[9]Non-additive valuation functions are discussed in Section 6.

[10]The connectedness constraint is common in the cake-cutting literature even for one-dimensional cakes. In subsections 4.6 and 5.2 we suggest two ways to relax this constraint.

pre-specified constant) and *area constraint* (each land plot should be a rectangle with an area of at least S square meters, where S is a pre-specified constant). The problem with these constraints is that they are not scalable. For example, if the land-cake is 200-by-200 meters and there is either a length constraint of $M \geq 10$ or an area constraint of $S \geq 100$, it is impossible to divide the land to more than 400 agents.

Governments often cope with this problem by putting an upper bound on the number of people allowed to settle in a certain location. However, this limitation prevents people from taking advantage of new possibilities that become available as the number of people grows. For example, while in rural areas a land-plot of less than 10-by-10 meters may be considered useless because it cannot be efficiently cultivated, in densely populated cities even a land-plot as small as 2-by-2 meters can be used as a parking lot for rent or as a lemonade selling spot. Limiting the number of agents assures that each agent gets a land-plot that can be cultivated efficiently, but it may prevent more profitable ways of using the land-plots. [11]

In light of the above problems, this paper suggests an alternative geometric constraint based on the following definition:

DEFINITION 3   An **R-fat rectangle** is a rectangle with a length/width ratio between R and $\frac{1}{R}$, where $R \geq 1$. [12]

Note that, if $R_2 \geq R_1$, then every R_1-fat rectangle is also R_2-fat. A 1-fat rectangle is a square, and an ∞-fat rectangle is an arbitrary rectangle.

The fatness constraint is scalable because it does not depend on the absolute size of the land-cake. It is equally meaningful in both densely and sparsely populated areas. Therefore, henceforth we focus on cake-cutting instances where each agent i decides on a certain ratio R_i and the set UP_i (Usable Plots) contains all R_i-fat axis-parallel rectangles in C.

As a fairness condition, we focus on *proportionality*: for every i in $\{1..n\}$, $V_i(P_i) \geq \frac{V_i(C)}{n}$. A cake-cutting instance with no geometric constraints always has a proportional solution [Steinhaus, 1948], but a cake-cutting instance with the constraint of R-fat rectangles may have no proportional solution, as shown in the following section.

3. THE IMPOSSIBILITY OF PROPORTIONAL DIVISION WITH FAT PIECES

To illustrate the crucial effect of the fatness constraint on the attainable proportionality, consider the following simple example: the land-cake is the square

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[11] A possible solution to the scalability problem is to allow shared ownership. This option is further discussed in Subection 5.4.

[12] The term "fat rectangle" comes from the fields of computer geometry and computer graphics [Agarwal et al., 2000, Tóth, 2008]. Length/width ratios (also known as *aspect ratios*) are also important in other fields, such as VLSI circuit design [Wimer et al., 1989].

$[-1, 1] \times [-1, 1]$, it has to be divided between $n = 2$ agents, each agent demands a square land-plot (i.e. $R = 1$ for both agents) and the agents share the same preferences - the value of each piece is equal to its area (hence the value of the entire land-cake is 4). If the area of any square land-plot is larger than 1, then its side-length must be larger than 1 and, therefore, its interior must contain the origin $(0, 0)$.[13] This means that there can be only one such land-plot; it is impossible to cut two disjoint square land-plots such that the area of each exceeds 1. Hence it is impossible to give each of the two agents a square land-plot worth more than $\frac{1}{4}$ of the total value! This is in contrast to the unconstrained case, in which it is always possible to give each agent at least $\frac{1}{2}$ of the total value of the land-cake, for example, by using the cut-and-choose procedure.

How severe is the loss of fairness (proportionality) in the general case? A quantitative answer to this question makes use of the following definition:[14]

DEFINITION 4   Given a land-cake C, the number of agents n and a bound on the length/width ratio of land-plots $R \geq 1$, $Prop(C, n, R)$ is defined as the highest value p, such that every cake-cutting instance on C with n agents has a solution $(P_1 \dots P_n)$ such that, for every i, P_i is an R-fat rectangle and $V_i(P_i) \geq p \cdot V_i(C)$.

In the example presented above: $Prop(square, \ n = 2, \ R = 1) \leq \frac{1}{4}$. [15]

In general, the maximum possible proportionality is $\frac{1}{n}$. Proportionality of $\frac{1}{n}$ means that it is possible to find a proportional division of C to any group of n agents. This maximum value is attainable if the land-cake is rectangular and all agents are willing to get arbitrary rectangles ($R = \infty$), for example, using the last-diminisher procedure [Steinhaus, 1948] or the recursive-halving procedure [Even and Paz, 1984]. Therefore:

$Prop(rectangle, \ n, \ R = \infty) = \frac{1}{n}$.

However, when R is finite and $n \geq 2$, *the proportionality value is always less than $\frac{1}{n}$*. We prove this negative result first for a square land-cake and then for general land-cakes.

3.1. *Square land-cakes*

Our upper bound on proportionality is proved by describing a specific example in which it is impossible that all n agents get more than a specified value. The example involves a desert land with several pools of water. All agents have the same valuation function, which depends only on the amount of water accessible

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[13]This fact is obvious when the land-plots are axis-parallel. It is less obvious, but still true, when the land-plots are rotated.

[14]The following discussion makes the assumption that all agents have the same geometric constraint (the same R). Subjective geometric constraints are discussed in Subsection 5.1.

[15]Note that the proportionality value depends only on the shape of C. Therefore, for the sake of brevity, instead of writing "for all C such that C is a square: $Prop(C, 2, 1) \leq \frac{1}{4}$", we use an abbreviated notation and write the shape of C directly inside the parentheses.

from their allocated land-plot. In order to construct the example we prove the following lemma:

LEMMA *(Pools Lemma). Let C be a land-cake and let $Pools$ be a finite set of disjoint squares in C (the elements of $Pools$ will be called "pools"). Define a "wetland" as a land-plot in C that intersects at least two different pools. For every set $Pools$ and $R \geq 1$, define $Wet(C, Pools, R)$ as the maximum number of disjoint wetlands in C that are R-fat rectangles.*

If there exists a set of pools $Pools$, such that $Wet(C, Pools, R) < n$, then $Prop(C, n, R) \leq \frac{1}{|Pools|}$ (where $|Pools|$ is the number of elements in $Pools$).

PROOF: Assume that, for a certain land-cake C and number R, there exists a set $Pools$ of pools, such that $Wet(C, Pools, R) < n$. Assume that the land-cake C should be divided to n agents such that each agent gets an R-fat rectangle. Assume that the valuation functions of all agents are identical such that the value of each pool is 1 (spread uniformly over the pool) and the value of the rest of C (i.e. all parts of C that are not contained in the pools) is 0. The total value of C is thus $|Pools|$.

By assumption, $Wet(C, Pools, R) < n$, so there is no set of n disjoint wetlands that are R-fat rectangles. Hence, in every set of n disjoint R-fat land-plots, at least one of them is not a wetland, i.e., it intersects at most a single pool. The value of this non-wetland is at most 1. Hence, by definition of proportionality,

$Prop(C, n, R) \leq \frac{1}{|Pools|}$. Q.E.D.

Using the Pools Lemma, all we have to do to establish an upper bound on the proportionality of a certain land-cake C is to find a certain set of pools $Pools$, such that $Wet(C, Pools, R) < n$, i.e., there is no set of n disjoint R-fat rectangles each of which intersects two or more pools.

CLAIM 1 For any $n \geq 2$ and $1 \leq R < \infty$, $Prop(square, n, R) \leq \frac{1}{2n-1}$. Moreover, when $1 \leq R < 2$, $Prop(square, n, R) \leq \frac{1}{2n}$.

PROOF: [16] Let $D = R + 1$ and let C be the square land-cake $[1, D^{n-1} + \epsilon] \times [1, D^{n-1} + \epsilon]$, for some small $\epsilon > 0$. Define the following set of pools: The set consists of $2n - 1$ squares with a side-length of ϵ. The south-western corners of the pools are:

$(1, 1), (1, D), (1, D^2), ..., (1, D^{n-1})$, $(D, 1), (D^2, 1), ..., (D^{n-1}, 1)$

If $1 \leq R < 2$, add one more pool at (D^{n-1}, D^{n-1}). See Figure 2 for an example.

To calculate $Wet(C, Pools, R)$, we have to calculate the maximal set of disjoint R-fat rectangles that intersect more than one pool. We call this set $WetRec$ (Wetland Rectangles).

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[16] We thank Prof. Boris Bukh for the helpful discussion which led to the construction in this proof.
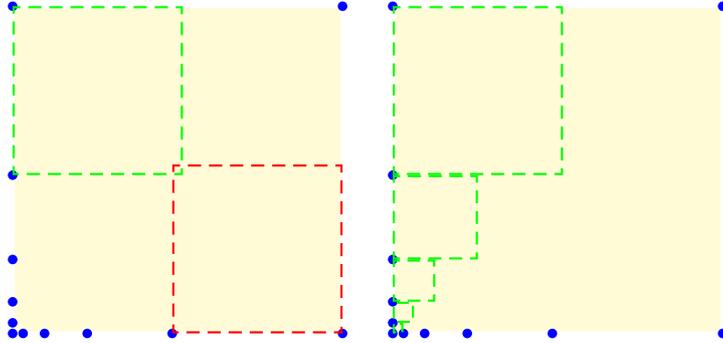
Figure 2: Upper bound on the proportionality of a square land-cake. In this figure, there are $n = 6$ agents that demand square pieces ($R = 1$). It is impossible to give each of them more than $\frac{1}{12}$ of their total value:

Left: The land-cake contains pools such that it is impossible to cut a square with two southern pools and the symmetric square with two western pools, because the interiors of the two squares intersect.

Right: Therefore, at most $n - 1$ square pieces can contain more than 1 pool. Hence, to guarantee that each of the n agents gets a piece with a value of at least 1, the total value of the cake should be at least the number of pools. In this case, it is $12 = 2n$.

First, consider a rectangle that intersects a pool from the western side ($x = 1$) and a pool from the southern side ($y = 1$). Every such rectangle must contain the point $(1 + \epsilon, 1 + \epsilon)$ in its interior. Therefore $WetRec$ can contain at most *one* such rectangle.

Next, consider a rectangle that intersects two pools from the western side, for example, the two adjacent pools $(1, D^i)$ and $(1, D * D^i)$, with $1 \leq i \leq n - 2$.[17] Its width must be at least $(D - 1)D^i - \epsilon$, therefore its length must be at least $\frac{(D-1)D^i - \epsilon}{R} = D^i - \frac{\epsilon}{R}$. Therefore (assuming that $\epsilon < 0.5$), the interior of this rectangle contains the point $(0.5 + D^i, 0.5 + D^i)$. The same holds for the symmetric rectangle on the southern side - the rectangle that intersects the pools at $(D^i, 1)$ and $(D * D^i, 1)$. Therefore, only *one* rectangle of each such pair can be a wetland, and $WetRec$ can contain at most $n - 2$ such rectangles.

Finally, consider the north-eastern pool (D^{n-1}, D^{n-1}), that appears on the island only when $1 \leq R < 2$. If this pool is used in a wetland rectangle, the side-length of that rectangle must be at least $\frac{(D^{n-1}-1)}{R} > \frac{(D^{n-1}-1)}{2}$, i.e., its interior must entirely cover either the eastern half or the northern half of the land-cake. Therefore its interior must contain the point $(0.5 + D^{n-2}, 0.5 + D^{n-2})$. Any set

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[17]There is no need to consider rectangles that intersect non-adjacent pools on the western side, since any such rectangle can be replaced by contained rectangles that intersect only adjacent pools.

of disjoint rectangles may contain *either* this north-eastern piece, *or* the north-western piece containing the pools $(1, D^{n-2})$ and $(1, D^{n-1})$, *or* the south-eastern piece containing the pools $(D^{n-2}, 1)$ and $(D^{n-1}, 1)$. Therefore, this north-eastern pool cannot increase the number of rectangles in $WetRec$.

In total, the number of R-fat rectangles in $WetRec$ is at most $n-1$. Therefore, $Wet(C, Pools, R) < n$, and by the Pools Lemma we get that:

$$Prop(C,\, n,\, R) \leq \frac{1}{|Pools|}$$

where $|Pools|$ is $2n - 1$ (when $R \geq 2$) or $2n$ (when $1 \leq R < 2$).   Q.E.D.

It is remarkable that the upper bounds in Claim 1 depend only weakly on R. As long as R is finite, no matter how large it is, we cannot guarantee that each agent gets more than $\frac{1}{2n-1}$ of his value of the land-cake.

3.2. *General land-cakes*

In general, when the land-cake is not a square, we can expect a reduction in the attainable proportionality. For example, assume that the land-cake is an L-by-1 rectangle. In some cases, even a single agent ($n = 1$) cannot get a square with a value of more than $\frac{1}{L}$ (e.g. when the valuation is uniform). Thus, we can expect that the proportionality of a general land-cake will be determined by two components: a component that depends on the number of agents n and a component that depends on the geometric shape of the land-cake. To define the geometric component, we need the following definition:

DEFINITION 5   Let C be a polygon and $R \geq 1$. An **R-independent set** in C is a set of disjoint squares contained in C no two of which are covered by the same R-fat rectangle contained in C.[18]

We are interested mainly in independent sets of maximum size:

DEFINITION 6   Let C be a polygon. The **R-independence number** of C, $R\text{-}IndepNum(C)$, is the maximum cardinality of an R-independent set in C.

Here are some examples: The $R\text{-}IndepNum$ of a square is 1 for all R. A 2-by-1 rectangle has a $1\text{-}IndepNum$ of 2 and a $2\text{-}IndepNum$ of 1 (and, obviously, an $R\text{-}IndepNum$ of 1 for every $R \geq 2$). The $R\text{-}IndepNum$ of an $L \times 1$ rectangle is $\left\lceil \frac{L}{R} \right\rceil$,[19] because we can place sufficiently small squares along its length such that

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[18]Albertson and O'Keefe [1981] call a 1-independent set an "anti-square". Chaiken et al. [1981] call a ∞-independent set an "anti-rectangle". We prefer the term "independent set", which in graph theory means: a set of vertices in a graph no two of which are connected by the same edge.

[19]$\lceil x \rceil$ is the *ceiling* of x - the smallest integer that is at least as large as x.

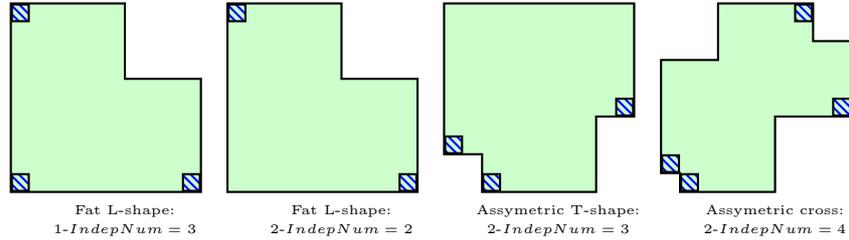| Fat L-shape: | Fat L-shape: | Assymetric T-shape: | Assymetric cross: |
| $1\text{-}IndepNum = 3$ | $2\text{-}IndepNum = 2$ | $2\text{-}IndepNum = 3$ | $2\text{-}IndepNum = 4$ |

Figure 3: Independence numbers: what is the largest size of an independent set of pools?
***Right**: An island with 3 pools, which are a 1-independent set (no two of the pools can be covered by a single square).*
***Middle, Left**: 2-independent sets: no two pools can be covered by a single 2-fat rectangle. In fact, they also cannot be covered by* any *rectangle. Therefore, the R-IndepNum of all polygons in this figure equals their 2-IndepNum for every $R \geq 2$.*

the distance between two adjacent squares is slightly larger than R. Some more examples are shown in Figure 3.

Some shapes have an infinite independence number. For example, if C is a circle, then we can build an arbitrarily large set of sufficiently small squares near the perimeter of C. In general, only rectilinear polygons (polygons with angles that are multiples of $90°$) have a finite independence number.

Using the independence number, we now generalize Claim 1 to land-cakes that are hole-free (simply-connected) polygons:

THEOREM 1   *Given a hole-free land-cake C and an integer $n \geq 1$:*
   (a) For every finite $R \geq 1$, $Prop(C, n, R)$ $\leq$ $\frac{1}{R\text{-}IndepNum(C) + 2n - 2}$
   (b) For $R = \infty$, $Prop(C, n, R = \infty)$ $\leq$ $\frac{1}{\infty\text{-}IndepNum(C) + n - 1}$

PROOF: Define a *corner square* as a square contained in C which has two adjacent sides contained in the boundary of C. Every hole-free polygon contains a maximum-size R-independent set that contains at least one corner square. [20] Choose an R-independent set in C with $R\text{-}IndepNum(C)$ squares that contains a corner square.

For part (a) of the theorem, replace that corner square with the set of $2n - 1$ smaller squares from the proof of Claim 1. Call the resulting set *Pools*. Note that $|Pools| = R\text{-}IndepNum(C) + 2n - 2$. By the definition of $R\text{-}IndepNum$, no R-fat rectangle can cover more than one of the large pools. By the proof of Claim 1, at most $n - 1$ R-fat rectangles can intersect more than one of the small pools. Therefore, $Wet(C, Pools, R) < n$. Therefore, by the Pools Lemma,

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[20] *Gill, Hum, and Pálvölgyi [2013]* proved this fact for $R = \infty$. Their proof is equally valid for every $R < \infty$.
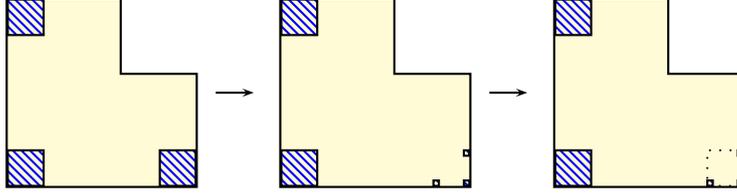
Figure 4: Upper bound on the proportionality of a general land-cake. In this figure, there are $n = 2$ agents that want square pieces ($R = 1$). It is impossible to give each of them more than $\frac{1}{5}$ of the total value.
Left: *A polygon with $1\text{-}IndepNum = 3$, containing 3 squares such that no two can be covered by a single square. In this case, all 3 squares are corner squares.*
Middle: *Take one corner square and replace it with 3 smaller squares. Define the resulting set of 5 squares as "pools".*
Right: *By the definition of $1\text{-}IndepNum$, there is no square that contains one of the bigger pools and another pool. Also, there is at most one square that contains more than one of the smaller pools. Therefore, $Wet(C, Pools, R) < 2$, and by the Pools Lemma, the proportionality of C is at most $\frac{1}{5}$.*

$Prop(C, n, R) \leq \frac{1}{R\text{-}IndepNum(C) + 2n - 2}$. Figure 4 illustrates the proof for $n = 2$ agents.

For part (b) of the theorem, replace the corner square with a set of n small squares along its diagonal and call the resulting set *Pools*. At most $n - 1$ rectangles can intersect more than one of these pools, and no rectangle can intersect more than one of the large pools. $|Pools| = R\text{-}IndepNum(C) + n - 1$ so by the Pools Lemma, $Prop(C, n, R = \infty) \leq \frac{1}{\infty\text{-}IndepNum(C) + n - 1}$.   *Q.E.D.*

This upper bound is valid even for $n = 1$:

$Prop(C, n = 1, R) \leq \frac{1}{R\text{-}IndepNum(C)}$.

Indeed, even Robinson Crusoe on a lonely island cannot always have the entire value of the island (at least if he wants this value contained in a square). As an extreme example, assume that Robinson Crusoe lives on a circular island of which he values only the beach, but still he wants a rectangular land-plot. The value contained in any rectangle contained in the circle might be infinitesimally small. This is in sharp contrast to the case of unconstrained cake-cutting, where a single agent can always have the entire cake.

When the whole land-cake is rectangular, the upper bound on the attainable proportionality has a more explicit formula:

COROLLARY   *For every $L \geq 1$, $n \geq 1$ and finite $R \geq 1$:*

$Prop(L\text{-ratio rectangle}, n, R) \leq \frac{1}{\lceil \frac{L}{R} \rceil + 2n - 2}$

Similarly to subsection 3.1, the bound is slightly tighter when $n \geq 2$, $L \leq R$ and $RL < 2$ (the proof details are omitted):

$Prop(L\text{-ratio rectangle}, n, R) \leq \frac{1}{2n}$

The results in this section were negative: we calculated an upper bound on the value that can be guaranteed to each agent when the land-plots must be R-fat rectangles. These upper bounds imply that the classic fairness requirement (proportionality) is incompatible with the fatness constraint.

However, this does not mean that we should entirely abandon one of these two requirements in favor of the other. As we will see in the following section, the fatness constraint is compatible with a *partial* proportionality requirement, by which each agent is guaranteed a predefined proportion (albeit smaller than $\frac{1}{n}$) of the total land-cake value.

4. PROCEDURES FOR PARTIAL PROPORTIONALITY

We start with the case of a single agent ($n = 1$), which is simple but not entirely trivial.

DEFINITION 7   Let C be a polygon. $R\text{-}CoverNum(C)$ is the minimum number of R-fat rectangles whose union is C.

Note that the rectangles in a cover need not be disjoint. Hence, for example, the $R\text{-}CoverNum$ of an *L-ratio rectangle* is $\lceil \frac{L}{R} \rceil$, because it can be covered by $\lceil \frac{L}{R} \rceil$ partially-overlapping R-fat rectangles.

CLAIM 2   *For every land-cake C and ratio R:*
$$Prop(C,\, n = 1,\, R) \quad \geq \quad \frac{1}{R\text{-}CoverNum(C)}$$

PROOF: C is the union of $R\text{-}CoverNum(C)$ R-fat rectangles. By the additivity of the valuation functions, at least one of the R-fat rectangles must have a value of at least $\frac{V(C)}{R\text{-}CoverNum(C)}$. A single agent can just take this rectangle and attain the specified proportionality value.[21] $Q.E.D.$

We proceed with a division procedure for n agents.

4.1. *Overview of the division procedure*

The division procedure is made of *rules* and *methods*. The rules are the protocol by which the division proceeds. The methods are the strategies that each agent is advised to follow during the division process in order to guarantee his/her fair share.[22] The distinction between rules and methods has several advantages:

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[21]The question how to find a cover with a minimal number of R-fat rectangles is beyond the scope of the current paper. The land-cakes we handle in this paper are simple enough so that the minimal cover is obvious.

[22]Steinhaus [1948] uses the terms "rules" and "methods". Even and Paz [1984] use the terms "protocol" and "winning strategies".
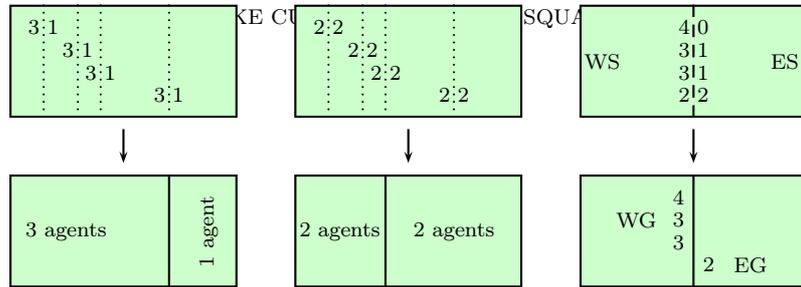
Figure 5: Basic partition steps used in division procedures. In all illustrations there are n = 4 agents. Dashed lines are marks made by the procedure. Dotted lines are drawn by agents. Numbers are Partners values declared by the agents.
Left: *Agent-bipartition (*k=3*).*
Middle: *Agent-bipartition (*k=2*).*
Right: *Cake-bipartition.*

1. It emphasizes that the guaranteed share to each agent does not depend on the actions of other agents or on the system being in equilibrium. Every agent who follows the recommended method will always get the promised value, regardless of whether the other agents follow the same method or not.[23]

2. It minimizes the role of the arbitrator that runs the division process (the "government"). The arbitrator only has to make sure that the rules are kept; it doesn't have to know the actual valuations of the agents. The valuation of each agent is private information used only for following the recommended method.

3. It makes it easy for different agents to use different methods, depending on their subjective geometric preferences. Subsection 5.1 explains this point in greater detail.

The division procedure is recursive, similar to the last diminisher procedure of Steinhaus [1948] and the recursive halving procedure of Even and Paz [1984]. We name the procedure "fair-and-square recursive-halving". In each step, the land-cake is divided into two parts and the agents are divided into two groups; one group is sent to one part and the second group to the second part. The two groups are let to divide their part using the same procedure, until each group contains a single agent. The two basic steps in the procedure are *agent-bipartition* and *cake-bipartition*, which we describe next.

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[23]In Steinhaus' words: "The greed, the ignorance, and the envy of other partners can not deprive him of the part due to him in his estimation; he has only to keep to the methods described above. Even a conspiracy of all other partners with the only aim to wrong him, even against their own interests, could not damage him" [Steinhaus, 1948].

4.1.1. *Agent-bipartition*

An agent-bipartition step splits the group of n agents to two groups of k and $n - k$ agents, where $k \in 1, .., n - 1$ is a pre-specified integer.

The *rules* of this step require each agent i to draw a land-plot P_i on the current land-cake. The plots are constrained such that, for every i, j, either $P_i \subseteq P_j$ or $P_j \subseteq P_i$. For example, assuming the land-cake is the rectangle $C = [0, L] \times [0, 1]$, each agent i may be asked to draw a rectangle adjacent to the western boundary: $P_i = [0, x_i] \times [0, 1]$.

Next, the agents are ordered in an increasing order of their plots, such that $P_1 \subseteq P_2 \subseteq ... \subseteq P_n$. The k agents that drew the smallest plots $(1..k)$ are given the piece P_{k+1} to divide recursively among them. The other $n - k$ agents are given the piece $C - P_{k+1}$ to divide recursively among them.

The agent-bipartition step is the basic step in both the last diminisher procedure [Steinhaus, 1948] and the recursive halving procedure [Even and Paz, 1984]. In the former, $k = n - 1$ (Figure 5/Left), and in the latter, $k = \frac{n}{2}$ (Figure 5/Middle).

The *methods* advise each agent i to draw a land-plot P_i such that $V_i(P_i)$ is large enough for sharing it with k agents and $V_i(C - P_i)$ is large enough for sharing it with $n - k$ agents. The definition of "large enough" depends of course on the subjective valuation of the agent. For example, in the recursive halving procedure (without geometric constraints), each agent is advised to draw a piece with a value of half of the total cake value, such that both P_i and $C - P_i$ are sufficiently valuable for dividing with $\frac{n}{2}$ agents. It is easy to prove by induction that any agent who follows these methods will get at least $\frac{1}{n}$ of the total value of the cake.

As explained in the introduction, the agent-bipartition step might be insufficient when there are geometric constraints, because an agent might be unable to draw a piece such that both P_i and $C - P_i$ are sufficiently valuable. For example, suppose an agent values only the northern shore of an island because he plans to build a square hotel adjacent to the sea (Figure 1/right). If the rules require to divide the land-cake using an east-west line, then this agent will be unable to do the division such that both the northern and the southern parts are sufficiently valuable. Therefore in the next subsection we introduce a complementary step.

4.1.2. *Cake-bipartition*

A cake-bipartition step splits the land-cake to two pieces using a cut in a pre-specified location. For example, assuming the land-cake is the rectangle $C = [0, L] \times [0, 1]$ when $L \geq 1$, it makes sense to cut in the middle of its longer side, and get two identical-sized pieces: $WS\,(WestSide) = [0, \frac{L}{2}] \times [0, 1]$ and $ES\,(EastSide) = [\frac{L}{2}, 1] \times [0, 1]$. This step assures that both of the resulting pieces have a "nice" shape, i.e., they are not too narrow.

The *rules* of this step require each agent i to declare two integers, e.g. $Partners_i(WS)$

and $Partners_i(ES)$, such that:

$$Partners_i(WS) + Partners_i(ES) \geq n$$

The *methods* advise agent i to declare $Partners_i(WS)$ such that $Val_i(WS)$ is large enough for sharing it with $Partners_i(WS)$ agents and similarly for *ES*.

Next, the agents are partitioned to two groups, *WG* (*WestGoers*) and *EG* (*EastGoers*), such that, for every agent i in *WG*, $Partners_i(WS) \geq |WG|$, and similarly, for every agent i in *EG*, $Partners_i(ES) \geq |EG|$. This can be done in the following way [Culter, 2013]: The agents are ordered in a decreasing order of $Partners_i(WS)$. For every agent i, if $Partners_i(WS) > |WG|$ then add agent i to *WG*. The ordering implies that every agent j already in *WG* had $Partners_j(WS)$ at least as large. Continue adding agents to *WS* until you reach an agent i for which $Partners_i(WS) \leq |WG|$. Now, because of the ordering, the latter inequality is satisfied for all the remaining agents. Hence, for all remaining agents $Partners_i(ES) \geq n - |WG|$. Luckily, there are only $n - |WG|$ remaining agents, so they can all be put in *EG*.

Next, if both *WG* and *EG* are non-empty, then the agents in *WG* (who all declared $Partners(WS) \geq |WG|$) are given the piece *WS* to divide recursively among them. Similarly, the agents in *EG* (who all declared $Partners(ES) \geq |EG|$) are given the piece *ES* to divide recursively among them (Figure 5/Right). Every agent i who follows the methods can be guaranteed that the value of the piece he has to divide recursively is sufficiently large for sharing it with the given number of other agents. The case when either *WG* or *EG* is empty requires special treatment, which will be described in the detailed account of the division rules.

The two partitioning steps just described - agent bipartition and cake bipartition - are parallel to the two steps of the standard cake cutting model of Robertson and Webb [1998] - *cut* and *eval*, except that instead of using the agents' valuation functions directly, they use it indirectly through the *Partners* function. This can also be seen as measure of *privacy*: our procedure does not require the agents to reveal their exact valuation function and their specific geometric constraints; it only requires them to reveal the approximate *Partners* value that is a combination of their valuation function and their geometric constraints.

4.2. *2-fat rectangular land-cakes*

Our division procedure is described in detail for land-cakes that are 2-fat rectangles.[24] The procedure is illustrated in Figure 6. The division *rules* are

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[24] An extension of the procedure for arbitrary rectangular land-cakes is briefly described in the next subsection. We start with land-cakes that are 2-fat rectangles (and not squares, as in the previous section) because a 2-fat rectangle can always be divided to two smaller 2-fat rectangles (by halving its longer side). This allows us to use a recursive division procedure. A square is, of course, a 2-fat rectangle.

1 *(input)*. The procedure begins with a group of $n \geq 2$ agents and a land-cake C with a certain length/width ratio $1 \leq L \leq 2$. Define a coordinate system in which the longer side of the land-cake is parallel to the x axis and the land-cake is $[0, L] \times [0, 1]$.

2 *(cake-bipartition)*. Define: $WS = WestSide = [0, \frac{L}{2}] \times [0, 1]$, $ES = EastSide = [\frac{L}{2}, L] \times [0, 1]$. Ask each agent i to declare $Partners_i(WS)$ and $Partners_i(ES)$ such that $Partners_i(WS) + Partners_i(ES) \geq n$. Divide the agents to two disjoint groups: $WG = WestGoers$ and $EG = EastGoers$, such that, for every agent $i \in WG$, $Partners_i(WS) \geq |WG|$, and similarly for EG. Proceed according to $|WG|$ and $|EG|$:
* If $1 \leq |WG| \leq n - 1$ (and of course $1 \leq |EG| \leq n - 1$), then make a vertical cut at $x = \frac{L}{2}$. Divide WS to the agent/s in WG and divide ES to the agent/s in EG. Use a recursive application of the procedure if the number of agents is more than 1.
* Otherwise, either $|WG| = n$ and $|EG| = 0$, or vice versa. These two cases are symmetric. W.l.o.g, assume $|WG| = n$ and proceed to the next step.

3 *(agent-bipartition)*. Ask each agent i to draw a rectangle adjacent to the western boundary, $P_i := [0, x_i] \times [0, 1]$, such that $x_i \in [0, \frac{L}{2}]$. Proceed according to the x_i-s:
* If there is at least one $x_i \geq \frac{1}{2}$, then select an arbitrary maximal x_m (such that $x_m \geq x_i$ for all i), divide the rectangle P_m (which is 2-fat because $x_m \geq \frac{1}{2}$) to the $n - 1$ agents with $i \neq m$, and give the remainder ($[x_m, L] \times [0, 1]$) to agent m.
* If for all agents, $x_i < \frac{1}{2}$, proceed to the next step.

4 *(cake-bipartition)*. Let $NS = NorthSide := [0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, $SS = SouthSide = [0, \frac{1}{2}] \times [0, \frac{1}{2}]$. Ask each agent i to provide $Partners_i(SS)$ and $Partners_i(NS)$ that sum to at least n. Divide the n agents to two disjoint groups $SG = SouthGoers$ and $NG = NorthGoers$, such that every agent i in NG has $Partners_i(NS) \geq |NG|$ and similarly for SG. Proceed according to $|SG|$ and $|NG|$:
* If $1 \leq |SG| \leq n - 1$ (and $1 \leq |NG| \leq n - 1$), make a horizontal cut at $y = \frac{1}{2}$. Divide SS to the agent/s in SG and NS to those in NG, using recursion if needed.
* Otherwise, either $|SG| = n$ and $|NG| = 0$ or vice versa. W.l.o.g, assume $|SG| = n$ and proceed to the next step.

5 *(agent-bipartition)*. Ask each agent i to draw a square on the south-western corner, $P_i := [0, x_i] \times [0, x_i]$, such that $x_i \in [0, 1]$. Select an arbitrary maximal x_m. Divide P_m to the $n - 1$ agent/s with $i \neq m$ and give the remaining L-shape ($C - P_m$) to agent m.

TABLE II

FAIR-AND-SQUARE RECURSIVE-HALVING DIVISION RULES.

detailed in Table II. These division rules allow us to prove the following claim:

CLAIM 3   For any $n \geq 2$ and $R \geq 1$: $Prop(2\,fat\,rectangle, n, R) \geq \frac{1}{6n-8}$.
            Moreover, when $R \geq 2$: $Prop(2\,fat\,rectangle, n, R) \geq \frac{1}{4n-5}$.

The proof of the claim is based on the *methods* detailed in Table III. The methods for each agent depend on his/her geometric constraint R and subjective valuation function V (the subscript i is omitted because, by definition, methods are for a single agent). Without loss of generality, assume that the function V is scaled such that the value of the whole land-cake is $An - B$, where A and B match the bounds in Claim 3, namely: $A = 4$, $B = 5$ if $R \geq 2$; $A = 6$, $B = 8$ if $R < 2$.

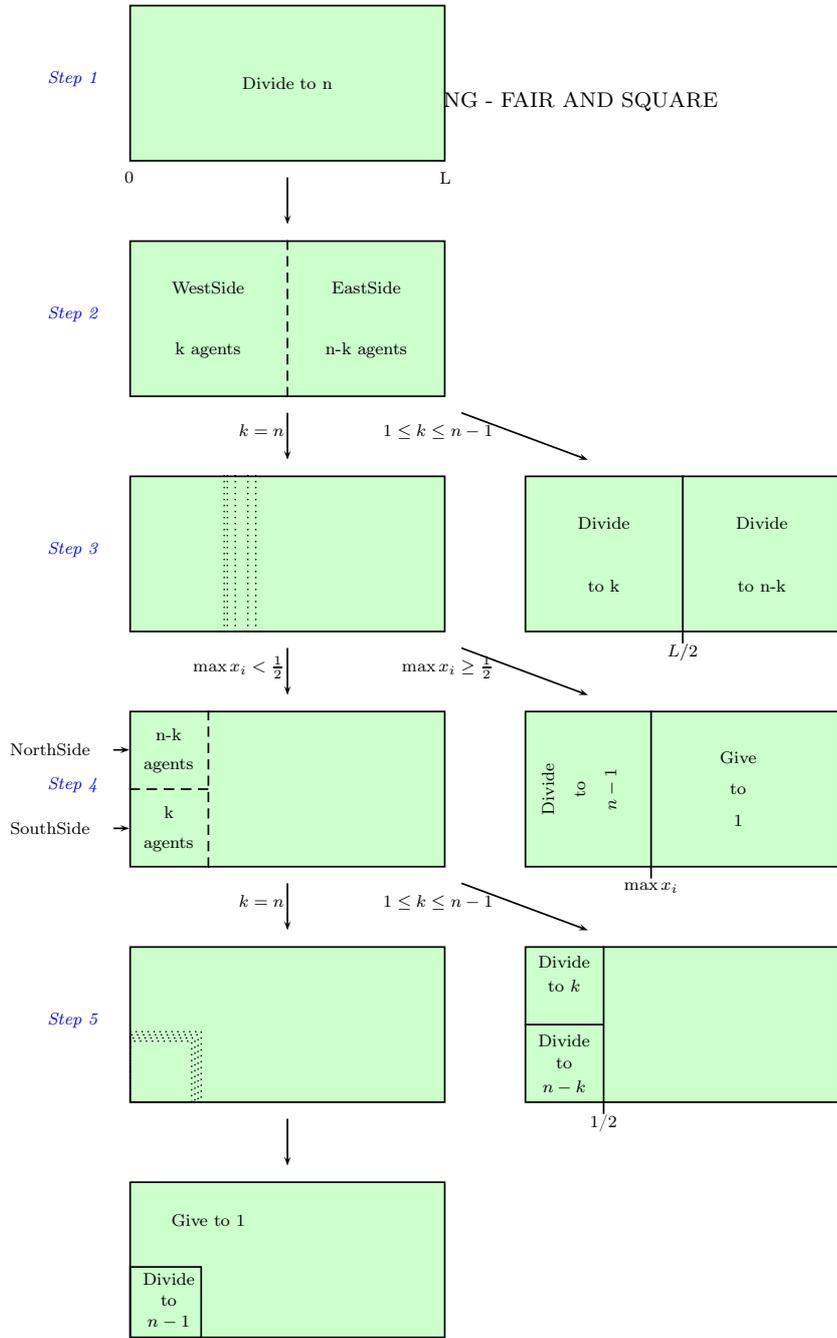*Figure 6: Fair-and-square recursive-halving rules, illustrated on a 2-by-1 rectangular land-cake. Two arrows originating from the same rectangle denote two alternative cases. Some cases were omitted because they are symmetric to the cases shown. Inside the rectangles, Dotted lines are drawn by the agents; dashed lines are marks made by the procedure; solid lines are cuts made by the procedure.*

Method for step 2 (cake-bipartition):

Let c be a part of the current land-cake (WS or ES) and $n \geq 2$ the total number of agents dividing the current land-cake.

- *If $V(c) < \lceil \frac{2}{R} \rceil$, then declare $Partners(c) = 0$.*
- *Else, if $Ak - B \leq V(c) < A(k+1) - B$, for some k in $\{1, .., n-2\}$, then declare $Partners(c) = k$ (Note that this case is possible only for $n \geq 3$).*
- *Else, if $A(n-1) - B \leq V(c) \leq An - B - \lceil \frac{2}{R} \rceil$, declare $Partners(c) = n-1$ (note that the condition is not empty because in both cases $A > \lceil \frac{2}{R} \rceil$).*
- *Else, $An - B - \lceil \frac{2}{R} \rceil < V(c)$; declare $Partners(c) = n$.*

Method for step 3 (agent-bipartition):

Draw a rectangle $P = [0, x] \times [0, 1]$ with $x \in [0, \frac{L}{2}]$, such that the value of the remaining rectangle is $V([x, L] \times [0, 1]) = \lceil \frac{2}{R} \rceil$.

Method for step 4 (cake-bipartition):

Let c be a part of the current land-cake (NS or SS) and let $n \geq 2$ be the total number of agents who divide the current land-cake.

- *If $V(c) < 1$, then declare $Partners(c) = 0$.*
- *Else, if $Ak - B \leq V(c) < A(k+1) - B$, for some k in $\{1, .., n-2\}$, then declare $Partners(c) = k$ (Note that this case is possible only for $n \geq 3$).*
- *Else, if $A(n-1) - B \leq V(c) \leq An - B - \lceil \frac{2}{R} \rceil - 1$, declare $Partners(c) = n-1$ (the condition is not empty because $A > \lceil \frac{2}{R} \rceil + 1$).*
- *Else, $An - B - \lceil \frac{2}{R} \rceil - 1 < V(c)$; declare $Partners(c) = n$.*

Method for step 5 (agent-bipartition):

Draw a corner-square P such that the value of the remaining L-shape is: $V(C - P) = 1 + \lceil \frac{2}{R} \rceil$.

TABLE III

FAIR-AND-SQUARE RECURSIVE-HALVING METHODS FOR A 2-FAT LAND-CAKE.

The following statements are proved in Appendix A:
1. An agent whose value of the entire land-cake is at least $An - B$ can follow the methods of Table III.
2. An agent who follows the methods of Table III is guaranteed to get an R-fat land-plot with a value of at least 1.

The combination of these two statements proves Claim 3.

A nice property of the proposed division rules is that they are anonymous - they treat all agents equally; this is in contrast to other division procedures, such as divide-and-choose, where each agent has a different role (See Nicolò and Yu [2008] for a discussion on the importance of anonymity).

4.3. *General rectangular land-cakes*

When the land-cake is a rectangle with a length/width ratio larger than 2, the proportionality is slightly lower, but its form is similar:

CLAIM 4   For every $L \geq 2$, $n \geq 2$ and $R \geq 1$:
$$Prop(L\text{-}ratio\ rectangle,\ n,\ R) \;\geq\; \frac{1}{6n-10+\lceil \frac{L}{R} \rceil}.$$
Moreover, when $R \geq 2$:
$$Prop(L\text{-}ratio\ rectangle,\ n,\ R) \;\geq\; \frac{1}{4n-6+\lceil \frac{L}{R} \rceil}.$$

Note that the only change is in the constant factor in the denominator, which changes from $-B$ to $-B' + \lceil \frac{L}{R} \rceil$, where $B' = B + \lceil \frac{2}{R} \rceil$. Also note that for $L = 2$ Claim 4 coincides with Claim 3.

The proof uses the same division rules as in Table II. Some of the methods of Table III should slightly change to take the larger length/width ratio into account. The new methods are detailed in Table V in Appendix B.

Claims 3 and 4 are summarized in the following theorem:

THEOREM 2   *For every $L \geq 1$, $n \geq 2$ and $R \geq 1$:*
$$Prop(L\text{-}ratio\ rectangle,\ n,\ R) \;\geq\; \frac{1}{\lceil \frac{max(L,2)}{R} \rceil + 6n - 10}.$$
Moreover, when $R \geq 2$:
$$Prop(L\text{-}ratio\ rectangle,\ n,\ R) \;\geq\; \frac{1}{\lceil \frac{L}{R} \rceil + 4n - 6}.$$

4.4. *Between the upper and lower bounds*

For the special case $n = 2$, when $R \geq 2$, the lower bound of Theorem 2 coincides with the upper bound of Theorem 1 and its corollary:
$$Prop(L\text{-}ratio\ rectangle,\ n = 2,\ R \geq 2) \;=\; \frac{1}{\lceil \frac{L}{R} \rceil + 2}.$$
For most other cases the bounds do not coincide - there is a gap between them. For $R \geq 2$:
$$\frac{1}{\lceil \frac{L}{R} \rceil + 2n - 2} \;\leq\; Prop(L\text{-}ratio\ rectangle,\ n,\ R) \;\leq\; \frac{1}{\lceil \frac{L}{R} \rceil + 4n - 6}$$

And for $R < 2$:

$$\frac{1}{\lceil \frac{L}{R} \rceil + 2n - 2} \quad \leq \quad Prop(\text{L-ratio rectangle}, n, R) \quad \leq \quad \frac{1}{\lceil \frac{max(L,2)}{R} \rceil + 6n - 10}$$

For example, for $R \geq 2$, $n = 9$ and $L \leq R$:

$$\frac{1}{31} \quad \leq \quad Prop(\text{L-ratio rectangle}, n = 9, R \geq 2) \quad \leq \quad \frac{1}{17}$$

This means that we know how to give each of the 9 agents an R-fat land plot containing at least $\frac{1}{31}$ of the total value of the land-cake, we are aware of the impossibility of guaranteeing more than $\frac{1}{17}$ of that value, but it is an open question whether it is possible to guarantee any proportion between these bounds. [25]

There is also a gap between the upper and lower bounds for $n = 1$. Combining Claim 2 with Theorem 1 for $n = 1$ yields:

$$\frac{1}{R\text{-}CoverNum(C)} \quad \leq \quad Prop(C, n = 1, R) \quad \leq \quad \frac{1}{R\text{-}IndepNum(C)}$$

The cover number (Definition 7) and the independence number (Definition 6) are closely related but are not always identical.[26]

4.5. *The special case of uniform valuations*

Theorem 2 specifies the proportion an agent can get in the worst case, but the actual proportion may be much higher. An interesting special case is when an agent cares only about the total area of the land-plot, i.e., his valuation function is uniform over the land-cake. Such an agent can often approach the attainment of the proportion $\frac{1}{n}$, regardless of the valuations of other agents:

THEOREM 3   *Let C be an L-by-1 rectangle to be divided among $n \geq 2$ agents using the fair-and-square recursive-halving rules (Table II). Define $N = 2^{\lceil log_2 n \rceil}$ = the smallest power of two which is at least as large as n ($n \leq N < 2n$).*

Then for every finite $R \geq 1$, each agent can get an R-fat land-plot with an area of at least $\min(R, \frac{L}{2N})$.

Moreover, when $R \geq 2$, each agent can get an R-fat land-plot with an area of at least $\min(R, \frac{L}{N})$.

PROOF: Consider an agent whose valuation function, V, is equal to the area function. It is easy to verify that in this case, the methods of Table III reduce to the following single recommendation for step 2: "Declare $Partners(WS) = Partners(ES) = \lceil \frac{n}{2} \rceil$".

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[25] In a study that is yet unpublished, we developed a division procedure that achieves the upper bound of the corollary to Theorem 1 for every n and every $R \geq 3$, provided that all agents share the same valuation function, i.e., the land value is determined by an objective assessor. This procedure guarantees to each agent a 3-fat rectangle with a value of at least $\frac{1}{\lceil \frac{L}{R} \rceil + 2n - 2}$ of the total value of the land-cake. It is an open question if and how this procedure can be generalized to the case of subjective valuations.

[26] Albertson and O'Keefe [1981] prove that if C is a hole-free (simply-connected) polygon, then $1\text{-}CoverNum(C) = 1\text{-}IndepNum(C)$. Chaiken et al. [1981] prove that if C is a linearly-convex polygon then $\infty\text{-}CoverNum(C) = \infty\text{-}IndepNum(C)$. It is an open question under what conditions $R\text{-}CoverNum(C) = R\text{-}IndepNum(C)$ when $1 < R < \infty$.

Since $\lceil \frac{n}{2} \rceil$ is always larger than 0, EG and WG in step 2 always have at least one member. Therefore the procedure always recurses at step 2(a) and never gets to steps 3-5. After k recursive steps, our agent is in a group of at most $\lceil \frac{n}{2^k} \rceil$ agents dividing a land-cake with an area of $\frac{L}{2^k}$. Therefore after at most $k = \lceil log_2 n \rceil$ steps our agent receives a piece with an area of at least $\frac{L}{N}$.

Now if the number of steps is small relative to L (such that $\frac{L}{N} > 2$), then this piece is $\frac{L}{N}$-fat and the largest R-fat land-plot contained in it is R-by-1. But when the number of steps is large enough (such that $\frac{L}{N} \leq 2$), the piece given to our agent is 2-fat, so he can get a 2-fat land-plot with an area of $\frac{L}{N}$ or a square with an area of $\frac{L}{2N}$.   Q.E.D.

Thus, when the number of agents is a sufficiently large power of 2, each agent can get a 2-fat land-plot with the largest possible area of $\frac{1}{n}$ of the total available area.

4.6. *Utilitarian social welfare*

Theorem 2 shows that in the worst case each agent gets approximately $\frac{1}{4n}$ or $\frac{1}{6n}$ of the total value of the cake. Thus in the worst case only $\frac{1}{4}$ or $\frac{1}{6}$ of the potential utilitarian welfare (the sum of allocated values) is realized, which is highly inefficient.

It is possible to substantially improve the efficiency of the division procedure (in terms of the utilitarian welfare), but this requires several compromises.

First, the freedom of the agents must be compromised and it must be assured that all of them follow the methods of Table III. Otherwise, some agents might make mistakes which will reduce their value to zero and, in turn, the sum of values may be considerably diminished.

Second, the subjectivity of agents must be compromised and it must be assumed that all of them assign the same value to the entire land-cake, $V = V(C)$. Otherwise, the sum of the values allocated to different agents is meaningless.[27] We also assume that all agents have the same geometric constraint R.

Third, the privacy of agents must be compromised. In the cake-bipartition steps (Subsection 4.1.2), instead of telling only their $Partners$ values (e.g. $Partners_i(WS)$ and $Partners_i(ES)$ in step 2), the agents must tell their exact valuation (e.g. $V(WS)$ and $V(ES)$). The agents should be ordered according to their valuations of WS, such that those whose value of WS is higher are assigned to WG first. This guarantees that every agent in WG values WS at least as much as every agent in EG and every agent in EG values ES at least as much as every agent in WG. This, in turn, guarantees that no value is lost in the cake-bipartition steps.

Under these three assumptions, the division procedure results in forgone value only in step 3 (the agent-bipartition step) in case all agents draw their vertical

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[27]This assumption is standard in the cake-cutting literature, but it was not required until this point.

lines to the west of $x = \frac{1}{2}$. This means that all agents can get from the eastern part, $[\frac{1}{2}, L] \times [0, 1]$, less than the guaranteed value. This in turn means that all agents value the eastern part less than a certain constant that depends on the geometric constraints. This loss happens at most $n - 1$ times, which allows us to bound the loss of value. These statements are formalized in the following claim:

CLAIM 5 *Let C be a 2-fat land-cake to be divided among $n \geq 2$ agents using the fair-and-square recursive-halving rules (Table II). Assume that all agents use the methods of Table III with the same geometric constraint R and assign the same value to the entire land-cake.*

Then for $1 \leq R < 2$ the sum of all values enjoyed by the agents is at least: $\frac{2n-2}{6n-8}$ of the total value.

When $R \geq 2$ the sum is at least: $\frac{3n-4}{4n-5}$ of the total value.

The proof is in Appendix C.

Thus, when n is large, the lower bound on the sum of values is approximately $\frac{1}{3}$ of the total value for $R < 2$ and approximately $\frac{3}{4}$ of the total value for $R \geq 2$. In other words, the average value per agent is approximately $\frac{1}{3n}$ of the total value for $R < 2$ and approximately $\frac{3}{4n}$ for $R \geq 2$.

The large gap between the average value of Claim 5 and the minimal value of Claim 3 comes from the cake-bipartition steps (steps 2 and 4). Although on average no value is lost in these steps, some of the agents may lose value in favor of other agents.

The following example illustrates the difference between the two types of losses. Consider a square land-cake divided among $n = 4$ agents with $R = 2$. All agents value the land-cake as $4n - 5 = 11$. In step 2, they value ES as 0.8 and WS as 10.2 so they declare $Partners(ES) = 0$ and $Partners(WS) = 4$. In step 3 all division lines are to the west of $\frac{1}{2}$ so the value of 0.8 in the eastern side is lost. This is a loss of *average value*, caused by the *agent-bipartition* step. Now in step 4 all agents value NS as 6.8 and SS as 3.4 so they declare $Partners(WS) = Partners(ES) = 2$. The procedure assigns two agents arbitrarily to NS and the other two to SS. Suppose that all agents value NS and SS uniformly, so each agent gets half its value: two agents get 3.4 and two agents get 1.7. The average value is 2.55, which is more than the guaranteed average of $\frac{3n-4}{n} = 2$, but the two agents assigned to the south get a much lower value. This is a loss of *minimal value*, caused by the *cake-bipartition* step. The deep cause for this loss is that the cake-bipartition step must cut the land-cake in a specified location in order to keep the two pieces fat. This location might force the agents to split their value unproportionally and lose some of the value.

The proof to Claim 5 also implies that the unallocated land consists of at most $n - 1$ disconnected pieces. Thus the value of these pieces need not be considered as wasted, because the unallocated pieces can be assigned to some of the agents as a second land-plot. This is a common practice in many settlements, where some or all of the settlers receive two land-plots - a high-valued land-plot for

building their home and a lower-valued land-plot for agricultural or industrial uses. The second division can be carried out by any procedure for indivisible item assignment, including an auction or a lot. The effects of such a second division on fairness, no matter how it is carried out, are relatively small, because the value of the remaining pieces is relatively small. [28] Another possibility is to use the remaining pieces for building public facilities such as gardens, roads etc., for which length/width ratio is of less importance.

4.7. *A plausible standard constraint*

In the previous subsection we analyzed three aspects of the division procedure: value per agent, area per agent and sum of the allocated values. In all three aspects, the lower bounds when $R \geq 2$ are significantly higher than those obtained when $R < 2$: the value per agent is at least$\approx \frac{1}{4n}$ (instead of $\frac{1}{6n}$), the area per agent can be at as high as $\approx \frac{1}{n}$ (instead of $\frac{1}{2n}$) and the average value per agent is at least $\approx \frac{3}{4n}$ (instead of $\frac{1}{3n}$).

On the other hand, increasing the value of R above 2 does not significantly improve the lower bounds when the number of agents is large. A possible practical policy recommendation implied by this analysis is to apply $R = 2$ (length/width ratio of at most 2) as a standard geometric constraint when dividing land-plots to a large number of people.

5. EXTENSIONS

In this section we briefly discuss several possible extensions to the basic problem of fair division with fat rectangles. The main purpose of the discussion is not to offer complete solutions to these extended problems but rather to illustrate the challenging range of possibilities for future research on cake-cutting with geometric constraints.

5.1. *Subjective geometric constraints*

The division procedure described in section 4 allows different agents to have different geometric constraints. For example, suppose the land-cake is a 1-by-30 rectangle to be divided among n agents, one of whom (say, Alice) wants a 1-fat rectangle and another one (George) wants a 3-fat rectangle. Each of them can follow the methods of Table III with her/his own value of R and get the guaranteed value: Alice will get a square with at least $\frac{1}{30+6n-10} = \frac{1}{6n+20}$ of her total value of the cake and George will get a 3-fat rectangle with at least $\frac{1}{10+4n-6} = \frac{1}{4n+4}$ of his total value of the cake.

However, this subjectivity is limited. If George wants an ∞-fat rectangle (i.e. he insists on a rectangle but does not care about the length/width ratio), our

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[28]In subsection 5.2 we consider the option of giving each person two land-plots already in the initial division.
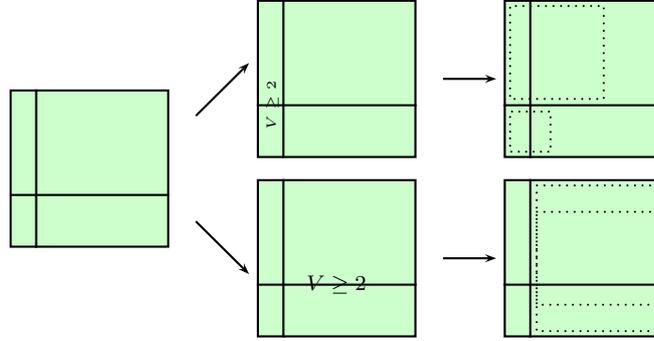
Figure 7: Fair cake-cutting with subjective geometric preferences. Alice and George value the square land-cake as 4. George wants a rectangle with a value of at least 2 and Alice wants a square with a value of at least 1.
Left: *George marks two perpendicular lines. He is advised to mark each line such that it divides the land-cake to two rectangles with equal value (2).*
Now it's Alice's turn (V is V_{Alice}):
Top: *If the value of the thinnest rectangle (on the west) is at least 2, then at least one of its parts (either the north or the south) has a value of 1. Alice either takes the northern dotted square and leaves the southern rectangle to George or vice versa.*
Bottom: *Otherwise, the value of the fattest rectangle (on the east) is at least 2. This is a 2-fat rectangle so Alice can cut from it a square with a value of at least 1. George receives the entire western rectangle.*

procedure can guarantee to him at most $\frac{1}{4n-5}$ (using Theorem 2 with $\left\lceil \frac{L}{\infty} \right\rceil = \lim_{R \to \infty} \left\lceil \frac{L}{R} \right\rceil = 1$), while under the classic recursive-halving procedure [Even and Paz, 1984] he could get at least $\frac{1}{n}$. This might lead to disagreements about the division procedure: agents who want arbitrary rectangles will demand the use of the classic recursive halving procedure, while agents who want fat rectangles will prefer the use of the fair-and-square recursive-halving procedure of Section 4.

Is it possible to design a procedure that satisfies both the arbitrary-rectangle agents and the fat-rectangle agents? For the case of $n = 2$ agents, the answer is yes.

For concreteness, suppose we have a square land-cake that both Alice and George value as 4. Alice wants a square piece ($R_A = 1$) and is willing to settle for a piece worth 1, because she knows that this is the highest value that can be guaranteed to her. On the other hand, George wants any rectangle ($R_G = \infty$) demanding to get a piece worth 2, because this is the proportion he can get under the classic cake-cutting procedures. The following division rules allow the demands of both Alice and George to be satisfied:

1. George makes a horizontal mark and a vertical mark.
2. Alice selects one of the two marks, cuts the land-cake by that mark and

selects one of the resulting rectangles.

George's method for getting a value of 2 is obvious - he just makes each mark such that the rectangles in each side of the mark have a value of 2. This is possible because the valuation functions are infinitely divisible in all directions (Section 2). Alice's method for getting a value of 1 in a square land-cake is described in Figure 7. It is easy to generalize Alice's method to L-ratio rectangles and to any value of R_A.

When there are more than 2 agents, any agent who wants an arbitrary rectangle can make a slight modification to the methods of Table III for steps 2 and 4 and declare $Partners(c) \geq 2$ when $V(c) \geq 2$ (instead of when $V(c) \geq 3$, as implied by the original methods). This allows the agent to modify one of the conditions on the values of the constants A and B (see Appendix A) such that $2A - B \geq 2$ (instead of $2A - B \geq 3$). This, in turn, translates to a slightly higher proportionality of $\frac{1}{3n-4}$ (instead of $\frac{1}{4n-5}$). This is still much lower than the $\frac{1}{n}$ guaranteed by the classic cake-cutting procedures. Closing this gap is an interesting topic for future research.

5.2. *More than one land-plot per agent*

In the cake-cutting literature there are two types of divisions:
1. Divisions in which each agent gets a single connected piece;
2. Divisions in which each agent receives an arbitrary number of disjoint pieces.

So far we assumed, both in the negative result of section 3 and in the positive result of section 4, that divisions are of type #1.

When divisions of type #2 are considered, the geometric constraints studied in this paper are not interesting, because every geometric shape can be approximated by a sufficiently large number of squares. Therefore, every division of a cake (particularly a proportional division) can be approximated to an arbitrary precision as a type #2 division where each agent gets a sufficiently large number of disjoint squares.

The common practice in land division is a compromise between these two extremes. Each agent can receive more than one land-plot, but not an unlimited number of small land-plots - typically 2 or 3 land-plots per agent. As far as we know, this kind of compromise has not been studied in the cake-cutting literature. In the context of geometric constraints, this compromise allows us to improve the proportionality guarantee, as shown in the following paragraphs.

For simplicity, we focus on the case of a 2-fat land-cake and R-fat land-plots with $R \geq 2$.[29] First consider the case of two agents, Alice and George, both of

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[29]The land-plots of a single agent may overlap, so in particular, an agent with a certain geometric constraint R may be satisfied with a single $2R$-fat rectangle. Therefore, for $R < 2$ the proportionality of the division can be improved simply by using the original division methods with $2R$ instead of R. But for $R \geq 2$, proportionality can be improved even more using the modified methods we describe.
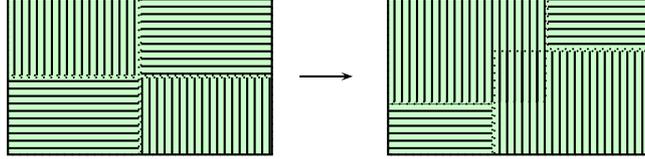
Figure 8: Proportional cake-cutting with two land-plots per agent.
Left: *George divides the 2-fat land-cake to four quarters that are grouped to two pairs of 2-fat rectangles: the "horizontal lines" pair and the "vertical lines" pair.*
Right: *George shrinks the pair with the larger value (in this case, the "horizontal lines" pair) until its value is exactly half of the total value, in George's eyes. The remaining shape is composed of two overlapping 2-fat rectangles and its value is also exactly half of the total. Now Alice selects one pair and George selects the other pair.*

whom want a pair of 2-fat rectangles. The two rectangles of a single agent may overlap, but in that case, the value of the overlapping part is counted only once when calculating the value enjoyed by an agent, since we still hold the additivity assumption made in Section 2.

The following division rules (illustrated in Figure 8) can be used to attain a proportional division in this case:

1. George divides the land-cake to two pairs of disjoint 2-fat rectangles: one pair contains the north-east and the south-west corners and the other pair contains the north-west and the south-east corners.
2. George may shrink one of the pairs (i.e. the NE-SW pair) towards the corners and enlarge the other pair accordingly.
3. Alice picks either the NE-SW or the NW-SE pair. George gets the other pair.

Alice's method for getting at least $\frac{1}{2}$ of the total value is obvious - she just picks the pair that is more valuable for her. George's method is to shrink the pair which is more valuable for him until the values of both pairs are identical and equal to $\frac{1}{2}$ of the total value of the land-cake. Thus, for $n = 2$ agents, a proportional division with pairs of 2-fat rectangles is possible.

If only one agent (say, Alice) wants two land-plots while the other agent (George) insists on getting a single land-plot, then Alice should get at least $\frac{1}{2}$ of her total value and George should get at least $\frac{1}{3}$ of his total value. This can be attained by a different set of division rules. We leave it as an exercise to the reader to find these rules.

When there are more than 2 agents, any agent who wants a pair of 2-fat rectangles can use modified methods similar to those described at the end of subsection 5.1, namely, declare $Partners(c) \geq 2$ when $V(c) \geq 2$ (instead of when $V(c) \geq 3$). This allows each agent to get a pair of 2-fat rectangles with a combined value of at least $\frac{1}{3n-4}$ (instead of $\frac{1}{4n-5}$) of the total land-cake value.
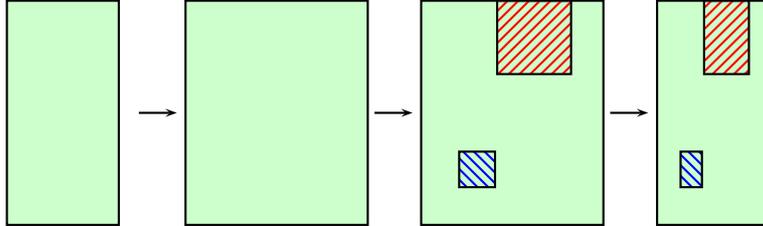
Figure 9: How to give each agent a fair golden-ratio rectangle.
Left: *original land-cake.*
Middle-Left: *land-cake in a scaled coordinate system.*
Middle-Right: *square land-plots in a scaled coordinate system.*
Right: *golden-ratio land-plots in the original coordinate system.*

Apparently, this slight modification does not take full advantage of the possibility of splitting land-plots to two disjoint rectangles - it splits land-plots only in a final step when there are $n=2$ agents. Is it possible to achieve a fully proportional division when the pieces are allowed to be pairs of R-fat rectangles? This is currently an open question.

5.3. *Exact length/width ratios*

The division procedure described in Section 4 handles only upper bounds on the length/width ratios. But in some cases it may be desirable to get rectangles with an exact length/width ratio. For example, when dividing advertisement areas in a newspaper or website, for aesthetic reasons it may be desirable to give each agent a piece with a length/width ratio equal to the golden ratio $\Phi = 1.618...$ If all agents agree on a single optimal length/width ratio, then the problem can be solved easily by scaling one axis of the coordinate system in that ratio. For example, if the original land-cake is a 2-by-1 rectangle and all agents want golden-ratio rectangles, then we can define a coordinate system in which the land-cake is a 2-by-Φ rectangle and then divide the land-cake using the procedure of Section 4. An agent who uses the methods of Table III with $R = 1$ will get a square with a value of at least $\frac{1}{6n-10+\lceil \frac{2}{\Phi} \rceil} = \frac{1}{6n-8}$ of the total value. In the original coordinate system, each such square is a golden-ratio rectangle (see Figure 9).[30]

An agent who uses the Methods with $R > 1$ is guaranteed to get an $x \times y$ rectangle, such that $\frac{\Phi}{R} \leq \frac{x}{y} \leq \Phi R$. Thus, it is also possible to require a range of

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[30]The resulting rectangles are all aligned such that their long side is parallel to the long side of the original land-cake. If we want the opposite, we should scale the other axis of the coordinate system, such that the land-cake is a $2\Phi \times 1$ rectangle.

length/width ratios, as long as the geometric mean of the range is equal for all agents.

What if different agents want length/width ratio ranges with different geometric means? For example, is it possible that some agents get golden-ratio rectangles while other agents get squares with the same partial-proportionality guarantee? This question is still open.

5.4. *Absolute size constraints, shared ownership and uniform preference externalities*

Our discussion so far focused on a specific geometric constraint, namely the fatness constraint. As explained in Section 2, this constraint is scalable and can be used with any size of land-cake and any number of agents. This is in contrast to absolute size constraints, such as minimum length or minimum area, which might be unsatisfiable when the number of agents is large and the land-cake is small.

If one insists on using absolute size constraints, a possible solution to the scalability problem is to consider *shared ownership*, i.e., allow a single land-plot to belong to several agents. This is actually a common practice in crowded cities, when several people live in an apartment building on a land-plot which is their shared property.

The division rules of subsection 4.2 can be adapted to support absolute size constraints (in addition to the fatness constraint) in the following way: whenever the rules specify that the current land-cake has to be cut to two pieces, and one of the resulting pieces is smaller than the minimal allowed size - the cut is not performed; instead, the current land-cake is given as a shared property to the current group of agents. This procedure is proposed only as a proof of concept - it is not necessarily optimal.

To analyze such procedures, we need to define utility functions for shared land-plots. Suppose an agent i belongs to a group of k agents who own a land-plot P. The utility of agent i can be written as:

$$U_i(P, k) = \frac{V_i(P)}{g_i(k)}$$

where $g_i(k)$ is the "congestion function" of agent i, describing the effect on his utility of sharing a land-plot with $k-1$ agents. [31] The utility of owning a private land-plot is just the value of that land-plot, $V_i(P)$, so $g_i(1) = 1$. If the agent is congestion-averse then $g_i(k) > k$, i.e., the agent prefers having a small private

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[31] This expression assumes that the utilities are multiplicatively seperable. A similar assumption has been studied by Milchtaich [2009] in the similar context of congestion games. Dall'Aglio and Maccheroni [2009] also discuss fractional ownerships, but their model is very general and allows arbitrary preference relations between sets of fractional shares.

land-plot with a value of $\frac{V_i(P)}{k}$ to sharing the entire large land-plot with $k-1$ agents.

Obviously, a proportional or even partially-proportional division cannot be guaranteed. For example, consider a land-cake of 10-by-10 meters with a value of 1, that has to be divided among n agents with an absolute length constraint of 10 meters and a congestion function of $g_i(k) = k^2$. Obviously no division is possible, so the entire land-cake is given to the n agents as a shared property, giving each agent a utility of $\frac{1}{n^2}$.

A fairness concept that can be used as an alternative to partial proportionality is *Uniform Preference Externalities* (UPE). It requires that the value given to each agent is at least as large as he could get if all n agents had the same preferences as his own. This concept has been studied mainly in scenarios of homogenous resource allocation [Moulin, 1990] and indivisible item assignment [Budish, 2011, Procaccia and Wangy, 2013], but it can easily be adapted to our scenario. For example, in the problem of division with R-fat rectangles without absolute size constraints, the UPE requirement is the following: *Each agent should receive at least as much value as he could get by cutting n R-fat rectangles and receiving the least valuable one.*

Is it always possible to attain a UPE division with only the fatness constraint? With only the absolute size constraint? With both the fatness and the absolute size constraint? We leave these interesting questions to future research.

6. AN ALTERNATIVE MODELLING: NON ADDITIVE VALUATIONS

In Section 2, a cake-cutting instance was defined as a tuple:

$$(C,\ n,\ (V_1,...,V_n),\ (UP_{1,...,}UP_n),\ FA)$$

using n additive valuation functions V_i and n sets UP_i (Usable Plots) that represent the geometric constraints. The problem can be defined in an alternative way by combining the constraints with the valuations:

DEFINITION 8   For every agent i and land-plot c:
$$UV_i(c) = \max_{P \subseteq c \ and \ P \in UP_i} V_i(P)$$

$UV_i(c)$ is the Usable Value of c - the maximum value that agent i can attain from a usable land-plot in c. When the set of usable plots (UP_i) is the set of R-fat rectangles, $UV_i(c)$ is the largest value of an R-fat rectangle contained in c. Based on the UV functions, we suggest the following alternative to Definition 2:

DEFINITION 9   A **cake-cutting instance** is a tuple:

$$(C,\ n,\ (UV_1,...,UV_n),\ FA)$$

where C, n and FA are the same as in Definition 2, and the list of UV functions replaces the list of V functions and UP sets.

A **solution** to a cake-cutting instance in the alternative formulation is an n-allocation $(P_1, \dots P_n)$, such that $(UV_1(P_1), \dots UV_n(P_n)) \in FA$.

In contrast to the valuation functions V_i, the usable value functions UV_i are not measures (i.e. they are not additive). For example, suppose that C is a 2-by-2 land-cake and there is a single agent whose valuation function V is equal to the area function (so $V(C) = 4$) and that he insists on a square land-plot ($R = 1$). Then, $UV(C) = 4$, but if the land-cake is halved to two 1-by-2 rectangles, then the UV of each half is 1 and the sum is 2, which is less than the UV of the whole land-cake. On the other hand, if one of the 1-by-2 rectangles is halved to two 1-by-1 squares, then the UV of each half is 1 and the sum is 2, which exceeds the UV of the whole 1-by-2 rectangle.

The alternative problem formulation allows us to relate our work to other studies about cake-cutting with non-additive utilities. There are three types of studies on this subject:

(a) Studies that consider only sub-additive, or concave, valuation functions, in which the sum of the values of the parts is larger than the value of the whole [Berliant et al., 1992, Maccheroni and Marinacci, 2003]. These valuations are typical in situations of decreasing marginal utility,[32] but are unapplicable in our scenario because, as illustrated above, the UV function with the fatness constraint is not necessarily sub-additive - the sum of the values of the parts might be less than the value of the whole.

(b) Studies that consider general non-additive valuation functions but provide only existence proofs [Sagara and Vlach, 2005, Dall'Aglio and Maccheroni, 2009, Hüsseinov and Sagara, 2013]. These proofs are not constructive - no procedure is given for finding the fair divisions that are guaranteed to exist - hence they are not comparable to the current work.

(c) Studies that propose practical procedures for fair cake-cutting with non-additive valuations, but impose other restrictions on the cake. In particular, Su [1999] describes a procedure (attributed to Forest Simmons) that converges to an envy-free division. It does not assume that the valuations are additive, but it does assume that the cake is 1-dimensional.[33] Caragiannis et al. [2011] study a special kind of non-additive valuations - piecewise-uniform with minimal length. An agent's value for a piece of cake is proportional to the total length of

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[32] Maccheroni and Marinacci [2003] give an example of cutting a pizza: the more you eat, the more stuffed you become and the smaller the value you get from an enlargement of your piece.

[33] If the procedure of Su [1999] is applied to a two-dimensional cake, the resulting pieces are short narrow strips (see Figure 1/middle). The division is indeed envy-free, but the UV of each agent might be very low. It is a challenging task to find a division that is both envy-free and keeps the partial proportionality guarantee of Theorem 2. In a work that is yet unpublished, we developed an algorithm for finding such a division for $n = 2$ agents. The challenge is still open for $n > 2$.

its intersection with the agent's desired intervals, excluding sub-intervals in the intersection that are shorter than the minimum length specified by the agent. In their setting, similarly to ours, proportionality cannot be achieved. They give tight bounds on the fraction of value that can be guaranteed and achieve the optimal degree of proportionality together with perfect envy-freeness. They, too, consider only 1-dimensional cakes. The procedure of Iyer and Huhns [2009] allows any valuation function that can be represented as a set of n desired rectangles. However, as explained in the introduction, the success of this procedure is not guaranteed.

Our study is thus the first to provide fair cake-cutting division procedures with non-additive valuations in two dimensions that are guaranteed to succeed.

7. CONCLUSION

In this paper, we made the first steps in the uncharted territory of fair division with specific geometric constraints. We calculated upper and lower bounds on the degree of proportionality that can be achieved in various cases; the bounds are summarized in Table IV. The upper bounds imply the impossibility of the implementation of common fairness (proportionality) as well as the quantification of the minimal loss of proportionality due to the imposition of the fatness constraint. The lower bounds represent the extent of possible alleviation of the first impossibility result - the attainable partial fairness (proportionality).

Some of our open questions are:

More general land-cakes. The division procedure of Section 4 assumes that the land-cake is rectangular. How can it be generalized to non-rectangular polygons, in a way that matches the upper bound of Theorem 1?

More general pieces. The impossibility result of Section 3 assumes that the pieces are axis-parallel rectangles. When the pieces are allowed to be rotated rectangles, or even other commonly used shapes such as fat trapezoids, this lower bound is no longer valid. Is it always possible to find a proportional division with fat rotated rectangles? With fat trapezoids?

Unequal shares. Often, the agents who participate in a cake division have unequal rights. For example, Alice may be entitled to $\frac{2}{3}$ of the cake while George is entitled to $\frac{1}{3}$. Robertson and Webb [1998] generalize the notion of proportional cake-cutting to this case, showing how to give each agent a piece of cake which he values as at least his due proportion of the entire cake. What is the meaning of such a generalized proportional division in the presence of geometric constraints, where each agent must receive less than his due share? How much should each agent receive and how should such a division be carried out?

Additional properties of divisions. In addition to proportionality, the cake-cutting literature deals with many other desirable properties of cake divisions, such as: envy-freeness, Pareto efficiency, equitability, maxmin (Rawlsian), maxsum (utilitarian), egalitarian-equivalence, strategy-proofness and more. Each of these desirable properties raises an interesting issue for future research, namely,

Cake → Plots ↓	Square ($L = 1$)	Rectangle with length/width ratio $L > 1$	General polygon
Squares ($R = 1$)	$F_1 = 1$ $F_2 = 4$ $6n - 8 \geq F_{n \geq 2} \geq 2n$	$F_1 = \lceil L \rceil$ $F_2 = \lceil L \rceil + 2$ $max(\lceil L \rceil, 2) + 6n - 10 \geq F_{n \geq 2} \geq$ $\lceil L \rceil + 2n - 2$	$Cover \geq F_1 \geq Indep$ (equality when C is hole-free). $F_{n \geq 2} \geq Indep + 2n - 2$
R-fat **rectangles** ($1 < R < 2$)		$F_1 = \left\lceil \frac{L}{R} \right\rceil$ If $1 \leq L \leq R$ and $RL < 2$: * $F_2 = 4$ * $6n - 8 \geq F_{n \geq 2} \geq 2n$ If $1 \leq R < L$ or $RL \geq 2$: * $F_2 = \left\lceil \frac{L}{R} \right\rceil + 2$ * $\left\lceil \frac{max(L,2)}{R} \right\rceil + 6n - 10 \geq F_{n \geq 2} \geq$ $\left\lceil \frac{L}{R} \right\rceil + 2n - 2$	$Cover \geq F_1 \geq Indep$ $F_{n \geq 2} \geq Indep + 2n - 2$
R-fat **rectangles** ($2 \leq R$)	$F_1 = 1$ $F_2 = 3$ $4n - 5 \geq F_{n \geq 2} \geq 2n - 1$	$F_1 = \left\lceil \frac{L}{R} \right\rceil$ $F_2 = \left\lceil \frac{L}{R} \right\rceil + 2$ $\left\lceil \frac{L}{R} \right\rceil + 4n - 6 \geq F_{n \geq 2} \geq \left\lceil \frac{L}{R} \right\rceil + 2n - 2$	
Rectangles ($R = \infty$)	$F_n = n$ (a proportional division is possible)		$Cover \geq F_1 \geq Indep$ (equality when C is linearly convex). $F_{n > 1} \geq Indep + n - 1$

TABLE IV

Summary of results - proportionality values for various land-cakes and land-plots. Some of the information is redundant, but was included anyway for the sake of clarity. Also, to save space in the table, the table contains the inverse of the proportionality, instead of the proportionality - F_n is a short-hand for $\frac{1}{Prop(C,n,R)}$, where C is the land-cake, n is the number of agents dividing it and R is the bound on the length/width ratio of the pieces. $Cover$ is a shorthand for $R\text{-}CoverNum(C)$ - the minimum number of R-fat rectangles whose union equals C. $Indep$ is a shorthand for $R\text{-}IndepNum(C)$ - the maximum number of disjoint squares in C such that no two of them are covered by the same R-fat rectangle.

the possibility of implementing it in conjunction with partially fair (proportional) division in the presence of geometric constraints.

Geometric constraints shed new light on the theory of fair cake-cutting. Many of the interesting challenges in this field become even more exciting in the presence of geometric constraints. Hopefully this paper will inspire further research providing insight regarding the relationships between geometry and economics, both within the settings proposed in this paper and other realistic settings.

APPENDIX A: PARTIAL PROPORTIONALITY OF FAIR-AND-SQUARE RECURSIVE HALVING

This appendix proves Claim 3, which deals with dividing a 2-fat land-cake. The proof is by induction on the number of agents n. The induction base is $n = 2$. Most of the proof is identical for $n = 2$ and $n > 2$. Therefore, instead of two separate proofs for $n = 2$ and $n > 2$, we present a single proof and indicate the places in which a special treatment is needed for the case $n = 2$.

The proof follows a single agent, Alice, throughout the steps of the division procedure described in Table II. Suppose that Alice values the entire land-cake as $An - B$, where $n \geq 2$ is the total number of agents dividing that land-cake and the constants A and B match the bounds in Claim 3, namely: $A = 4$, $B = 5$ if $R \geq 2$; $A = 6$, $B = 8$ if $R < 2$.

The following claims are proved:

(1) Alice can follow the methods of Table III.

(2) If Alice follows these methods, then, either she is given an R-fat land-plot which she values as at least 1, or she enters the procedure recursively to divide with k agents a land-cake which she values as at least $Ak - B$ (where $2 \leq k < n$).

Hence, by induction, the division procedure terminates after a finite number of steps and gives Alice a land-plot she values as at least 1.

The proof uses the following inequalities on A and B:[34]

- $B \geq A + \lceil \frac{2}{R} \rceil$

- $2A - B \geq \lceil \frac{2}{R} \rceil + 2 \geq 2\lceil \frac{2}{R} \rceil$

In Step 2, Alice declares $Partners(WS)$ and $Partners(ES)$. To verify that the values specified by the Methods satisfy the requirements of the Rules ($Partners(WS) + Partners(ES) \geq n$), consider each of the 4 cases in turn:

- $V(WS) < \lceil \frac{2}{R} \rceil$ and Alice declares $Partners(WS) = 0$. By the additivity of V, $V(ES) > (An - B) - \lceil \frac{2}{R} \rceil$. Additionally $2A - B \geq 2\lceil \frac{2}{R} \rceil$ so $V(ES) > \lceil \frac{2}{R} \rceil$ even in the case $n = 2$. Therefore Alice declares $Partners(ES) = n$.

- $Ak - B \leq V(WS) < A(k + 1) - B$ for some $k \in \{1, .., n - 2\}$ (Note that this case is possible only for $n \geq 3$). Alice declares $Partners(WS) = k$. By additivity, $V(ES) \geq (An - B) - (A(k + 1) - B) = A(n - k) - A$. Because $B > A$, $V(ES) \geq A(n - k) - B$ and Alice declares $Partners(ES) \geq n - k$.

- $A(n - 1) - B \leq V(WS) \leq An - B - \lceil \frac{2}{R} \rceil$. Alice declares $Partners(WS) = n - 1$. By additivity, $V(ES) \geq \lceil \frac{2}{R} \rceil$ and Alice declares $Partners(ES) \geq 1$.

- $An - B - \lceil \frac{2}{R} \rceil < V(WS)$. Alice declares $Partners(WS) = n$. In any case $Partners(ES) \geq 0$.

In all cases, indeed, $Partners(WS) + Partners(ES) \geq n$. Now the division can proceed in one of several ways (we describe them for WG and WS; the proof for EG and ES is symmetric):

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[34]In fact, A and B were calculated as the minimal constants that satisfy these inequalities.

- Alice is the single agent in WG. Alice is given the 2-fat rectangle WS, for which she declared $Partners(WS) \geq 1$. Assuming Alice followed the Methods, this means $V(WS) \geq \lceil \frac{2}{R} \rceil$, so by Claim 2 she can get an R-fat rectangle with a value of at least 1.
- Alice is in WG and $|WG| = k \in \{2,..,n-1\}$. This group is sent to recursively divide the 2-fat rectangle WS, for which Alice declared $Partners(WS) \geq k \geq 2$, which implies $V(WS) \geq Ak - B$. Hence the induction assumption holds.
- $|WG| = n$. This means that Alice has reported $Partners(WS) = n$, which implies $V(WS) > An - B - \lceil \frac{2}{R} \rceil$ and the division proceeds at step 3.

In Step 3, Alice draws a rectangle $P = [0,x] \times [0,1]$ with $x \in [0, \frac{L}{2}]$, such that the value of the remaining rectangle is $V([x,L] \times [0,1]) = \lceil \frac{2}{R} \rceil$. Now the division can proceed in one of several ways:

- Alice's rectangle is selected as the largest and she receives its complement $[x,L] \times [0,1]$. This is a 2-fat rectangle which Alice values as $\lceil \frac{2}{R} \rceil$, so by Claim 2 she can get an R-fat rectangle with a value of at least 1.
- Alice's rectangle is not selected as the largest, $n = 2$, and Alice is given the rectangle $[0,x_m] \times [0,1]$, where $x_m \geq x$ and $x_m \geq \frac{1}{2}$. This is a 2-fat rectangle which Alice values as at least $(2A - B) - \lceil \frac{2}{R} \rceil \geq \lceil \frac{2}{R} \rceil$, so by Claim 2 she can get an R-fat rectangle with a value of at least 1.
- Alice's rectangle is not selected as the largest, $n > 2$, and Alice is sent to recursively divide a rectangle $[0,x_m] \times [0,1]$ with $n-1$ agents, where $x_m \geq x$ and $x_m \geq \frac{1}{2}$. This is a 2-fat rectangle which Alice values as at least $(An - B) - \lceil \frac{2}{R} \rceil > A(n-1) - B$ (because $A > \lceil \frac{2}{R} \rceil$). Hence the induction assumption holds.
- All agents, including Alice, declared $x < \frac{1}{2}$. This means that $V([0, \frac{1}{2}] \times [0,1]) \geq An - B - \lceil \frac{2}{R} \rceil$ and the division proceeds at step 4.

In step 4, Alice declares $Partners(NS)$ and $Partners(SS)$, where $NS \cup SS = [0, \frac{1}{2}] \times [0,1]$. Again we verify that the values specified by the Methods satisfy $Partners(NS) + Partners(SS) \geq n$ in all 4 cases:

- $V(NS) < 1$ and Alice declares $Partners(NS) = 0$. By the additivity of V, $V(SS) > (An - B - \lceil \frac{2}{R} \rceil) - 1$. Additionally, $2A - B \geq \lceil \frac{2}{R} \rceil + 2$ so $V(ES) > 1$ even in the case $n = 2$. Therefore Alice declares $Partners(SS) = n$.
- $Ak - B \leq V(NS) < A(k+1) - B$ for some $k \in \{1,..,n-2\}$. Alice declares $Partners(NS) = k$. By additivity, $V(SS) \geq (An - B - \lceil \frac{2}{R} \rceil) - (A(k+1) - B) = A(n-k) - A - \lceil \frac{2}{R} \rceil$. Because $B \geq A + \lceil \frac{2}{R} \rceil$, $V(SS) \geq A(n-k) - B$ and Alice declares $Partners(SS) \geq n-k$.
- $A(n-1) - B \leq V(NS) \leq An - B - \lceil \frac{2}{R} \rceil - 1$. Alice declares $Partners(NS) = n - 1$. By additivity, $V(SS) \geq 1$ and Alice declares $Partners(SS) \geq 1$.
- $An - B - \lceil \frac{2}{R} \rceil - 1 < V(NS)$. Alice declares $Partners(NS) = n$. In any case $Partners(SS) \geq 0$.

In all cases, indeed, $Partners(NS) + Partners(SS) \geq n$. Now the division can proceed in one of several ways (we describe them for NG and NS; the case for SG and SS is symmetric):

- Alice is the single agent in NG. Alice is given the square NS, for which she declared $Partners(NS) \geq 1$. Assuming Alice followed the Methods, $V(NS) \geq 1$.
- Alice is in NG and $|NG| = k \in \{2,..,n-1\}$. This group is sent to recursively divide the square NS, for which Alice declared $Partners(NS) \geq k \geq 2$, which implies $V(NS) \geq Ak - B$. Hence the induction assumption holds.
- $|NG| = n$. This means that Alice has declared $Partners(NS) = n$, which implies $V(NS) > An - B - \lceil \frac{2}{R} \rceil - 1$ and the division proceeds at step 5.

In Step 5, Alice draws a corner square P such that the value of the remaining L-shape is $V(C - P) = 1 + \lceil \frac{2}{R} \rceil$. Now the division can proceed in one of several ways:

- Alice's square is selected as the largest and she receives its complement L-shape. Assuming Alice followed the methods, she values this L-shape as $1 + \lceil \frac{2}{R} \rceil$. This L-shape can be covered by one square on the north/south and additional $\lceil \frac{2}{R} \rceil$ R-fat rectangles on the east/west, for a total of $1 + \lceil \frac{2}{R} \rceil$. Therefore by Claim 2 Alice can get an R-fat rectangle with a value of at least 1.

- Alice's square is not selected as the largest, and $n = 2$. Alice is given a square which she values as at least $(2A - B) - (1 + \lceil \frac{2}{R} \rceil) \geq 1$.

- Alice's square is not selected as the largest, and $n > 2$. Alice is sent to recursively divide a corner square with a side-length $x_m \geq x$ with $n - 1 \geq 2$ agents. This is a square which Alice values as at least $(An - B) - (1 + \lceil \frac{2}{R} \rceil) \geq A(n-1) - B$ (because $A \geq 1 + \lceil \frac{2}{R} \rceil$). Hence the induction assumption holds.

This completes the proof of Claim 3.

APPENDIX B: PARTIAL PROPORTIONALITY FOR GENERAL RECTANGULAR CAKES

A modified set of methods for agents dividing a general rectangular cake is given in Table V.

Similarly to the proof of Claim 3 in Appendix A, Claim 4 is proved by induction on the number of agents n, following a single agent, Alice, throughout the steps of the division procedure described in Table II. Suppose that Alice values the entire land-cake as $An - B' + \lceil \frac{L}{R} \rceil$, where $L \geq 1$ is the length/width ratio of the land-cake and $n \geq 2$ is the total number of agents dividing that land-cake. The constants A and B' match the bounds in Claim 4, namely: $A = 4$, $B' = 6$ if $R \geq 2$; $A = 6$, $B' = 10$ if $R < 2$.

We prove the following claims:

(1) Alice can follow the methods of Table V.

(2) If Alice follows these methods, then, either she is given an R-fat land-plot which she values as at least 1, or she enters the procedure recursively to divide a land-cake of length/width ratio $L' > 2$ which she values as at least $Ak - B' + \lceil \frac{L'}{R} \rceil$ with k agents, or she enters the procedure recursively to divide a 2-fat land-cake which she values as at least $Ak - B$ with k agents (where $2 \leq k < n$).

Hence, by induction and using Claim 3, the division procedure terminates after a finite number of steps and gives Alice a land-plot she values as at least 1.

The proof uses the following relations on A and B':

- $B' \geq A + 2\lceil \frac{2}{R} \rceil$
- $2A - B' \geq 2$
- $B' = B + \lceil \frac{2}{R} \rceil$, where B is the corresponding constant from Claim 3.

The proof also uses the following arithmetic lemma:

LEMMA (Ceiling Lemma) For every positive $L \geq 1$, $R \geq 1$ and $x \in [0, L]$: $\left\lceil \frac{\max(x,2)}{R} \right\rceil + \left\lceil \frac{\max(L-x,2)}{R} \right\rceil \leq \left\lceil \frac{L}{R} \right\rceil + 2\lceil \frac{2}{R} \rceil - 1$.

Hence: $\lceil \frac{L}{R} \rceil - \left\lceil \frac{\max(x,2)}{R} \right\rceil \geq \left\lceil \frac{\max(L-x,2)}{R} \right\rceil + 1 - 2\lceil \frac{2}{R} \rceil$.

PROOF: There are several cases:

1. $x \leq 2$ and $L - x \leq 2$. The LHS is $2\lceil \frac{2}{R} \rceil$. Subtracting $2\lceil \frac{2}{R} \rceil$ from both sides, we get: $0 \leq \left\lceil \frac{L}{R} \right\rceil - 1$, which is always true because L and R are positive.

2. $x \leq 2$ and $L - x > 2$. Subtracting $\lceil \frac{2}{R} \rceil$ from both sides, we get: $\left\lceil \frac{L-x}{R} \right\rceil \leq \left\lceil \frac{L}{R} \right\rceil + \lceil \frac{2}{R} \rceil - 1$, which is always true because $\lceil \frac{2}{R} \rceil \geq 1$.

New method for step 2 (cake-bipartition):

Let c be a part of the current land-cake (WS or ES), L_c its length/width ratio (note that by the division rules ES and WS have the same dimensions so their length/width ratio is identical) and $n \geq 2$ the total number of agents who divide the current land-cake.

- If $V(c) < \lceil \frac{L_c}{R} \rceil$ then declare $Partners(c) = 0$.

- Else, if $Ak - B' + \lceil \frac{\max(2, L_c)}{R} \rceil \leq V(c) < A(k+1) - B' + \lceil \frac{\max(2, L_c)}{R} \rceil$, for some k in $\{1, .., n-2\}$, then declare $Partners(c) = k$ (Note that this case is possible only for $n \geq 3$).

- Else, if $A(n-1) - B' + \lceil \frac{\max(2, L_c)}{R} \rceil \leq V(c) \leq An - B' + \lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil$, declare $Partners(c) = n-1$.

- Else, $An - B' + \lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil < V(c)$; declare $Partners(c) = n$.

New method for step 3 (agent-bipartition):

Draw a rectangle $P = [0, x] \times [0, 1]$ with $x \in [0, \frac{L}{2}]$, such that the value of the remaining rectangle is $V([x, L] \times [0, 1]) = \left\lceil \frac{L-x}{R} \right\rceil$.

New method for step 4 (cake-bipartition):

Let c be a part of the current land-cake (NS or SS) and let $n \geq 2$ be the total number of agents who divide the current land-cake.

- If $V(c) < 1$, then declare $Partners(c) = 0$.

- Else, if $Ak - B' + \lceil \frac{2}{R} \rceil \leq V(c) < A(k+1) - B' + \lceil \frac{2}{R} \rceil$, for some k in $\{1, .., n-2\}$, then declare $Partners(c) = k$.

- Else, if $A(n-1) - B' + \lceil \frac{2}{R} \rceil \leq V(c) \leq An - B' - 1$, declare $Partners(c) = n-1$.

- Else, $An - B' - 1 < V(c)$; declare $Partners(c) = n$.

New method for step 5 (agent-bipartition):

Draw a corner-square P such that the value of the remaining L-shape is: $V(C - P) = 1 + \left\lceil \frac{L}{R} \right\rceil$.

TABLE V

FAIR-AND-SQUARE RECURSIVE-HALVING METHODS FOR A GENERAL RECTANGULAR LAND-CAKE WITH A LENGTH/WIDTH RATIO OF L (SEE SUBSECTION 4.3).

3. $x > 2$ and $L - x > 2$. The LHS is: $\lceil \frac{x}{R} \rceil + \lceil \frac{L-x}{R} \rceil$, and by the properties of the ceiling operator, it is at most $\lceil \frac{L}{R} \rceil + 1 \geq \lceil \frac{L}{R} \rceil + 2 \lceil \frac{2}{R} \rceil - 1$. *Q.E.D.*

Back to Alice and the division rules:

In Step 2, Alice declares $Partners(WS)$ and $Partners(ES)$. According to the division rules, the length/width ratio of each of these pieces is $L_c = \frac{L}{2} \geq 1$ (since $L \geq 2$). Hence $\lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil \geq \lceil \frac{L_c}{R} \rceil - 1$ and $\lceil \frac{L}{R} \rceil - \lceil \frac{\max(2,L_c)}{R} \rceil \geq \lceil \frac{\max(2,L_c)}{R} \rceil + 1 - 2 \lceil \frac{2}{R} \rceil$.

To assure that the values specified by the Methods satisfy the requirements of the Rules ($Partners(WS) + Partners(ES) \geq n$), consider each of the following 4 cases:

- $V(WS) < \lceil \frac{L_c}{R} \rceil$ and Alice declares $Partners(WS) = 0$. By additivity, $V(ES) > An - B' + \lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil \geq An - B' + \lceil \frac{L}{R} \rceil - 1$. Additionally $2A - B' \geq 2$ so $V(ES) > \lceil \frac{L_c}{R} \rceil + 1$ even in the case $n = 2$. Therefore Alice declares $Partners(ES) = n$.

- $Ak - B' + \lceil \frac{\max(2,L_c)}{R} \rceil \leq V(c) < A(k+1) - B' + \lceil \frac{\max(2,L_c)}{R} \rceil$ for some $k \in \{1,..,n-2\}$ (Note that this case is possible only for $n \geq 3$). Alice declares $Partners(WS) = k$. By additivity, $V(ES) \geq (An - B') - (A(k+1) - B') + (\lceil \frac{L}{R} \rceil - \lceil \frac{\max(2,L_c)}{R} \rceil)$. By the ceiling lemma, $V(ES) \geq A(n-k) - A + (\lceil \frac{\max(2,L_c)}{R} \rceil + 1 - 2 \lceil \frac{2}{R} \rceil)$. Because $B' > A + 2 \lceil \frac{2}{R} \rceil - 1$, $V(ES) > A(n-k) - B' + (\lceil \frac{\max(2,L_c)}{R} \rceil)$ and Alice declares $Partners(ES) \geq n - k$.

- $A(n-1) - B' + \lceil \frac{\max(2,L_c)}{R} \rceil \leq V(WS) \leq An - B' + \lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil$. Alice declares $Partners(WS) = n-1$. By additivity, $V(ES) \geq \lceil \frac{L_c}{R} \rceil$ and Alice declares $Partners(ES) \geq 1$.

- $An - B' + \lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil < V(WS)$. Alice declares $Partners(WS) = n$. In any case $Partners(ES) \geq 0$.

In all cases, indeed, $Partners(WS) + Partners(ES) \geq n$. Now the division can proceed in one of several ways (we describe them for WG and WS; the case for EG and ES is symmetric):

- Alice is the single agent in WG. Alice is given the rectangle WS for which she reported $Partners(WS) \geq 1$. Assuming Alice followed the Methods, this implies $V(WS) \geq \lceil \frac{L_c}{R} \rceil$, where L_c is the length/width ratio of WS. Hence by Claim 2 Alice can get an R-fat rectangle with a value of at least 1.

- Alice is in WG and $|WG| = k \in \{2,..,n-1\}$. This group is sent to recursively divide the rectangle WS, for which Alice reported $Partners(WS) \geq k \geq 2$, which implies $V(WS) \geq Ak - B' + \lceil \frac{\max(2,L_c)}{R} \rceil$. If $L_c \leq 2$ then this value equals $Ak - B$ and by Claim 3 Alice will get at least 1. Otherwise $L_c > 2$ and the induction assumption holds.

- $|WG| = n$. This implies that Alice reported $Partners(WS) = n$, which implies $V(WS) > An - B' + \lceil \frac{L}{R} \rceil - \lceil \frac{L_c}{R} \rceil \geq An - B' + \lceil \frac{L}{R} \rceil - 1$ and the division proceeds at step 3.

In Step 3, Alice draws a rectangle $P = [0, x] \times [0, 1]$ with $x \in [0, \frac{L}{2}]$, such that the value of the remaining rectangle is $V([x, L] \times [0, 1]) = \lceil \frac{L-x}{R} \rceil$. Now the division can proceed in one of several ways:

- Alice's rectangle is selected as the largest and she receives its complement $[x, L] \times [0, 1]$. This is a rectangle with an aspect ratio of $L - x \geq \frac{L}{2} \geq 1$, which Alice values as $\lceil \frac{L-x}{R} \rceil$. Hence by Claim 2 Alice can get an R-fat rectangle with a value of at least 1.

- The largest rectangle is $P_m = [0, x_m] \times [0, 1]$, where $x_m \geq x$ and $x_m \geq \frac{1}{2}$. Define the rectangle $P' = [0, \max(\frac{1}{2}, x)] \times [0, 1]$. P' contains P so its value is at least $(An - B' + \lceil \frac{L}{R} \rceil) - \lceil \frac{L-x}{R} \rceil \geq An - B' + \lceil \frac{x}{R} \rceil - 1$. There are several sub-cases:

  - $n = 2$. Alice Receives P_m which contains P' which she values as at least $(2A - B') + \lceil \frac{x}{R} \rceil - 1 \geq 2 + \lceil \frac{x}{R} \rceil - 1 \geq \lceil \frac{\max(x,2)}{R} \rceil$. The length/width ratio of P' is at

most $\max(x, 2)$ so by Claim 3 it contains an R-fat rectangle with a value of at least 1.

- $n > 2$ and $x_m \leq 2$. Alice is sent to divide P_m, which is a 2-fat rectangle, with $n - 1$ agents. Alice values P_m as at least $An - B' + \lceil \frac{x}{R} \rceil - 1 \geq An - B' = A(n-1) + A - B - \lceil \frac{2}{R} \rceil \geq A(n-1) - B$ so by Claim 3 she gets a value of at least 1.

- $n > 2$ and $x_m > 2$. Alice is sent to divide P_m with $n - 1$ agents. P_m contains P' which she values as at least $An - B' + \lceil \frac{x}{R} \rceil - 1 \geq A(n-1) + A - B' + \lceil \frac{\max(x,2)}{R} \rceil - 2 \geq A(n-1) - B' + \lceil \frac{\max(x,2)}{R} \rceil$. The length/width ratio of P' is at most $\max(x, 2)$ so by the induction assumption Alice gets a value of at least 1.

- All agents, including Alice, declared $x < \frac{1}{2}$. This means that $V([0, \frac{1}{2}] \times [0, 1]) \geq (An - B' + \lceil \frac{L}{R} \rceil) - \lceil \frac{L-x}{R} \rceil \geq An - B' = An - B - \lceil \frac{2}{R} \rceil$ and the division proceeds at step 4.

In step 4, Alice declares $Partners(NS)$ and $Partners(SS)$, where $NS \cup SS = [0, \frac{1}{2}] \times [0, 1]$. Again we must make sure that the values specified by the Methods satisfy $Partners(NS) + Partners(SS) \geq n$ in all 4 cases:

- $V(NS) < 1$ and Alice declares $Partners(NS) = 0$. By the additivity of V, $V(SS) > An - B' - 1$. Additionally, $2A - B' \geq 2$ so $V(ES) > 1$ even in the case $n = 2$. Therefore Alice declares $Partners(SS) = n$.

- $Ak - B' + \lceil \frac{2}{R} \rceil \leq V(NS) < A(k+1) - B' + \lceil \frac{2}{R} \rceil$ for some $k \in \{1, .., n-2\}$. Alice declares $Partners(NS) = k$. By additivity, $V(SS) \geq (An - B') - (A(k + 1) - B' + \lceil \frac{2}{R} \rceil) = A(n - k) - A - \lceil \frac{2}{R} \rceil$. Because $B' \geq A + 2\lceil \frac{2}{R} \rceil$, $V(SS) \geq A(n-k) - B' + \lceil \frac{2}{R} \rceil$ and Alice declares $Partners(SS) \geq n - k$.

- $A(n-1) - B' + \lceil \frac{2}{R} \rceil \leq V(NS) \leq An - B' - 1$. Alice declares $Partners(NS) = n - 1$. By additivity, $V(SS) \geq 1$ and Alice declares $Partners(SS) \geq 1$.

- $An - B' - 1 < V(NS)$. Alice declares $Partners(NS) = n$. In any case $Partners(SS) \geq 0$.

In all cases, indeed, $Partners(NS) + Partners(SS) \geq n$. Now the division can proceed in one of several ways (we describe them for NG and NS; the case for SG and SS is symmetric):

- Alice is the single agent in NG. Alice is given the square NS, for which she reported $Partners(NS) \geq 1$. Assuming Alice followed the Methods, $V(NS) \geq 1$.

- Alice is in NG and $|NG| = k \in \{2, .., n-1\}$. This group is sent to recursively divide the square NS, for which Alice reported $Partners(NS) \geq k \geq 2$, which implies $V(NS) \geq Ak - B' + \lceil \frac{2}{R} \rceil = Ak - B$. Hence by Claim 3 Alice gets a value of 1.

- $|NG| = n$. This implies that Alice reported $Partners(NS) = n$, which implies $V(NS) > An - B' - 1$ and the division proceeds at step 5.

In Step 5, Alice draws a corner square P such that the value of the remaining L-shape is $V(C - P) = 1 + \lceil \frac{L}{R} \rceil$. Now the division can proceed in one of several ways:

- Alice's square is selected as the largest and she receives its complement L-shape. Assuming Alice followed the methods, she values this L-shape as $1 + \lceil \frac{L}{R} \rceil$. This L-shape can be covered by one square at the north/south and additional $\lceil \frac{L}{R} \rceil$ R-fat rectangles at the east/west, for a total of $1 + \lceil \frac{L}{R} \rceil$. Hence by Claim 2 Alice can get an R-fat rectangle with a value of at least 1.

- Alice's square is not selected as the largest, and $n = 2$. Alice is given a square which she values as at least $(2A - B' + \lceil \frac{L}{R} \rceil) - (1 + \lceil \frac{L}{R} \rceil) = 2A - B' - 1 \geq 1$.

- Alice's square is not selected as the largest, and $n > 2$. Alice is sent to recursively divide a corner square with a side-length $x_m \geq x$ with $n - 1 \geq 2$ agents. This is a square which Alice values as at least $(An - B' + \lceil \frac{L}{R} \rceil) - (1 + \lceil \frac{L}{R} \rceil) = An - B' - 1 = An - B - \lceil \frac{2}{R} \rceil - 1 \geq A(n-1) - B$ (because $A \geq 1 + \lceil \frac{2}{R} \rceil$). Hence by Claim 2 Alice can get an R-fat rectangle with a value of at least 1.

This completes the proof of Claim 4. $\square$

APPENDIX C: SUM OF VALUES IN FAIR-AND-SQUARE RECURSIVE HALVING

This appendix proves Claim 5, which makes the following assumptions:

- There is a 2-fat land-cake C that is to be divided among $n \geq 2$ agents.
- All agents follow the methods of Table III with the same R.
- All agents assign the same value V to the entire land-cake.

For the sake of the proof, assume that all valuation functions are normalized so that $V(C) = V = An - B$, where the constants A and B match the bounds in Claim 3, namely $V = 4n - 5$ for $R \geq 2$ and $V = 6n - 8$ for $R < 2$. The proof is different for $R \geq 2$ and $R < 2$.

C.1. $R \geq 2$

We first prove by induction that if all $n \geq 1$ agents value the entire land-cake as V then the sum of allocated values is $V - (n - 1)$.

PROOF: The base is $n = 1$. Indeed a single agent can get the entire value.

For $n > 1$, consider the four places where the procedure can terminate with either an allocation or a recursive application:

- In Step 2, when there are agents in both WG and EG (i.e. $|WG| \geq 1$ and $|EG| \geq 1$). Let V_{WS} be the minimal value assigned to WS by an agent from WG and V_{ES} the minimal value assigned to ES by an agent from EG. By the division procedure, all agents from EG assign to WS at most V_{WS} and all agents from WG assign to ES at most V_{ES}. Hence $V_{ES} \geq V - V_{WS}$. By the induction assumption, the sum of values from the division of WS is at least $V_{WS} - (|WG| - 1)$ and the sum of values from the division of ES is at least $V_{ES} - (|EG| - 1)$. The sum of allocated values is at least $V_{WS} - (|WG| - 1) + V_{ES} - (|EG| - 1) = V - (n - 2) > V - (n - 1)$ and the induction step is satisfied.

- In Step 3, when the eastmost division line is at $x_m \geq \frac{1}{2}$. Agent m receives $[x_m, L] \times [0, 1]$ and gets its entire value. The other $n - 1$ agents recursively divide $[0, x_m] \times [0, 1]$. By the induction assumption, the total value allocated is at least $V - (n - 2) > V - (n - 1)$.

If $x_m < \frac{1}{2}$ then all agents value WS as more than $V - 1$ and give away the eastern part $(C - WS)$ which has a value of less than 1 and the division proceeds at step 4.

- In Step 4, when there are agents in both NG and SG. Let V_{NS} be the minimal value assigned to NS by an agent from NG and V_{SS} the minimal value assigned to SS by an agent from SG. By the division procedure, $V_{NS} \geq V - 1 - V_{ES}$. Similarly to step 2, the sum of allocated values is at least $V_{NS} - (|NG| - 1) + V_{SS} - (|SG| - 1) = V - 1 - (n - 2) = V - (n - 1)$ and the induction step is satisfied.

- In Step 5. The largest square is selected and divided to $n - 1$ agents, who value it as $V - 2$. By the induction assumption, the sum of values allocated from that square is at least $V - 2 - (n - 2)$. Adding the value of at least 1 allocated to the n-th agent gives the sum of $V - (n - 1)$ and the induction step is satisfied.   *Q.E.D.*

Initially all agents value the entire land-cake as $V = 4n - 5$ so the sum of allocated values is $V - (n - 1) = 3n - 4$.

C.2. $R < 2$

We first prove by induction that if all $n \geq 2$ agents value the entire land-cake as V then the sum of allocated values is $\frac{V-2(n-2)}{2}$.

PROOF: The base is for $n = 2$. Indeed when there are two agents each of them can have at least a quarter of the total value, so the sum of values is at least $\frac{V}{2}$.

For $n > 2$, consider the four places where the procedure can terminate with either an allocation or a recursive application:

- In Step 2, when there are agents in both WG and EG (i.e. $|WG| \geq 1$ and $|EG| \geq 1$). Let V_{WS} be the minimal value assigned to WS by an agent from WG and V_{ES} the minimal value assigned to ES by an agent from EG. By the division procedure, all agents from EG assign to WS at most V_{WS} and all agents from WG assign to ES at most V_{ES}. Hence $V_{ES} \geq V - V_{WS}$. Now there are two sub-cases for WG: If $|WG| \geq 2$ then WS is divided recursively among $|WG|$ agents. By the induction assumption, the sum of values from this division is at least $\frac{V_{WS}-2(|WG|-2)}{2}$. Otherwise, $|WG| = 1$ and WS is given to a single agent who receives at least $\frac{V_{WS}}{2}$. There are similar sub-cases for EG. The worst case (when $n > 2$) is when $|WG| = 1$ and $|EG| = n-1 \geq 2$ (or vice versa). The sum of allocated values is at least $\frac{V_{WS}}{2} + \frac{V_{ES}-2(n-3)}{2} = \frac{V-2(n-3)}{2} > \frac{V-2(n-2)}{2}$ and the induction step is satisfied.

- In Step 3, when the eastmost division line is at $x_m \geq \frac{1}{2}$. Agent m receives $[x_m, L] \times [0,1]$ and gets at least half its value. The other $n-1$ agents recursively divide $[0, x_m] \times [0,1]$. By the induction assumption, the total value allocated is at least $\frac{V-2(n-3)}{2} > \frac{V-2(n-2)}{2}$.

If $x_m < \frac{1}{2}$ then all agents value WS as more than $V - 2$ and give away the eastern part $(C - WS)$ which has a value of less than 2 and the division proceeds at step 4.

- In Step 4, when there are agents in both NG and SG. Let V_{NS} be the minimal value assigned to NS by an agent from NG and V_{SS} the minimal value assigned to SS from SG. By the division procedure, $V_{NS} \geq V - 2 - V_{ES}$. Similarly to step 2, the sum of allocated values is at least $\frac{V-2-2(n-3)}{2} = \frac{V-2(n-2)}{2}$ and the induction step is satisfied.

- In Step 5. The largest square is selected and divided among $n-1$ agents, who value it as $V - 3$. By the induction assumption, the sum of values allocated from that square is at least $\frac{V-3-2(n-3)}{2}$. Adding the value of at least 1 allocated to the n-th agent gives $\frac{V-1-2(n-3)}{2} > \frac{V-2(n-2)}{2}$ and the induction step is satisfied. $Q.E.D.$

Initially all agents value the entire land-cake as $V = 6n - 8$ so the sum of allocated values is $\frac{V-2(n-2)}{2} = 2n - 2$.

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