A GAME WITH NO BAYESIAN APPROXIMATE EQUILIBRIA

ZIV HELLMAN

ABSTRACT. Harsányi (1967) showed that Bayesian games over finite games of payoff uncertainty with finite sets of belief types always admit Bayesian equilibria. That still left the question of whether Bayesian games over finite games of payoff uncertainty with infinitely many types are guaranteed to have equilibria. Simon (2003) presented an example of a Bayesian game with no measurable Bayesian equilibria, even though the underlying game of payoff uncertainty is finite. We present a new and shorter proof of Simon’s result using a simpler Bayesian game that moreover does not even have measurable approximate equilibria. That game in turn is used as the basis for constructing another Bayesian game which has no Bayesian equilibria at all, even in non-measurable strategies, in a construction complementary to one appearing in Friedenberg and Meier (2012).

1. INTRODUCTION

One of the seminal contributions of Harsányi (1967) was the analysis of Bayesian games for studying games of incomplete information, which included showing that every finite Bayesian game has a Bayes-Nash, or Bayesian, equilibrium. The fact that modellers could safely assume the existence of at least one equilibrium was undoubtedly an element in the widespread acceptance of Bayesian games in modelling a wide range of economic situations. Indeed, at this point it is impossible to imagine modern economic modelling and game theory without the theory of Bayesian games.

Harsányi studied Bayesian games whose underlying games of payoff uncertainty are finite, that is, the sets of players, actions and payoff relevant parameters are all assumed to be finite. The breakthrough idea of Harsányi (1967) was appending to such games belief types. Cross products of belief types form states of the world, with the original payoff parameters forming states of nature. Belief hierarchies unfold from the belief types. Harsányi’s Theorem then shows that when the set of belief types is finite, Bayesian equilibria always exist.

There are many interesting models that can readily be formed, however, in which even against a fixed background of a finite game of payoff uncertainty, the set of belief types have the cardinality of the continuum. This may occur, for example, when the identity of the types is tied to continuous variables, such as profits, time points, accumulated resources, population percentages, and so forth. For such models, the question of the existence of Bayesian equilibria was open for a long time.
A major step in answering this question was attained in Simon (2003). Simon con-
structed a finite game of payoff uncertainty and an associated Bayesian game that (a) has
no measurable Bayesian equilibria but (b) does have some Bayesian equilibrium. This
showed that measurable Bayesian may not exist even when the underlying game of payoff
uncertainty is finite.

This paper presents a new proof of Simon’s result. In particular, in Section 3, a finite
game of payoff uncertainty is constructed along with an associated Bayesian game that
(a) does not have a measurable Bayesian equilibrium but (b) does have some Bayesian
equilibrium.

Although many elements used here are similar to those used in Simon (2003)’s game,
most prominently type spaces based on a Cantor space construction and the application
of measure-preserving operations such as involutions and the Bernoulli shift, the game
of payoff uncertainty here as well as the associated Bayesian game are both distinct\(^1\) from
Simon (2003)’s original game. The Bayesian game constructed here is significantly simpler
and therefore hopefully will be easier for many to follow and make use of for further
insights. It has already been used as an element for further research studies in Friedenberg
and Meier (2012a) and Hellman and Levy (2013). In addition, unlike Simon (2003)’s
game, the Bayesian game here does not even admit measurable approximate Bayesian
equilibria.

Having completed the construction of the new Bayesian game with no measurable equi-
lbria, we go on in Section 4 to use that as the basis for constructing another Bayesian game
which has no Bayesian equilibrium at all, even in non-measurable strategies. This is a re-
sult that was already previously attained in Friedenberg and Meier (2012a), building on the
game presented in this paper in Section 3. The game with no Bayesian equilibria of Sec-
tion 4 here is complementary to that of Friedenberg and Meier (2012a), providing another
approach to resolving the question of equilibrium non-existence.

Extensive use is made in this paper of techniques and results developed in Levy (2013)’s
paper on stochastic games without stationary Nash equilibria. In fact, one can in principle
show in a formal manner that the Bayesian game presented in Section 3 is in a sense
derived directly from the stochastic game in Section 3 of Levy (2013) by converting the
inter-temporal interactions of the players in the stochastic games into ‘belief interactions’
of the types in the Bayesian game. The fact that techniques from Levy (2013) have been
shown to be useful for generating counter-examples in separate subject fields (stochastic
games and Bayesian games) may indicate that they have additional potential applications
in several fields of interest in future research.

2. Preliminaries and Notation

2.1. Measures and Measurability.

We assume without further specification that we are working with standard Borel spaces,
with each such \( \Omega \) endowed with its Borel \( \sigma \)-algebra \( \Sigma(\Omega) \); we will often simply write \( \Sigma \) in
place of \( \Sigma(\Omega) \) when it is unambiguous. \( \Delta(\Omega) \) denotes the set of probability measures over

\(^1\) At the technical level, some of the most important distinctions are that the game here involves only two
players, not three, the common knowledge components here are composed of singly-infinite sequences rather
than doubly-infinite sequences, and the measure-preserving operations of this game form a semi-group of actions
in contrast to the group of actions in Simon (2003)’s game.
$\Omega$, measurable with respect to $\Sigma$. Given $\mu \in \Delta(\Omega)$, $\Sigma(\Omega; \mu)$ denotes the completion of the Borel $\sigma$-algebra with respect to $\mu$.

A function $f : \Omega_1 \rightarrow \Omega_2$ is measurable if it is $(\Sigma(\Omega_1), \Sigma(\Omega_2))$-measurable, and $\mu$-measurable if it is $(\Sigma(\Omega_1, \mu), \Sigma(\Omega_2))$-measurable. A function $f$ is $\mu$-integrable if it is Lebesgue integrable with respect to $\mu$, which, if $f$ is bounded, holds iff $f$ is $\mu$-integrable.

### 2.2. Games of Payoff Uncertainty

Let $\Theta$ be a finite set that will be interpreted as a set of states of nature (or alternatively, payoff-relevant parameters). Let $I$ be a finite set of players. A generic player in $I$ will be denoted by $i$.

Given $I$, let $\mathfrak{A}_i$ for each $i$ be a finite set that represents player $i$’s action set. Writing $\mathfrak{A} := \times_{i \in I} \mathfrak{A}_i$, player $i$’s payoff function is a mapping $\rho_i : \Theta \times \mathfrak{A} \rightarrow \mathbb{R}$. The payoff functions $\rho_i$ are extended to $\Theta \times \prod_{i \in I} \Delta(\mathfrak{A}_i)$ in the usual way.

A finite\(^2\) $\Theta$-based game of payoff uncertainty $\Gamma$ is then a structure

$$\Gamma := (\Theta, (\mathfrak{A}_i)_{i \in I}, (\rho_i)_{i \in I}),$$

where $(\mathfrak{A}_i)_{i \in I}$ and $(\rho_i)_{i \in I}$ represent actions and payoffs as in the previous paragraph.

### 2.3. Type Spaces

A $\Theta$-based space of states is a triplet $(\Omega, \Sigma, \nu)$ composed of a standard Borel set $\Omega$ that is interpreted as a set of states of the world, a $\sigma$-algebra $\Sigma$ of measurable subsets (events) of $\Omega$, and a mapping $\nu : \Omega \rightarrow \Theta$ that associates a state of nature with every state of the world. An information structure over a state space $(\Omega, \Sigma, \nu)$ is then given by profiles of partitions $(\Pi_i)_{i \in I}$ of $\Omega$, where each element of each partition $\Pi_i$ is an event in $\Sigma$.

The meet of the partition profile $(\Pi_i)_{i \in I}$ of the players is the finest partition that is coarser than the partition of each player. Each element of the meet is called a common knowledge component.

A type function $t_i$ of player $i$ for $(\Omega, \Sigma, \nu, (\Pi_i)_{i \in I})$ is a measurable function $t_i : \Omega \rightarrow \Delta(\Omega)$ from states to probability measures over $(\Omega, \Sigma)$ such that for all $\omega$ the mapping $t_i(\cdot)$ satisfies:

\begin{align*}
(1) \quad & t_i(\omega)(\Pi_i(\omega)) = 1; \\
(2) \quad & t_i(\omega) = t_i(\omega') \text{ for all } \omega' \in \Pi_i(\omega).
\end{align*}

A $\Theta$-based type space $\mathcal{T}$ is given by $\mathcal{T} := (\Theta, I, \Omega, \Sigma, \nu, (\Pi_i)_{i \in I}, (t_i)_{i \in I})$, where the triplet $(\Omega, \Sigma, \nu)$ forms a $\Theta$-based space of states and for each $i$, $\Pi_i$ is a partition of $\Omega$ and $t_i$ is a type function.

A probability measure $\mu_i$ over $(\Omega, \Sigma)$ is a prior for a type function $t_i$ if for each event $E$

$$\mu_i(E) = \int_{\Omega} t_i(\omega)(E) \, d\mu_i(\omega).$$

A probability measure $\mu$ that is a prior for each of the players’ type function in a type space is a common prior.

\(^2\) In a more general setting, $\Theta$ and $\mathfrak{A}_i$ may be any Polish space. This paper concentrates solely on the finite case, in which $\Theta$, $I$ and $\mathfrak{A}_i$ are all finite sets.
2.4. Bayesian Games and Bayesian Equilibria. A \( \Theta \)-based Bayesian game \( BG \) consists of a pair \((\Gamma, \mathcal{T})\), where \( \Gamma \) is a \( \Theta \)-based game and \( \mathcal{T} \) is a \( \Theta \)-based type space.

A Bayesian game induces strategies, where a strategy \( \sigma^i \) for player \( i \) is a map \( \sigma^i : \Omega \to \Delta(\mathcal{A}_i) \), satisfying the constraint that \( \sigma^i(\omega) \) is constant over \( \Pi_i(\omega) \). As standard, given a profile of strategies \( \sigma = (\sigma^i)_{i \in I} \), the notation \( \sigma^{-i} \) stands for the tuple containing the strategies in \( \sigma \) of all players other than \( i \). Note that \( \sigma^{-i} \) is a function of \( \omega \), just as \( \sigma \) is.

As a Bayesian game inherits the payoff functions \((\rho_i)_{i \in I}\) as well as the mapping \( \nu : \Omega \to \Theta \), for each pairing of an action \( a_i \in \mathcal{A}_i \) and a strategy profile \( \sigma \) we can define mappings \( \overline{p}_i[a_i, \sigma^{-i}] : \Omega \to \mathbb{R} \) by
\[
(2.2) \quad \overline{p}_i[(a_i, \sigma^{-i})](\omega) := \rho_i(\nu(\omega), a_i, \sigma_1(\omega), \ldots, \sigma_{i-1}(\omega), \sigma_{i+1}(\omega), \ldots, \sigma_{|I|}(\omega)).
\]

**Definition 1.** A Bayesian \( \varepsilon \)-equilibrium, for \( \varepsilon \geq 0 \), is a profile of strategies \( \sigma = (\sigma^i)_{i \in I} \) such that for each \( i \in I \), each \( \omega \in \Omega \) and each \( a_i \in \mathcal{A}_i \)

1. \( \overline{p}_i[a_i, \sigma^{-i}] \) is \( t_i(\omega) \)-integrable; and
2. \[
\int_{\Pi_i(\omega)} \overline{p}_i[\sigma_i(\omega), \sigma^{-i})(\omega) \geq \int_{\Pi_i(\omega)} \overline{p}_i[a_i, \sigma^{-i}](\omega) - \varepsilon.
\]

Intuitively, condition (1) states that at each possible state of the world, each player \( i \) can compute his or her expected payoff for each possible action assuming that all other players choose the equilibrium strategy. Condition (2) says that at each state of the world \( \omega \) each player \( i \) is playing an \( \varepsilon \)-best reply given his or her interim stage expected payoff in \( \Pi_i(\omega) \), assuming the strategies of the other players are fixed. Both conditions are needed, because clearly a player needs to be able to compute expected payoffs in order to know whether or not a strategy is an \( \varepsilon \)-best reply to the opponent’s strategy.

**Definition 2.** Given a Bayesian game with a common prior \( \mu \), a Bayesian \( \varepsilon \)-equilibrium \( \sigma = (\sigma^i)_{i \in I} \) of that game is called measurable if for all \( i \in I \) the strategy \( \sigma^i \) is \( \mu \)-measurable.

3. A Bayesian Game With No Measurable Bayesian Equilibria

We define in this section a game \( BG \) that admits no measurable Bayesian \( \varepsilon \)-equilibria.

3.1. The Game of Payoff Uncertainty.

The set of players \( I \) contains two players, denoted Alice and Bob. We will sometimes refer to a generic player as \( i \), in which case \(-i\) will refer to the other player. In addition, for the sake of brevity in subscripts Alice and Bob will often be identified simply by A and B, respectively.

Both players have available to them the same pair of actions, namely, \( L \) and \( R \).

The set of states of nature is given by \( \Theta := \{A, B\} \times \{1, -1\} \), that is, \( \Theta \) is the set of pairs \( \{(A, 1), (A, -1), (B, 1), (B, -1)\} \).

The payoff functions \( \rho_A \) and \( \rho_B \) of Alice and Bob, respectively, are given in the matrices in Table 1, where Alice is the row player and Bob is the column player.

Note the following:

- At each A (respectively, B) state, Bob (respectively, Alice) gets payoff 0 no matter what actions are played by him (respectively, her) or the other player. Each player...
is indifferent with respect to the actions at the states of nature whose letter differs from the first letter of his or her name, because s/he always gets a zero payoff at those states in any event.

- At \((i, 1)\), player \(i\) prefers to match player \(-i\)'s pure actions, while at \((i, -1)\) player \(i\) prefers to mis-match player \(-i\)'s pure actions. Ideally, \(i\) wants both \(i\) and \(-i\) to play \(L\) when the state of nature is \((i, 1)\), and wants \(-i\) to play \(L\) while \(i\) plays \(R\) if the state of nature is \((i, -1)\).

This completes the construction of a \(\Theta\)-based game of payoff uncertainty \(\Gamma\).

3.2. States of The World.

Let \(Y\) be the Cantor space, that is, \(Y := \{-1, 1\}^{\mathbb{N} \ge 0}\), the set of infinite sequences of 1 and \(-1\). We will sometimes denote a generic element \(y \in Y\) by a sequence \(y_0, y_1, y_2, \ldots\), with \(y_i\) the \(i\)-th coordinate of \(y\). Endowing \(Y\) with the Borel \(\sigma\)-algebra \(\Sigma(Y)\) generated by the cylinder sets, let \(\chi\) be the standard Lebesgue measure over \(\Sigma(Y)\). Denote by \(\overline{S} : Y \to Y\) the usual Bernoulli shift, i.e., \(\overline{S}(y_0, y_1, y_2, \ldots) = (y_1, y_2, y_3, \ldots)\)

Next, define the following two sets:

\[
\Omega_A = \{A\} \times Y
\]

and

\[
\Omega_B = \{B\} \times Y,
\]

that is, a generic element of \(\Omega_A\) is \((A, y_0, y_1, y_2, \ldots)\) and a generic element of \(\Omega_B\) is \((B, y_0, y_1, y_2, \ldots)\).

Our set of states is \(\Omega := \Omega_A \cup \Omega_B\). Define \(\zeta(A) = \zeta(B) = \frac{1}{2}\) and then take \(\mu\) to be the measure over \(\Sigma(\Omega)\) given by \(\mu := \zeta \times \chi\).

Define \(C : \Omega \to \{A, B\}\) to be projection on the first coordinate, i.e, when \(\omega = (A, y_0, y_1, y_2, \ldots)\) then \(C(\omega) = A\), and if \(\omega = (B, y_0, y_1, y_2, \ldots)\) then \(C(\omega) = B\). We may think of \(C(\omega)\) as the ‘character’ of the state of the world \(\omega\) (because this relates it to the characters of our story, Alice and Bob).

Define \(Z : \Omega \to \{-1\}\) to be projection on the second coordinate, i.e, if \(\omega = (i, y_0, y_1, y_2, \ldots)\) then \(Z(\omega) = y_0\). We may think of \(Z\) as mapping \(\omega\) to the ‘zero’ element of its Cantor space component.

These two projection operators together give us the mapping \(\nu : \Omega \to \Theta\) we need to relate states of the world to states of nature, by setting \(\nu(\omega) := (C(\omega), Z(\omega))\). The triplet \((\Omega, \Sigma(\Omega), \nu)\) completes the description of a \(\Theta\)-based space of states of the world.

### Table 1. The parameter-dependent payoff matrices.

<table>
<thead>
<tr>
<th></th>
<th>(L)</th>
<th>(R)</th>
<th></th>
<th>(L)</th>
<th>(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta = (A, 1))</td>
<td>1.0</td>
<td>0.0</td>
<td>(\theta = (A, -1))</td>
<td>0.7</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
<td>0.3</td>
<td></td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(\theta = (B, 1))</td>
<td>0.1</td>
<td>0.3</td>
<td>(\theta = (B, -1))</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.3</td>
<td></td>
<td>0.7</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.0</td>
<td></td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
3.3. The Type Space.

We proceed to define the partitions of $\Omega$ associated with Alice and Bob, denoted respectively by $\Pi_A$ and $\Pi_B$. A generic element of $\Pi_A$ is of the form

$$\{(A, y_1, y_2, y_3, \ldots), (B, 1, y_1, y_2, \ldots), (B, -1, y_1, y_2, \ldots)\}.$$ 

A generic element of $\Pi_B$ is of the form

$$\{(B, y_1, y_2, y_3, \ldots), (A, 1, y_1, y_2, \ldots), (A, -1, y_1, y_2, \ldots)\}.$$ 

Note that each of Alice’s partition elements contains one $A$ state and two $B$ states, while each of Bob’s partition elements contains one $B$ state and two $A$ states. This observation plays an important role in this construction.

We can express the content of the partition elements more compactly using the following notation:

Denote by $\iota$ the operator on $\Omega$ defined by

$$\iota(A, y_0, y_1, y_2, \ldots) := (A, -1 \cdot y_0, y_1, y_2, \ldots)$$

$$\iota(B, y_0, y_1, y_2, \ldots) := (B, -1 \cdot y_0, y_1, y_2, \ldots).$$

Note that $\iota$ is a measure preserving and ‘character preserving’ involution of $\Omega$.

Denote by $S$ the operator on $\Omega$ defined by

$$S(A, y_0, y_1, y_2, \ldots) := (B, y_1, y_2, y_3 \ldots)$$

$$S(B, y_0, y_1, y_2, \ldots) := (A, y_1, y_2, y_3 \ldots).$$

$S$ is the product of the measure preserving involution $A \leftrightarrow B$ and the Bernoulli shift $\Xi$, which is well-known to be measure preserving,\(^3\) hence $S$ is measure preserving. Denote

$$S^{-1}_+(A, y_0, y_1, y_2, \ldots) := \{(B, 1, y_0, y_1, y_2, \ldots)\}$$

and

$$S^{-1}_-(A, y_0, y_1, y_2, \ldots) := \{(B, -1, y_0, y_1, y_2, \ldots)\}$$

with the same operator defined for $S^{-1}_+(B, y_0, \ldots)$ and $S^{-1}_-(B, y_0, \ldots)$ with the letters reversed. Then $S^{-1}(\omega) = S^{-1}_+(\omega) \cup S^{-1}_-(\omega)$, hence it maps each point to two points.

The partition elements can now be re-expressed using the above-defined operators. If $\omega$ is an $A$ state (i.e., $C(\omega) = A$) then

$$\Pi_A(\omega) = \{\omega\} \cup S^{-1}_-(\omega)$$

$$\Pi_B(\omega) = \{S(\omega), \omega, \iota(\omega)\}$$

If $\omega$ is a $B$ state (i.e., $C(\omega) = B$) then the partition elements are the same with the letters reversed

$$\Pi_B(\omega) = \{\omega\} \cup S^{-1}_-(\omega)$$

$$\Pi_A(\omega) = \{S(\omega), \omega, \iota(\omega)\}$$

To complete the description of the types, all that is left is to give the probabilities. It suffices to describe $t_A(\omega)$ (resp. $t_B(\omega)$) over $\Pi_A(\omega)$ (resp. $\Pi_B(\omega)$) when $C(\omega) = A$ (resp. $C(\omega) = B$). This is accomplished by

$$t_A(\omega) = \frac{1}{2}; t_A(S^{-1}_+(\omega)) = \frac{1}{4}; t_A((S^{-1}_-(\omega)) = \frac{1}{4}$$

with exactly the same probability distribution for $t_B$, mutatis mutandis.

\(^3\) See, for example, Halmos (1956).
In words, Alice ascribes probability 1/2 to the single A state in each of her partition elements and 1/4 to each of the two B states. Bob ascribes probability 1/2 to the single B state in each of his partition elements and 1/4 to each of the two A states.

**Lemma 3.1.** The functions \( t_A \) and \( t_B \) satisfy the conditions for being type functions with \( \mu \) as their common prior.

The proof of Lemma 3.1 is in the appendix. This completes the construction of the Bayesian game \( \mathcal{B} \).

### 3.4 Comments on Equilibrium Strategies

Suppose that \( \sigma \) is a profile of strategies in \( \mathcal{B} \). Since, for \( i \in I \), each \( \sigma_i \) is a mapping from \( \Omega \) to \( \Delta(\{L, R\}) \), we can derive a mapping \( \ell_i : \Omega \to [0, 1] \) by setting \( \ell_i(\omega) := \sigma_i(\omega)[L] \), i.e., the probability that the strategy \( \sigma_i \) gives to playing \( L \) at the state \( \omega \). We call \( \ell_i \) the derived strategy of player \( i \). In essence, both \( \sigma_i \) and \( \ell_i \) contain the same information, as one can be entirely constructed from the other.

Note that for each partition element \( \pi \) of player \( i \in I \) there is a unique \( \omega = (i, y_0, x_1, \ldots) \in \Omega \) such that

\[
\pi = \{(i, y_0, y_1, \ldots), (-i, 1, y_0, y_1, \ldots), (-i, -1, y_0, y_1, \ldots)\}.
\]

There is thus a bijective mapping \( \tau_i : \Pi_i \to \Omega_i \) such that \( \tau_i(\pi) = (i, y_0, y_1, \ldots) \) iff \( \pi = \{(i, y_0, y_1, \ldots), (-i, 1, y_0, y_1, \ldots), (-i, -1, y_0, y_1, \ldots)\} \).

It follows from this, and the fact that \( \ell_i(\omega) \) must be constant over \( \Pi_i(\omega) \), that from the pair of derived strategies \( \ell_A \) and \( \ell_B \) we can form a single derived strategy \( \ell : \Omega \to [0, 1] \) containing all the information in both \( \ell_A \) and \( \ell_B \) by taking

\[
\ell(\omega) := \ell_i(\omega) \quad \text{if} \quad \omega \in \Omega_i.
\]

**Lemma 3.2.** If \( \sigma = (\sigma_A, \sigma_B) \) is a measurable strategy profile, i.e., both \( \sigma_A \) and \( \sigma_B \) are \( \mu \)-measurable, then both \( \ell_A \) and \( \ell_B \) are \( \mu \)-measurable, and then \( \ell \) is also \( \mu \)-measurable.

**Proof.** For the first conclusion, suppose that \( B \subset [0, 1] \) is Borel. Then \( B \times [0, 1] \) is a Borel subset of \( [0, 1] \times [0, 1] \). By the assumption of the measurability of \( \sigma_A \), the set \( \sigma_A^{-1}(B \times [0, 1]) \) is \( \mu \)-measurable. As \( \ell_A^{-1}(B) = \sigma_A^{-1}(B \times [0, 1]) \), the measurability of \( \ell_A \) follows. The proof for \( \ell_B \) is identical.

For the second conclusion, again let \( B \subset [0, 1] \) be Borel. For each \( i \in I \), the set \( \ell_i^{-1}(B) = \{\omega \in \Omega \mid \ell_i(\omega) \in B\} \) is \( \mu \)-measurable by what was just shown. Since \( \sigma_i \) is constant over each partition element \( \Pi_i(\omega) \) and \( |\Pi_i(\omega) \cap \Omega_i| = 1 \),

\[
\{\omega \in \Omega_i \mid \ell_i(\omega) \in B\} = \{\omega \in \Omega \mid \ell_i(\omega) \in B\} \cap \Omega_i
\]

hence \( \{\omega \in \Omega_i \mid \ell_i(\omega) \in B\} \) is also \( \mu \)-measurable.

Finally, \( \ell_i^{-1}(B) = \{\omega \in \Omega_i \mid \ell_i(\omega) \in B\} \cup \{\omega \in \Omega_j \mid \ell_j(\omega) \in B\} \), which is also \( \mu \)-measurable.

In addition, note that although by definition a player’s payoff at a state \( \omega \) depends solely on the payoff relevant parameters, namely the state of nature \( \nu(\omega) = (C(\omega), Z(\omega)) \) which takes into account only two out of the infinite elements comprising \( \omega \), under equilibrium strategies the rest of the infinite sequence of digits in \( \omega \) do play an extremely important role.
To see this, consider for example Alice’s perspective when the true state of the world is a B state $\omega$. The relevant partition element for Alice is $\Pi_A(\omega) = \{\omega, \ell(\omega), S(\omega)\}$. By construction, $S(\omega)$ is an A state and is the only A state in $\Pi_A(\omega)$.

Now, by the type mapping $t_A$ ascribes probability $1/2$ to $S(\omega)$ and probability $1/4$ to each of the other states in $\Pi_A(\omega)$. Hence it might at first glance seem that she ought to consider the relevant information at all those states. But since Alice’s action choice can lead to a non-zero payoff for her only at $S(\omega)$, for calculating her expected payoff at $\Pi_A(\omega)$ she need only take into account the state of nature associated with $S(\omega)$; Alice can ‘strategically ignore’ the B states. At the same time, she does needs to plan a best reply to what she ascribes will be Bob’s actions at $S(\omega)$.

By similar considerations, at state $S(\omega)$ Bob need only plan a best reply to Alice’s ascribed action at $S^2(\omega)$. This can be continued ad infinitum to show that Alice’s best-reply action at $\omega$ depends on the probabilities ascribed to the actions to be played by the players at each of $S(\omega), S^2(\omega), \ldots, S^n(\omega), \ldots$. By symmetry, the same can be said for Bob’s actions.

3.5. Nonexistence of Measurable $\varepsilon$-Equilibria.

From here, set $0 < \varepsilon < \frac{1}{50}$ and $\delta = 10\varepsilon < \frac{1}{5}$.

**Lemma 3.3.** Under any Bayesian $\varepsilon$-equilibrium of $BG$, the following hold at any state $\omega \in \Omega$.

If $Z(\omega) = 1$, then

\begin{align}
(3.2) & \quad \ell(S(\omega)) < 1/5 \Rightarrow \ell(\omega) < \delta \\
(3.3) & \quad \ell(S(\omega)) > 2/5 \Rightarrow \ell(\omega) > 1 - \delta
\end{align}

If $Z(\omega) = -1$, then

\begin{align}
(3.4) & \quad \ell(S(\omega)) < 3/5 \Rightarrow \ell(\omega) < 1 - \delta \\
(3.5) & \quad \ell(S(\omega)) > 4/5 \Rightarrow \ell(\omega) < \delta
\end{align}

**Proof.** Suppose that $Z(\omega) = 1$ and that $C(\omega) = i$. Then the expected payoff of player $i$, when the probability of $i$ playing $L$ is $\ell(\omega)$ and the probability of player $-i$ playing $L$ is $\ell(S(\omega))$, is

\begin{equation}
\ell(\omega)\ell(S(\omega)) + (1 - \ell(\omega))0.3 = \ell(\omega)\left(\ell(S(\omega)) - 0.3\right) + 0.3
\end{equation}

(1) If $\ell(S(\omega)) < 0.3$ then the 0-best reply is to choose $\ell(\omega) = 0$ (i.e., play purely $R$). Playing $\ell(\omega) > 0$ means paying a ‘penalty’ of $\ell(\omega)(0.3 - \ell(S(\omega)))$, which is deducted from the optimal payoff of 0.3. If $\ell(S(\omega)) < 1/5$, then $0.3 - \ell(S(\omega)) > 1/10$. Then if $\ell(\omega) \geq \delta$, the difference in expected payoff between playing purely $R$ and playing $\ell(\omega)$ is greater than $\delta/10 = \varepsilon$. It follows that an $\varepsilon$-best reply in this case requires $\ell(\omega) < \delta$.

(2) If $\ell(S(\omega)) > 0.3$ then the 0-best reply is to choose $\ell(\omega) = 1$ (i.e., play purely $L$). We can again interpret deviation from this optimal play as leading to a linear loss, this time from the maximal payoff of $\ell(S(\omega))$ down to the worst payoff of 0.3 (when playing purely $R$).

If $\ell(S(\omega)) > 2/5$, then $\ell(S(\omega)) - 0.3 > 1/10$. Then if $\ell(\omega) \leq 1 - \delta$, the difference in expected payoff between playing purely $L$ and playing $\ell(\omega)$ is greater than $\delta/10 = \varepsilon$. It follows that an $\varepsilon$-best reply in this case requires $\ell(\omega) > 1 - \delta$. 

Suppose that $Z(\omega) = -1$ and that $C(\omega) = i$. Then the expected payoff of player $i$ is $(1 - \ell(\omega))\ell(S(\omega)) + \ell(\omega)0.7$. Writing $\bar{\ell}(\omega) := 1 - \ell(\omega)$ enables us to rewrite this in a way that is more reminiscent of Equation (3.6):

\begin{equation}
\bar{\ell}(\omega)\ell(S(\omega)) + (1 - \bar{\ell}(\omega))0.7 = \bar{\ell}(\omega)\left(\ell(S(\omega)) - 0.7\right) + 0.7
\end{equation}

The analysis now proceeds in parallel to that immediately following Equation (3.6).

1. If $\ell(S(\omega)) < 0.7$ then the 0-best reply is to choose $\overline{\ell}(\omega) = 0$ (i.e., play purely $L$). Playing $\bar{\ell}(\omega) > 0$ means paying a linear penalty from the optimal payoff of 0.7.
   
   If $\ell(S(\omega)) < 3/5$, then $0.7 - \ell(S(\omega)) > 1/10$. Then if $\bar{\ell}(\omega) \geq \delta$, the difference in expected payoff between playing purely $L$ and playing $\bar{\ell}(\omega)$ is greater than $\delta/10 = \varepsilon$. It follows that an $\varepsilon$-best reply in this case requires $\ell(\omega) > 1 - \delta$.

2. If $\ell(S(\omega)) > 0.7$ then the 0-best reply is to choose $\bar{\ell}(\omega) = 1$ (i.e., play purely $R$). We can again interpret deviation from this optimal play as leading to a linear loss, this time from the maximal payoff of $\ell(S(\omega))$ down to the worst payoff of 0.7 (when playing purely $L$).
   
   If $\ell(S(\omega)) > 4/5$, then $\ell(S(\omega)) - 0.7 > 1/10$. Then if $\bar{\ell}(\omega) \leq 1 - \delta$, the difference in expected payoff between playing purely $R$ and playing $\bar{\ell}(\omega)$ is greater than $\delta/10 = \varepsilon$. It follows that any $\varepsilon$-best reply in this case requires $\ell(\omega) < \delta$.

**Definition 3.** Given a strategy profile of the players, a state $\omega \in \Omega$ is $L$-quasi-pure (respectively, $R$-quasi-pure) if $\ell(\omega) > 1 - \delta$ (respectively, $\ell(\omega) < \delta$). If a state is either $L$-quasi-pure or $R$-quasi-pure, we may simply refer to it as being quasi-pure. (Recall that $\delta$ depends on $\varepsilon$, and that Lemma 3.3 explicitly relates $\varepsilon$-equilibria to $\delta$ and the derived strategies, hence this definition is of relevance to studying $\varepsilon$ equilibria.)

**Lemma 3.4.** Suppose that a Bayesian $\varepsilon$-equilibrium of $BG$ is played. Let $\omega \in \Omega$ be $L$-quasi-pure (respectively, $R$-quasi-pure). Then:

- $S^{-1}_n(\omega)$ is $L$-quasi-pure (respectively, $R$-quasi-pure);
- $S^{-1}_n(\omega)$ is $R$-quasi-pure (respectively, $L$-quasi-pure).

**Proof.** The proof, similar to that of Lemma 3.4 of Levy (2013), uses repeated application of Lemma 3.3, as follows:

- If $S(\omega)$ is $L$-quasi-pure and $Z(\omega) = 1$ then $\ell(S(\omega)) < \delta < \frac{1}2$, hence $\ell(\omega) < \delta$.
- If $S(\omega)$ is $L$-quasi-pure and $Z(\omega) = -1$ then $\ell(S(\omega)) < \delta < \frac{2}{5}$, hence $\ell(\omega) > 1 - \delta$.
- If $S(\omega)$ is $R$-quasi-pure and $Z(\omega) = 1$ then $\ell(S(\omega)) > 1 - \delta > \frac{2}{5}$, hence $\ell(\omega) > 1 - \delta$.
- If $S(\omega)$ is $R$-quasi-pure and $Z(\omega) = -1$ then $\ell(S(\omega)) > 1 - \delta > \frac{4}{5}$, hence $\ell(\omega) < \delta$.

Lemma 3.4 shows what happens when a state $\omega$ is quasi-pure: the effect propagates ‘backwards’ forever along the inverse of the $S$ mapping, with every state $\omega'$ such that there exists an integer $m$ such that $\omega' \in S^{-m}(\omega)$ also quasi-pure.

**Lemma 3.5.** Suppose that a Bayesian $\varepsilon$-equilibrium of $BG$ is played. Then for all $\omega \in \Omega$, either $\omega$ is quasi-pure or $\ell(\omega)$ is quasi-pure.

**Proof.** The proof is based on Lemma 3.5 of Levy (2013). Assume that $Z(\omega) = 1$. Symmetric reasoning applies if $Z(\omega) = -1$. 

If \( \ell(S(\omega)) > 2/5 \) then by Lemma 3.3, \( \ell(\omega) > 1 - \delta \), i.e., \( \omega \) is quasi-pure. If \( \ell(S(\omega)) < 3/5 \) then, since \( S(\ell(\omega)) = S(\omega) \) and \( Z(\ell(\omega)) = -Z(\omega) \), by Lemma 3.3 \( \ell(\epsilon(\omega)) > 1 - \delta \), i.e., \( \epsilon(\omega) \) is quasi-pure. But \( \ell(S(\omega)) > 2/5 \) or \( \ell(S(\omega)) < 3/5 \) (or both).

Putting together the output of Lemmas 3.4 – 3.5 yields a function \( \ell : \Omega \rightarrow [0, 1] \): satisfying the following properties:

**P1.** If \( Z(\omega) = 1 \) and \( S(\omega) \) is \( R \)-quasi-pure (resp. \( L \)-quasi-pure) then \( \omega \) is \( R \)-quasi-pure (resp. \( L \)-quasi-pure).

**P2.** If \( Z(\omega) = -1 \) and \( S(\omega) \) is \( R \)-quasi-pure (resp. \( L \)-quasi-pure) then \( \omega \) is \( L \)-quasi-pure (resp. \( R \)-quasi-pure).

**P3.** For all \( \omega \) either \( \omega \) is quasi-pure or \( \ell(\omega) \) is quasi-pure.

At this point, we make use of Levy’s Theorem, whose proof can be found in Lemma 3.8 and Proposition 3.9 of Levy (2013), or Theorem A.2 of Friedenberg and Meier (2012b).

**Levy’s Theorem.** If \( \ell : \Omega \rightarrow [0, 1] \) satisfies properties P1-P2-P3 then \( \ell \) is not \( \mu \)-measurable.

**Theorem 1.** The game \( BG \) has no \( \mu \)-measurable Bayesian \( \epsilon \)-equilibria.

**Proof.** Suppose that \( (\sigma_A, \sigma_B) \) is a \( \mu \)-measurable Bayesian \( \epsilon \)-equilibrium. Since \( (\sigma_A, \sigma_B) \) is \( \mu \)-measurable, by Lemma 3.2, \( \ell \) is \( \mu \)-measurable.

At the same time, as \( (\sigma_A, \sigma_B) \) is a Bayesian \( \epsilon \)-equilibrium, Lemmas 3.4 – 3.5 imply that \( \ell \) satisfies properties P1-P2-P3, hence by Levy’s Theorem \( \ell \) is not \( \mu \)-measurable. This is a contradiction.

### 3.6. A Non-measurable Equilibrium.

Theorem 1 shows that the game \( BG \) has no measurable Bayesian \( \epsilon \)-equilibria. It does, however, have non-measurable Bayesian equilibria.

This can be established by noting that \( BG \) is a locally finite game, as defined in Simon (2003), because there are a finite number of players, each player \( i \) has a finite set of actions and the support of each type \( t_i(\omega) \) is finite. Proposition 1 of Simon (2003) then shows that the game must admit a Bayesian equilibrium.

### 4. A Bayesian Game with no Bayesian \( \epsilon \)-Equilibria

Section 3 presented a game with no measurable Bayesian approximate equilibria but one with non-measurable equilibria. In the interim stage, of that game both players can measurably compute their expected payoffs and best reply strategies.

Here we consider the question of whether there is a game with no Bayesian approximate equilibria at all, whether measurable or non-measurable. That question has in fact been answered in the affirmative by Friedenberg and Meier (2012a), who show that there must exist a game with no Bayesian equilibria by embedding a variation of the game \( BG \) of Section 3 above in an associated universal Bayesian game (essentially meaning that the type structure of the game is a universal type structure). This section presents another approach, complementary to that of Friedenberg and Meier (2012a), to constructing a game with no Bayesian equilibria.

The intuitive argument used here runs as follows: a third player Eve, joins our list of *dramatis personae*. Eve is effectively laying bets as to whether Alice will choose \( L \) or Bob will choose \( R \) in the game \( BG \). Alice and Bob do not know of Eve’s existence, hence they simply play as above. Eve, however, does not get the signals that Alice and Bob
Eve indifferent between strategic perspectives, Alice and Bob are literally playing exactly the same game in respective derived strategies (4.1) \( \Pi \). Theorem 2. The game \( BG' \) is the trivial partition, \( \Pi \). Throughout, we continue to use \( \Omega, \Pi, (\omega), \mu, \rho_A, \rho_B, \rho_E \) and similar notations to denote the constructions labelled as such above in Section 3.

The three players in \( BG' \) are labelled Alice, Bob and Eve. For the sake of brevity, in subscripts Alice and Bob may be identified by A and B as before, while Eve may be identified by E.

The action set is identical at every state for all players, and as before is \( \{L, R\} \). The parameters, or set of states of nature, are exactly the same as in \( BG \), namely \( \Theta = \{A, B\} \times \{1, -1\} \).

The payoff functions \( \rho'_A, \rho'_B \) and \( \rho'_E \) are given as follows. Writing \((y, z) \in \{A, B\} \times \{1, -1\} \) for a generic state of nature and \( (a_A, a_B, a_E) \in \{R, L\} \times \{R, L\} \times \{R, L\} \) for a generic action profile, and using \( \rho_A \) and \( \rho_B \) as in \( BG \):

\[
\begin{align*}
\rho_A((y, z), (a_A, a_B, a_E)) &:= \rho_A((y, z), (a_A, a_B)) \\
\rho_B((y, z), (a_A, a_B, a_E)) &:= \rho_B((y, z), (a_A, a_B)) \\
\rho_E((y, z), (a_A, a_B, a_E)) &:= \begin{cases} 
1 & \text{if } a_E = L \text{ and } a_A = L \\
1 & \text{if } a_E = R \text{ and } a_B = R \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

In words, Alice and Bob regard Eve as an outside observer whose presence has no effect on their payoffs and whose actions can therefore be ignored against the background of each others’ actions. From their perspective, they are playing the same \( \Theta \)-based game as in Section 3.

If Eve plays \( L \) she receives a positive payoff iff Alice matches her and also plays \( L \). If Eve plays \( R \) she receives a positive payoff iff Bob matches her and also plays \( R \). Otherwise she receives zero. These values hold regardless of the state of nature.

To complete the description of the Bayesian game we now need to define the states of the world, the player partitions and the types. The set of states of the world of \( BG' \) is \( \Omega \), the same set of states as that of \( BG \), with the measure \( \mu \) as before.

The partitions \( \Pi_A, \Pi_B \) and \( \Pi_E \) of Alice, Bob and Eve are as follows. \( \Pi_A' = \Pi_A \) and \( \Pi_B' = \Pi_B \), i.e., Alice and Bob have exactly the same partitions as they have in \( BG \). Eve’s partition is the trivial partition,

\[
(4.1) \quad \Pi_E'(\omega') := \Omega'.
\]

The type functions follow in the same vein. Namely, \( t_A'(\omega) = t_A(\omega) \) and \( t_B'(\omega) = t_B(\omega) \) for all \( \omega \in \Omega \), while \( t_E'(\omega)(\cdot) := \mu(\cdot) \) for all \( \omega \in \Omega \).

**Theorem 2.** The game \( BG' \) has no Bayesian \( \varepsilon \)-equilibrium.

**Proof.** Suppose that Alice and Bob are playing \( \mu \)-measurable strategies \( \sigma_A \) and \( \sigma_B \), with respective derived strategies \( t_A \) and \( t_B \). Then Eve’s \( \varepsilon \)-optimal reply is easily calculated. If \( \int_{\Omega} t_A d\mu > \int_{\Omega} (1 - t_B) d\mu + \varepsilon \) then Eve ought to play pure action \( L \). If \( \int_{\Omega} (1 - t_B) d\mu > \int_{\Omega} t_A d\mu + \varepsilon \), then pure action \( R \) is called for, while \( |\int_{\Omega} t_A d\mu - \int_{\Omega} (1 - t_B) d\mu| \leq \varepsilon \) leaves Eve indifferent between \( L \) and \( R \).

Now, suppose that \( (\sigma_A, \sigma_B, \sigma_E) \) forms a Bayesian \( \varepsilon \)-equilibrium of \( BG' \). From their strategic perspectives, Alice and Bob are literally playing exactly the same game in \( BG' \) as
in $BG$, given their payoffs at each state and the fact that Eve’s actions have no effect at all on them.

In other words, if $(\sigma_\alpha, \sigma_\beta, \sigma_\epsilon)$ is a Bayesian $\varepsilon$-equilibrium of $BG'$, then $(\sigma_\alpha, \sigma_\beta)$ is a Bayesian $\varepsilon$-equilibrium of $BG$. By Theorem 1 we deduce that at least one of $\ell_\alpha$ and $\ell_\beta$ is not $\mu$-measurable. But then Eve cannot calculate at least one of $\int_{\Omega} \ell_\alpha d\mu$ or $\int_{\Omega} (1 - \ell_\beta) d\mu$, and therefore cannot assess the expected payoffs of her actions. Condition (1) of Definition 1 is not met and therefore there can be no $\varepsilon$-equilibrium.

5. Appendix

Proof of Lemma 3.1. By construction, $t_\epsilon(\omega)(\Pi(\omega)) = 1$ for all $\omega$ and $t_\epsilon(\omega) = t_\epsilon(\omega')$ for $\omega' \in \Pi(\omega)$. Two more items need to be checked: that for each event $E$, $t_\epsilon(\omega)(E)$ is measurable and that $\mu(E) = \int_{\Omega} t_\epsilon(\omega)(E) d\mu(\omega)$. We will prove these for $i = \alpha$, with the proof for $i = \beta$ conducted similarly.

For the rest of this proof, denote by $1_E(\omega)$ the indicator function that returns 1 if $\omega \in E$ and 0 if $\omega \notin E$. Fix an event $E \in \Sigma(\Omega)$. Then:

$$t_\alpha(\omega)(E) = \begin{cases} \frac{1}{4}1_E(\omega) + \frac{1}{2}1_E(S(\omega)) & \text{if } \omega \in \Omega_B \\ \frac{1}{4}1_E(S^{-1}(\omega)) + \frac{1}{2}1_E(S^{-1}(\omega')) & \text{if } \omega \in \Omega_A \end{cases}$$

from which we conclude that $t_\alpha(\omega)(E)$ is measurable.

Next, we divide up $E$ as follows:

$$E_1 := \{\omega \in E \mid \omega \in \Omega_B \text{ and } \iota(\omega), S(\omega) \notin E\},$$

$$E_2 := \{\omega \in E \mid \omega \in \Omega_B \text{ and } \iota(\omega) \in E, S(\omega) \notin E\},$$

$$E_3 := \{\omega \in E \mid \omega \in \Omega_B \text{ and } \iota(\omega), S(\omega) \in E\},$$

$$E_4 := \{\omega \in E \mid \omega \in \Omega_A \text{ and } S^{-1}(\omega) \subset E^c\},$$

$$E_5 := \{\omega \in E \mid \omega \in \Omega_A \text{ and } S^{-1}(\omega) \cap E = \emptyset, S^{-1}(\omega) \notin E\}$$

$$\cup \{\omega \in E \mid \omega \in \Omega_B \text{ and } S(\omega) \in E, \iota(\omega) \notin E\},$$

$$E_6 := \{\omega \in E \mid \omega \in \Omega_A \text{ and } S^{-1}(\omega) \subset E\}.$$  

The sets $(E_j)$ are all disjoint from each other. The proof proceeds by showing that $\mu(E_j) = \int_{\Omega} t_\alpha(\omega)(E_j) d\mu(\omega)$ for each $1 \leq j \leq 6$, which is straightforward but tedious. We show how it is accomplished in two of the cases, trusting that the technique for the rest of the cases will be clear enough.

Case $E_1$. Define $D := E_1 \cup \iota(E_1) \cup S(E_1 \cup \iota(E_1))$. By the measure-preserving properties of $\iota$ and $S$, one has $\mu(\iota(E_1)) = \mu(E_1)$ and

$$\mu(S(E_1 \cup \iota(E_1))) = \mu(E_1 \cup \iota(E_1)),$$

hence $\mu(D) = 4\mu(E_1)$. On the other hand,

$$\int_{\Omega} t_\alpha(\omega)(E_1) d\mu(\omega) = \int_{D} t_\alpha(\omega)(E_1) d\mu(\omega) = \int_{D} \frac{1}{4} d\mu(\omega) = \mu(D)/4,$$

leading to the conclusion that $\int_{\Omega} t_\alpha(\omega)(E_1) d\mu(\omega) = \mu(E_1)$.

Case $E_5$. Define $C := E_5 \cup \iota(E_5 \cap \Omega_B)$. Using similar reasoning as in the previous case, relying on measure-preserving properties, one deduces that $\mu(C) = \frac{3}{4} E_5$. On the other hand,

$$\int_{\Omega} t_\alpha(\omega)(E_5) d\mu(\omega) = \int_{C} t_\alpha(\omega)(E_5) d\mu(\omega) = \int_{C} \frac{3}{4} d\mu(\omega) = \frac{3}{4} \mu(C),$$

hence $\int_{\Omega} t_\alpha(\omega)(E_5) d\mu(\omega) = \mu(E_5)$. 


REFERENCES

Harsányi, J. C. (1967), Games with Incomplete Information Played by Bayesian Players, Management Science, 14, 159–182.