Static Stability in Games
Part II: Asymmetric Games

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Static stability in strategic games differs from dynamic stability in only considering the players' incentives to change their strategies. It does not rely on any assumptions about the players' reactions to these incentives and it is not linked with any particular dynamics. This paper introduces a general notion of static stability of strategy profiles that is applicable to any N-player strategic game. It examines several important classes of games, with strategy spaces or payoff functions that have special structures, where this general notion takes a simple, concrete form. The paper also explores the relations between static stability and specific kinds of dynamic stability, and connects static stability in general, asymmetric games with the related, but essentially weaker, notion of static stability of strategies in symmetric games. *JEL Classification:* C72.

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1 The basic idea

A system is at an equilibrium state if there is no (net) force pushing it towards a different state. Stability differs from equilibrium in also considering the forces acting at states that are (usually, only slightly) different from the one under consideration and, roughly speaking, requiring that these forces push the system in the direction of that state. More precisely, this description concerns *static stability*, as it does not involve a law of motion that specifies how forces translate into actual movement of the system. For example, a ball at the bottom of a pit is stable but one at the top of a hill is not. In both cases, the net force acting on the ball vanishes, but any displacement would result in a non-zero force that is directed towards the equilibrium point in the first case and away from it in the second case. This description is static rather than kinetic. It does not involve motion, and therefore does not invoke Newton's second law.

In game theory, where forces may be equated with incentives, a (Nash) equilibrium point is a strategy profile where there is no incentive for any player to change his strategy unilaterally. A *strict* equilibrium has a flavor of a stable point, as it is defined by the condition that if any player deviated to another strategy, reverting to the original strategy would make him better off. However, consideration of deviations by only one player may be too narrow a perspective. For example, in the finite two-player game

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$$\begin{array}{ccc} & L & R \\ \begin{matrix} T \\ B \end{matrix} & \multicolumn{2}{c}{\begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 100,100 \end{pmatrix}} \end{array}$$

(where only pure strategies are considered) both (T, L) and (B, R) are strict equilibria but arguably only the latter is stable. The sense in which (T, L) fails to be stable is that if one player deviates from it, his incentive to return is weaker than the other player's incentive to also deviate. Or, from a somewhat different perspective, if the players start at the strategy profile (B, R), their overall incentive to go to (T, L) is negative, as the loss for the first mover outweighs the second mover's gain. Of course, in general, the sum of the movers' gains or losses may depend on the path taken. For example, this is so in a similar game where in both equilibria the payoffs are $(1,100)$. Here, going from (T, L) to (B, R) via (B, L) is relatively "easy" but going via (T, R) is "hard" (and going in the opposite direction is easy).

A simple way of combining the incentives of moving along different paths is averaging them. Consider then a strategy profile x that differs from a given strategy profile y in $k \geq 2$ coordinates, so that going from y to x requires (at least) k unilateral moves, which can be performed in $k!$ different orders. The players' overall incentive to move is quantified by the average over all $k!$ paths from y to x of the sum of the k individual payoff increments along each path. The strategy profile y may be considered *globally stable*, *globally weakly stable* or *globally definitely unstable* if this average is negative, nonpositive or positive, respectively, for all $x \neq y$. In the first case, y is necessarily a strict equilibrium, and in the second case, it is an equilibrium. However, these necessary conditions are not sufficient for global stability or weak stability.

Example 1. *Games in the plane.* The strategy sets of player 1 and player 2 are the real line $\mathbb{R}$. Their payoff functions are

$$h_1(x_1, x_2) = -x_1^2 + 3x_1x_2 \text{ and } h_2(x_1, x_2) = -\frac{1}{2}x_2^2 - x_1x_2. \tag{1}$$

It is not difficult to see that the origin is the unique equilibrium, and it is moreover strict. A path from $(0,0)$ to any other strategy profile (x_1, x_2) goes through either $(x_1, 0)$ or $(0, x_2)$. In the first case, where player 1 is the first to move, the sum of the movers' payoff increments is $-x_1^2 - x_2^2/2 - x_1x_2$, and when player 2 move first, the sum is $-x_2^2/2 - x_1^2 + 3x_1x_2$. The average of the two sums is $-x_1^2 + x_1x_2 - x_2^2/2 = -(x_1 - x_2/2)^2 - x_2^2/4$. This expression is negative for all $(x_1, x_2) \neq (0,0)$, which proves that the equilibrium is globally stable. By contrast, in the game obtained by dropping the second term in h_2, where the payoff functions are

$$h_1(x_1, x_2) = -x_1^2 + 3x_1x_2 \text{ and } h_2(x_1, x_2) = -\frac{1}{2}x_2^2, \tag{2}$$

the corresponding average is $-x_1^2 + 3x_1x_2/2 - x_2^2/2$. This expression is positive for any $(x_1, x_2) \neq (0,0)$ that is a multiple of $(2,3)$, which means that the strict equilibrium $(0,0)$ is not even globally weakly stable.

The notion of static stability outlined above does not assume or require sophisticated players. The moves considered are all myopic and opportunistic, there is no foresight or an attempt to manipulate the other player, and there is no cooperation or coordination – including the incidental coordination that might result from a restriction on the times at which players can move – which effectively rules out simultaneous deviations. The above definitions do however involve a strong assumption, namely, that the players' utilities – more specifically, the changes of payoffs resulting from unilateral moves – are comparable. The comparability reflects an interpretation of these changes as a measure of how readily or willingly the moves are made. This interpretation goes against an entrenched line of thought in economics and game theory, which views utilities merely as a convenient representation of preferences. For example, von Neumann-Morgenstern utilities are unique only up to arbitrary increasing affine transformations. Here, by contrast, only shifts by additive constants and scaling of all players' payoffs by a common factor would be inconsequential. Nevertheless, comparability of utilities is not a totally radical idea. When people hear someone taking about a "strong" or "weak" incentive to do something, they have an intuitive understanding of what the speaker means. It may not be a qualitative leap to go from such descriptive terms to a common quantitative measure of the incentive to act. In fact, under normal conditions, monetary incentives may provide just such a common scale.

Another strong assumption underlying the definitions is that moves that decrease the mover's payoff are not impossible, although they are less likely to occur, or are less readily made, than payoff-increasing moves. The definitions of (static) stability and instability do not however rely on any concrete assumptions about when and where such move, or any other ones, take place. For comparisons with certain kinds of dynamic stability, which do specify a precise law of motion, see Section 4.1.

2 Formal framework

Stability of a strategy profile y becomes a local concept when consideration is restricted to strategy profiles x that are close to y. Closeness is meaningful when the strategy set X_i of each player i is a topological space. The product topology on the set $X = \prod_i X_i$ of all strategy profiles then gives a meaning to a *neighborhood* of a strategy profile x: it is any set of strategy profiles whose interior includes x. In an N-player game with such strategy spaces, where the payoff function of player i is $h_i: X \to \mathbb{R}$, consider for any two strategy profiles x and y and a permutation π of $(1,2,\ldots,N)$ the path from y to x in which the players change their strategies in the order specified by π: player $\pi(1)$ moves first, from $y_{\pi(1)}$ to $x_{\pi(1)}$ (which may or may not be the same strategy), then player $\pi(2)$ moves, and so on. Summation of the movers' changes of payoff and averaging over the set Π of all permutations gives the expression

$$\frac{1}{N!}\sum_{\pi\in\Pi}\sum_{j=1}^{N}\Big(h_{\pi(j)}(y \mid x_{\{\pi(1),\pi(2),\ldots,\pi(j)\}}) - h_{\pi(j)}(x \mid y_{\{\pi(j),\pi(j+1),\ldots,\pi(N)\}})\Big), \tag{3}$$

where $y \mid x_S$ denotes the strategy profile where the players in a set S play according to the strategy profile x and those outside S plays according to y. This expression quantifies the overall incentive to move from y to x. The incentive to move in the opposite direction, from x to y, is given by the negative of (3).

Definition 1. A strategy profile y in an N-player game is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where (3) is negative, nonpositive or positive, respectively, for all $x \neq y$.

In principal, for these notions to be well defined, the topologies on the players' strategy sets need to be specified. In practice, the topologies can often be inferred from the context, as there are unique natural ones (usually determined by the Euclidean distance between strategies). However, regardless of the latter, an important special case of the definition involves the *trivial topology* on X, where the only neighborhood of any strategy profile is the entire set of strategy profiles. It is not difficult to see that stability, weak stability or definite instability with respect to this topology implies the same condition for any other topology and coincides with the *global* version of the property, which is defined in the previous section.

The somewhat unwieldy expression (3) can be put into a simpler form, which also suggests an alternative interpretation of the inequality defining stability. As the next lemma shows, this inequality roughly means that, when players only play according to x or according to y, those doing the former fare worse on average. Specifically, for strategy profiles x and y in an N-player game, define the *payoff of x players when playing against y players* as the quantity

$$I(x,y) = \sum_{j=1}^{N}\left[\frac{1}{\binom{N}{j}}\sum_{\substack{S\\|S|=j}}\bar{h}_S(\,y \mid x_S\,)\right] = \sum_S \frac{1}{\binom{N}{|S|}}\bar{h}_S(\,y \mid x_S\,) = \sum_S \frac{1}{\binom{N}{|S|}}\bar{h}_{S^{\mathrm{C}}}(\,x \mid y_S\,), \tag{4}$$

where, for a set of players $S \subseteq \{1,2,\dots,N\}$, $|S|$ is the number of players in S and $\bar{h}_S = (1/|S|)\sum_{i\in S} h_i$ is their average payoff, which is defined as 0 if $S = \emptyset$. Note that the expression in square brackets is the average of $\bar{h}_S(\,y \mid x_S\,)$ over all sets S of size j. The last equality in (4) is obtained by replacing the summation variable S with the complementary set S^{C}.

Lemma 1. Expression (3) is equal to $I(x,y) - I(y,x)$.

Proof. Each of the payoffs in (3) has the form $h_i(\,y \mid x_S\,)$ or $h_i(\,x \mid y_S\,)$, with $i \in S$. Specifically, the pair (i,S) is given by the two equations $i = \pi(j)$ and either $S = \{\pi(1),\pi(2),\dots,\pi(j)\}$ or $S = \{\pi(j),\pi(j+1),\dots,\pi(N)\}$. In both cases, for every pair (i,S) with $i \in S$ there are precisely $(|S|-1)!\,(N-|S|)!$ pairs (π,j) satisfying the two equations (as j is uniquely determined by $|S|$). Therefore, (3) is equal to

$$\sum_{S\neq\emptyset}\sum_{i\in S}\frac{(|S|-1)!\,(N-|S|)!}{N!}\big(h_i(\,y \mid x_S\,) - h_i(\,x \mid y_S\,)\big) = I(x,y) - I(y,x).$$

■

A strategy profile y that is stable but not globally stable is not necessarily an equilibrium. However, it is still a "local strict equilibrium" in the sense that

$$h_i(\,y \mid x_i\,) - h_i(y) < 0, \tag{5}$$

for every player i and all $x_i \neq y_i$ in some neighborhood of y_i, where $y \mid x_i$ denotes the strategy profile that differs from y only in that player i uses strategy x_i. This fact, which follows from the definition of stability by examining the special case of a strategy profile that differs from y in only one coordinate, may be interpreted as the requirement that, when the players move one by one from any nearby strategy x to y, the *last* move benefits the mover. This requirement is weaker than stability, which considers all the steps from x to y rather than only the last one. By contrast, the requirement that the *first* mover gains, at least on average, turns out to be a stronger condition than stability. This condition is formalized by the next definition and is analyzed by the proposition following it.

Definition 2. A strategy profile y in an N-player game is *locally superior* if it has a neighborhood where for all $x \neq y$

$$\frac{1}{N}\sum_{i=1}^{N}\left(h_i(x) - h_i(x \mid y_i)\right) < 0. \tag{6}$$

Proposition 1. Every locally superior strategy profile is stable, but not conversely.

Proof. A locally superior strategy y has a rectangular neighborhood where inequality (6) holds for all $x \neq y$. In that neighborhood, a similar inequality holds with the strategy profile x replaced by $y \mid x_S$, for any set of players S such that $y \mid x_S$ is different from y. Division by $\binom{N-1}{|S|-1}$ and summation over all nonempty sets S give

$$\begin{aligned}
0 &> \sum_{S\neq\emptyset}\frac{1}{\binom{N-1}{|S|-1}}\frac{1}{N}\sum_{i\in S}\left(h_i(y \mid x_S) - h_i(y \mid x_{S\setminus\{i\}})\right) \\
&= \sum_{S\neq\emptyset}\frac{1}{|S|\binom{N}{|S|}}\sum_{i\in S}h_i(y \mid x_S) - \sum_i\sum_{\substack{S\\ i\in S}}\frac{1}{N\binom{N-1}{|S\setminus\{i\}|}}h_i(y \mid x_{S\setminus\{i\}}) \\
&= \sum_S\frac{1}{\binom{N}{|S|}}\bar{h}_S(y \mid x_S) - \sum_i\sum_{\substack{S\\ i\notin S}}\frac{1}{N\binom{N-1}{|S|}}h_i(y \mid x_S) \\
&= I(x,y) - \sum_S\frac{1}{\binom{N}{|S|}}\bar{h}_{S^{\mathrm{c}}}(y \mid x_S) = I(x,y) - I(y,x).
\end{aligned}$$

By Lemma 1, this proves that y is stable.

To see that (even global) stability is not a sufficient condition for local superiority, note that $y = (0,0)$ is not locally superior in the game (1), because the left-hand side of (6) is equal to the expression $-x_1^2/2 + x_1x_2 - x_2^2/4$, which is positive if $x_1 = x_2 \neq 0$. ■

3 Comparison with stability in symmetric games

A symmetric N-player game is specified by the players' common strategy space X, which in the present context is assumed to be a topological space, and a single payoff function $g: X^N \to \mathbb{R}$ that is invariant to permutations of its second through Nth arguments. If one player uses strategy x and the other players use $y, z, \ldots, w$, in any order, the first player's payoff is $g(x, y, z, \ldots, w)$. A strategy y is an *equilibrium strategy*, with the equilibrium payoff $g(y, y, \ldots, y)$, if

$$g(y, y, \dots, y) \geq g(x, y, \dots, y), \qquad x \in X. \tag{7}$$

Static stability for symmetric games (Milchtaich 2017) differs from that for general N-player games, referred to below as *asymmetric* games, in that the concept is applied to strategies rather than strategy profiles. A strategy y is considered stable if, when the players move one-by-one from y to any nearby strategy x, their moves harm them on average.

Definition 3. A strategy y in a symmetric N-player game with payoff function g is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where, for every strategy $x \neq y$, the inequality

$$\frac{1}{N}\sum_{j=1}^{N}\Big(g(x, \underbrace{x, \dots, x}_{j-1\text{ times}}, \underbrace{y, \dots, y}_{N-j\text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1\text{ times}}, \underbrace{y, \dots, y}_{N-j\text{ times}})\Big) < 0, \tag{8}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds. If the corresponding inequality holds for *all* strategies $x \neq y$, then y is *globally* stable, weakly stable or definitely unstable, respectively.

This definition of stability generalizes a number of more special concepts of statics stability, such as evolutionarily stable strategy, or ESS (Milchtaich 2017), that are applicable only to specific classes of symmetric games. Conceptually, it is similar to Definition 1, and in a sense, the latter can be derived from it. The link between the two definitions is provided by the concept of *symmetrization* of an asymmetric game. An N-player game h, where the strategy space X_i of each player i is a topological space, is symmetrized by letting the players switch roles, with all possible permutations considered. This gives a symmetric N-player game where the players' common strategy space is the space $X = X_1 \times X_2 \times \cdots \times X_N$ of all strategy profiles in the asymmetric game, with the product topology. For a player in g, a strategy $x = (x_1, x_2, \dots, x_N) \in X$ specifies the strategy x_i the player will use when called to assume the role of any player i in h, and the payoff is defined as his average payoff in the $N!$ possible assignments of players in g to roles in h. Formally, for any N strategies in X, $x^1 = (x_1^1, x_2^1, \dots, x_N^1)$, $x^2 = (x_1^2, x_2^2, \dots, x_N^2)$, …, $x^N = (x_1^N, x_2^N, \dots, x_N^N)$,

$$g(x^1, x^2, \dots, x^N) = \frac{1}{N!}\sum_{\pi \in \Pi} h_{\pi(1)}\left(x_1^{\pi^{-1}(1)}, x_2^{\pi^{-1}(2)}, \dots, x_N^{\pi^{-1}(N)}\right), \tag{9}$$

where Π is the set of all permutation of $(1,2, \dots, N)$ and h_i denotes the payoff function of player i in h. (Note that superscripts in this formula index players' strategies in the symmetric game g while subscripts refer to roles in the asymmetric game h. For $\pi \in \Pi$, player i in g is assigned to role $\pi(i)$ in h.)

Proposition 2. A strategy profile in an asymmetric N-player game h is stable, weakly stable or definitely unstable if and only if it has the same property as a strategy in the game g obtained by symmetrizing h. Similarly, a strategy profile is an equilibrium in h if and only if it is an equilibrium strategy in g. In this case, the equilibrium payoff in g is equal to the players' average equilibrium payoff in h.

Proof. To prove the first part of the theorem, it suffices to show that the sum in (8) is equal to expression (3). By (9), that sum can be written as

$$\sum_{j=1}^{N} \frac{1}{N!} \sum_{\pi \in \Pi} \left(h_{\pi(1)}(y \mid x_{\{\pi(1),\pi(2),\dots,\pi(j)\}}) - h_{\pi(1)}(x \mid y_{\{\pi(1),\pi(j+1),\pi(j+n),\dots,\pi(N)\}}) \right). \qquad (10)$$

Since the inner sum in (10) is over the set of all permutations, is it left unchanged by replacing the summation variable π with $\pi \circ \pi_j$, for any (j-specific) permutation π_j. For π_j that is the transposition switching 1 and j, this replacement transforms (10) into (3).

A strategy profile y is an equilibrium strategy in g if and only if the expression obtained from the right-hand side of (9) by setting $x^2 = x^3 = \cdots = x^N = y$ is maximized by choosing $x^1 = y$. That expression can be simplified by partitioning the set of permutations Π into N equal-size parts, each of cardinality $(N-1)!$, according to the value i of $\pi(1)$. Thus, the expression under consideration is equal to

$$\frac{1}{N} \sum_{i=1}^{N} h_i(y_1, y_2, \dots, x_i^1, \dots, y_N).$$

Clearly, choosing $x^1 = y$ maximizes this sum if and only if, for each i, the ith term is maximized by choosing $x_i^1 = y_i$. The latter is also the condition for y to be an equilibrium in h. If it holds, then the maximum (obtained by setting $x_i^1 = y_i$ in each of the terms) is the players' average equilibrium payoff in h. ■

Another notion of static stability in symmetric games is *local superiority* (or strong uninvadability; Bomze 1991). Its definition differs from that of stability in that inequality (8) is replaced with

$$g(x, x, \dots, x) - g(y, x, \dots, x) < 0. \qquad (11)$$

Thus, a change of strategy from x to y is required to benefit the *first* player who makes this move. It is easy to see that a strategy profile y in an asymmetric game h is locally superior if and only if y is locally superior as a strategy in the symmetric game g obtained by symmetrizing h. Indeed, the left-hand sides of (6) and (11) are equal.

In a symmetric game g, a strategy y that is stable or even globally stable is not always an equilibrium strategy of even a "local equilibrium strategy". That is, the inequality in (7) may not hold for x arbitrary close to y. For example, in the symmetric two-player game where the strategy space is the real line and $g(x, y) = x^2 - 3xy$, the origin 0 is globally stable but it is not a local equilibrium strategy. This contrasts with the situation for asymmetric games, where a stable strategy profile is always a local strict equilibrium. This difference suggests that, in some sense, the first kind of stability is weaker than the second kind.

In some classes of symmetric games, a stable strategy *is* automatically an equilibrium strategy. For example, this is so for symmetric $n \times n$ games, where a strategy is stable if and only if it is an ESS (Milchtaich 2017). However, even in this case, the stability condition is in a sense weaker than the corresponding one for asymmetric games, as an ESS y is not necessarily a pure strategy and therefore the (symmetric) equilibrium (y, y) specified by it is not necessarily strict. This contrasts with the situation for asymmetric $m \times n$, or *bimatrix games*, as Theorem 3 in Section 6 shows that a strategy profile in a bimatrix game is stable if and only if it is a strict (hence, pure) equilibrium.

By the last fact and Proposition 2, the stable strategies in the game g obtained by symmetrizing a bimatrix game h are the strict equilibria in h. This conclusion is similar, and closely related, to the well-known fact that a strategy profile y in g is an ESS if and only if it is a strict equilibrium in h (Selten 1980). The similarity reflects (indeed, it proves) the fact that in a game obtained by symmetrizing a bimatrix game, a strategy is stable if and only if it is an ESS. Thus, these symmetric games are similar in this respect to a symmetric $n \times n$ games (although they are generally *not* $n \times n$ games, for any n).

3.1 Essentially symmetric games

A direct comparison between the concepts of stability of a strategy in a symmetric game and stability of a strategy profile in an asymmetric game is provided by the essentially symmetric games. An asymmetric N-player game h is *essentially symmetric* if the players share a common strategy space and for every strategy profile $(x_1, x_2, \dots, x_N)$ and permutation π of $(1,2, \dots, N)$

$$h_i(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(N)}) = h_{\pi(i)}(x_1, x_2, \dots, x_N), \qquad i = 1,2, \dots, N. \tag{12}$$

Thus, if the players' strategies are shuffled, such that each player i takes the strategy of some other player $\pi(i)$, the latter's old payoff becomes player i's new payoff. In other words, the rules of the game are indifferent to the players' identities and are therefore completely specified by the payoff function of any single player, and in particular by h_1. The latter may be viewed as the payoff function in a symmetric game. In fact, for fixed strategy space and number of players N, the mapping $h \mapsto h_1$ is a one-to-one correspondence between the set of essentially symmetric games and the set of symmetric games. It may thus seem that there is little difference between the two concepts. And, indeed, essentially symmetric games are usually referred to simply as symmetric games (von Neumann and Morgenstern 1953). However, there is in fact a substantive, non-technical difference between describing a particular situation as a symmetric game and describing it as an essentially symmetric one, with each alternative corresponding to a different interpretation of the situation. This fact is well recognized in the biological game theory literature, where essential symmetry is often referred to by other names such as uncorrelated asymmetry (Maynard Smith and Parker 1976; the correlation referred to here is that between the players' traits and their payoff functions) and inessential asymmetry (Eshel 2005). A symmetric pairwise contest with identical contestants, such as two equal-size males seeking to obtain a newly vacated territory, is best modeled as a symmetric game such as Chicken, or the Hawk–Dove game. Precedence or other perceivable asymmetries between the contestants, which do not by themselves change the payoffs (i.e., the stakes or the opponents' fighting abilities), make the contest an essentially symmetric one, and, in reality, may significantly affect the contestants' behavior (Maynard Smith 1982, Riechert 1998).

The differences between a symmetric game and the corresponding essentially symmetric one are reflected by the differences between the corresponding notions of stability: stability of a strategy in the first case and stability of a strategy profile in the second case. The second notion is more general, in that it is applicable also to asymmetric strategy profiles, where not all players are using the same strategy. However, even in the case of a symmetric strategy profile, in which all players use the same strategy y, and even if y is an equilibrium strategy in the symmetric game (i.e., it satisfies (7)), stability of y in the symmetric game and stability of the symmetric equilibrium $(y, y, \dots, y)$ in the essentially symmetric game are not the same thing. In fact, the second requirement is stronger.[1] As the proof of the following proposition clearly shows, the reason is that the second requirement takes into consideration a larger set of alternatives than the first one. An alternative to a strategy y is another (nearby) strategy x, to which all the players switch. The alternatives to a symmetric strategy profile include (nearby) strategy profiles that are not symmetric, which means that only some of the players may move to x while the others stick with y or move to other strategies.

Proposition 3. If a symmetric strategy profile $\vec{y} = (y, y, \dots, y)$ in an essentially symmetric N-player game h is stable, then strategy y is stable in the corresponding symmetric game g $(= h_1)$. However, the converse is false even if $\vec{y}$ is an equilibrium and $N = 2$. The strategy profile $\vec{y}$ is an equilibrium in h if and only if y is an equilibrium strategy in g.

Proof. To prove the first assertion, consider the strategy profile $\vec{y}$, another symmetric strategy profile $\vec{x} = (x, x, \dots, x)$, a player i and a set of players S with $i \in S$. Let π be a permutation that maps 1 to i (that is, $\pi(1) = i$) and maps $2, 3, \dots, |S|$ to the other elements of S (if any). By (12),

$$h_i(\vec{y} \mid \vec{x}_S) = h_1(\underbrace{x, \dots, x}_{|S| \text{ times}}, \underbrace{y, \dots, y}_{|S^{\mathrm{c}}| \text{ times}}).$$

It follows from this equality and (4) that

$$\begin{aligned} I(\vec{x}, \vec{y}) - I(\vec{y}, \vec{x}) &= \sum_{j=1}^{N} (h_1(\underbrace{x, \dots, x}_{j \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - h_1(\underbrace{y, \dots, y}_{j \text{ times}}, \underbrace{x, \dots, x}_{N-j \text{ times}})) \\ &= \sum_{j=1}^{N} (g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{N-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}})), \end{aligned}$$

where the second equality once again uses (12) (for the second term). As the last sum is easily seen to be equal to that in (8), y is stable in g if and only if $I(\vec{x}, \vec{y}) - I(\vec{y}, \vec{x}) < 0$ for all $x \neq y$ in some neighborhood of y. By Lemma 1, a sufficient condition for this is that $\vec{y}$ is stable in h.

To see that the last condition is not necessary, consider any symmetric 2×2 game, with payoff matrix A, that has a completely mixed (that is, not pure) ESS y. As indicated, an ESS is a stable strategy. But in the corresponding essentially symmetric bimatrix game (A, A^{T}), the symmetric equilibrium $\vec{y} = (y, y)$ is not stable, because it is not locally strict: any unilateral deviation leaves the deviator's payoff unchanged.

The last assertion in the proposition follows from the fact that, in a symmetric strategy profile in an essentially symmetric game, a player may gain from a unilateral change of strategy if and only if player 1 would gain from making the same move. ■

[1] For bimatrix games, a related difference holds for the index and degree of the symmetric equilibrium, which may depend on whether it is viewed as an equilibrium in the essentially symmetric bimatrix game or in the corresponding symmetric $n \times n$ one (Demichelis and Germano 2000).

4 Games with differentiable payoffs

Consider an N-player game where the strategy space of each player i is a set in a Euclidean space $\mathbb{R}^{n_i}$, with (possibly, player-specific) $n_i \geq 1$, where the topology is given by the Euclidean distance. Strategies are written as column vectors. Correspondingly, a strategy profile $x = (x_1, x_2, \dots, x_N)$ is an n-dimensional column vector, where $n = \sum_i n_i$. It is an *interior* strategy profile if each x_i is an interior strategy in the sense that it lies in the *relative interior* of player i's strategy space (which coincides with the interior if the strategy space has affine dimension n_i, that is, it is of full affine dimension). The gradient with respect to the components of player i's strategy is denoted ∇_i and is written as an n_i-dimensional row vector (of first-order differential operators). For any i and j, $\nabla_i^{\mathrm{T}} \nabla_j$ is therefore an $n_i \times n_j$ matrix (of second-order differential operators). In particular, $\nabla_i^{\mathrm{T}} \nabla_i h_i$ is the Hessian matrix of player i's payoff function with respect to the player's own strategy. These Hessian matrices are the diagonal blocks in the $n \times n$ block matrix

$$H = \begin{pmatrix} \nabla_1^{\mathrm{T}} \nabla_1 h_1 & \cdots & \nabla_1^{\mathrm{T}} \nabla_N h_1 \\ \vdots & \ddots & \vdots \\ \nabla_N^{\mathrm{T}} \nabla_1 h_N & \cdots & \nabla_N^{\mathrm{T}} \nabla_N h_N \end{pmatrix}. \tag{13}$$

The value that the matrix H attains when its entries are evaluated at a strategy profile x is denoted $H(x)$. The following result, which extends Proposition 7 in Milchtaich (2012), connects this value with the stability of the strategy profile.

Theorem 1. Let h be an N-player game where the strategy space of each player is a set in a Euclidean space, and y an equilibrium with a neighborhood where the players' payoff functions are twice continuously differentiable. If the players' strategy spaces are convex, then a sufficient condition for y to be stable is that the matrix $H(y)$ is negative definite. The same is true also without the convexity assumption if y is an interior equilibrium. If the strategy spaces are of full dimension, then a necessary condition for an interior equilibrium y to be weakly stable is that $H(y)$ is negative semidefinite.[2]

Proof. By Lemma 1, and with χ_S denoting the characteristic function of a set of players S, expression (3) can be written as

$$\frac{1}{N} \sum_S \sum_i \frac{1}{\binom{N-1}{|S \setminus \{i\}|}} \Big(\chi_S(i)\, h_i(y \mid x_S) - \chi_{S^c}(i)\, h_i(y \mid x_S) \Big). \tag{14}$$

For x tending to y, that is, with $\epsilon_i = x_i - y_i \to 0$ for all i, (14) can be written as

$$\frac{1}{N} \sum_S \sum_i \frac{1}{\binom{N-1}{|S \setminus \{i\}|}} \big(\chi_S(i) - \chi_{S^c}(i) \big) \left(h_i + \sum_{j \in S} \nabla_j h_i \, \epsilon_j + \frac{1}{2} \sum_{j \in S} \sum_{k \in S} \epsilon_k^{\mathrm{T}} \, \nabla_k^{\mathrm{T}} \nabla_j h_i \, \epsilon_j \right) \tag{15}$$
$$+ o(\|\epsilon\|^2),$$

where the payoff functions h_i and their partial derivatives are evaluated at the point y and $\|\epsilon\|$ is the (Euclidean) length of the vector $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_N) = x - y$. For each player i, the

[2] H is said to be negative definite or semidefinite if the symmetric matrix $(1/2)(H + H^{\mathrm{T}})$ has the same property, equivalently, if the latter's eigenvalues are all negative or nonpositive, respectively.

coefficient of h_i in (15), which is

$$\frac{1}{N}\sum_{S}\frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\left(\chi_S(i)-\chi_{S^{\mathrm{c}}}(i)\right)=\frac{1}{N}\sum_{\substack{S\\ i\notin S}}\frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\left[\left(\chi_S(i)-\chi_{S^{\mathrm{c}}}(i)\right)+\left(\chi_{S\cup\{i\}}(i)-\chi_{(S\cup\{i\})^{\mathrm{c}}}(i)\right)\right],$$

is equal to zero, because the condition $i\notin S$ implies that the expression in square brackets is zero. For each i and j, the coefficient of $\nabla_j h_i\,\epsilon_j$ in (15) is

$$\frac{1}{N}\sum_{\substack{S\\ j\in S}}\frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\left(\chi_S(i)-\chi_{S^{\mathrm{c}}}(i)\right),$$

which by a similar argument is zero if $j\neq i$, and is equal to

$$\frac{1}{N}\sum_{\substack{S\\ i\in S}}\frac{1}{\binom{N-1}{|S|-1}}=\frac{1}{N}\sum_{l=1}^{N}\frac{\binom{N-1}{l-1}}{\binom{N-1}{l-1}}=1 \tag{16}$$

if $i=j$. For each i, j and k, the coefficient of $(1/2)\epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_j h_i\,\epsilon_j$ is

$$\frac{1}{N}\sum_{\substack{S\\ j,k\in S}}\frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\left(\chi_S(i)-\chi_{S^{\mathrm{c}}}(i)\right),$$

which again is zero if j and k are both different from i. If $j=k=i$, then, by (16), the coefficient is equal to 1, and if $k=i$ but $j\neq i$ or vice versa, then it is equal to

$$\frac{1}{N}\sum_{\substack{S\\ i,j\in S}}\frac{1}{\binom{N-1}{|S|-1}}=\frac{1}{N}\sum_{l=2}^{N}\frac{\binom{N-2}{l-2}}{\binom{N-1}{l-1}}=\frac{1}{N}\sum_{l=2}^{N}\frac{l-1}{N-1}=\frac{1}{2}.$$

Therefore, (15) reduces to

$$\begin{aligned}\sum_i \nabla_i h_i\,\epsilon_i+\sum_i\left(\frac{1}{4}\sum_j \epsilon_i^{\mathrm{T}}\,\nabla_i^{\mathrm{T}}\nabla_j h_i\,\epsilon_j+\frac{1}{4}\sum_k \epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_i h_i\,\epsilon_i\right)+o(\|\epsilon\|^2)\\ =\sum_i \nabla_i h_i\,\epsilon_i+\frac{1}{2}\epsilon^{\mathrm{T}}H(y)\epsilon+o(\|\epsilon\|^2),\end{aligned} \tag{17}$$

where the equality holds because, at y, the first-order partial derivatives of h_i commute and therefore $\epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_i h_i\,\epsilon_i=\epsilon_k^{\mathrm{T}}\,(\nabla_i^{\mathrm{T}}\nabla_k h_i)^{\mathrm{T}}\epsilon_i=\epsilon_i^{\mathrm{T}}\,\nabla_i^{\mathrm{T}}\nabla_k h_i\,\epsilon_k$.

If the strategy set of a player i is convex, then every convex combination of x_i and y_i is a strategy of i. The one-sided limit $\lim_{\lambda\to 0^+}(1/\lambda)(h_i(y\mid\lambda x_i+(1-\lambda)y_i)-h_i(y))$ exists, is equal to $\nabla_i h_i(y)(x_i-y_i)=\nabla_i h_i\,\epsilon_i$, and is necessarily nonpositive, because y is an equilibrium. The same is true if the strategy set is not necessarily convex but y_i lies in its relative interior. Moreover, in this case, the one-sided limit with $\lambda\to 0^-$ also exists. As this limit is necessarily nonnegative, but it is also equal to $\nabla_i h_i\,\epsilon_i$, the latter must be zero. If $H(y)$ is negative definite, then $\epsilon^{\mathrm{T}}H(y)\epsilon\leq-|\lambda_0|\|\epsilon\|^2$, where λ_0 ($\neq 0$) is the eigenvalue closest to 0 of $(1/2)(H(y)+H(y)^{\mathrm{T}})$. Therefore, (17) is negative for $\epsilon\neq 0$ sufficiently close to 0, which proves that (3) is negative for $x \neq y$ sufficiently close to y, so that y is stable. If $H(y)$ is not negative semidefinite, then $(1/2)(H(y) + H(y)^{\mathrm{T}})$ has an eigenvector v with eigenvalue $\lambda > 0$, so that $v^{\mathrm{T}}H(y)v$ is positive and equal to $\lambda\|v\|^2$. If y is an interior equilibrium (hence, the first term in (17) is zero) and the strategy spaces are of full dimension, this means that there are strategy profiles x arbitrarily close to y for which (3) is positive, so that y is not weakly stable. ∎

Negative definiteness of H is also connected with the *uniqueness* of the equilibrium (Resen 1965). In particular, it follows from the next proposition that an equilibrium is necessarily unique if the players' strategy sets are convex and H is negative definite everywhere.

Proposition 4. Let h be an N-player game where the strategy space of each player is a set in a Euclidean space. If $H(x)$ is negative definite for all strategy profiles x in a convex set X', which has a neighborhood where the payoff functions are twice continuously differentiable, then X' includes at most one equilibrium.

Proof. As shown in the proof of Theorem 1, for every equilibrium $y \in X'$ and strategy profile $x \in X'$ the inequality $\nabla_i h_i(y)(x_i - y_i) \leq 0$ holds for all i. If x is also an equilibrium, then a similar inequality holds with x and y interchanged, so that $(\nabla_i h_i(x) - \nabla_i h_i(y))(x_i - y_i) \geq 0$. Summation over i gives

$$0 \leq \sum_i \left(\int_0^1 \frac{d}{d\lambda} \nabla_i h_i(\lambda x + (1-\lambda)y)\, d\lambda \right)(x_i - y_i)$$
$$= \int_0^1 \sum_{i,j} (x_j - y_j)^{\mathrm{T}} \nabla_j^{\mathrm{T}} \nabla_i h_i(\lambda x + (1-\lambda)y)(x_i - y_i)\, d\lambda.$$

However, the integrand is equal to $(x-y)^{\mathrm{T}}H(\lambda x + (1-\lambda)y)(x-y)$, and therefore $x \neq y$ cannot hold, for otherwise the nonnegativity of the integral would contradict the assumed negative definiteness of H at the points on the line segment connecting x and y. ∎

4.1 Comparison with dynamic stability

The notion of static stability, as defined in this paper, is based on incentives rather than motion. Dynamic stability, by contrast, is based on explicit assumptions about the way that incentives to move translate into actual changes of strategies. For example, if the players' strategy spaces are unidimensional (i.e., $n_i = 1$ for all i), the law of motion may take the form

$$\frac{dx_i}{dt} = d_i h_{i,i}(x_1, x_2, \dots, x_N), \qquad i = 1,2, \dots, N, \tag{18}$$

where $d_i > 0$ for all i and $h_{i,i}$ is a shorthand for the partial derivative $\partial h_i / \partial x_i$. This system of differential equations, where t is the time variable, expresses the assumption that the rate of change of each strategy x_i is proportional to the corresponding marginal payoff. With these dynamics, the condition for asymptotic stability of an interior equilibrium y with a neighborhood where the players' payoff functions are twice continuously differentiable is that, at y, the (Jacobian) matrix

$$\begin{pmatrix} d_1 h_{1,11} & \cdots & d_1 h_{1,1N} \\ \vdots & \ddots & \vdots \\ d_N h_{N,N1} & \cdots & d_N h_{N,NN} \end{pmatrix}$$

(where $h_{i,jk} = \partial^2 h_i / \partial x_k \partial x_j = \partial^2 h_i / \partial x_j \partial x_k$) is stable, that is, all its eigenvalues have negative real parts. The condition is usually required to hold for all positive adjustment speeds $d_1, d_2, \dots, d_N$ (Dixit 1986). This requirement is known as *D-stability* of the matrix obtained by setting $d_1 = d_2 = \cdots = d_N = 1$, which is the matrix H.

D-stability is a weaker condition than negative definiteness: every negative definite matrix is *D*-stable, but not conversely. For example, in the two-player case ($N = 2$), the matrix H is *D*-stable if and only if

$$h_{1,11} \leq 0 \text{ and } h_{2,22} < 0 \text{ or vice versa, and } h_{1,11} h_{2,22} > h_{1,12} h_{2,21} \tag{19}$$

(Hofbauer and Sigmund 1998),[3] but it is negative definite if and only if it satisfies the stronger condition

$$h_{1,11}, h_{2,22} < 0 \text{ and } h_{1,11} h_{2,22} > \frac{1}{4}\left(h_{1,12} + h_{2,21}\right)^2. \tag{20}$$

Moreover, as the following example shows, *D*-stability of $H(y)$ is not a sufficient condition for static stability of an equilibrium y.

Example 1 (continued). The origin $(0,0)$ is an asymptotically stable equilibrium in the two-player game (1), for which

$$H = \begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix},$$

because (19) holds. As shown, the equilibrium is also (statically) stable, and this fact also follows from Theorem 1, because (20) holds. By contrast, in the game (2), where

$$H = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix},$$

the equilibrium $(0,0)$ is not even weakly stable, because one eigenvalue of $(1/2)(H + H^{\mathrm{T}})$ is positive. However, since (19) still holds, the equilibrium is asymptotic stable.

While asymptotic stability with respect to the dynamics (18) is an essentially weaker condition than static stability, the same is not necessarily true for other kinds of dynamic stability. In particular, static stability does not imply asymptotic stability with respect to another natural adjustment process, where the two players alternate in myopically playing a best response to their opponent's strategy. As seen in Figure 1, starting from any other strategy profile, these dynamics quickly bring the players to the origin in the game (2) but take them increasingly farther away from it in (1). Thus, the situation is the opposite of that for static stability, as the equilibrium $(0,0)$ is stable for (1) but not for (2), and it is also

[3] Unlike negative definiteness, for which a number of useful characterizations are known, necessary and sufficient conditions for *D*-stability of $n \times n$ matrices are known only for small n (Impram et al. 2005), and they are reasonably simple only for $n = 2$.

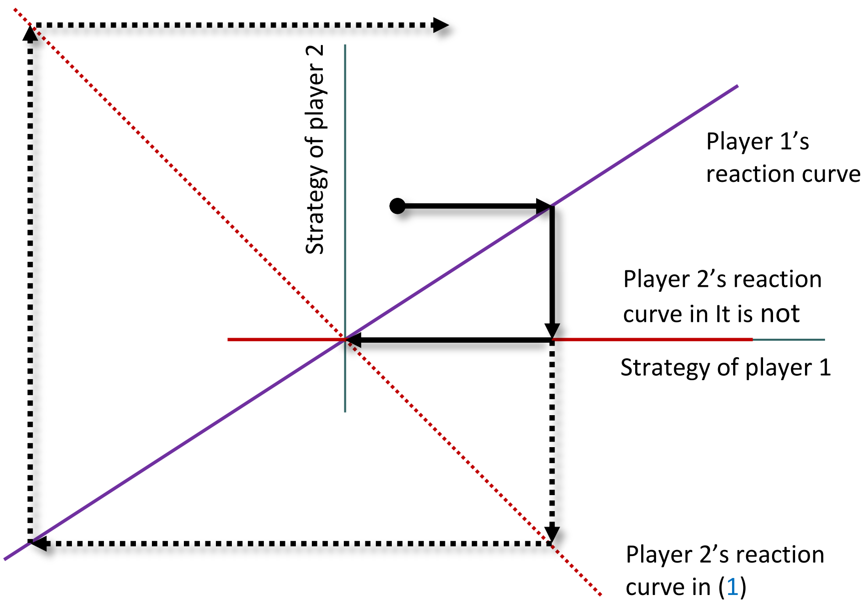

Figure 1. The players' reaction curves in the two games in Example 1. Player 1's reaction curve (upward sloping line) is the same in both games, but those of player 2 (horizontal and downward sloping lines) are different. The arrows show possible trajectories under the alternating-best-responses dynamics, in which player 1 moves first, then player 2, then player 1 again, and so on. For the game given by (2) (solid arrows), the trajectory ends at the equilibrium point $(0,0)$. For the game in (1) (dotted arrows), it spirals away.

different from the situation for the simultaneous and continuous adjustment process (18), for which the equilibrium is asymptotically stable in both games.

These differences between the different kinds of stability can be understood by noting that, if both inequalities in the first part of (19) are strict, then the second part can be written as

$$\left(-\frac{h_{2,21}}{h_{2,22}}\right)\left(-\frac{h_{1,12}}{h_{1,11}}\right) < 1.$$

Thus, asymptotic stability of an interior equilibrium y with respect to the dynamics (18) essentially requires that, at that point, the product of the slope of player 2's reaction curve and the reciprocal of the slope of player 1's curve be less than 1. (The two reaction, or best-response, curves lie in the space where the horizontal and vertical axes correspond to the strategies of player 1 and player 2, respectively, as in Figure 1.) This condition is similar to, but weaker than, the condition for asymptotic stability of the equilibrium with respect to alternating best responses, which is that the *absolute value* of the product be less than 1 (Fudenberg and Tirole 1995). The stronger condition, which means that player 1's reaction curve is steeper than that of player 2, is not implied by (19). The condition is also not implied by, and it does not imply, negative definiteness of H, as demonstrated by the fact that it does not hold for the game in (1) but does hold for (2).

A general lesson that can be learned from the above analysis is that there is no single, general notion of dynamic stability with which static stability can be meaningfully compared. Even for a specific, simple class of games, one kind of dynamic stability may be weaker than static stability while another may be incomparable with it.

An exception to the above general conclusion is provided by the essentially symmetric games (see Section 3.1) with unidimensional strategy spaces. In such games, the matrix $H(y)$ is symmetric at any symmetric strategy profile y where the second-order derivatives exist. A symmetric matrix is negative definite if and only if it is *D*-stable. This means that static stability of a symmetric strategy profile is essentially equivalent to asymptotic stability with respect to the dynamics (18). For example, in the two-player case ($N = 2$), the essential symmetry condition (12) implies that, at any interior symmetric strategy profile,

$$h_{1,11} = h_{2,22} \text{ and } h_{1,12} = h_{2,21}.$$

With these equalities, (19) and (20) are both equivalent to

$$h_{1,11}, h_{2,22} < 0 \text{ and } \left|\frac{h_{2,21}}{h_{2,22}}\right| < 1. \tag{21}$$

At any interior equilibrium, the second-order maximization condition $h_{i,ii} \leq 0$ holds automatically for $i = 1,2$, and the first part of (21) only adds the requirement that the inequalities are strict. As indicated, the inequality in the second part of (21) means that the equilibrium is asymptotically stable with respect to alternating best responses. Thus, for an interior symmetric equilibrium, this kind of (dynamic) stability, asymptotic stability with respect to the continuous dynamics (18), and static stability are all essentially equivalent to one another and to the condition that, at the equilibrium point, the slope of player 2's reaction curve is less than 1 but greater than -1. On the other hand, the last pair of inequalities is stronger than the condition for static stability of an equilibrium strategy in a symmetric game, which consists of the first inequality only (Milchtaich 2017). This difference is another example of the more lenient nature of the stability condition in symmetric games in comparison with corresponding essentially symmetric ones.

5 Potential games

An N-player game is a *potential game* (Monderer and Shapley 1996) if it admits an (*exact*) *potential*, which is a function $P: X \to \mathbb{R}$ (on the set of strategy profiles) such that, whenever a single player i changes his strategy, the resulting change in i's payoff is equal to the change in P. Thus,

$$h_i(y \mid x_i) - h_i(y) = P(y \mid x_i) - P(y), \qquad x_i \in X_i, y \in X.$$

For potential games, static stability and instability of strategy profiles have particularly simple characterizations in terms of the extremum points of the potential.

Theorem 2. A strategy profile y in an N-player game with a potential P is stable, weakly stable or definitely unstable if and only if y is a strict local maximum point, local maximum point or strict local minimum point of P, respectively. A *global* maximum point of P is both globally weakly stable (and if it is a strict global maximum point, globally stable) and an equilibrium.

Proof. The first part of the theorem is an immediate corollary of the fact that, by the definition of P, expression (3) can be written as

$$\frac{1}{N!}\sum_{\pi\in\Pi}\sum_{j=1}^{N}\left(P(y \mid x_{\{\pi(1),\pi(2),\dots,\pi(j)\}}) - P(x \mid y_{\{\pi(j),\pi(j+1),\dots,\pi(N)\}})\right) = P(x) - P(y).$$

The special case where the topology is the trivial one and the definition of P immediately give the second part of the theorem. ■

For symmetric games (Milchtaich 2017), the notion of potential game has essentially the same meaning as for asymmetric games. A symmetric function $F: X^N \to \mathbb{R}$ is a *potential* for a symmetric game g with strategy space X if, for any $N+1$ strategies $x, y, z, \dots, w$,

$$F(x,z,\dots,w) - F(y,z,\dots,w) = g(x,z,\dots,w) - g(y,z,\dots,w).$$

Symmetrization maps potential games to potential games. Indeed, as the proof of the following lemma shows, it essentially also maps potentials to potentials.

Proposition 5. An N-player h, where the set of all strategy profiles is X, is a potential game if and only if this is so for the symmetric game g obtained by symmetrizing h.

Proof. If P is a potential for h, then the symmetric function $F: X^N \to \mathbb{R}$ defined by

$$F(x^1, x^2, \dots, x^N) = \frac{1}{N!}\sum_{\pi\in\Pi} P(x_1^{\pi^{-1}(1)}, x_2^{\pi^{-1}(2)}, \dots, x_N^{\pi^{-1}(N)})$$

is a potential for g. This is because, by (9), for $x^1, x^2, \dots, x^N$ and y in X,

$$\begin{aligned}
&g(x^1,x^2,\dots,x^N) - g(y,x^2,\dots,x^N) = \\
&= \frac{1}{N!}\sum_{\pi\in\Pi}\left(h_{\pi(1)}(x_1^{\pi^{-1}(1)},\dots,x_{\pi(1)}^1,\dots,x_N^{\pi^{-1}(N)}) - h_{\pi(1)}(x_1^{\pi^{-1}(1)},\dots,y_{\pi(1)},\dots,x_N^{\pi^{-1}(N)})\right) \\
&= \frac{1}{N!}\sum_{\pi\in\Pi}\left(P(x_1^{\pi^{-1}(1)},\dots,x_{\pi(1)}^1,\dots,x_N^{\pi^{-1}(N)}) - P(x_1^{\pi^{-1}(1)},\dots,y_{\pi(1)},\dots,x_N^{\pi^{-1}(N)})\right) \\
&= F(x^1,x^2,\dots,x^N) - F(y,x^2,\dots,x^N).
\end{aligned}$$

Conversely, if F is a potential for g, then the function P defined by

$$P(x) = F(x,x,\dots,x)$$

is a potential for h. This is because, for $x, y \in X$ such that $x = y \mid z_i$ for some player i and strategy z_i,

$$\begin{aligned}
P(y \mid z_i) - P(y) &= F(x,x,\dots,x) - F(y,y,\dots,y) \\
&= \sum_{j=1}^{N}(F(x,\underbrace{y,\dots,y}_{j-1\text{ times}},\underbrace{x,\dots,x}_{N-j\text{ times}}) - F(y,\underbrace{y,\dots,y}_{j-1\text{ times}},\underbrace{x,\dots,x}_{N-j\text{ times}})) \\
&= \sum_{j=1}^{N}(g(x,\underbrace{y,\dots,y}_{j-1\text{ times}},\underbrace{x,\dots,x}_{N-j\text{ times}}) - g(y,\underbrace{y,\dots,y}_{j-1\text{ times}},\underbrace{x,\dots,x}_{N-j\text{ times}})) \\
&= \frac{1}{N!}\sum_{j=1}^{N}\sum_{\substack{\pi\in\Pi\\ \pi(1)=i}}\left(h_{\pi(1)}(y_1,\dots,x_{\pi(1)},\dots,y_N) - h_{\pi(1)}(y_1,\dots,y_{\pi(1)},\dots,y_N)\right) \\
&= h_i(y \mid z_i) - h_i(y),
\end{aligned}$$

where the second equality uses the symmetry of the function F and the fourth equality uses the fact that $x_k = y_k$ for $k \neq i$. ∎

6 Multilinear games

Multilinear games are the mixed extensions of finite games. Put differently, they are the multiplayer generalization of bimatrix games. In a multilinear N-player game, the strategy space X_i of each player i is the unit simplex in some Euclidean space $\mathbb{R}^{n_i}$, with $n_i \geq 1$ (and the topology is given by the Euclidean distance), and the payoff function h_i is linear in each of the N arguments. As the following theorem shows, in this class of games stability has a simple, strong meaning.

Theorem 3. For a strategy profile y in a multilinear N-player game the following conditions are equivalent:

(i) y is stable,
(ii) y is locally superior,
(iii) y is a strict equilibrium.

Proof. (i) ⇒ (iii). If y is stable, then for every player i inequality (5) holds for all $x_i \neq y_i$ in some neighborhood of y_i. Therefore, for *every* $x_i \neq y_i$, a similar inequality in which x_i is replaced with $\epsilon x_i + (1-\epsilon)y_i$ holds for sufficiently small $\epsilon > 0$. However, by linearity of h_i in player i's own strategy, that inequality is equivalent to (5), which proves that y is a strict equilibrium.

(iii) ⇒ (ii). Suppose that y is a strict equilibrium, so that (5) holds for all i and $x_i \neq y_i$. For each player i, let Z_i be the collection of all strategies $z_i = (z_i^1, z_i^2, \ldots, z_i^{n_i}) \in X_i$ that satisfy $z_i^j = 0$ for some j with $y_i^j > 0$. This is a compact subset of X_i that does not include y_i, and therefore the expression on the left-hand side of (5) is bounded away from zero for $x_i \in Z_i$. Thus, there is some $\delta_i > 0$ such that

$$h_i(y \mid z_i) - h_i(y) \leq -\delta_i, \qquad z_i \in Z_i. \tag{22}$$

Since Z_i is compact, it follows from (22) that there is a neighborhood of y where for every strategy profile x

$$h_i(x \mid z_i) - h_i(x) \leq -\delta_i/2, \qquad z_i \in Z_i. \tag{23}$$

For every strategy $x_i \neq y_i$, there is a unique $0 < \epsilon_i \leq 1$ (which depends on x_i) such that for some (indeed, a unique) $z_i \in Z_i$

$$x_i = (1-\epsilon_i)y_i + \epsilon_i z_i.$$

By linearly of h_i in the ith coordinate, the last equation and (23) imply that $(1-\epsilon_i)(h_i(x) - h_i(x \mid y_i)) = \epsilon_i(h_i(x \mid z_i) - h_i(x)) < 0$. The conclusion proves that there is a neighborhood of y where (6) holds for all $x \neq y$.

(ii) ⇒ (i). Proposition 1. ∎

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