Static Stability in Games
Part II: Asymmetric Games

Igal Milchtaich[*]

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Static stability in strategic games differs from dynamic stability in only considering the players' incentives to change their strategies. It does not rely on any assumptions about the players' reactions to these incentives and it is thus not linked with any particular dynamics. This paper introduces a general notion of static stability of strategy profiles that is applicable to any N-player strategic game. It examines several important classes of games, with strategy spaces or payoff functions that have special structures, where this general notion takes a simple, concrete form. The paper also explores the relations between static stability and specific kinds of dynamic stability, and connects static stability in general, asymmetric games with the related, but essentially weaker, notion of static stability of strategies in symmetric games. *JEL Classification:* C72.

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1 The framework

A system is at an equilibrium state if there is no (net) force pushing it towards a different state. In game theory, where forces may be equated with incentives, this idea is embodied by the Nash equilibrium solution concept, which requires that there is no incentive for any player to change his strategy unilaterally. If, moreover, any unilateral move would actually harm the mover, the equilibrium is said to be strict. Stability differs from equilibrium in also considering the forces acting at states that are (usually, only slightly) different from the one under consideration. Roughly speaking, it requires that these forces push the system in the direction of that state. More precisely, this description concerns *static stability*, as it does not involve a law of motion that specifies how forces translate into actual movement of the system. For example, a ball at the bottom of a pit is stable but one at the top of a hill is not. In both cases, the net force acting on the ball vanishes, but any displacement would result in a non-zero force, which is directed towards the equilibrium point in the first case and away from it in the second case. This description is static rather than kinetic. It does not involve motion, and therefore does not invoke Newton's second law.

In game theory, static stability of a strategy profile y can analogously be defined in terms of the players' incentives to move towards y when they start at a different strategy profile x. In particular, for x that differs from y only in the strategy of a single player i, a unilateral change of strategy from x_i to y_i must make player i better off. The challenge is to extend this requirement to x that differs from y in $k \geq 2$ coordinates, so that going from x to y

[*] Department of Economics, Bar-Ilan University, Ramat Gan 5290002, Israel
igal.milchtaich@biu.ac.il http://faculty.biu.ac.il/~milchti

requires (at least) k unilateral moves, which can be performed in $k!$ different orders. The extension is enabled by a particular view of the players' payoffs, more specifically, of the change in the payoff of a player who performs a unilateral move. The view is that a positive or negative payoff difference quantifies the player's willingness or reluctance to move (with reluctance identified with the willingness to make the opposite move) or the degree to which the move is readily made. This interpretation provides a conceptual framework within which different players' payoffs can be compared, and contrasts with the often taken view that payoffs are incomparable and, in particular, that each player's payoff function is determined (at most) up to arbitrary increasing affine transformations. The alternative view expressed above entails that only shifts by additive constants and scaling of all payoff functions by a common factor are inconsequential. The significance of this difference is that the overall willingness or reluctance to move can be quantified by the average over all $k!$ paths from x to y of the sum of the k individual payoff increments along each path. The strategy profile y is *globally stable*, *globally weakly stable* or *globally definitely unstable* if this average is positive, nonnegative or negative, respectively, for all $x \neq y$. Put differently, the expression corresponding to the reverse paths, from y to x, is required to be negative, nonpositive or positive, respectively. In the first case, y is necessarily a strict equilibrium, and in the second case, it is an equilibrium. However, as the following example shows, these necessary conditions are not sufficient for global stability or weak stability.

Example 1. *Games in the plane.* The strategy sets of player 1 and player 2 are the real line $\mathbb{R}$. Their payoff functions are

$$h_1(x_1, x_2) = -x_1^2 + 3x_1x_2 \text{ and } h_2(x_1, x_2) = -\frac{1}{2}x_2^2 - x_1x_2. \tag{1}$$

It is not difficult to see that the origin is the unique equilibrium, and it is moreover strict. A path from $(0,0)$ to any other strategy profile (x_1, x_2) goes through either $(x_1, 0)$ or $(0, x_2)$. In the first case, where player 1 is the first to move, the sum of the movers' payoff increments is $-x_1^2 - x_2^2/2 - x_1x_2$, and when player 2 move first, it is $-x_2^2/2 - x_1^2 + 3x_1x_2$. The average of the two expressions is $-x_1^2 + x_1x_2 - x_2^2/2 = -(x_1 - x_2/2)^2 - x_2^2/4$, which is negative for all $(x_1, x_2) \neq (0,0)$. This proves that the equilibrium in this game is globally stable. By contrast, in the game obtained by dropping the second term in h_2, where the payoff functions are

$$h_1(x_1, x_2) = -x_1^2 + 3x_1x_2 \text{ and } h_2(x_1, x_2) = -\frac{1}{2}x_2^2, \tag{2}$$

the corresponding average is $-x_1^2 + 3x_1x_2/2 - x_2^2/2$. This expression is positive for any $(x_1, x_2) \neq (0,0)$ that is a multiple of (2,3), which implies that the strict equilibrium $(0,0)$ is not even globally weakly stable.

Stability becomes a local concept when the requirement described above is restricted to strategy profiles x that are close to y. The restriction is meaningful when the strategy set X_i of each player i is a topological space. The product topology on the set $X = \prod_i X_i$ of all strategy profiles then gives a meaning to a *neighborhood* of a strategy profile x: it is any set whose interior includes x. In an N-player game with such strategy spaces, where the payoff function of player i is $h_i: X \to \mathbb{R}$, consider for any two strategy profiles x and y and a permutation π of $(1,2,\ldots,N)$ the path from y to x in which the players change their strategies in the order specified by π. Thus, player $\pi(1)$ moves first, from $y_{\pi(1)}$ to $x_{\pi(1)}$ (which may or may not be the same strategy), then player $\pi(2)$ moves, and so on. Summation of the movers' changes of payoff and averaging over the set Π of all permutations gives the expression

$$\frac{1}{N!}\sum_{\pi\in\Pi}\sum_{j=1}^{N}\Big(h_{\pi(j)}(y \mid x_{\{\pi(1),\pi(2),\ldots,\pi(j)\}}) - h_{\pi(j)}(x \mid y_{\{\pi(j),\pi(j+1),\ldots,\pi(N)\}})\Big), \quad (3)$$

where $y \mid x_S$ denotes the strategy profile where the players in and outside the set S play according to the strategy profiles x and y, respectively (and similarly with x and y interchanged). This expression quantifies the overall incentive to move from y to x. The incentive to move in the opposite direction, from x to y, is given by the negative of (3).

Definition 1. A strategy profile y in an N-player game is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where (3) is negative, nonpositive or positive, respectively, for all $x \neq y$.

In principal, for these notions to be well defined, the topologies on the players' strategy sets need to be specified. In practice, the topologies can often be inferred from the context, as there are unique natural ones. However, regardless of the latter, an important special case of the definition involves the *trivial topology* on X, where the only neighborhood of any strategy is the entire strategy set. It is not difficult to see that stability, weak stability or definite instability with respect to this topology implies the same condition for any other topology, and it coincides with the *global* version of the property, as defined above.

The somewhat unwieldy expression (3) can be put into a simpler form, which also suggests an alternative interpretation of the inequality defining stability. As the next lemma shows, this inequality roughly means that, when players only play according to x or according to y, those doing the former fare worse on average. Specifically, for strategy profiles x and y in an N-player game, define

$$I(x,y) = \sum_S \frac{1}{\binom{N}{|S|}} \bar{h}_S(y \mid x_S) = \sum_S \frac{1}{\binom{N}{|S|}} \bar{h}_{S^\mathrm{C}}(x \mid y_S) = \sum_{j=1}^{N}\left[\frac{1}{\binom{N}{j}} \sum_{\substack{S\\|S|=j}} \bar{h}_S(y \mid x_S)\right], \quad (4)$$

where, for a set of players $S \subseteq \{1,2,\ldots,N\}$, $|S|$ is the number of players in S and $\bar{h}_S = (1/|S|)\sum_{i\in S} h_i$ is their average payoff, which is defined as 0 if $S = \emptyset$. The second equality in (4) is obtained by replacing the summation variable S with the complementary set S^C. $I(x,y)$ may be termed the *total payoff of x players when playing against y players*. Note that the expression in square brackets is the average of $\bar{h}_S(y \mid x_S)$ over all sets S of size j.

Lemma 1. Expression (3) is equal to $I(x,y) - I(y,x)$.

Proof. Each of the payoffs in (3) has the form $h_i(y \mid x_S)$ or $h_i(x \mid y_S)$, with $i \in S$. Specifically, i is given by the equation $i = \pi(j)$ and S is given either by the equation

$S = \{\pi(1), \pi(2), \dots, \pi(j)\}$ or by $S = \{\pi(j), \pi(j+1), \dots, \pi(N)\}$. In both cases, for every pair (S, i) with $i \in S$ there are precisely $(|S|-1)!\,(N-|S|)!$ pairs (π, j) satisfying the two equations (as j is uniquely determined by $|S|$). Therefore, (3) is equal to

$$\sum_{S \neq \emptyset} \sum_{i \in S} \frac{(|S|-1)!\,(N-|S|)!}{N!} \big(h_i(\, y \mid x_S\,) - h_i(\, x \mid y_S\,)\big) = I(x, y) - I(y, x).$$

■

A strategy profile y that is stable but not globally stable is not necessarily an equilibrium. However, it is still a "local strict equilibrium" in the sense that for every player i and all $x_i \neq y_i$ in some neighborhood of y_i

$$h_i(\, y \mid x_i\,) - h_i(y) < 0, \tag{5}$$

where $y \mid x_i$ denotes the strategy profile that differs from y only in that player i uses strategy x_i. This conclusion, which follows from the definition of stability by examining the special case of a strategy profile that differs from y in only one coordinate, may also be interpreted as the requirement that, when the players move one by one to y from any nearby strategy x, the last mover gains from his move. This requirement is weaker than stability, which considers all the steps from x to y rather than only the last step. By contrast, the requirement that the *first* mover gains, at least on average, turns out to be a stronger condition than stability. This condition is formalized by the next definition and is analyzed by the proposition following it.

Definition 2. A strategy profile y in an N-player game is *locally superior* if it has a neighborhood where for all $x \neq y$

$$\frac{1}{N} \sum_{i=1}^{N} \big(h_i(x) - h_i(\, x \mid y_i\,)\big) < 0. \tag{6}$$

Proposition 1. Every locally superior strategy profile is stable, but not conversely.

Proof. A locally superior strategy y has a rectangular neighborhood where inequality (6) holds for all $x \neq y$. In that neighborhood, a similar inequality holds with the strategy profile x replaced by $y \mid x_S$, for any set of players S such that $y \mid x_S$ is different from y. Division by $\binom{N-1}{|S|-1}$ and summation over all nonempty sets S give

$$\begin{aligned}
0 &> \sum_{S \neq \emptyset} \frac{1}{\binom{N-1}{|S|-1}} \frac{1}{N} \sum_{i \in S} \Big(h_i(\, y \mid x_S\,) - h_i(\, y \mid x_{S \setminus \{i\}}\,)\Big) \\
&= \sum_{S \neq \emptyset} \frac{1}{\binom{N}{|S|}|S|} \sum_{i \in S} h_i(\, y \mid x_S\,) - \sum_i \sum_{\substack{S \\ i \in S}} \frac{1}{\binom{N-1}{|S \setminus \{i\}|} N} h_i(\, y \mid x_{S \setminus \{i\}}\,) \\
&= \sum_S \frac{1}{\binom{N}{|S|}} \bar{h}_S(\, y \mid x_S\,) - \sum_i \sum_{\substack{S \\ i \notin S}} \frac{1}{\binom{N-1}{|S|} N} h_i(\, y \mid x_S\,) \\
&= I(x, y) - \sum_S \frac{1}{\binom{N}{|S|}} \bar{h}_{S^{\mathrm{c}}}(\, y \mid x_S\,) = I(x, y) - I(y, x).
\end{aligned}$$

By Lemma 1, this proves that y is stable.

To see that (even global) stability is not a sufficient condition for local superiority, note that $y = (0,0)$ is not locally superior in the game (1), because the left-hand side of (6) is equal to the expression $-x_1^2/2 + x_1x_2 - x_2^2/4$, which is positive if $x_1 = x_2 \neq 0$. ■

2 Comparison with stability in symmetric games

A symmetric N-player game is specified by the players' common strategy space X, which in the present context is assumed to be a topological space, and a single payoff function $g: X^N \to \mathbb{R}$ that is invariant to permutations of its second through Nth arguments. If one player uses strategy x and the other players use $y, z, \dots, w$, in any order, the first player's payoff is $g(x, y, z, \dots, w)$. A strategy y is an *equilibrium strategy*, with the equilibrium payoff $g(y, y, \dots, y)$, if

$$g(y, y, \dots, y) \geq g(x, y, \dots, y), \qquad x \in X. \tag{7}$$

Static stability for symmetric games (Milchtaich 2017) differs from that for general N-player games, referred to below as *asymmetric* games, in that the concept is applied to strategies rather than strategy profiles. A strategy y is considered stable if, when the players move one-by-one from y to any nearby strategy x, their moves harm them on average.

Definition 3. A strategy y in a symmetric N-player game with payoff function g is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where, for every strategy $x \neq y$, the inequality

$$\frac{1}{N}\sum_{j=1}^{N}\Big(g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})\Big) < 0, \tag{8}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds. If the corresponding inequality holds for *all* strategies $x \neq y$, then y is *globally* stable, weakly stable or definitely unstable, respectively.

This definition of stability generalizes a number of more special concepts of statics stability that are applicable only to specific classes of symmetric games, such as evolutionarily stable strategy, or ESS (Milchtaich 2017). Conceptually, it is similar to Definition 1, and in a sense, the latter can be derived from it. The link between the two definitions is provided by the concept of *symmetrization* of an asymmetric game. An N-player game h, where the strategy space X_i of each player i is a topological space, is symmetrized by letting the players switch roles, with all possible permutations considered. This yields a symmetric N-player game where the players' common strategy space is the space $X = X_1 \times X_2 \times \cdots \times X_N$ of all strategy profiles in the asymmetric game, with the product topology. For a player in g, a strategy $x = (x_1, x_2, \dots, x_N) \in X$ specifies the strategy x_i the player will use when called to assume the role of any player i in h, and the payoff is defined as his average payoff in the $N!$ possible assignments of players in g to roles in h. Formally, for any N strategies in X, $x^1 = (x_1^1, x_2^1, \dots, x_N^1), x^2 = (x_1^2, x_2^2, \dots, x_N^2), \dots, x^N = (x_1^N, x_2^N, \dots, x_N^N)$,

$$g(x^1, x^2, \dots, x^N) = \frac{1}{N!} \sum_{\pi \in \Pi} h_{\pi(1)}(x_1^{\pi^{-1}(1)}, x_2^{\pi^{-1}(2)}, \dots, x_N^{\pi^{-1}(N)}), \tag{9}$$

where Π is the set of all permutation of $(1,2, \dots, N)$ and h_i denotes the payoff function of player i in h. (Note that superscripts in this formula index players' strategies in the symmetric game g while subscripts refer to roles in the asymmetric one h. For $\pi \in \Pi$, player i in g is assigned to role $\pi(i)$ in h.)

Proposition 2. A strategy profile y in an asymmetric N-player game h is stable, weakly stable or definitely unstable if and only if it has the same property as a strategy in the game g obtained by symmetrizing h. A strategy profile is an equilibrium in h if and only if it is an equilibrium strategy in g. In this case, the equilibrium payoff in g is equal to the players' average equilibrium payoff in h.

Proof. To prove the first part of the theorem, it suffices to shows that the sum in (8) is equal to expression (3). By (9), that sum can be written as

$$\frac{1}{N!} \sum_{\pi \in \Pi} \sum_{j=1}^{N} \left(h_{\pi(1)}(y \mid x_{\{\pi(1),\pi(2),\dots,\pi(j)\}}) - h_{\pi(1)}(x \mid y_{\{\pi(1),\pi(j+1),\pi(j+n),\dots,\pi(N)\}}) \right). \tag{10}$$

The payoffs in (10) have the form $h_i(y \mid x_S)$ or $h_i(x \mid y_S)$, with $i \in S$. Specifically, i is given by the equation $i = \pi(1)$ and S is given either by the equation $S = \{\pi(1), \pi(2), \dots, \pi(j)\}$ or by $S = \{\pi(1), \pi(j+1), \pi(j+n), \dots, \pi(N)\}$. In both cases, for every pair (S, i) with $i \in S$ there are precisely $(|S|-1)!\,(N-|S|)!$ pairs (π, j) satisfying the two equations (as j is uniquely determined by $|S|$). Therefore, (10) is equal to

$$\sum_{S \neq \emptyset} \sum_{i \in S} \frac{(|S|-1)!\,(N-|S|)!}{N!} \left(h_i(y \mid x_S) - h_i(x \mid y_S) \right).$$

By Lemma 1, this expression is equal to (3).

The strategy profile y is an equilibrium strategy in g if and only if the expression obtained by setting $x^2 = x^3 = \dots = x^N = y$ in the right-hand side of (9) is maximized by choosing $x^1 = y$. That expression can be simplified by partitioning the set of permutations Π into N equal-size parts, each of cardinality $(N-1)!$, according to the value i of $\pi(1)$. Thus, the expression under consideration is equal to

$$\frac{1}{N} \sum_{i=1}^{N} h_i(y_1, y_2, \dots, x_i^1, \dots, y_N).$$

Clearly, choosing $x^1 = y$ maximizes this sum if and only if, for each i, the ith term is maximized by choosing $x_i^1 = y_i$. The latter is also the condition for y to be an equilibrium in h. If it holds, then the maximum (obtained by setting $x_i^1 = y_i$ in each of the terms) is the players' average equilibrium payoff in h. ■

Another notion of static stability in symmetric games is *local superiority* (or strong uninvadability; Bomze 1991). Its definition differs from that of stability in that inequality (8) is replaced with

$$g(x, x, \ldots, x) - g(y, x, \ldots, x) < 0. \quad (11)$$

Thus, a change of strategy from x to y is required to benefit the *first* player who makes this move. It is easy to see that a strategy profile y in an asymmetric game h is locally superior if and only if y is locally superior as a strategy in the symmetric game g obtained by symmetrizing h. Indeed, the left-hand sides of (6) and (11) are equal.

In a symmetric game g, a strategy y that is stable or even globally stable is not always an equilibrium strategy of even a "local equilibrium strategy". That is, the inequality in (7) may not hold for x arbitrary close to y. For example, in the symmetric two-player game where the strategy space is the real line and $g(x, y) = x^2 - 3xy$, the origin 0 is globally stable but it is not a local equilibrium strategy. This contrasts with the situation for asymmetric games, where a stable strategy profile is always a local strict equilibrium. This difference suggests that, in some sense, the first kind of stability is weaker than the second kind.

In some classes of symmetric games, a stable strategy *is* automatically an equilibrium strategy. For example, this is so for symmetric $n \times n$ games, where a strategy is stable if and only if it is an ESS (Milchtaich 2017). However, even in this case, the stability condition is in a sense weaker than the corresponding one for asymmetric games, as an ESS y is not necessarily a pure strategy and therefore the (symmetric) equilibrium (y, y) specified by it is not necessarily strict. This contrasts with the situation for asymmetric $m \times n$, or *bimatrix games*, as Theorem 3 in Section 5 shows that a strategy profile in a bimatrix game is stable if and only if it is a strict (hence, pure) equilibrium.

By the last fact and Proposition 2, the stable strategies in the game g obtained by symmetrizing a bimatrix game h are the strict equilibria in h. This conclusion is similar, and closely related, to the well-known fact that a strategy profile y in g is an ESS if and only if it is a strict equilibrium in h (Selten 1980). The similarity reflects (indeed, it proves) the fact that in a game obtained by symmetrizing a bimatrix game, a strategy is stable if and only if it is an ESS. Thus, these symmetric games are similar in this respect to a symmetric $n \times n$ games (although they are generally *not* $n \times n$ games, for any n).

2.1 Essentially symmetric games

A direct comparison between the concepts of stability of a strategy in a symmetric game and stability of a strategy profile in an asymmetric game is provided by the essentially symmetric games. An asymmetric N-player game h is *essentially symmetric* if the players share a common strategy space and for every strategy profile $(x_1, x_2, \ldots, x_N)$ and permutation π of $(1,2, \ldots, N)$

$$h_i(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(N)}) = h_{\pi(i)}(x_1, x_2, \ldots, x_N), \qquad i = 1,2, \ldots, N. \quad (12)$$

Thus, if the players' strategies are shuffled, such that each player i takes the strategy of some other player $\pi(i)$, the latter's old payoff becomes player i's new payoff. In other words, the rules of the game ignore the players' identities and are therefore completely specified by the payoff function of any single player, and in particular by h_1. The latter may be viewed as the payoff function in a symmetric game. In fact, for fixed strategy space and number of players N, the mapping $h \mapsto h_1$ is a one-to-one correspondence between the set of essentially symmetric games and the set of symmetric games. It may thus seem that there is little difference between the two concepts. And, indeed, essentially symmetric games are usually referred to simply as symmetric games (von Neumann and Morgenstern 1953). However, there is in fact a substantive, non-technical difference between describing a particular situation as a symmetric game and describing it as an essentially symmetric one, with each alternative corresponding to a different interpretation of the situation. This fact is well recognized in the biological game theory literature, where essential symmetry is often referred to by other names such as uncorrelated asymmetry (Maynard Smith and Parker 1976; the correlation referred to here is that between the players' traits and their payoff functions) and inessential asymmetry (Eshel 2005). A symmetric pairwise contest with identical contestants, such as two equal-size males seeking to obtain a newly vacated territory, is best modeled as a symmetric game such as Chicken, or the Hawk–Dove game. Precedence or other perceivable asymmetries between the contestants, which do not by themselves change the payoffs (i.e., the stakes or the opponents' fighting abilities), make the contest an essentially symmetric one, and, in reality, may significantly affect the contestants' behavior (Maynard Smith 1982, Riechert 1998).

The differences between a symmetric game and the corresponding essentially symmetric one are reflected by the differences between the corresponding notions of stability: stability of a strategy in the first case and stability of a strategy profile in the second case. The second notion is more general, in that it is applicable also to asymmetric strategy profiles, where not all players are using the same strategy. However, even in the case of a symmetric strategy profile, in which all players use the same strategy y, and even if y is an equilibrium strategy in the symmetric game (i.e., it satisfies (7)), stability of y in the symmetric game and stability of the symmetric equilibrium $(y, y, \dots, y)$ in the essentially symmetric game are not the same thing. In fact, the second requirement is stronger.[1] As the proof of the following proposition clearly shows, the reason is that this requirement takes into consideration a larger set of alternatives than the first one. An alternative to a strategy y is another (nearby) strategy x, to which all the players switch. The alternatives to a symmetric strategy profile include (nearby) strategy profiles that are not symmetric, which means that only some of the players may move to x while the others stick with y or move to other strategies.

Proposition 3. If a symmetric strategy profile $\vec{y} = (y, y, \dots, y)$ in an essentially symmetric N-player game h is stable, then strategy y is stable in the corresponding symmetric game g $(= h_1)$. However, the converse is false even if $\vec{y}$ is an equilibrium and $N = 2$. The strategy profile $\vec{y}$ is an equilibrium in h if and only if y is an equilibrium strategy in g.

Proof. To prove the first assertion, consider another symmetric strategy profile $\vec{x} = (x, x, \dots, x)$, a player i and a set of players S with $i \in S$. Let π be a permutation that maps 1 to i (that is, $\pi(1) = i$) and maps $2, 3, \dots, |S|$ to the other elements of S (if any). By (12),

[1] For bimatrix games, a related difference holds for the index and degree of the symmetric equilibrium, which may depend on whether it is viewed as an equilibrium in the essentially symmetric bimatrix game or in the corresponding symmetric $n \times n$ one (Demichelis and Germano 2000).

$$h_i(\vec{y} \mid \vec{x}_S) = h_1(\underbrace{x, \dots, x}_{|S| \text{ times}}, \underbrace{y, \dots, y}_{|S^{\mathrm{C}}| \text{ times}}).$$

It follows from this equality and (4) that

$$\begin{aligned} I(\vec{x}, \vec{y}) - I(\vec{y}, \vec{x}) &= \sum_{j=1}^{N} (h_1(\underbrace{x, \dots, x}_{j \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - h_1(\underbrace{y, \dots, y}_{j \text{ times}}, \underbrace{x, \dots, x}_{N-j \text{ times}})) \\ &= \sum_{j=1}^{N} (g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{N-j \text{ times}}, \underbrace{y, \dots, y}_{j-1 \text{ times}})), \end{aligned}$$

where the second equality once again uses (12) (for the second term). As the last sum is easily seen to be equal to that in (8), y is stable in g if and only if $I(\vec{x}, \vec{y}) - I(\vec{y}, \vec{x}) < 0$ for all $x \neq y$ in some neighborhood of y. By Lemma 1, a sufficient condition for this is that $\vec{y}$ is stable.

To see that the last condition is not necessary, consider any symmetric 2×2 game, with payoff matrix A, that has a completely mixed (that is, not pure) ESS y. As indicated, an ESS is a stable strategy. But in the corresponding essentially symmetric bimatrix game (A, A^{T}), the symmetric equilibrium $\vec{y} = (y, y)$ is not stable, because it is not locally strict: any unilateral deviation leaves the deviator's payoff unchanged.

The last assertion in the proposition follows from the fact that, in a symmetric strategy profile in an essentially symmetric game, a player may gain from a unilateral change of strategy if and only if player 1 would gain from making the same move. ∎

3 Games with differentiable payoffs

Consider an N-player game where the strategy space of each player i is a set in a Euclidean space $\mathbb{R}^{n_i}$, with (possibly, player-specific) $n_i \geq 1$, and the topology is given by the Euclidean distance. Strategies are viewed as column vectors. Correspondingly, a strategy profile $x = (x_1, x_2, \dots, x_N)$ is viewed as an n-dimensional column vector, where $n = \sum_i n_i$. It is an *interior* strategy profile if each strategy x_i lies in the interior of player i's strategy space (equivalently, if x lies in the interior of the product space). The gradient with respect to the components of player i's strategy is denoted ∇_i and is viewed as an n_i-dimensional row vector (of first-order differential operators). For any i and j, $\nabla_i^{\mathrm{T}} \nabla_j$ is therefore an $n_i \times n_j$ matrix (of second-order differential operators). In particular, $\nabla_i^{\mathrm{T}} \nabla_i h_i$ is the Hessian matrix of player i's payoff function with respect to the player's own strategy. These Hessian matrices are the diagonal blocks in the $n \times n$ block matrix

$$H = \begin{pmatrix} \nabla_1^{\mathrm{T}} \nabla_1 h_1 & \cdots & \nabla_1^{\mathrm{T}} \nabla_N h_1 \\ \vdots & \ddots & \vdots \\ \nabla_N^{\mathrm{T}} \nabla_1 h_N & \cdots & \nabla_N^{\mathrm{T}} \nabla_N h_N \end{pmatrix}. \tag{13}$$

The value that the matrix H attains when its entries are evaluated at a strategy profile x is denoted $H(x)$. The following result, which is a general version of Proposition 7 in Milchtaich (2012), connects this value with the stability or instability of the strategy profile.

Theorem 1. In an N-player game where the strategy space of each player is a set in a Euclidean space, let y be an interior equilibrium with a neighborhood where the players' payoff functions are twice continuously differentiable. A sufficient condition for y to be stable or definitely unstable is that $H(y)$ is negative definite or positive definite, respectively, and a necessary condition for weak stability is that it is negative semidefinite.[2]

Proof. By Lemma 1, and with χ_S denoting the characteristic function of a set of players S, expression (3) can be written as

$$\frac{1}{N}\sum_S\sum_i \frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\Big(\chi_S(i)\, h_i(y \mid x_S) - \chi_{S^{\mathrm{c}}}(i)\, h_i(y \mid x_S)\Big). \tag{14}$$

For x tending to y, so that $\epsilon_i = x_i - y_i \to 0$ for all i, (14) can be written as

$$\frac{1}{N}\sum_S\sum_i \frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\big(\chi_S(i) - \chi_{S^{\mathrm{c}}}(i)\big)\left(h_i + \sum_{j\in S}\nabla_j h_i\,\epsilon_j + \frac{1}{2}\sum_{j\in S}\sum_{k\in S}\epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_j h_i\,\epsilon_j\right) + o(\|\epsilon\|^2), \tag{15}$$

where the payoff function h_i and its partial derivatives are evaluated at the point y and $\|\epsilon\|$ is the (Euclidean) length of the vector $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_N) = x - y$. For each player i, the coefficient of h_i in (15) is

$$\frac{1}{N}\sum_S \frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\big(\chi_S(i) - \chi_{S^{\mathrm{c}}}(i)\big) = \frac{1}{N}\sum_{\substack{S\\ i\notin S}} \frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\Big[\big(\chi_S(i) - \chi_{S^{\mathrm{c}}}(i)\big) + \big(\chi_{S\cup\{i\}}(i) - \chi_{(S\cup\{i\})^{\mathrm{c}}}(i)\big)\Big] = 0,$$

where the second equality holds because the condition $i \notin S$ implies that the expression in square brackets is zero. For each i and j, the coefficient of $\nabla_j h_i\,\epsilon_j$ in (15) is

$$\frac{1}{N}\sum_{\substack{S\\ j\in S}} \frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\big(\chi_S(i) - \chi_{S^{\mathrm{c}}}(i)\big),$$

which by a similar argument is zero if $j \neq i$. For each i, j and k, the coefficient of $\epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_j h_i\,\epsilon_j$ is

$$\frac{1}{2N}\sum_{\substack{S\\ j,k\in S}} \frac{1}{\binom{N-1}{|S\setminus\{i\}|}}\big(\chi_S(i) - \chi_{S^{\mathrm{c}}}(i)\big), \tag{16}$$

which again is zero if j and k are both different from i. If $j = k = i$, then (16) is equal to

$$\frac{1}{2N}\sum_{\substack{S\\ i\in S}} \frac{1}{\binom{N-1}{|S|-1}} = \frac{1}{2N}\sum_{l=1}^{N} \frac{\binom{N-1}{l-1}}{\binom{N-1}{l-1}} = \frac{1}{2},$$

[2] H is said to be negative definite, negative semidefinite or positive definite if the symmetric matrix $(1/2)(H + H^{\mathrm{T}})$ has the same property, equivalently, if the latter's eigenvalues are all negative, nonpositive or positive, respectively.

and if $k = i$ but $j \neq i$ or vice versa, then it is equal to

$$\frac{1}{2N}\sum_{\substack{S\\ j,k\in S}}\frac{1}{\binom{N-1}{|S|-1}} = \frac{1}{2N}\sum_{l=2}^{N}\frac{\binom{N-2}{l-2}}{\binom{N-1}{l-1}} = \frac{1}{2N}\sum_{l=2}^{N}\frac{l-1}{N-1} = \frac{1}{4}.$$

Finally, $\nabla_i h_i = 0$ for each player i because the interior equilibrium y necessarily satisfies these first-order maximization conditions. Therefore, (15) reduces to

$$\sum_i\left(\frac{1}{4}\sum_j \epsilon_i^{\mathrm{T}}\,\nabla_i^{\mathrm{T}}\nabla_j h_i\,\epsilon_j + \frac{1}{4}\sum_k \epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_i h_i\,\epsilon_i\right) + o(\|\epsilon\|^2) = \frac{1}{2}\epsilon^{\mathrm{T}}H(y)\epsilon + o(\|\epsilon\|^2), \quad (17)$$

where the equality holds because, at y, the first-order partial derivatives of h_i commute and therefore $\epsilon_k^{\mathrm{T}}\,\nabla_k^{\mathrm{T}}\nabla_i h_i\,\epsilon_i = \epsilon_k^{\mathrm{T}}\,(\nabla_i^{\mathrm{T}}\nabla_k h_i)^{\mathrm{T}}\,\epsilon_i = \epsilon_i^{\mathrm{T}}\,\nabla_i^{\mathrm{T}}\nabla_k h_i\,\epsilon_k$. If $H(y)$ is negative definite or positive definite and $\epsilon \neq 0$, then $\epsilon^{\mathrm{T}}H(y)\epsilon$ is negative or positive, respectively, and its absolute value is at least $|\lambda_0|\|\epsilon\|^2$, where $\lambda_0 \neq 0$ is the eigenvalue closest to 0 of $(1/2)(H(y) + H(y)^{\mathrm{T}})$. Therefore, in the first or second case, (17) is positive or negative for ϵ sufficiently close to 0, which proves that (3) is negative or positive, respectively, for x sufficiently close to y. Thus, y is stable or definitely unstable, respectively. If $H(y)$ is not negative semidefinite, then $(1/2)(H(y) + H(y)^{\mathrm{T}})$ has an eigenvector η with eigenvalue $\lambda > 0$, so that $\eta^{\mathrm{T}}H(y)\eta$ is positive and equal to $\lambda\|\eta\|^2$. This means that there are strategy profiles x arbitrarily close to y for which (3) is positive, so that y is not weakly stable. ■

3.1 Comparison with dynamic stability

The notion of static stability, as defined in this paper, is based on incentives rather than motion. Dynamic stability, by contrast, is based on explicit assumptions about the way that incentives to move translate into actual changes of strategies. For example, if the players' strategy spaces are unidimensional (i.e., $n_i = 1$ for all i), the law of motion may take the form

$$\frac{dx_i}{dt} = d_i h_{i,i}(x_1, x_2, \ldots, x_N), \qquad i = 1,2,\ldots,N, \quad (18)$$

where $d_i > 0$ for all i and $h_{i,i}$ is a shorthand for the partial derivative $\partial h_i/\partial x_i$. This system of differential equations, where t is the time variable, expresses the assumption that the rate of change of each strategy x_i is proportional to the corresponding marginal payoff. With these dynamics, the condition for asymptotic stability of an interior equilibrium y with a neighborhood where the players' payoff functions are twice continuously differentiable is that, at y, the (Jacobian) matrix

$$\begin{pmatrix} d_1 h_{1,11} & \cdots & d_1 h_{1,1N} \\ \vdots & \ddots & \vdots \\ d_N h_{N,N1} & \cdots & d_N h_{N,NN} \end{pmatrix}$$

(where $h_{i,jk} = \partial^2 h_i/\partial x_k \partial x_j = \partial^2 h_i/\partial x_j \partial x_k$) is stable, that is, all its eigenvalues have negative real parts. This condition is usually required to hold for all positive adjustment speeds $d_1, d_2, \ldots, d_N$ (Dixit 1986), a requirement known as *D-stability* of the matrix obtained by setting $d_1 = d_2 = \cdots = d_N = 1$, which is the matrix H.

D-stability is a weaker condition than negative definiteness; every negative definite matrix is *D*-stable but not conversely. For example, in the two-player case ($N = 2$), the matrix H is *D*-stable if and only if[3]

$$h_{1,11} \leq 0 \text{ and } h_{2,22} < 0 \text{ or vice versa, and } h_{1,11}h_{2,22} > h_{1,12}h_{2,21} \tag{19}$$

(Hofbauer and Sigmund 1998), but it is negative definite if and only if it satisfies the stronger condition

$$h_{1,11}, h_{2,22} < 0 \text{ and } h_{1,11}h_{2,22} > \frac{1}{4}\left(h_{1,12} + h_{2,21}\right)^2. \tag{20}$$

Moreover, as the following example shows, *D*-stability of $H(y)$ is not a sufficient condition for static stability of an equilibrium y.

Example 1 (continued). The origin $(0,0)$ is an asymptotically stable equilibrium of the two-player game (1), for which

$$H = \begin{pmatrix} -2 & 3 \\ -1 & -1 \end{pmatrix},$$

because (19) holds. As shown, the equilibrium is also (statically) stable, and this fact also follows from Theorem 1, because (20) holds. By contrast, in the game (2), where

$$H = \begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix},$$

the equilibrium $(0,0)$ is not even weakly stable, because one eigenvalue of $(1/2)(H + H^{\mathrm{T}})$ is positive. But the equilibrium is asymptotic stable, because (19) still holds.

While asymptotic stability with respect to the dynamics (18) is an essentially weaker condition than static stability, the same is not necessarily true for other kinds of dynamic stability. In particular, static stability does not imply asymptotic stability with respect to another natural adjustment process, in which the two players alternate in myopically playing a best response to their opponent's strategy. As seen in Figure 1, starting from any other strategy profile, these dynamics quickly bring the players to the origin in the game (2) but take them increasingly farther away from it in (1). Thus, the situation is the opposite of that for static stability, as the equilibrium $(0,0)$ is stable for (1) but not for (2), and it is also different from the situation for the simultaneous and continuous adjustment process (18), for which the equilibrium is asymptotically stable in both games.

These differences between the different kinds of stability can be understood by noting that, if both inequalities in the first part of (19) are strict, the second part can be written as

$$\left(-\frac{h_{2,21}}{h_{2,22}}\right)\left(-\frac{h_{1,12}}{h_{1,11}}\right) < 1.$$

[3] Unlike negative definiteness, for which a number of useful characterizations are known, necessary and sufficient conditions for *D*-stability of $n \times n$ matrices are known only for small n (Impram et al. 2005), and they are reasonably simple only for $n = 2$.

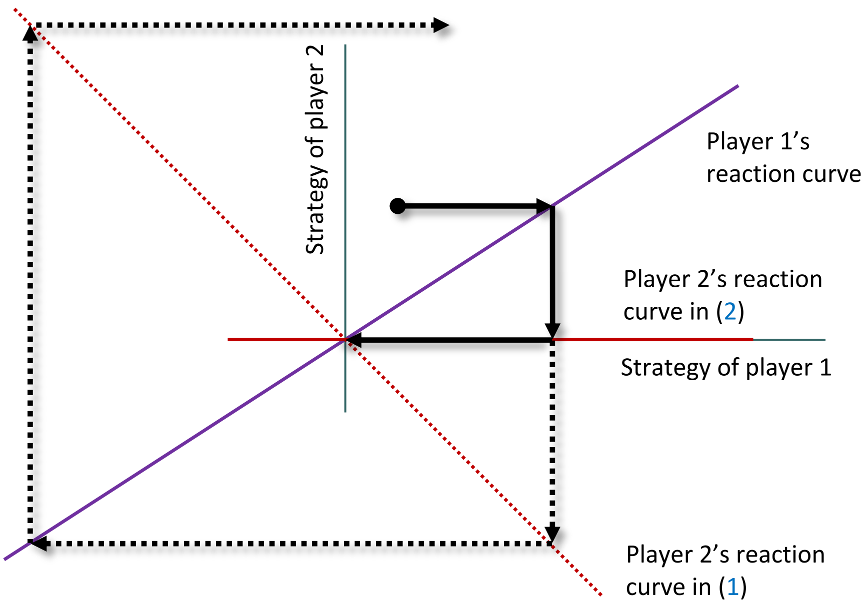

Figure 1. The players' reaction curves in the two games in Example 1. Player 1's reaction curve (upward sloping line) is the same in both games, but those of player 2 (horizontal and downward sloping lines) are different. The arrows show possible trajectories under the alternating-best-responses dynamics, in which player 1 moves first, then player 2, then player 1 again, and so on. For the game given by (2) (solid arrows), the trajectory ends at the equilibrium point $(\mathbf{0},\mathbf{0})$. For the game in (1) (dotted arrows), it spirals away.

Thus, asymptotic stability of an interior equilibrium y with respect to the dynamics (18) essentially requires that, at that point, the product of the slope of player 2's reaction curve and the reciprocal of the slope of player 1's curve be less than 1. This condition is similar to, but weaker than, the condition for asymptotic stability of the equilibrium with respect to alternating best responses, which is that the *absolute value* of the product be less than 1 (Fudenberg and Tirole 1995). The stronger condition, which means that player 1's reaction curve is steeper than that of player 2, is not implied by (19). The condition is also not implied by, and it does not imply, negative definiteness of H, as demonstrated by the fact that it does not hold for the game in (1) but does hold for (2).

A general lesson that can be learned from the above analysis is that there is no single, general notion of dynamic stability with which static stability can be meaningfully compared. Even for a specific, simple class of games, one kind of dynamic stability may be weaker than static stability while another may be incomparable with it.

An exception to the above general conclusion is provided by the essentially symmetric (see Section 2.1) games with unidimensional strategy spaces. In such games, the matrix H is symmetric at any symmetric strategy profile where the second-order derivatives exist. A symmetric matrix is negative definite if and only if it is *D*-stable. This means that static stability of a symmetric strategy profile is essentially equivalent to asymptotic stability with respect to the dynamics (18). For example, in the two-player case ($N = 2$), the essential symmetry condition (12) implies that, at any interior symmetric strategy profile,

$$h_{1,11} = h_{2,22} \text{ and } h_{1,12} = h_{2,21}.$$

With these equalities, (19) and (20) are both equivalent to

$$h_{1,11}, h_{2,22} < 0 \text{ and } \left|\frac{h_{2,21}}{h_{2,22}}\right| < 1. \tag{21}$$

At any interior equilibrium, the second-order maximization condition $h_{i,ii} \leq 0$ holds automatically for $i = 1,2$, and the first part of (21) only adds the requirement that the inequalities are strict. As indicated, the inequality in the second part means that the equilibrium is asymptotically stable with respect to alternating best responses. Thus, for an interior symmetric equilibrium, this kind of (dynamic) stability, asymptotic stability with respect to the continuous dynamics (18), and static stability are all essentially equivalent to one another and to the condition that, at the equilibrium point, the slope of player 2's reaction curve is less than 1 but greater than -1. On the other hand, this pair of inequalities is stronger than the condition for static stability of an equilibrium strategy in a symmetric game, which consists of the first inequality only (Milchtaich 2017). This difference is another example of the more lenient nature of the stability condition in symmetric games in comparison with corresponding essentially symmetric ones.

4 Potential games

An N-player game is a *potential game* (Monderer and Shapley 1996) if it admits an (*exact*) *potential*, which is a function $P: X \to \mathbb{R}$ (on the set of strategy profiles) such that, whenever a single player i changes his strategy, the resulting change in i's payoff is equal to the change in P. Thus,

$$h_i(x \mid y_i) - h_i(x) = P(x \mid y_i) - P(x), \qquad x \in X, y_i \in X_i.$$

For potential games, stability and instability of strategy profiles have particularly simple characterizations in terms of the extremum points of the potential.

Theorem 2. A strategy profile y in an N-player game with a potential P is stable, weakly stable or definitely unstable if and only if y is a strict local maximum point, local maximum point or strict local minimum point of P, respectively. A *global* maximum point of P is both globally weakly stable (and if it is a strict global maximum point, globally stable) and an equilibrium.

Proof. The first part of the theorem is an immediate corollary of the fact that, by the definition of P, expression (3) can be written as

$$\frac{1}{N!}\sum_{\pi\in\Pi}\sum_{j=1}^{N}\left(P(y \mid x_{\{\pi(1),\pi(2),\dots,\pi(j)\}}) - P(x \mid y_{\{\pi(j),\pi(j+1),\dots,\pi(N)\}})\right) = P(x) - P(y).$$

The special case where the topology is the trivial one and the definition of P immediately give the second part of the theorem. ∎

For symmetric games (Milchtaich 2017), the notion of potential game has essentially the same meaning as for asymmetric games. A symmetric function $F: X^N \to \mathbb{R}$ is a *potential* for a symmetric game g with strategy space X if, for any $N+1$ strategies $x, y, z, \dots, w$,

$$F(x, z, \dots, w) - F(y, z, \dots, w) = g(x, z, \dots, w) - g(y, z, \dots, w).$$

Symmetrization maps potential games to potential games. Indeed, as the proof of the following lemma shows, it essentially also maps potentials to potentials.

Proposition 4. An N-player h, where the set of all strategy profiles is X, is a potential game if and only if this is so for the symmetric game g obtained by symmetrizing h.

Proof. If P is a potential for h, then the symmetric function $F: X^N \to \mathbb{R}$ defined by

$$F(x^1, x^2, \dots, x^N) = \frac{1}{N!} \sum_{\pi \in \Pi} P(x_1^{\pi^{-1}(1)}, x_2^{\pi^{-1}(2)}, \dots, x_N^{\pi^{-1}(N)})$$

is a potential for g. This is because, for $x^1, x^2, \dots, x^N$ and y in X,

$$\begin{aligned}
&g(x^1, x^2, \dots, x^N) - g(y, x^2, \dots, x^N) = \\
&= \frac{1}{N!} \sum_{\pi \in \Pi} \left(h_{\pi(1)}(x_1^{\pi^{-1}(1)}, \dots, x_{\pi(1)}^1, \dots, x_N^{\pi^{-1}(N)}) - h_{\pi(1)}(x_1^{\pi^{-1}(1)}, \dots, y_{\pi(1)}, \dots, x_N^{\pi^{-1}(N)}) \right) \\
&= \frac{1}{N!} \sum_{\pi \in \Pi} \left(P(x_1^{\pi^{-1}(1)}, \dots, x_{\pi(1)}^1, \dots, x_N^{\pi^{-1}(N)}) - P(x_1^{\pi^{-1}(1)}, \dots, y_{\pi(1)}, \dots, x_N^{\pi^{-1}(N)}) \right) \\
&= F(x^1, x^2, \dots, x^N) - F(y, x^2, \dots, x^N).
\end{aligned}$$

Conversely, if F is a potential for g, then the function P defined by

$$P(x) = F(x, x, \dots, x)$$

is a potential for h. This is because, for $x, y \in X$ such that $y = x \mid z_i$ for some player i and strategy z_i,

$$\begin{aligned}
P(x \mid z_i) - P(x) &= F(y, y, \dots, y) - F(x, x, \dots, x) \\
&= \sum_{i=1}^{N} (F(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - F(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})) \\
&= \sum_{i=1}^{N} (g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})) \\
&= \frac{1}{N!} \sum_{i=1}^{N} \sum_{\substack{\pi \in \Pi \\ \pi(1) = i}} \left(h_{\pi(1)}(x_1, \dots, y_{\pi(1)}, \dots, x_N) - h_{\pi(1)}(x_1, \dots, x_{\pi(1)}, \dots, x_N) \right) \\
&= h_i(x \mid z_i) - h_i(x),
\end{aligned}$$

where the second equality uses the symmetry of the function F. ∎

5 Multilinear games

Multilinear games are the mixed extensions of finite games. Put differently, they are the multiplayer generalization of bimatrix games. In a multilinear N-player game, the strategy space X_i of each player i is the unit simplex in some Euclidean space $\mathbb{R}^{n_i}$, with $n_i \geq 1$, the topology on X_i is given by the Euclidean distance, and the payoff function h_i is linear in each of the N arguments. As the following theorem shows, in this class of games stability has a simple, strong meaning.

Theorem 3. For a strategy profile y in a multilinear N-player game the following conditions are equivalent:

(i) y is stable,
(ii) y is locally superior,
(iii) y is a strict equilibrium.

Proof. (i) ⇒ (iii). If y is stable, then for every player i inequality (5) holds for all $x_i \neq y_i$ in some neighborhood of y_i. Therefore, for *every* $x_i \neq y_i$, a similar inequality in which x_i is replaced with $\epsilon x_i + (1-\epsilon)y_i$ holds for sufficiently small $\epsilon > 0$. However, by linearity of h_i in player i's own strategy, that inequality is equivalent to (5), which proves that y is a strict equilibrium.

(iii) ⇒ (ii). Suppose that y is a strict equilibrium, so that (5) holds for all i and $x_i \neq y_i$. For each player i, let Z_i be the collection of all strategies $z_i = (z_i^1, z_i^2, \dots, z_i^{n_i}) \in X_i$ that satisfy $z_i^j = 0$ for some j with $y_i^j > 0$. This is a compact subset of X_i that does not include y_i, and therefore the expression on the left-hand side of (5) is bounded away from zero for $x_i \in Z_i$. Thus, there is some $\delta_i > 0$ such that

$$h_i(y \mid z_i) - h_i(y) \leq -\delta_i, \qquad z_i \in Z_i. \tag{22}$$

Since Z_i is compact, it follows from (22) that there is a neighborhood of y where for every strategy profile x

$$h_i(x \mid z_i) - h_i(x) \leq -\delta_i/2, \qquad z_i \in Z_i. \tag{23}$$

For every strategy $x_i \neq y_i$, there is a unique $0 < \epsilon_i \leq 1$ (which depends on x_i) such that for some (indeed, a unique) $z_i \in Z_i$

$$x_i = (1-\epsilon_i)y_i + \epsilon_i z_i.$$

By linearly of h_i in the ith coordinate, the last equation and (23) imply that $(1-\epsilon_i)(h_i(x) - h_i(x \mid y_i)) = \epsilon_i(h_i(x \mid z_i) - h_i(x)) < 0$. This conclusion proves that there is a neighborhood of y where (6) holds for all $x \neq y$.

(ii) ⇒ (i). Proposition 1. ■

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