Games and Economic Behavior 82 (2013) 132–142

Contents lists available at SciVerse ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Spectrum value for coalitional games ☆

Mikel Álvarez-Mozos [a], Ziv Hellman [b,*], Eyal Winter [c,d]

[a] Department of Statistics and Operations Research, University of Santiago de Compostela, Spain
[b] Department of Statistics and Operations Research, School of Mathematical Sciences, Tel Aviv University, Israel
[c] Department of Economics and the Center for the Study of Rationality, Hebrew University of Jerusalem, Israel
[d] Department of Economics, University of Leicester, Leicester, United Kingdom

ARTICLE INFO

Article history:
Received 17 August 2012
Available online 8 July 2013

JEL classification:
C71
D46
D72

Keywords:
Coalitional games
Shapley value
Restricted cooperation
Political spectrum

ABSTRACT

Assuming a 'spectrum' or ordering of the players of a coalitional game, as in a political spectrum in a parliamentary situation, we consider a variation of the Shapley value in which coalitions may only be formed if they are connected with respect to the spectrum. This results in a naturally asymmetric power index in which positioning along the spectrum is critical. We present both a characterization of this value by means of properties and combinatoric formulae for calculating it. In simple majority games, the greatest power accrues to 'moderate' players who are located neither at the extremes of the spectrum nor in its center. In supermajority games, power increasingly accrues towards the extremes, and in unanimity games all power is held by the players at the extreme of the spectrum.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

The Shapley value (Shapley, 1953) has for decades been one of the main indices used in the literature for measuring the relative power of players in coalitional game situations. The Shapley–Shubik power index (Shapley and Shubik, 1954), the restriction of the Shapley value to simple games, has in particular found wide application for studying voting situations. A voting situation is characterized by the set of agents participating in it and the subsets of this set that have enough power to pass a bill. These two elements together define a simple game.

A comprehensive overview on simple games and power indices can be found in Felsenthal and Machover (1998). For just two examples of the extent to which the Shapley–Shubik and related power indices have been used to measure the power of the agents in major institutions around the world, see Bilbao et al. (2002), which studies the implications of the enlargement of the European Union, and Alonso-Meijide and Bowles (2005), which studies the distribution of power in the International Monetary Fund.

In practice, however, in many political situations the Shapley–Shubik index and its variants have often failed to capture what one would consider a realistic power measure. We put forward here the claim that this is because many papers on the subject do not take into account the relative ideologies of the players, which is of key importance in political situations.

☆ We wish to thank the referees and the editor for their comments and suggestions which substantially improved a previous version of the document. M. Álvarez-Mozos acknowledges the financial support of the Spanish Ministry of Economy and Competitiveness through Project MTM2011-27731-C03-02. Z. Hellman acknowledges support by the European Research Council under the European Commission's Seventh Framework Programme (FP7/2007–2013)/ERC grant agreement no. 249159 and Israel Science Foundation grants 538/11 and 212/09. E. Winter is grateful to the German–Israeli Foundation for Research for its financial support.

* Corresponding author.
E-mail addresses: ziv.hellman@gmail.com (Z. Hellman), eyal.winter@huji.ac.il (E. Winter).

0899-8256/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.geb.2013.06.011

Consider for example a parliamentary situation in which there are n players, with the same number of votes, and a simple majority of them is required to form a government. A straightforward application of the Shapley value grants each player $1/n$, using symmetry considerations. In real-life parliaments, however, it is intuitively clear to all observers that not all members have equal power. It is highly unusual to see, for example an extreme right party joining an extreme left party in a coalition without any center parties also included in the coalition to bridge political differences between them.

As the above discussion indicates, part of the problem is that the standard Shapley approach assumes that all possible permutations of the players be used in forming coalitions. That means that even highly unlikely coalitions, such as those formed by an extreme left party joining with an extreme right party while by-passing all the parties in between, including their most natural political allies, must be counted equally along with every other coalition.

Different approaches have been proposed in the literature to study situation in which not all coalitions are feasible or equally likely. In many papers, the problem is tackled by considering some structure on the set of players to describe the way in which players can form coalitions. Coalitional games together with these kind of structures are usually denoted games with restricted cooperation.

One of the most widely-studied model of games with restricted cooperation is the restricted communication model proposed by Myerson (1977). In Myerson's approach, in addition to the game itself one considers an undirected graph that describes communication possibilities between the players. A modification of the Shapley value is then proposed under the assumption that coalitions that are not connected in this graph are split into connected components. In contrast, in our approach we focus on permutations, that is, on the way in which coalitions are formed, instead of imposing restrictions directly on possible coalitions.

We propose here an intuitive way to modify the Shapley value by taking the political spectrum explicitly into account. The incorporation of the ideological positions of the agents for the study of the power distribution of a decision-making body was first introduced by Owen (1972). In that work, agents' political positions are given as points in a high-dimensional Euclidean space, and a probability distribution on the set of all permutations is inferred from them. Then, a modification of the Shapley value is proposed based on two properties, namely that an ordering and its reverse ordering should have the same probability and that the removal of a subset of agents should not affect the probabilities assigned to the relative orderings of the remaining agents.

Shapley (1977) proposed taking into account the political positions of the agents as well, using this to develop an asymmetric generalization of the Shapley value. This modification of the original Shapley value was also considered in Owen and Shapley (1989) to study the optimal ideological position of candidates. More recently, Alonso-Meijide et al. (2011) introduced what they termed the distance index. This value for simple games is another modification of the Shapley value that takes into account the ideological positions of agents. Based on Euclidean distances between agents, a probability distribution is constructed that gives high probability to coalitions whose total distance is relatively small.

Even though it is based on ideas similar to the above-cited works, our approach is much simpler. Firstly, we consider only ordinal positions in a one-dimensional space, without further exogenous specification of distances. Secondly, we assign equal probability to all the permutations that are admissible in our model. This simplicity allows for a characterization of the value by means of a set of properties and eases computation of the value. With regard to the properties of the Owen (1972) value, the value introduced here shares the first of those properties but not the second one.

In this work we assume that there exists a spectrum, from 'left to right' according to which the players are ordered linearly. We then impose the condition that as coalitions are formed *à la* Shapley, they must be connected with respect to the spectrum. Hence, we propose a novel way to generalize the Shapley value to games with restricted cooperation in which the restrictions arise from the position of the agents in a one-dimensional spectrum. This leads to an interesting new value that may shed light on relative power measures in situations in which there is a natural ordering of the players.

Perhaps the paper with the most similar general motivation to ours is Gilles and Owen (1999), in which an exogenously given hierarchy amongst players is assumed (as opposed to the exogenously given spectrum as in our paper). A player in the Gilles and Owen (1999) model may join a coalition only if s/he received permission from one or more 'supervisors'. As in this paper, this has the result of limiting the admissible coalitions that may be formed, thus affecting the value. The value in Gilles and Owen (1999), however, differs from the spectrum value because of the different assumptions regarding which coalitions are admissible. In particular, there is no way to define a clear hierarchy in the model of this paper; for any pair of players i and j, there are admissible coalitions that i can join before j joins (hence i cannot depend on 'permission' from j) and admissible coalitions that j can join before i joins.

Nowak and Radzik (1994) introduce a value called the solidarity value by adding a new postulate to the Shapley properties based on the average marginal contributions of the members of coalitions that may be formed. In its basic interpretation, if a coalition S is formed then the players who contribute more to S than the average marginal contribution members of S contribute to supporting the 'weaker' partners in S. Consideration of the solidarity value, however, does not involve any restriction on the admissible coalitions that can be formed, in contrast with the model used in this paper.

Calvo and Gutierrez (2013) start from similar suppositions to those in Owen (1977), namely that coalitional games are endowed with a coalitional structure, an exogenously given partition of the players. When coalitions are formed, the players interact at two levels: first, bargaining takes place among the unions and then bargaining takes place inside each union. Calvo and Gutierrez (2013) make use of the solidarity value in this model: first, unions play a quotient game among themselves according to the Shapley value, and then the outcome obtained by the union is shared among its members by paying the solidarity value in the internal game. The coalitional value that they define is then applied in political situations to explain why government coalitions are sometimes larger than minimal winning coalitions.

2. Spectra and values

2.1. Definitions

As usual, a *coalitional game* is a pair (N, v) where N is a finite set of players and v, the characteristic function, is a real-valued function over the set of all coalitions, i.e., $v : 2^N = \{S\colon\ S \subseteq N\} \to \mathbb{R}$ with the convention that $v(\emptyset) = 0$. Denote the set of all coalitional games by $\mathcal{G}$. A *value on* $\mathcal{G}$ is a map φ that associates to every coalitional game (N, v) a payoff vector $\varphi(N, v) \in \mathbb{R}^N$. The cardinality of a finite set N will be denoted by $|N|$ or simply by a lower case n. We also denote the set of all permutations over a finite set N by Ω_N.

Definition 1. The *Shapley value*, ϕ^{Sh}, is the value on $\mathcal{G}$ defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$\phi_i^{Sh}(N, v) = \frac{1}{n!} \sum_{\omega \in \Omega_N} \bigl[v\bigl(P_i^\omega \cup \{i\}\bigr) - v\bigl(P_i^\omega\bigr)\bigr],$$

where P_i^ω stands for the set of predecessors of i at $\omega \in \Omega_N$, i.e.,

$$P_i^\omega = \bigl\{j \in N\colon\ \omega(j) < \omega(i)\bigr\}.$$

For every bijection $\sigma : N \to \{1, \ldots, |N|\}$, define a *spectrum on* N to be a strict total ordering $\prec^\sigma$ of N as follows: for every $i, j \in N$, $i \prec^\sigma j$ if and only if $\sigma(i) < \sigma(j)$. The name spectrum is, of course, chosen because of the intention of modeling a political spectrum in a parliament, from left to right. For this reason, if $j \prec^\sigma i$, we will sometimes say that j comes before i in the spectrum ordering, or that j is to the left of i (and i is to the right of j).

A *coalitional game with a spectrum* will be formally denoted by $(N, v, \prec^\sigma)$, where $(N, v) \in \mathcal{G}$ and $\prec^\sigma$ is a spectrum on N. Furthermore, we will usually assume that the relative positions of the players in N are compatible with the given spectrum, i.e., $N = \{1, \ldots, n\}$ and for every $i \in N$, $\sigma(i) = i$. In this case for every $i, j \in N$, $i \prec^\sigma j$ if and only if $i < j$.

Denote the set of all coalitional games with a spectrum by $\mathcal{GS}$. A *value on* $\mathcal{GS}$, φ, is a map that associates to every coalitional game with a spectrum, $(N, v, \prec^\sigma)$, a payoff vector, $\varphi(N, v, \prec^\sigma) \in \mathbb{R}^N$. A coalition $S \subseteq N$ is *connected* (with respect to $\prec^\sigma$) if for all $i, j \in S$, $i \prec^\sigma k \prec^\sigma j$ implies that $k \in S$.[1] Denote the set of all connected coalitions with respect to $\prec^\sigma$ by $C^{\prec^\sigma}(N)$ (or simply by $C(N)$ if $\prec^\sigma$ is obvious from the context). Given a coalition $S \subseteq N$, we can always identify the player at the left end $\min(S)$, who is the player $i \in N$ such that $i \prec^\sigma k$ for all $k \in S \setminus \{i\}$, and similarly, the player at the right end $\max(S)$, who is the player $j \in N$ such that $k \prec^\sigma j$ for all $k \in S \setminus \{j\}$.

Every connected coalition $S \in C(N)$ with $|S| > 1$ contains two distinguished 'extreme' players $i = \min(S)$ and $j = \max(S)$, and all the players in between. We will sometimes write $[i \ldots j]$ to denote the connected coalition of all the players between i and j, inclusive, with the understanding that if $i > j$ then $[i \ldots j] := \emptyset$.

Definition 2. For every coalitional game with a spectrum, $(N, v, \prec^\sigma) \in \mathcal{GS}$, the associated *connected-reduced game*, $(N, v^{\prec^\sigma}) \in \mathcal{G}$ is defined for every $S \subseteq N$ by

$$v^{\prec^\sigma}(S) = \begin{cases} v(S) & \text{if } S \text{ is connected with respect to } \prec^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Associating a connected-reduced coalitional game to every coalitional game with a spectrum allows one to define a value on $\mathcal{GS}$ *à la* Myerson, i.e., defining the value that is obtained by applying the Shapley value to the connected-reduced game. Note, however, that the connected-reduced game of a monotone game may not be monotone; this may lead to negative payoffs in monotone games. We follow a different approach here that does not suffer from this problem.

A permutation $\omega \in \Omega_N$ is *admissible* (with respect to a spectrum $\prec^\sigma$) if P_i^ω is connected (with respect to $\prec^\sigma$) for all $i \in N$. Denote the set of all admissible permutations with respect to $\prec^\sigma$ by $\Omega_{\prec^\sigma}$ (or simply by Ω if $\prec^\sigma$ is obvious by context).

We may interpret this in the spirit of one of the interpretations of the Shapley value. Regard each permutation as an ordered queue of the players, who enter a room one by one according to their number in the queue. Each permutation thus defines a dynamic way of forming a coalition, which grows by one player at a time, thus enabling us to measure the contribution of each player to the coalition formed by the players who preceded him or her in entering the room. A connected admissible permutation, as defined here, relates this queue to the spectrum ordering, in the sense that each player that enters the room must be the player who is immediately to the left or immediately to the right of the set of predecessors. The coalitions that are thus dynamically formed are always connected, as defined above. Therefore, $\Omega_{\prec^\sigma}$

[1] The empty set and singleton coalitions vacuously satisfy the condition of being connected.

represents the set of all possible ways of dynamically forming the grand coalition while ensuring that the sub-coalitions formed in each step are connected.

Furthermore, for every $i \in N$ denote the set of coalitions that can be expressed as sets of predecessors of i at a connected admissible permutation by $C_i(N)$, i.e., $C_i(N) = \{P_i^\omega\colon\ \omega \in \Omega\}$. Note that we can write $C_i(N)$ as a disjoint union of the empty set, the set of connected coalitions 'immediately to the left of' player i, and the set of connected coalitions 'immediately to the right of' player i. Formally, $C_i(N) = L_i(N) \cup R_i(N) \cup \emptyset$ where $L_i(N) = \{[k \ldots i-1]\colon\ 1 \leqslant k < i\}$ and $R_i(N) = \{[i+1 \ldots k]\colon\ i < k \leqslant n\}$.

2.2. The spectrum value

We are now in the position to introduce the formal definition of the spectrum value for games with a spectrum.

Definition 3. The *spectrum value*, ϕ, is the value on $\mathcal{GS}$ defined for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and $i \in N$ by

$$\phi_i\left(N, v, \prec^\sigma\right) := \frac{1}{|\Omega_{\prec^\sigma}|} \sum_{\omega \in \Omega_{\prec^\sigma}} \left[v\left(P_i^\omega \cup i\right) - v\left(P_i^\omega\right)\right].$$

When N, $\prec^\sigma$ and v are clear from context, we will sometimes save space by writing $\phi_i(v)$ or simply ϕ_i instead of $\phi_i(N, v, \prec^\sigma)$.

2.3. Combinatoric analysis of the spectrum value

The spectrum value lies within several families of values studied in the literature. Before proceeding, we look at some combinatorial calculations that are tedious but straightforward to calculate. It is instructive to compare them to their standard analogues.

$$\left|C(N)\right| = \frac{n(n+1)}{2} + 1 \tag{1}$$

(in contrast, the cardinality of the set of all coalitions is 2^n);

$$|\Omega_{\prec^\sigma}| = 2^{n-1} \tag{2}$$

(in contrast, the cardinality of Ω_N is $n!$); finally, when k is the k-th player in the ordering given by $\prec^\sigma$,

$$|\Omega_k| = \binom{n-1}{k-1} \tag{3}$$

where $\binom{n}{k}$ is the standard binomial coefficient. Hence, we may rewrite the spectrum value as follows

$$\phi_i\left(N, v, \prec^\sigma\right) := \frac{1}{2^{n-1}} \sum_{\omega \in \Omega_{\prec^\sigma}} \left[v\left(P_i^\omega \cup i\right) - v\left(P_i^\omega\right)\right]. \tag{4}$$

For an arbitrary set A, denote by $\Delta(A)$ the set of all probability distributions on A. Denote by E_p the expectation operator with respect to $p \in \Delta(M)$. Then, a value φ is a *random order value* (Weber, 1988) if for every $(N, v) \in \mathcal{G}$ and $i \in N$ there is a probability distribution $b \in \Delta(\Omega_N)$ such that,

$$\varphi_i(N, v) = E_b\left(v\left(P_i^\omega \cup \{i\}\right) - v\left(P_i^\omega\right)\right) = \sum_{\omega \in \Omega_N} b(\omega)\left[v\left(P_i^\omega \cup \{i\}\right) - v\left(P_i^\omega\right)\right].$$

It is well known that the standard Shapley value is a random order value for the uniform distribution over all permutations of Ω_N. In analogy, from Definition 3, it is easy to check that the spectrum value is also a random order value for the uniform distribution over the set of connected admissible permutations $\Omega_{\prec^\sigma}$ (which gives zero probability to $\Omega_N \setminus \Omega_{\prec^\sigma}$).

Recall that for any finite set N, 2^N stands for the set of all subsets of N. Then, a value ϕ is known as a *probabilistic value* (Weber, 1988) if for every $(N, v) \in \mathcal{G}$ and $i \in N$ there is a probability distribution $p^i \in \Delta(2^{N \setminus \{i\}})$ such that,

$$\phi_i(N, v) = E_{p^i}\left(v\left(S \cup \{i\}\right) - v(S)\right) = \sum_{S \in 2^{N \setminus \{i\}}} p^i(S)\left[v\left(S \cup \{i\}\right) - v(S)\right].$$

The Shapley value is the probabilistic value defined by the probability distributions $p^i \in \Delta(2^{N \setminus \{i\}})$ such that for every $S \subseteq N \setminus \{i\}$, $p^i(S) = 1/(n\binom{n-1}{|S|})$. The spectrum value can similarly be expressed as a probabilistic value.

Claim 1.

$$\phi_i(N, v, \prec^\sigma) = \frac{1}{2^{n-1}}\left\{\binom{n-1}{i-1} v(\{i\}) + \sum_{S \in L_i(N)} 2^{|S|-1}\binom{n-(|S|+1)}{\min(S)-1}\bigl[v(S \cup \{i\}) - v(S)\bigr] + \sum_{T \in R_i(N)} 2^{|T|-1}\binom{n-(|T|+1)}{i-1}\bigl[v(T \cup \{i\}) - v(T)\bigr]\right\}. \tag{5}$$

Proof. For each player i, the only relevant coalitions for calculating $\phi_i(N, \prec^\sigma, v)$ are the connected coalitions in $C_i(N)$. That is, coalitions immediately to i's left and immediately to i's right, along with the empty set. Hence, we only need to count how many times each coalition is in $L_i(N) \cup R_i(N) \cup \emptyset$ appearing in the expression of Eq. (4).

Let $S \in L_i(N)$. By Eq. (2) the number of different ways in which the coalition S can be formed in a connected manner under an admissible permutation is $2^{|S|-1}$. Next, we only need to count the number of admissible permutations of $N \setminus S$ that start with i. By Eq. (3) it is easy to check that this number is precisely $\binom{n-|S|-1}{\min(S)-1}$.

The proof for the connected coalitions immediately to i's right and the empty set follows similar lines. □

Claim 2.

$$\phi^i(N, v, \prec^\sigma) = \frac{1}{2^{n-1}}\left\{\binom{n-1}{i-1} v(\{i\}) + \sum_{k<i} 2^{i-k-1}\binom{n-1-(i-k)}{k-1}\bigl[v([k \ldots i]) - v([k \ldots i-1])\bigr] + \sum_{k>i} 2^{k-i-1}\binom{n-1-(k-i)}{i-1}\bigl[v([i \ldots k]) - v([i+1 \ldots k])\bigr]\right\}. \tag{6}$$

Proof. This is the previous claim re-written in terms of the identities of the players as determined by the spectrum ordering. □

3. Majority games

Next, we study aspects of the spectrum value by analyzing the payoffs it prescribes in simple voting situations.

3.1. Simple majority games

Consider a decision-making body in which all players have the same weight (1 for instance) and a simple majority of votes is needed to pass a bill. The situation may be described by a simple majority game $(N, v) \in \mathcal{G}$ where $N = \{1, \ldots, n\}$ and $v(S) = 1$ if $|S| > n/2$ and $v(S) = 0$ otherwise. The standard Shapley value in the simple majority game is symmetric for all the players – it grants each player an equal payoff or power $1/n$. However, if we take the political spectrum into account the result is quite different. As usual, suppose that the spectrum is consistent with the relative positions of the agents in N.

Definition 4. A player is $i \in M$ is *left-moderate* with respect to a spectrum if $i \in \{LM_1, LM_2\}$ where

$$LM_1 = \left\lceil \frac{n}{4} \right\rceil \quad \text{and} \quad LM_2 = \left\lfloor \frac{n}{4} \right\rfloor + 1 \quad \text{if } n \text{ is even},$$
$$LM_1 = \left\lceil \frac{n+1}{4} \right\rceil \quad \text{and} \quad LM_2 = \left\lfloor \frac{n+1}{4} \right\rfloor + 1 \quad \text{if } n \text{ is odd}$$

and is *right-moderate* with respect to a spectrum if $i \in \{RM_1, RM_2\}$ where

$$RM_1 = \left\lceil \frac{3}{4}n \right\rceil \quad \text{and} \quad RM_2 = \left\lfloor \frac{3}{4}n \right\rfloor + 1 \quad \text{if } n \text{ is even},$$
$$RM_1 = \left\lceil \frac{3}{4}(n+1) \right\rceil \quad \text{and} \quad RM_2 = \left\lfloor \frac{3}{4}(n+1) \right\rfloor + 1 \quad \text{if } n \text{ is odd}.$$

Observe that if $\frac{n+1}{2}$ is odd then $LM_1 = LM_2$ and $RM_1 = RM_2$. Note also that moderate players are located virtually half-way between the center and one of the most extreme players (when n is large enough).

Claim 3. *In a simple majority game with a spectrum LM_1, LM_2, RM_1, and RM_2 are the players earning the highest payoff. Furthermore, ϕ^i (as a function of i) is*

- *strictly increasing in the interval* $[1, LM_1]$,
- *strictly decreasing in the interval* $[LM_2, \lceil n/2 \rceil]$,
- *strictly increasing in the interval* $[\lfloor n/2 \rfloor + 1, RM_1]$,
- *strictly decreasing in the interval* $[RM_2, n]$.

Proof. If n is even, the vector of spectrum payoffs looks like

$$\frac{1}{2^{n-1}}\left(\binom{\frac{n}{2}-1}{0}, \binom{\frac{n}{2}-1}{1}, \ldots, \binom{\frac{n}{2}-1}{\frac{n}{2}-1}, \binom{\frac{n}{2}-1}{\frac{n}{2}-1}, \ldots, \binom{\frac{n}{2}-1}{1}, \binom{\frac{n}{2}-1}{0}\right)$$

or in other words like two copies of the corresponding row of Pascal's triangle concatenated one after another.

If n is odd, the vector of spectrum payoffs looks like

$$\frac{1}{2^{n-1}}\left(\binom{\frac{n-1}{2}}{0}, \binom{\frac{n-1}{2}}{1}, \ldots, \binom{\frac{n-1}{2}}{\frac{n-1}{2}-1}, 2\binom{\frac{n-1}{2}}{\frac{n-1}{2}}, \binom{\frac{n-1}{2}}{\frac{n-1}{2}-1}, \ldots, \binom{\frac{n-1}{2}}{1}, \binom{\frac{n-1}{2}}{0}\right)$$

which almost (apart from the middle term) looks like two copies of the corresponding row of Pascal's triangle concatenated one after another.

Given the well-known fact that the maximum of each row of Pascal's triangle is attained at its central position (or central positions, depending on whether the number of elements is odd or even), the claim follows immediately. □

3.2. Supermajority games

Let $1/2 < \alpha \leqslant 1$. A coalitional game (N, v) is an α-*supermajority game* if each player has one vote and a quota of $\lceil \alpha n \rceil$ is needed to pass a bill, i.e., if for every $S \subseteq N$, $v(S) = 1$ if $|S| \geqslant \lceil \alpha n \rceil$ and $v(S) = 0$ otherwise.

Let $0 \leqslant k \leqslant n-1$. Denote by A_k^l the set of players whose distance to the left extreme is greater than or equal to k and by A_k^r the set of players whose distance to the right extreme is greater than or equal to k, i.e., $A_k^l = [k+1 \ldots n]$ and $A_k^r = [1 \ldots n-k]$.

Claim 4. *Let (N, v) be an α-supermajority game with a spectrum and $k = \lceil \alpha n \rceil$. Then the spectrum value of each player i is*

$$\phi^i = \begin{cases} 2^{\frac{k-2}{n-1}}\binom{n-k}{i} & i \in A_{k-1}^r, \\ 2^{\frac{k-2}{n-1}}\binom{n-k}{n-i} & i \in A_{k-1}^l, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $i \notin A_{k-1}^r \cup A_{k-1}^l$, then there is no connected coalition of size $k-1$ either to the right or the left of i in the spectrum, hence i can never be the k-th to join a coalition and therefore he receives a 0 payoff.

The calculations of the values for players $i \in A_{k-1}^r \cup A_{k-1}^l$ are straightforward combinatorial calculations. □

Given the result of Claim 4, we may call the set $N \setminus A_{k-1}^r \cup A_{k-1}^l$ the *null zone* of a supermajority game, because players in the null zone all get spectrum value 0. The null zone, if non-empty, always includes the central players with respect to the spectrum. The maximum spectrum value goes to the players about half-way between the extremes of the null zone and the extremes of the spectrum, one on each side of the null zone.

In general, as α increases the null zone grows larger. When $\alpha = 1$, the game is the unanimity game of the grand coalition in which the worth 1 is attained only by the grand coalition N. In that case, the null zone includes all the players except for the leftmost extreme player and the rightmost extreme player; players 1 and n each receive 1/2, and all other players receive zero. This fits with an intuition that when unanimity is required to pass a measure it is the most extreme elements who usually wield the strongest veto power.

4. Axiomatic characterization of the spectrum value

Fix $(N, v, \prec^\sigma) \in \mathcal{GS}$. For each player $i \in N$, define i's *reverse twin* (*with respect to* $\prec^\sigma$)[2] to be the player $j = n+1-i$. For every $S \subseteq N$, the *reverse coalition* of S is the coalition S^{-1} composed by the reverse twins of S, i.e., $S^{-1} = \{j \in N\colon\ j = n+1-i \text{ for some } i \in S\}$.

Two players $i, j \in N$ are *connected symmetric* to each other if i and j are reverse twins and $v(S \cup \{i\}) = v(S^{-1} \cup \{j\})$ for all $S \in C_i(N)$. A player $i \in N$ is a *veto player* if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. A player $i \in N$ is a *connected null player* if $v(S \cup \{i\}) = v(S)$ for all $S \in C_i(N)$. Hence, a standard null player is in particular a connected null player.

[2] A player i is his own reverse twin iff n is odd and $i = (n+1)/2$.

Finally, for every permutation $\sigma : N \to \{1, \ldots, n\}$ and every $i \in N$ with $\sigma(i) < n$ we define $\sigma_{i\to} = \sigma_{\leftarrow(i+1)}$ as the permutation obtained from σ when player i swaps his position with player $(i+1)$.

Using these concepts, we can list properties that a value on $\mathcal{GS}$ might be expected to satisfy.

EFF A value on $\mathcal{GS}$, φ, satisfies *efficiency* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$,

$$\sum_{i\in N} \varphi_i\big(N, v, \prec^\sigma\big) = v(N).$$

ADD A value on $\mathcal{GS}$, φ, satisfies *additivity* if for every $(N, v, \prec^\sigma)$ and $(N, w, \prec^\sigma) \in \mathcal{GS}$,

$$\varphi\big(N, v+w, \prec^\sigma\big) = \varphi\big(N, v, \prec^\sigma\big) + \varphi\big(N, w, \prec^\sigma\big).$$

CNP A value on $\mathcal{GS}$, φ, satisfies the *connected null player property* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every connected null player $i \in N$,

$$\varphi_i\big(N, v, \prec^\sigma\big) = 0.$$

CSP A value on $\mathcal{GS}$, φ, satisfies the *connected symmetric player property* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every pair of connected symmetric players $i, j \in N$,

$$\varphi_i\big(N, v, \prec^\sigma\big) = \varphi_j\big(N, v, \prec^\sigma\big).$$

BCVP A value on $\mathcal{GS}$, φ, satisfies *balanced contributions for veto players* if for every $(N, v, \prec^\sigma) \in \mathcal{GS}$ and every pair of veto players $i, j \in N$ with $i < j$[3]

$$\varphi_i\big(N, v, \prec^\sigma\big) - \varphi_i\big(N, v, \prec^{\sigma_{j\to}}\big) = \varphi_j\big(N, v, \prec^\sigma\big) - \varphi_j\big(N, v, \prec^{\sigma_{\leftarrow i}}\big).$$

The first two properties, EFF and ADD, are the standard efficiency and additivity properties stated in the framework of games with a spectrum. CNP and CSP are based on the standard null player and symmetric player properties. Note, however, that the definition of connected null player is less demanding than its standard counterpart and hence, CNP is a stronger property than the standard null player property. In the case of CSP the definition of a connected symmetric player is more demanding than the definition of a standard symmetric player. CSP is therefore a weaker property than the standard symmetry property.

Finally, BCVP is a property that only applies to veto players; it is therefore not overly demanding. Furthermore, is a reciprocity property in the spirit of various versions of the balanced contributions property that have been used in the literature. It measures the change in the payoff of one veto player when the other veto player becomes more extreme in the spectrum. The gain or loss that a veto player can inflict on the other veto player by moving one position towards the more extreme end of the spectrum is equal to the gain or loss that the second veto player can inflict to the first one[4] by doing the same.

The property BCVP plays an important role in our axiomatization in the following way. As is well known, a value satisfying additivity is essentially determined by its specification on unanimity games. The standard proof of the uniqueness of the Shapley value, for example, uses the efficiency, anonymity and dummy player properties to establish that the Shapley value grants each member of the carrier of a unanimity game S a payoff $1/|S|$, with uniqueness following. For the spectrum value, what matters is not the absolute size of the carrier of a unanimity game but the positions of the extreme veto players with respect to the full spectrum. From that perspective, BCVP is a very concise axiom for expressing how the values of veto players in unanimity games change as their positions are shifted. Used inductively, this is exactly what is needed to determine the spectrum value for all unanimity games, as detailed in the proof of Theorem 1.

Lemma 1. *The spectrum value, ϕ, satisfies* EFF, ADD, CNP, CSP, *and* BCVP.

Proof. The proof for EFF and ADD follows directly from Definition 3. CNP follows from Claim 1 since the definition of the spectrum value depends only on the connected coalitions in $C_i(N)$.

To show that ϕ satisfies CSP let i and j be reverse twins and $r : C_i(N) \to C_j(N)$ be given by $r : S \mapsto S^{-1}$. Then, it is easy to check that r is a bijection. Moreover, $r(L_i(N)) = R_j(N)$, $r(R_i(N)) = L_j(N)$, $r(L_j(N)) = R_i(N)$, and $r(R_j(N)) = L_i(N)$. Then, by Claim 1 ϕ satisfies CSP.

Finally, let $i, j \in N$ be two veto players. Taking into account Claim 2 if $i < j < n$, then

$$\phi^i\big(N, v, \prec^\sigma\big) - \phi^i\big(N, v, \prec^{\sigma_{j\to}}\big) = \frac{2^{j-i-1}}{2^{n-1}}\binom{n-j-i+1}{i-1} v\big([i \ldots j]\big), \tag{7}$$

[3] Note that, if $i = 1$ then $\sigma_{\leftarrow i}$ is undefined, and similarly if $j = n$ then $\sigma_{j\to}$ is undefined. For convenience, we extend every value on $\mathcal{GS}$ to these undefined permutations as follows: for every $k \in N$, $\varphi_k(N, v, \prec^{\sigma_{\leftarrow 1}}) = \varphi_k(N, v, \prec^{\sigma_{n\to}}) = 0$.

[4] When one of the players is already at the extreme edge, he cannot become more extreme. Then the gain or loss that a non-extreme veto player can inflict on an extreme veto player is precisely what the first player gets in his original position.

and if $1 < i < j$, then

$$\phi^j(N, v, \prec^\sigma) - \phi^j(N, v, \prec^{\sigma_{\leftarrow i}}) = \frac{2^{j-i-1}}{2^{n-1}} \binom{n-j-i+1}{i-1} v([i \dots j]), \tag{8}$$

and BCVP follows. The two cases in which one of the veto players is at the extreme edge can be shown as follows: let $i = 1$, then the right hand side of Eq. (7) becomes $2^{j-n} v([1 \dots j])$. Taking into account the convention $\phi_j(N, v, \prec^{\sigma_{\leftarrow 1}}) = 0$ the desired result follows. The case $j = n$ can be shown similarly using Eq. (8) instead of Eq. (7).

Theorem 1. *The spectrum value is the unique value on $\mathcal{GS}$ satisfying* EFF, ADD, CNP, CSP, *and* BCVP.

Proof. Given the result of Lemma 1 we need only prove uniqueness. Let φ be a value satisfying the properties above. By the assumption of ADD we only need to check uniqueness in unanimity games. Let $S \subseteq N$ and let u_S be the unanimity game with carrier $S \neq \emptyset$.

First of all, note that by CNP $\varphi_k(N, v, \prec^\sigma) = 0$ for every $k \in N \setminus \{\min(S), \max(S)\}$. Define the *span* of S to be

$$s(S) := \max(S) + 1 - \min(S) \in \{1, \dots, n\}.$$

We prove uniqueness by backward induction on the span of S.

First step of the induction. If $s(S) = n$ then $1, n \in S$. Hence by CSP $\varphi_1(N, u_S, \prec^\sigma) = \varphi_n(N, u_S, \prec^\sigma)$. Uniqueness then follows by EFF.

Inductive hypothesis. Suppose that $\varphi(N, u_S, \prec^\sigma)$ is uniquely determined for every $S \subseteq N$ such that $s(S) > k$ with $k < n$.

Induction step. Let $S \subseteq N$ satisfy $s(S) = k$. If $1 \in S$ (that is, $\min(S) = 1$ and $\max(S) = k$), then by BCVP, $\varphi_1(N, u_S, \prec^\sigma) - \varphi_k(N, u_S, \prec^\sigma) = \varphi_1(N, u_S, \prec^{\sigma_{k\rightarrow}})$. Note that $(N, u_S, \prec^{\sigma_{k\rightarrow}})$ is a unanimity game with carrier of span $k + 1$. Hence, it follows that $\varphi_1(N, u_S, \prec^\sigma) - \varphi_k(N, u_S, \prec^\sigma)$ is uniquely determined. Next, by EFF $\varphi_1(N, u_S, \prec^\sigma) + \varphi_k(N, u_S, \prec^\sigma) = 1$ and the uniqueness of φ follows. In case $n \in S$, we can repeat the lines above. Thus, we may assume that $1 < \min(S) < \max(S) < n$. Let $i = \min(S)$ and $j = \max(S)$, then by BCVP, $\varphi_i(N, u_S, \prec^\sigma) - \varphi_j(N, u_S, \prec^\sigma) = \varphi_i(N, u_S, \prec^{\sigma_{j\rightarrow}}) - \varphi^j(N, u_S, \prec^{\sigma_{\leftarrow i}})$. Note that both $(N, u_S, \prec^{\sigma_{j\rightarrow}})$ and $(N, u_S, \prec^{\sigma_{\leftarrow i}})$ are unanimity games with carriers of span $k + 1$, hence the difference $\varphi_i(N, u_S, \prec^\sigma) - \varphi_j(N, u_S, \prec^\sigma)$ is unique by the induction hypothesis. Finally, $\varphi_i(N, u_S, \prec^\sigma) + \varphi_j(N, u_S, \prec^\sigma) = 1$ by EFF, which completes the proof. □

5. Case study: The Israeli general elections of 1981 and 1984

5.1. Background

We present here a case study analysis of the results of the Israeli general elections for the Knesset (Parliament) in 1981 and 1984 and the subsequent government coalitions using the spectrum value.

We chose to look at Knesset elections in the 1980s mainly because national political discourse in Israel in that decade was dominated by a single issue: the Arab–Israeli conflict and the future disposition of the West Bank and the Gaza Strip. This meant that there was general agreement on a one-dimensional spectrum that could be applied to virtually all the political parties in the Knesset. The Labour-Alignment party was considered to the left of the Likud party because it advocated a more conciliatory negotiating position in peace talks. Parties that were more dovish than Labour-Alignment comprised the left wing of the Israeli political spectrum in a fairly linear ordering of positions with respect to the extent of peace negotiation concessions proposed by the parties. Similarly, parties that were more hawkish than Likud comprised the right wing. We will list here the political parties winning representation in the Knesset along a spectrum from left to right, with the leftmost party appearing in the first position of the list and the rightmost party in the last position.

It was difficult if not impossible to imagine a party to the left of Labour-Alignment joining a coalition led by the Likud if that coalition did not include Labour-Alignment, and conversely it was difficult if not impossible to imagine a party to the right of Likud joining a coalition led by Labour-Alignment if that coalition did not include Likud, thus justifying the main assumption of coalitions being restricted to spectrum connectedness. The main exception to this rule chiefly involved ultra-Orthodox religious parties, such as Agudah and Shas, which were open to inclusion in either right-wing or left-wing coalitions. We therefore place them in the center of the spectrum between Labour-Alignment and Likud.

There are 120 seats in the Israeli Knesset. The coalitional game that is being played immediately after each general election is a weighted majority game, with the weights determined by the number of seats held by each party and the quota for forming a governing coalition being 61 members.

5.2. The Israeli general election of 1981

Ten political parties won seats in the Knesset elections conducted on 30 June 1981.[5]

[5] Cf. a detailed analysis of the 1981 Israeli elections in Rapoport and Golan (1985). In that paper the analysis is conducted using six indices, the Shapley–Shubik index, the Banzhaf index, the Deegan–Packel index, the generalized Shapley–Shubik index, the generalized Banzhaf index and the generalized

	Party	Seats	Seats (%)	Spectrum (%)	Shapley (%)
1	Hadash	4	3.33	0	6.9841
2	Ratz	1	0.833	0	1.5873
3	Shinui	2	1.66	0	3.0952
4	Labour	47	39.166	31.25	26.4286
5	Agudah	4	3.33	7.8125	6.9841
6	Telem	2	1.66	1.5625	3.0952
7	Likud	48	40.0	34.375	29.9603
8	NRP[a]	6	5.0	15.625	11.7857
9	Tami	3	2.5	7.8125	5.0397
10	Tehiya	3	2.5	1.5625	5.0397

[a] National Religious Party.

Fig. 1. The Israeli Knesset election results of 1981. The quota for forming a governing coalition is 61 seats.

The election results are presented in Fig. 1, with the parties listed in order according to their positioning on the spectrum starting from the extreme left to the extreme right.

As the table in Fig. 1 shows, Likud and Labour-Alignment together won 95 seats, representing approximately 80 percent of the 120 Knesset seats. Individually they were of almost exactly equal size, with Labour-Alignment holding 47 seats (about 39 percent of seats) to Likud's 48 (40 percent of seats). Their spectrum values are also quite close: Labour-Alignment's spectrum value is 0.3125 compared to Likud's 0.34375. Looking at those numbers alone one might consider the election results a near dead heat.

An analysis of the spectrum value reveals a different result. The key to this analysis involves looking not only at the absolute spectrum value but also regarding in detail the connected coalitions leading to that number. In this case, the bulk of the weight of the spectrum value is on the right side of the spectrum, and this comes about because of the relatively large number of ways that right-wing or mostly right-wing connected coalitions can be formed with right-wing parties playing pivotal roles given these election results. In contrast, parties such as Ratz and Shinui get 0 under the spectrum value because they can never be pivotal relative to a connected coalition: any connected coalition that adds Ratz or Shinui to it will already have contained more members than the quota. Note that the Shapley value does not capture this at all. For obvious reasons, it grants equal values to Agudah and Hadash because they are symmetric, both having 4 seats. The spectrum value here strongly distinguishes between the two, giving Agudah nearly 8 percent but giving Hadash zero, since there is no connected coalition for which Hadash is pivotal.

The governing coalition formed on 5 August 1981 was a right-wing government comprised of the connected coalition Agudah, Telem, Likud, NRP and Tami.[6]

5.3. The Israeli general election of 1984

Fifteen political parties won seats in the Knesset elections conducted on 23 July 1984. However, one of those parties, Yahad, with three seats, later merged with Labour-Alignment while Ometz (1 seat) and Tami (1 seat) merged with Likud. We will therefore regard the election results as including only 12 parties, with the seats held by the small merging parties counted among the total seats held by the larger parties. Based on that the election results were as presented in Fig. 2, with the parties listed in order according to their positioning on the spectrum starting from the extreme left to the extreme right.

As in 1981, in 1984 the two largest parties split about 80 percent of the Knesset seats between them. The situation, however, was very different when one considers the spectrum values of the parties. Two extreme parties, Hadash on the left and Kach on the right, garnered non-zero spectrum values because the only way to compose a purely left-wing coalition (respectively, a purely right-wing coalition) would involve including Hadash (respectively, Kach) in a pivotal position. Neither of these options was politically feasible given public opinion.

Note the extreme change in Hadash's payoff between 1981 and 1984 despite it garnering the same number of seats in the parliament. This change is due to the increase in the number of seats for Labour-Alignment which turned Hadash into a pivotal player within a potential connected left-wing coalition.

Disregarding Hadash and Kach leaves positive values concentrated in the center of the spectrum. Furthermore, a careful analysis reveals that the large spectrum value of Labour-Alignment arises solely from considering potential coalitions that contain both Labour-Alignment and Likud, and similarly the large spectrum value of Likud arises solely from considering potential coalitions that contain both Labour-Alignment and Likud. Again, the Shapley value, which does not take into

Deegan–Packel index. The generalized indices, which are based on an analysis of a multi-dimensional "ideological space" that considers how the players in a coalitional game will vote with respect to various issues (which are combinations of parameters of the ideological space) appear to give weights that are closer to observed political power than the non-generalized indices. The generalized Shapley–Shubic index in that paper, in particular, indicates a preponderance of political weight on the right side of the political spectrum following the 1981 Knesset elections, as does the spectrum value index of this paper.

[6] Tehiya later joined the governing coalition on 26 August 1981.

	Party	Seats	Seats (%)	Spectrum (%)	Shapley (%)
1	PLP[a]	2	1.66	0	2.8319
2	Hadash	4	3.3	7.421875	6.0137
3	Ratz	3	2.5	0	4.347
4	Shinui	3	2.5	0	4.347
5	Labour	47	39.166	46.875	33.759
6	Shas	4	3.33	5.078	6.0137
7	Agudah	2	1.66	0	2.8319
8	Likud	43	35.833	39.0625	21.829
9	NRP	4	3.33	0	6.0137
10	Morasha	2	1.66	0	2.8319
11	Tehiya	5	4.166	0	7.8283
12	Kach	1	0.833	1.5625	1.3528

[a] Progressive List for Peace.

Fig. 2. The Israeli Knesset election results of 1984. The quota for forming a governing coalition is 61 seats.

account the relationships between the parties in terms of their ideological positioning relative to each other, fails to capture any of these subtleties.

The governing coalition formed on 13 September 1984 was an unprecedented 'rotation government' that gave both Labour-Alignment and Likud equal power; the prime minister during the first half of the Knesset's four-year term was the leader of Labour-Alignment and the prime minister during the second half of the Knesset's four-year term was the leader of Likud. The coalition itself was a connected coalition that included Shinui, Labour-Alignment, Shas, Agudah, Likud, NRP and Morasha.

6. Extending the model

The model presented here is a simple one, leaving room for further extensions, in particular for the sake of applications in studying more complex situations.

Most obviously, a major simplification of this model is that it is entirely one dimensional. It reduces all differences between players of a coalitional game to a single position along a strict linear ordering. The analysis of most political situations is much more complex and multi-dimensional. For example, a political party may be to the right of a rival party with respect to foreign affairs issues and to the left with respect to social policy. Furthermore, the exact position of a player in the spectrum may be critical for the measure of his or her power under the spectrum value. This is especially clear in unanimity games, where there are only two players in each carrier, at its extreme edges, who receive positive values.

There are several directions in which a multi-dimensional analogue to the spectrum value introduced here could be studied, which we intend to follow up in future research. One possibility would be to consider a different unidimensional spectrum for every voting issue, calculate a value for each such spectrum, and then take an appropriately weighted average of these values for calculating an overall value.

The model is also given to being extended to a fully general graph model in which each player is a vertex and a connected coalition is formed by a connected subset of the graph. This enables much more complex 'affinity' relationships between the players than a linear spectrum to be studied. Initial research in this direction appears in Hellman and Peretz (2013).

Another possible direction would be to allow different weights to the admissible coalitions instead of uniform weights. This might be accomplished by overcoming another simplification of our model: the fact that we use only ordinal ordering between players, without measuring any 'distances'. Taking cues from Alonso-Meijide et al. (2011), measures of distances between players could be added to the spectrum ordering which would then be used for deriving weights for admissible coalitions, leading to a weighted spectrum values.

Finally, we note that the simplicity of the model presented here has its advantages. It has enabled the introduction of the basic ideas and intuitions behind the new value we are suggesting in an uncluttered manner and yielded results that are clear and crisp while containing content.

Appendix A

Claim 5. *The properties used in the characterization of Theorem 1 are independent.*

Proof.

- Let φ^1 be a value on $\mathcal{GS}$ defined for every $(N, v, \prec^\sigma)$ and $i \in N$ by

$$\varphi_i^1(N, v, \prec^\sigma) = 0.$$

Then φ^1 satisfies all properties but EFF.

- Let φ^2 be a value on $\mathcal{GS}$ defined for every $(N, v, \prec^\sigma)$ and $i \in N$ by

$$\text{If } v(1) = v(n) = 0 \quad \varphi_i^2(N, v, \prec^\sigma) = \phi_i(N, v, \prec^\sigma)$$

$$\text{Otherwise} \qquad \varphi_i^2(N, v, \prec^\sigma) = \begin{cases} \frac{v(N)v(i)}{v(1)+v(n)} & \text{if } i \in \{1, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then φ^2 satisfies all properties but ADD.

- Let φ^3 be a value on $\mathcal{GS}$ defined for every $(N, v, \prec^\sigma)$ and $i \in N$ by

$$\varphi_i^3(N, v, \prec^\sigma) = \begin{cases} \frac{v(N)}{2} & \text{if } i \in \{1, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then φ^3 satisfies all properties but CNP.

- Let φ^4 be the additive extension of the following value defined for unanimity games:

$$\varphi^4(N, u_N, \prec^\sigma) = (1, 0, \ldots, 0) \quad \text{and} \quad \varphi^4(N, u_S, \prec^\sigma) = \phi(N, u_S, \prec^\sigma) \quad \text{for every } S \subsetneq N.$$

Then φ^4 satisfies all properties but CSP.

- Let φ^5 be a value on $\mathcal{GS}$ defined for every $(N, v, \prec^\sigma)$ and $i \in N$ by

$$\varphi_i^5(N, v, \prec^\sigma) = \frac{1}{2}\big(v([1 \cdots i]) - v([1 \cdots i-1]) + v([i \cdots n]) - v([i+1 \cdots n])\big)$$

Then φ^5 satisfies all properties but BCVP.

References

Alonso-Meijide, J.M., Bowles, C., 2005. Generating functions for coalitional power indices: An application to the IMF. Ann. Operations Res. 137, 21–44.

Alonso-Meijide, J., Fiestras-Janeiro, M., García-Jurado, I., 2011. A new power index for spatial games. In: Modern Mathematical Tools and Techniques in Capturing Complexity, pp. 275–285.

Bilbao, J., Fernández, J., Jiménez, N., López, J., 2002. Voting power in the European Union enlargement. Europ. J. Operational Res. 143, 181–196.

Calvo, E., Gutierrez, E., 2013. The Shapley-solidarity value for games with a coalition structure. Int. Game Theory Rev. 15, 1350002. 24 pp.

Felsenthal, D., Machover, M., 1998. The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes. Edward Elgar.

Gilles, R., Owen, G., 1999. Cooperative games and disjunctive permission structures. Technical report 999-20. Tilburg University, Center for Economic Research.

Hellman, Z., Peretz, R., 2013. Graph value for cooperative games. Technical report 50073. London School of Economics.

Myerson, R.B., 1977. Graphs and cooperation in games. Math. Operations Res. 2, 673–692.

Nowak, A., Radzik, T., 1994. A solidarity value for n-person transferable utility games. Int. J. Game Theory 23, 43–48.

Owen, G., 1972. Political games. Nav. Res. Logist. Q. 18, 345–355.

Owen, G., 1977. Values of games with a-priori unions. In: Henn, R., Moeschlin, O. (Eds.), Essays in Mathematical Economics and Game Theory. Springer-Verlag, pp. 765–788.

Owen, G., Shapley, L., 1989. Optimal location of candidates in ideological space. Int. J. Game Theory 18, 339–356.

Rapoport, A., Golan, E., 1985. Assessment of political power in the Israeli Knesset. Amer. Polit. Sci. Rev. 79, 225–229.

Shapley, L., 1953. A value for n-person games. In: Kuhn, H., Tucker, A. (Eds.), Contributions to the Theory of Games, vol. II. Princeton University Press.

Shapley, L., 1977. A comparison of power indices and a nonsymetric generalization. Technical report P-5872. RAND Corporation.

Shapley, L., Shubik, M., 1954. A method for evaluating the distribution of power in a committee system. Amer. Polit. Sci. Rev. 48, 787–792.

Weber, R.J., 1988. Probabilistic values for games. In: Roth, A.E. (Ed.), The Shapley Value. Essays in Honor of L.S. Shapley. Cambridge University Press, pp. 101–119.