Introduction to VARs and Structural VARs:
Estimation & Tests Using Stata

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Background: VAR

• **Background:**

• **Structural simultaneous equations**
  – Lack of Fit with the data
  – Lucas Critique (1976)

• **VAR: Vector Auto Regressions**
  – Simple
  – Non Structural
  – All Variables are treated identically
  – Better Fit with the Data
Simple VAR: Sims (1980)

- **Symmetric**
  - Lags of the dependent variables
  - Same Number of Lags

\[
y_{1,t} = \alpha_0 + \alpha_1 y_{1,t-1} + \alpha_2 y_{2,t-1} + \alpha_3 y_{3,t-1} + \alpha_4 y_{1,t-2} + \alpha_2 y_{2,t-2} + \alpha_3 y_{3,t-2} + \ldots + \varepsilon_{1,t}
\]
\[
y_{2,t} = \beta_0 + \beta_1 y_{1,t-1} + \beta_2 y_{2,t-1} + \beta_3 y_{3,t-1} + \beta_4 y_{1,t-2} + \beta_2 y_{2,t-2} + \beta_3 y_{3,t-2} + \ldots + \varepsilon_{2,t}
\]
\[
y_{31,t} = \gamma_0 + \gamma_1 y_{1,t-1} + \gamma_2 y_{2,t-1} + \gamma_3 y_{3,t-1} + \gamma_4 y_{1,t-2} + \gamma_2 y_{2,t-2} + \gamma_3 y_{3,t-2} + \ldots + \varepsilon_{3,t}
\]

\[
E(\varepsilon_{i,t}, \varepsilon_{j,t}) = \begin{cases} 
\sigma_{i,j} & \text{if } t = \tau \\
0 & \text{if } t \neq \tau 
\end{cases}
\]

\[i, j \in \{1, 2, 3\}\]
Simple VAR: Matrix Form

- In Matrix Form:

\[ y_t = \alpha + \Gamma_{t-1}y_{t-1} + \Gamma_{t-2}y_{t-2} + \ldots + \epsilon_t \]

or Simply:

\[ [I - \Gamma(L)]y_t = \Lambda + \epsilon_t \]

- \( y_t \) is a vector of the Dependent Variables
- \( \Gamma_{t-i} \) is a Matrix of Coefficients
- \( \Gamma(L) \) is a Matrix in Lagged Variables
- \( \epsilon_t \) is a Vector of White Noise Errors
- \( \Lambda \) is a Matrix of exogenous variables (constant, …)
Covariance Matrix

\[
\mathbf{e}_t \times \mathbf{e}_t' = \begin{pmatrix}
\varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{1,3} \\
\varepsilon_{2,1} & \varepsilon_{2,2} & \varepsilon_{2,3} \\
\varepsilon_{3,1} & \varepsilon_{3,2} & \varepsilon_{3,3}
\end{pmatrix} \times \begin{pmatrix}
\varepsilon_{1,1} & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{2,1} & \varepsilon_{2,2} & \varepsilon_{2,3} & \cdots & \varepsilon_{3,1} & \varepsilon_{3,2} & \varepsilon_{3,3} & \cdots
\end{pmatrix} =
\begin{pmatrix}
\sigma_{1,1} & 0 & 0 & \cdots & \sigma_{1,2} & 0 & \cdots & \sigma_{1,3} & 0 & \cdots \\
0 & \sigma_{1,1} & 0 & \cdots & 0 & \sigma_{1,2} & \cdots & 0 & \sigma_{1,3} & \cdots \\
0 & 0 & \sigma_{1,1} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Contemporary Variance Matrix

\[
\Omega = \begin{pmatrix}
\sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} \\
\sigma_{2,1} & \sigma_{2,2} & \sigma_{2,3} \\
\sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3}
\end{pmatrix}
\]
Issues Before Estimation

• Stationarity:
  – Constant expected value
  – Constant Variance
  – Constant Covariances

• Granger Exogeneity:
  – Order of variables

• Lag Length
  – Optimal lag length
Testing Stationarity

• We have data on Canada 1966Q1-2002Q1
  – GDP
  – Consumer Price Index (CPI)
  – Household Consumption (consumption)

<table>
<thead>
<tr>
<th>Consumption</th>
<th>CPI</th>
<th>GDP</th>
<th>Descriptor</th>
</tr>
</thead>
<tbody>
<tr>
<td>36.91</td>
<td>18.04</td>
<td>62.53</td>
<td>1966Q1</td>
</tr>
<tr>
<td>37.47</td>
<td>18.24</td>
<td>64.58</td>
<td>1966Q2</td>
</tr>
<tr>
<td>38.46</td>
<td>18.41</td>
<td>65.47</td>
<td>1966Q3</td>
</tr>
</tbody>
</table>
Declare: Time Series

• Define and format: time variable
  – date(var_name,"dmy") or
  – Quarterly(var_name, "yq")
  – format: format var_name %d

• Declare database as time series
  – Menu: statistics → time series → setup & utilities →
    declare dataset to be time series data

```
. generate time2 = quarterly(descriptor, "yq")
. format time2 %tq
. list time2 if _n<=3
```

<table>
<thead>
<tr>
<th>time2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1966q1</td>
</tr>
<tr>
<td>2. 1966q2</td>
</tr>
<tr>
<td>3. 1966q3</td>
</tr>
</tbody>
</table>
Declare Time Series

**tsset - Declare dataset to be time-series data**

- **Time variable:**  time2
- **Panel ID variable:** (optional)
- **Clear all settings**

**Display format for the time variable:**
- None specified
- Weekly
- Quarterly
- Yearly
- %tw

- Daily
- Monthly
- Half-yearly
- Generic
Transformations

• Convenience
  – taking logs: gen var_name=log(var_name)
    
    gen log_gdp=ln(gdp_sa)
    gen log_cons=log(household_cons)
    gen log_cpi=log(cpi)

• Differences are in percentage
Plots

- Menu: Graphics → easy graphs → line graph
- Follow the wizard…
Log of household consumption

Date

01 Jan 1960
09 Sep 1973
19 May 1987
25 Jan 2001
Stationarity

• Data don’t look stationary
• Formal test required

• Common tests (Greene, 636-646):
  – Dickey Fuller:
    • H0: Variable has a unit root
  – Philips Peron
    • H0: Variable has a unit root
  – Dickey Fuller – GLS
    • H0: Variable has a unit root
Testing:

• Menu: Statistics → Time Series → Tests

• Choose a test and follow the menu
  – Augmented Dickey Fuller
  – DF-GLS for a Unit root
  – Phillips-Peron unit root
Choosing a test

- Summaries, tables, & tests
- Linear regression and related
- Binary outcomes
- Ordinal outcomes
- Count outcomes
- Categorical outcomes
- Selection models
- Generalized linear models (GLM)
- Nonparametric analysis

- Time series
  - Multivariate time series
  - Cross-sectional time series

- Survival analysis
- Observational/Epi. analysis

- Survey data analysis

- ANOVA/MANOVA
- Cluster analysis
- Other multivariate analysis

- Setup & utilities
  - ARIMA models
  - ARCH/GARCH
  - Prais-Winsten regression
  - Regression with Newey-West std. errors
  - Smoothers/univariate forecasters

- Tests

- Graphs

- ytitle(log of gdp)

- 0.02, but with gaps

- Augmented Dickey-Fuller unit-root test
- Perform DF-GLS test for a unit root
- Phillips-Perron unit roots test
Running a Test

• Augmented Dickey-Fuller Test
  – 6 lags
  – Including Trend
Result

- Cannot reject the null at 5%

### Augmented Dickey-Fuller test for unit root

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>1% Critical Value</th>
<th>5% Critical Value</th>
<th>10% Critical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z(t)$</td>
<td>-1.074</td>
<td>-4.026</td>
<td>-3.445</td>
</tr>
</tbody>
</table>

* MacKinnon approximate p-value for $Z(t) = 0.9330$
Create First Differences

• Cannot Reject Unit root: Data is I(1)
• Create First Differences of the data:

```
. gen y=log_gdp-log_gdp[._n-1]
(1 missing value generated)

. gen inflation=log_cpi-log_cpi[._n-1]
(1 missing value generated)

. gen dcons=log_cons-L.log_cons
(1 missing value generated)
```
Check the new graphs

Log of differenced GDP
Log differenced household consumption
Is it stationary now? (PP test)

```
. pperron y, lags(6)
```

Phillips-Perron test for unit root

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>1% Critical Value</th>
<th>5% Critical Value</th>
<th>10% Critical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z(t)</td>
<td>-5.403</td>
<td>-3.495</td>
<td>-2.887</td>
</tr>
</tbody>
</table>

Number of obs = 146
Newey-West lags = 6

The differenced data seems to be stationary
Granger Causality

• X Does not Granger Cause Y if:

\[ E(y_t \mid y_{t-1}, y_{t-2}, \ldots, x_{t-1}, x_{t-2}, \ldots) = E(y_t \mid y_{t-1}, y_{t-2}, \ldots) \]

• Test (Greene, p.592):
  – Regress Y on lags of X and Y
  – Regress Y on lags of Y
  – Test if the restricted model is significantly outperformed by the non restricted model
  – Either \( \chi^2 \) or F test
Granger Test

• Run simple VAR between the variables of interest

• Menu: Statistics → multivariate time series → Basic Vector Autoregression Model

• Choose
  – Variables
  – Lag Length
Granger Test: Running VAR

The image shows a software interface for running a Vector Autoregression (VAR) model. The interface includes options for including lags, specifying the horizon for Impulse Response Functions (IRFs), and selecting whether to graph the results. The dependent variables are set to "dcon inflation." The lags are set to include up to 4 lags, and the horizon for IRFs, IRFs, and FEVDs is set to 12 periods.
Testing in Stata

- Statistics ➔ multivariate time series ➔ var diagnostics and tests ➔ Granger causality test
Granger Test

• Choose variables
Granger Test: Results

- We can reject that Inflation Granger Cause Household Consumption
- We cannot reject that Household Consumption Granger Cause Inflation

<table>
<thead>
<tr>
<th>Equation</th>
<th>Excluded</th>
<th>chi2</th>
<th>df</th>
<th>Prob &gt; chi2</th>
</tr>
</thead>
<tbody>
<tr>
<td>dcons</td>
<td>inflation</td>
<td>2.4424</td>
<td>4</td>
<td>0.6550</td>
</tr>
<tr>
<td>dcons</td>
<td>ALL</td>
<td>2.4424</td>
<td>4</td>
<td>0.6550</td>
</tr>
<tr>
<td>inflation</td>
<td>dcons</td>
<td>22.7235</td>
<td>4</td>
<td>0.0001</td>
</tr>
<tr>
<td>inflation</td>
<td>ALL</td>
<td>22.7235</td>
<td>4</td>
<td>0.0001</td>
</tr>
</tbody>
</table>
Optimal Lag Length

• Sometimes, we have theory to guide us
• Often, we do not
• Three common tests (Greene, 589):
  – Likelihood Ratio Test (LR)
  – Akaike Information Criterion
  – Bayesian (Schwartz) Information Criterion
Likelihood Ratio (LR) test

General to simple approach: Run VAR with p lags. Use the LR test. If the test rejects the null, then stop. Otherwise run p-1 lags and compare with p-2...

\[ \lambda = T \left( \ln |W_{res}| - \ln |W_{unres}| \right) \rightarrow \chi^2 \left( M^2 \right) \]

- \( W_{res} \) – restricted covariance matrix
- \( W_{unres} \) – unrestricted covariance matrix
- \( M \) – Number of equations
Information Criteria

• Two information Criteria: Akaike (AIC) and Bayesian (BIC). Find the information criteria for lag length 1 to \( p \). Choose the lag length that minimizes the information criteria that you chose.

\[
\lambda = \ln(|W|) + \frac{(pM^2 + M)IC(T)}{T}
\]

- \( W \) – The covariance matrix, \( p \) – number of lags,
- \( T \) – number of observations, \( M \) – number of equations,
- \( IC(T) = 2 \) for AIC, \( T \) for BIC,
Tests in Stata

- Menu: Statistics → multivariate time series → var diagnostics and tests → Lag-Order Selection statistics
Running test

- Choose Variables
- Choose maximum lags
### Lag Length: Results

<table>
<thead>
<tr>
<th>Lag</th>
<th>LL</th>
<th>LR</th>
<th>df</th>
<th>p</th>
<th>FPE</th>
<th>AIC</th>
<th>HQIC</th>
<th>SBIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>389.636</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>8.66e-07</td>
<td>-5.44557</td>
<td>-5.42019</td>
<td>-5.38312</td>
</tr>
<tr>
<td>1</td>
<td>1482.751</td>
<td>2186.231</td>
<td>9</td>
<td>0.000</td>
<td>2.02e-13</td>
<td>-20.7148</td>
<td>-20.6133</td>
<td>-20.465</td>
</tr>
<tr>
<td>2</td>
<td>1533.067</td>
<td>100.631</td>
<td>9</td>
<td>0.000</td>
<td>1.13e-13</td>
<td>-21.2967</td>
<td>-21.1191*</td>
<td>-20.8596*</td>
</tr>
<tr>
<td>4</td>
<td>1551.512</td>
<td>18.768</td>
<td>9</td>
<td>0.027</td>
<td>1.13e-13</td>
<td>-21.303</td>
<td>-20.9731</td>
<td>-20.4912</td>
</tr>
<tr>
<td>5</td>
<td>1562.889</td>
<td>22.755</td>
<td>9</td>
<td>0.007</td>
<td>1.09e-13</td>
<td>-21.3365</td>
<td>-20.9305</td>
<td>-20.3373</td>
</tr>
<tr>
<td>6</td>
<td>1575.778</td>
<td>25.778*</td>
<td>9</td>
<td>0.002</td>
<td>1.03e-13*</td>
<td>-21.3912*</td>
<td>-20.9091</td>
<td>-20.2047</td>
</tr>
</tbody>
</table>

We go with the LR and AIC and say 6 (why not?)
Run Simple VAR

- We run a simple VAR (not structural, no assumptions on order of variables) between Household Consumption, Inflation and GDP

- To do so:
  - Menu: Statistics → multivariate time series → Basic Vector Autoregression Model
Simple VAR

- Choose
  - Variables
  - Lag Length

- Choose how to plot the response functions:
  - Irf (simply uses the covariance matrix, minimum order)
  - Orf (orthogonalized the Covariance matrix to set order)
  - FEVD: Variance Decomposition Tables (In a graph form)
Simple VAR

[Image of a software interface for fitting a simple VAR and graphing IRFs]
## Results: Table of Coefficients

<table>
<thead>
<tr>
<th>Model Lag Order Selection Statistics</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>FPE</td>
<td>AIC</td>
<td>HQIC</td>
<td>SBIC</td>
<td>LL</td>
<td>Det(Sigma_ml)</td>
<td></td>
</tr>
</tbody>
</table>

| Coef. | Std. Err. | z  | P>|z| | [95% Conf. Interval] |
|-------|-----------|----|------|----------------------|
| y     |           |    |      |                      |
| L1    | .5240169  | .094267 | 5.56 | 0.000 | .3392569 | .7087769 |
| L2    | -.1716709 | .1008584 | -1.70 | 0.089 | -.3693498 | .026008 |
| L3    | .1312994  | .0988067 | 1.33 | 0.184 | -.0623581 | .3249569 |
| L4    | -.0964825 | .098346 | -0.98 | 0.327 | -.2892371 | .0962721 |
| L5    | -.0531829 | .0987135 | -0.54 | 0.590 | -.2466577 | .1402919 |
| L6    | .2467331  | .0968582 | 2.55 | 0.011 | .0568944 | .4365717 |
| dcons |           |    |      |                      |
|       |           |    |      |                      |
| —more—|           |    |      |                      |
Impulse Response Function

Graphs by irf name, impulse variable, and response variable.
## Simple VAR: Variance Decomposition Table

<table>
<thead>
<tr>
<th>step</th>
<th>&lt;9&gt; fevd</th>
<th>&lt;9&gt; Lower</th>
<th>&lt;9&gt; Upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>.997849</td>
<td>.982572</td>
<td>1.01313</td>
</tr>
<tr>
<td>2</td>
<td>.821666</td>
<td>.707423</td>
<td>.935909</td>
</tr>
<tr>
<td>3</td>
<td>.758885</td>
<td>.620901</td>
<td>.896868</td>
</tr>
<tr>
<td>4</td>
<td>.729669</td>
<td>.579145</td>
<td>.880193</td>
</tr>
<tr>
<td>5</td>
<td>.714909</td>
<td>.555406</td>
<td>.874413</td>
</tr>
<tr>
<td>6</td>
<td>.653683</td>
<td>.482896</td>
<td>.82447</td>
</tr>
<tr>
<td>7</td>
<td>.594682</td>
<td>.41942</td>
<td>.769945</td>
</tr>
<tr>
<td>8</td>
<td>.543658</td>
<td>.364066</td>
<td>.723249</td>
</tr>
<tr>
<td>9</td>
<td>.501872</td>
<td>.316881</td>
<td>.686863</td>
</tr>
<tr>
<td>10</td>
<td>.463959</td>
<td>.277265</td>
<td>.650654</td>
</tr>
<tr>
<td>11</td>
<td>.44034</td>
<td>.252745</td>
<td>.627934</td>
</tr>
</tbody>
</table>

- 5% lower and upper bounds reported
- 1) irfname = varbasic, impulse = y, and response = y
- 2) irfname = varbasic, impulse = y, and response = dcons
- 3) irfname = varbasic, impulse = y, and response = inflation
- 4) irfname = varbasic, impulse = dcons, and response = y
- 5) irfname = varbasic, impulse = dcons, and response = dcons
- 6) irfname = varbasic, impulse = dcons, and response = inflation
- 7) irfname = varbasic, impulse = inflation, and response = y
- 8) irfname = varbasic, impulse = inflation, and response = dcons
- 9) irfname = varbasic, impulse = inflation, and response = inflation
Generating After Estimation

- generate after estimation:
  - Choose:
    - Menu: Statistics → Multivariate time series → IRF & Variance Decomposition Analysis
    - Choose the table or impulse response function that you need
To get the results

• If you want to use some of the results:
  • Coefficients
  • Number of observations
  • Etc…
  – Stata keeps them under the ereturn command

  – To get them type e(variable_name)
  – To see all the variables that you can choose from:
    • ereturn list
Examples

```
. ereturn list

scalars:
    e(r2_3) = .760610955474849
    e(ll_3) = 567.8811540726108
    e(df_m3) = 18
    e(chi2_1) = 230.9532693071359
    e(chi2_2) = 161.690542764445
    e(chi2_3) = 447.9993849956065
    e(ll) = 1557.959742978141
    e(detsig_ml) = 5.07253323216e-14

. matrix list e(Sigma)

symmetric e(Sigma)[3,3]

    y   dcons   inflation
    y .00005872
    dcons .00002995 .00006185
    inflation 9.504e-07 1.555e-06 .00001859
```
More than simple VAR

• More than a simple VAR:
  – Adding Exogenous Variables
  – Constraining blocks of variables to equal zero

• Use Menu: Statistics ➔ multivariate time series ➔ Vector Autoregression Model

• Generating Impulse Responses:
  • Menu: Statistics ➔ Multivariate time series ➔ IRF & Variance Decomposition Analysis
More than simple VAR

- Adding constraints on the A or B matrix
  - A: y Matrix, B: errors matrix
  - Short and long run constraints
- skip lags
- Menu: Statistics ➔ multivariate time series ➔ Structural Autoregression Model
- Stata runs the VAR with the restrictions
- Caveat 1: Too many constraints can lead to failures in the convergence process
- Caveat 2: You need enough constraints to allow identification.
Structural VAR

Defining a matrix of constraints: the ‘.’ imply a free parameter

Using the constraints: Forcing the values in the constrained matrix
### Structural VAR: Results

<table>
<thead>
<tr>
<th>Equation</th>
<th>Obs</th>
<th>Parms</th>
<th>RMSE</th>
<th>R-sq</th>
<th>chi2</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>141</td>
<td>19</td>
<td>0.008238</td>
<td>0.6209</td>
<td>230.9533</td>
<td>0.0000</td>
</tr>
<tr>
<td>inflation</td>
<td>141</td>
<td>19</td>
<td>0.004635</td>
<td>0.7606</td>
<td>447.9994</td>
<td>0.0000</td>
</tr>
<tr>
<td>dcons</td>
<td>141</td>
<td>19</td>
<td>0.008455</td>
<td>0.5342</td>
<td>161.6905</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

### VAR Model lag order selection statistics

<table>
<thead>
<tr>
<th></th>
<th>FPE</th>
<th>AIC</th>
<th>HQIC</th>
<th>SBIC</th>
<th>LL</th>
<th>Det(Sigma_ml)</th>
</tr>
</thead>
</table>

|                    | Coef. | Std. Err. | z     | P>|z| | [95% Conf. Interval] |
|--------------------|-------|-----------|-------|------|----------------------|
| a_2 1  _cons       | 22.33818 | 270.0692 | 0.08 | 0.934 | -506.9877 - 551.664 |
| a 3 1  _cons       | 148.7274 | 41.47564 | 3.59 | 0.000 | 67.43667 - 230.0182 |
| a 1 2  _cons       | 231.9367 | 13.81163 | 16.79 | 0.000 | 204.8664 - 259.007 |
| a 2 2  _cons       | -10.76834 | 19.54312 | -0.55 | 0.582 | -49.07215 - 27.53546 |
Structural VAR: Results

Graphs by irf name, impulse variable, and response variable
Structural VARs

• Structural VAR: VAR that is the result of a structural model

• Goal: Obtaining the Structural parameters out of the Estimated Reduced Form

• Required: Number of Constraints
Model: Inflation and GDP

- Assume we have a simple model of the form:

\[ y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 \pi_{t-1} + \varepsilon_t \]

\[ \pi_t = \beta_0 + \beta_1 y_t + \beta_2 y_t + \beta_3 \pi_{t-1} + \nu_t \]

\[ y_t - GDP \]

\[ \pi_t - inflation \]

\[ \nu_t, \varepsilon_t - White noise, independent random variables \]
We can write it:

\[ y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 \pi_{t-1} + \varepsilon_t \]

\[ \pi_t - \beta_1 y_t = \beta_0 + \beta_2 y_t + \beta_3 \pi_{t-1} + \nu_t \]
In Matrix Form:

\[
\begin{pmatrix}
1 & 0 \\
-\beta_1 & 1
\end{pmatrix}
\begin{pmatrix}
y_t \\
p_t
\end{pmatrix}
= \begin{pmatrix}
\alpha_{0t} \\
\beta_0
\end{pmatrix}
+ \begin{pmatrix}
y_{t-1} \\
p_{t-1}
\end{pmatrix}
+ \begin{pmatrix}
\epsilon_t \\
\nu_t
\end{pmatrix}
\]

OR:

\[
\begin{pmatrix}
y_t \\
p_t
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
-\beta_1 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
\alpha_{0t} \\
\beta_0
\end{pmatrix}
+ \begin{pmatrix}
1 & 0 \\
-\beta_1 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
y_{t-1} \\
p_{t-1}
\end{pmatrix}
+ \begin{pmatrix}
1 & 0 \\
-\beta_1 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
\epsilon_t \\
\nu_t
\end{pmatrix}
\]
Inverting the Matrix gives

\[
\begin{pmatrix}
1 & 0 \\
-\beta_1 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
1 & 0 \\
\beta_1 & 1
\end{pmatrix}
\]

So we can substitute this in the equations:
We find:

\[ y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 \pi_{t-1} + \varepsilon_t \]
\[ \pi_t = (\beta_1 \alpha_0 + \beta_0) + (\beta_1 \alpha_1 + \beta_2) y_{t-1} + (\beta_1 \alpha_2 + \beta_3) \pi_{t-1} + (\beta_1 \varepsilon_t + \nu_t) \]

So we can write in VAR form:

\[ y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 \pi_{t-1} + \varepsilon_t \]
\[ \pi_t = \theta_0 + \theta_1 y_{t-1} + \theta_2 \pi_{t-1} + \eta_t \]
Almost there

• After estimating the VAR we can find:

\[
\begin{align*}
(\beta_1 \alpha_0 + \beta_0) &= \theta_0 \\
(\beta_1 \alpha_1 + \beta_2) &= \theta_1 \\
(\beta_1 \alpha_2 + \beta_3) &= \theta_2
\end{align*}
\]

So we have three equations and four unknowns…
Hakuna Matata

• We also have the covariance matrix:

\[
\begin{pmatrix}
\sigma_{\varepsilon,\varepsilon} & \sigma_{\varepsilon,\eta} \\
\sigma_{\varepsilon,\eta} & \sigma_{\eta,\eta}
\end{pmatrix}
= 
\begin{pmatrix}
\sigma_{\varepsilon,\varepsilon} & \beta_1\sigma_{\varepsilon,\varepsilon} \\
\beta_1\sigma_{\varepsilon,\varepsilon} & (\beta_1\sigma_{\varepsilon,\varepsilon} + \nu_t)^2
\end{pmatrix}
\]

• So we have a fourth equation:

\[
\beta_1\sigma_{\varepsilon,\varepsilon} = \sigma_{\varepsilon,\eta}
\]
Run the VAR

• Note that because we assume that the “real” covariance matrix has the triangular form:

\[
\begin{pmatrix}
\sigma_{\varepsilon,\varepsilon} & 0 \\
\beta_1\sigma_{\varepsilon,\varepsilon} & \sigma_{\varepsilon,\varepsilon}
\end{pmatrix}
\]

• We can use the OIRF that Stata gives us (Cholesky factorization) to watch the Structural impulse functions.
Run the VAR (1 lag)
Study the Impulse Responses

Graphs by irfname, impulse variable, and response variable
Get the coefficients

|               | Coef.  | Std. Err. |    z  | P>|z| | [95% Conf. Interval] |
|---------------|--------|-----------|------|-----|---------------------|
| y             |        |           |      |     |                     |
| y             |        |           |      |     |                     |
| L1            | 0.622347 | 0.070297 | 8.85 | 0.000 | 0.4845675, 0.756666 |
| inflation     |        |           |      |     |                     |
| L1            | 0.1421922 | 0.0999996 | 1.42 | 0.155 | -0.0538034, 0.338178 |
| _cons         | 0.0057446  | 0.0015036 | 3.82 | 0.000 | 0.0027976, 0.008684 |
| inflation     |        |           |      |     |                     |
| y             |        |           |      |     |                     |
| L1            | 0.1950256 | 0.040598 | 4.80 | 0.000 | 0.1154549, 0.275562 |
| inflation     |        |           |      |     |                     |
| L1            | 0.6247529 | 0.0577519 | 10.82 | 0.000 | 0.5115613, 0.738945 |
| _cons         | 0.0005972  | 0.0008684 | 0.69 | 0.492 | -0.0011047, 0.002292 |
Get the Errors matrix

```
. matrix list e(Sigma)

symmetric e(Sigma)[2,2]

\sigma_{\varepsilon,\varepsilon} y inflation
y .00008145
inflation 7.018e-06 .00002716
\beta_1\sigma_{\varepsilon,\varepsilon}
```
We find:

\[ \beta_1 = \frac{\text{cov}(\varepsilon, \eta)}{\sigma_{\varepsilon, \varepsilon}} = \frac{0.000007018}{0.00008145} = 0.086 \]

\[ \beta_0 = \theta_0 - \beta_1 \alpha_0 = 0.0005972 - 0.086 \times 0.0574 = -0.0043 \]

\[ \beta_2 = \theta_1 - \beta_1 \alpha_1 = 0.195 - 0.086 \times 0.622 = 0.142 \]

\[ \beta_3 = \theta_2 - \beta_1 \alpha_2 = 0.625 - 0.086 \times 0.142 = 0.614 \]
Conclusion

- Enough restrictions
- Exact Identification
- Possible to deduce the Structural Parameters
To test a restricted Model

- Run a non restricted model
- Test the null by using the LR test on the difference between the restricted and unrestricted model

\[
\lambda = T(\ln|W_{\text{res}}| - \ln|W_{\text{unres}}|) \rightarrow \chi^2(M)
\]

- $W_{\text{res}}$ – restricted covariance matrix
- $W_{\text{unres}}$ – unrestricted covariance matrix
- $M$ – Number of restrictions
Caveat

• With the data we used, it is likely that the variables are cointegrated (consumption and GDP)
• One should (theoretically) check for that option