BAYESIAN GAMES WITH A CONTINUUM OF STATES

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ABSTRACT. Negative results on the the existence of Bayesian equilibria when state spaces have the cardinality of the continuum have been attained in recent years. This has led to the natural question: are there conditions that characterise when Bayesian games over continuum state spaces have measurable Bayesian equilibria? We answer this in the affirmative. Assuming that each type has finite or countable support, Bayesian equilibria may fail to exist if and only if the underlying common knowledge σ-algebra is non-separable. Furthermore, anomalous examples with continuum state spaces have been presented in the literature in which common priors exist over entire state spaces but not over common knowledge components. There are also spaces in which *ex ante* there is no trading possible yet trade can occur in the interim stage. We show that when the common knowledge σ-algebra is separable all these anomalies disappear.

1. INTRODUCTION

When are Bayesian games guaranteed to have Bayesian equilibria?

One answer to that question was given by John Harsányi in his path-breaking work studying finite games of incomplete information in the late

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1960s. Harsányi (1967) famously introduced the concept of types for players, which enabled him to reduce the question of the existence of Bayesian equilibria to the question of existence of Nash equilibria in an associated game of complete information. As the latter always exist, so do Bayesian equilibria.

It is difficult to overstate the influence that Harsányi's ideas have had. They are now essential to the study of a wide range of subfields in economics and game theory, with subjects such as incomplete and asymmetric information models, signalling theory, principal-agent models, adverse selection and the provisions of public goods forming only a very partial list.

Harsányi's Theorem on the existence of Bayesian equilibria, however, was proved only for Bayesian games in which all variables are finite, that is, games with a finite number of players, finite action spaces, finite payoff parameters and a finite number of possible types. But there are many models that can readily be formed in which continuous variables, such as profits, time points, accumulated resources, population percentages, and so forth naturally require consideration of infinite states of the world and hence infinitely many types. Belief hierarchies can also dictate models with infinite types. Indeed, even a two-player game over a two-by-two matrix can require analysis using a Bayesian game over a type space with continuum many states, given sufficiently rich interactions between the beliefs of the players.

Over spaces with continuum-many states, Harsányi's Theorem on the existence of Bayesian equilibria no longer holds. Robert Simon, in Simon (2003), presented an example of a three-player Bayesian game over a continuum state space with no measurable Bayesian equilibrium.[1] Any hopes that positive results could be restored by considering approximate equilibria instead of exact equilibria were dashed when Hellman (2012) showed an example of a two-player Bayesian game over a continuum state space with no Bayesian ε-equilibrium for sufficiently small ε.

These negative results naturally lead to the question of what conditions *do* guarantee the positive existence of Bayesian equilibria in Bayesian games with type spaces containing continuum-many states. In this paper, in Theorem 1, we characterise when measurable Bayesian equilibria (MBE) exist

[1] Restricting attention to the question of the existence of measuable equilibria is not truly restrictive: given a game without measurable Bayesian equilibria one can always construct another game, with an additional player whose payoffs depend on the strategies of the players in the original game, that has no well-defined equilibria at all.

in the class of Bayesian games that satisfy the condition of having everywhere finite[2] support (note that every known negative example, that is, of a Bayesian game lacking MBE, has everywhere finite support). This result thus generalizes Harsányi's Theorem.

The main characterisation essentially comes down to the question of whether the common knowledge σ-algebra of a Bayesian game is separable. If it is separable then the existence of an MBE is guaranteed. Conversely, given a non-separable σ-algebra, one can construct a type space and a Bayesian game, whose common knowledge σ-algebra coincides with $\mathcal{F}$, that lacks an MBE.

Furthermore, separability helps resolve yet another anomaly first noted by Simon that occurs in type spaces with continuum-many states and is related to a concept introduced by Harsányi. The common prior assumption, which first appeared in Harsányi (1967), has been so pervasively adopted in the literature that an exhaustive list of the fields in which it is used would be too long to print here. It also plays a central role in the celebrated Aumann Agreement Theorem (Aumann (1976)), which, as extended by Sebenius and Geanakoplos (1983), Milgrom and Stokey (1982) and Feinberg (2000), states that agents will mutually accept bets or trades if and only if they do not share a common prior.

Aumann's Agreement Theorem was proved for finite type spaces. It does not fully extend to non-compact infinite spaces: there are examples of such spaces that have no common prior yet admit no agreement (see Feinberg (2000)).

'Even worse', even without asking about the relationship between common priors and agreements, Simon (2000) presented an example in which the very existence of a common prior depends on whether one is looking at the *ex ante* stage or the interim stage. That is, common priors exist globally in the full state space in the *ex ante* stage but do not exist in any common knowledge component (i.e., in the interim stage).

This is so counter intuitive that Heifetz (2006) conjectured (using the concept of common improper priors) that despite the lack of consistency in the existence of common priors in such examples there would still be *behavioural* consistency in terms of agreement, i.e., agents would consistently agree not to trade in both the *ex ante* stage and the interim stage. We show here in Section 3 that this conjecture is wrong: there *is* behavioural inconsistency, with agents unable to agree to trade *ex ante* but agreeing to trade in the interim stage.

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[2] This can, in fact, be extended to everywhere countable support, as we show in Section 5.1.

Once again, separability of the common knowledge σ-algebra comes to the rescue. As shown in Theorems 2 and 3, if it holds then a common prior in the *ex ante* stage is preserved in the interim stage and consistent no-trading behaviour is maintained. Without it, one can always construct a structure that recapitulates Simon (2003)'s paradox of inconsistency with regards to trading/no-trading.

The intuitive reason that separability of the common knowledge σ-algebra is so crucial is related to the ideas of transversality and measurable selections from descriptive set theory. Measurable selection enables one to 'stitch' together a 'local' property associated with each common knowledge component K – such as the existence of an equilibrium in the game restricted to K, or the existence of an acceptable bet at all states in K – into a corresponding measurable property that is 'global' in the sense of holding at all states (or, in some cases, almost all) in Ω.

We have, in fact, found that there are parallels between concepts used in game theory and descriptive set theory concepts that are surprisingly useful for arriving at conclusions in game theoretic models. Although we have kept these mathematical aspects in the appendices to allow the reader to more easily focus the main results, we do strive to make these parallels explicit when they are employed. Hopefully, these sorts of parallels can be deepened in future research, leading to more new results.

The outline of the paper is as follows: Section 2 briefly presents some mathematical concepts and recalls the standard notions and model of games of incomplete information. Section 3 presents motivating examples. Section 4 presents the results. Section 5 presents extensions and variations along with some discussion of the concepts of *belief induced* σ-algebras and *agreeing to agree*. The full details of the mathematical tools used in this paper and proofs of the theorems are in the appendix.

2. Preliminaries and the Model

This section briefly presents the concepts of *standard Borel spaces* and *proper regular conditional distributions* and then proceeds to recall the standard definitions (appropriately generalised to continuous state spaces) of knowledge spaces, Bayesian games, and Bayesian equilibrium.

2.1. **A Few Preliminaries.**

A *standard Borel* space is a topological space that is homeomorphic to a Borel subset of a Polish space.[3] Whenever we refer to a σ-algebra on a standard Borel space, we mean a sub σ-algebra of the Borel σ-algebra. Measurability without further qualification in this paper, in the context of a standard Borel space Ω, will be understood to mean measurability with respect to the Borel σ-algebra of Ω.

For a standard Borel space X, let $\Delta(X)$ denote the space of regular Borel probability distributions on X, with the topology of weak convergence of probability measures, and let $\Delta_f(X) \subseteq \Delta(X)$ (resp. $\Delta_a(X) \subseteq \Delta(X)$) denote the subspace of finitely supported (resp. purely atomic) measures. $\Delta_f(X), \Delta_a(X)$ are Borel subsets[4] of $\Delta(X)$.

If $(\Omega, \mathcal{B})$ is a measurable space and $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{B}$, then (see Blackwell and Ryll-Nardzewski (1963)) a *proper regular conditional distribution* (henceforth, proper RCD) given $\mathcal{F}$, is a mapping $t : \Omega \times \mathcal{B} \to [0,1]$ such that for each $B \in \mathcal{B}$, $t(\cdot)(B)$ is Borel, and such that:

$$\mu(B) = \int_\Omega t(x)(B) d\mu(x), \text{ for all } B \in \mathcal{B}$$

and

$$t(\omega)(A) = 1, \text{ if } \omega \in A \in \mathcal{F}$$

Note that in particular,

$$t(\omega)(T) = E_\mu[1_T \mid \mathcal{F}](\omega), \ \mu\text{-a.e. } \omega \in \Omega$$

(In terms that may be more familiar for game theorists, a proper RCD t of a probability measure μ may be thought of as the posterior t of a prior μ with respect to a knowledge structure $\mathcal{F}$.)

A very central concept in this paper is:

Definition 1. A σ-algebra $\mathcal{F}$ on a Borel space Ω is *separable*[5] if there is a countable[6] collection of Borel subsets $\{B_n\}_{n\in\mathbb{N}}$ of Ω that generates $\mathcal{F}$; that is, $\mathcal{F}$ is the smallest σ-algebra such that $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$. ♦

[3] Equivalently, a measurable space $(X, \mathcal{B})$ is standard Borel if there exists a metric on X that makes it a complete separable metric space in such a way that $\mathcal{B}$ is then the Borel σ-algebra, i.e., the smallest σ-algebra containing the open sets.

[4] $\Delta_f(X)$ can be viewed as $\cup_{n\in\mathbb{N}}\Delta_n(X)$, where $\Delta_n(X)$ consists of the probability measures supported on at most n points. $\Delta_n(X)$ can be viewed as the image in $\Delta(X)$ of $X^n \times \Delta_n$, where Δ_n is the n-simplex, under a finite-to-one map. Similarly, $\Delta_a(X)$ can be viewed as the image of $\{X \in X^{\mathbb{N}} \mid \forall n \neq m, x_n \neq x_m\} \times \{\alpha \in \mathbb{R}^{\mathbb{N}} \mid \alpha \geq 0, \sum_{n=1}^{\infty} \alpha_n = 1\}$ under a countable-to-one map. Hence, $\Delta_f(X)$ and $\Delta_a(X)$ are Borel.

[5] 'Separable' is synonymous with 'countably generated'.

[6] In this paper, 'countable' refers both to finite cardinalities and to countably infinite cardinalities.

2.2. **Knowledge Spaces.**

A *knowledge space* for a nonempty, finite set of players I is given by a triple $(\Omega, I, (\mathcal{F}^i)_{i\in I})$, where Ω is a standard Borel space of *states*, and $\mathcal{F}^i$ for each $i \in I$ is a σ-algebra over Ω, called i's *knowledge σ-algebra*. Intuitively, the elements in $\mathcal{F}^i$ represent the events that player i can identify, hence the name knowledge σ-algebra.

Let $\mathcal{F} := \cap_{i\in I}\mathcal{F}^i$; that is, $\mathcal{F}$ is the finest σ-algebra contained in all the players' knowledge σ-algebras. $\mathcal{F}$ is called the *common knowledge σ-algebra* of the knowledge space. The elements of $\mathcal{F}$ intuitively represent the events of which all the players can have common knowledge.

2.3. **Type Spaces.**

Most game theory models[7] work with partitionally generated type spaces. In such models, where Ω is finite or countable, each player i has a partition Π^i of Ω. Player i's knowledge σ-algebra $\mathcal{F}^i$ is the σ-algebra generated by Π^i.[8] A type function t^i is then a Borel probability measure on Ω such that

(a) for each $\omega \in \Omega$, $t^i(\omega)(\Pi^i(\omega)) = 1$;
(b) for each $\omega' \in \Pi^i(\omega)$, $t^i(\omega') = t^i(\omega)$.

Intuitively, a type function t^i represents the probability measure that player i ascribes to the states conditional on receiving a signal that ω is a possible true state.

We work in a more general framework. Fix a knowledge space $(\Omega, I, (\mathcal{F}^i)_{i\in I})$ (where $\mathcal{F}^i$ for each i is a σ-algebra that is not necessarily generated from a partition). For each $i \in I$, a *type function* t^i is a mapping $t^i : \Omega \to \Delta(\Omega)$ that is $\mathcal{F}^i$-measurable and satisfies $t^i(\omega)(A) = 1$ whenever $\omega \in A \in \mathcal{F}^i$.

A triple $(\Omega, I, (t^i)_{i\in I})$ is called a *type space*. One can always easily recover the underlying σ-algebras $(\mathcal{F}^i)_{i\in I}$ from $(t^i)_{i\in I}$, since for each i, $\mathcal{F}^i$ is the σ-algebra generated[9] by t^i.

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[7] This can be broadened to: nearly all models in the economics, game theory and decision theory literature.

[8] By the σ-algebra generated by Π^i, we mean the collection of Borel sets which contain all those elements of Π^i that they intersect.

[9] The σ-algebra generated by a mapping $f : X \to Y$ between standard Borel spaces is $\{f^{-1}(B) \mid B \subseteq Y \text{ is Borel}\}$.

Definition 2. A type space satisfying the conditions that $t^i(\omega) \in \Delta_f(\Omega)$,[10] for all $i \in I$ and all $\omega \in \Omega$ will be called a type space with *everywhere finite support*.

If $t^i(\omega)$ has countable support for all $i \in I$ and all $\omega \in \Omega$, the type space has *everywhere countable support*. ♦

Assumption 1: For most of this paper, unless otherwise specified, we will assume that type spaces have everywhere finite support. This assumption is relaxed in one example in Section 4.5. In Section 5 we show that none of the results of this paper would change if we allow for type spaces with everywhere countable support

Assumption 2: Unless otherwise specified, we will henceforth assume[11] that all type spaces satisfy *positivity*, i.e., that $t^i(\omega)[\omega] > 0$ for all $i \in I$ and $\omega \in \Omega$. Type spaces that do not satisfy this condition are *non-positive*. We will offer justification in Proposition 9 as to why we restrict attention to positive type spaces.

We note here that from the assumption that $t^i(\omega)$ is a measure with finite support for all $i \in I$ and all $\omega \in \Omega$, and that t^i is positive, it follows that each atom[12] of each player's knowledge σ-algebra $\mathcal{F}^i$, and hence also of the common knowledge σ-algebra $\mathcal{F}$, is countable.[13] The atom of the common knowledge σ-algebra $\mathcal{F}$ containing ω is called the *common knowledge component* containing ω, and is denoted $K^\infty(\omega)$.

Remark 1. The knowledge σ-algebra of a player with finite (or countably) supported types is always separable. This follows from Proposition 4 in the Appendix.

A measure $\mu^i \in \Delta(X)$ such that t^i is a proper RCD for μ^i given $\mathcal{F}^i$ is a *prior* for t^i. A *common prior* is a measure μ that is a prior for the type functions of all the players $i \in I$.

Definition 3. A σ-algebra $\mathcal{F}$ with countable atoms is *belief induced* if there are are finitely many σ-algebras with finite atoms, $\mathcal{F}^1, \ldots, \mathcal{F}^n$, such that

[10] Recall that $\Delta_f(\Omega)$ is the set of finitely supported measures over Ω, hence everywhere finite support is equivalent to each player ascribing positive probability only to a finite number of elements in all knowledge components. Type spaces with finite fanout, as defined in Simon (2003), in which each partition element of the underlying partionally-based knowledge space contains only a finite number of elements, are a special case of this.

[11] This assumption also appears in Samet (1998).

[12] An *atom* of a σ-algebra is an element of it which is non-empty and is not strictly contained in any other element.

[13] This fact appears even more transparent when the formalism of countable Borel equivalence relationship is introduced in Section 6.1.1.

$\mathcal{F} = \cap_{j=1}^{n} \mathcal{F}^j$. This is equivalent to stating that there is a finite set of finitely supported knowledge spaces whose common knowledge σ-algebra is $\mathcal{F}$.

Not all σ-algebras with countable atoms are belief induced, even those whose structures are 'nice' enough to induce Borel equivalence relationships (see Section 6.1.1).[14] We elaborate on this in Section 5.

2.4. **Bayesian Games and Bayesian Equilibria.**

A Bayesian game $\Gamma = (\Omega, I, (t^i)_{i\in I}, (A^i)_{i\in I}, (r^i)_{i\in I})$ consists of the following components:

- $(\Omega, I, (t^i)_{i\in I})$ forms a type space (with $(\mathcal{F}^i)_{i\in I}$ understood implicitly as generated by $(t^i)_{i\in I}$).
- A^i is a finite action set for each Player $i \in I$.
- $r : \Omega \times \prod_{i\in I} A^i \to \mathbb{R}^I$ is a bounded measurable payoff function.

As usual, we extend r multi-linearly to $r : \Omega \times \prod_{i\in I} \Delta(A^i) \to \mathbb{R}^I$, that is,

$$r(\omega, (x^i)_{i\in I}) = \sum_{(a^i)_{i\in I} \in \prod_{i\in I} A^i} \left(\prod_{i\in I} x^i[a^i] \right) r(\omega, (a^i)_{i\in I})$$

A *strategy* of a player $i \in I$ is a mapping $\sigma^i : \Omega \to \Delta(A^i)$ which is constant on each player's knowledge component: I.e., if $\omega, \omega' \in \Omega$ with $t(\omega)[\omega'] > 0$ - equivalently, ω, ω' are in the same atom of $\mathcal{F}^i$ - then $\sigma^i(\omega) = \sigma^i(\omega')$. A *Bayesian ε-equilibrium*, with $\varepsilon \geq 0$, is a profile of strategies $\sigma = (\sigma^i)_{i\in I}$ such that for each $i \in I$, each atom A of $\mathcal{F}^i$, and each alternative strategy $x \in \Delta(A^i)$ of player i,

$$\int_A r^i(\omega, \sigma(\omega)) dt^i(\omega) + \varepsilon \geq \int_A r^i(\omega, x, \sigma^{-i}(\omega)) dt^i(\omega)$$

[14] We are grateful to Benjamin Weiss for pointing this out to us.

When a Bayesian ε-equilibrium σ satisfies the condition that each σ^i is Borel measurable,[15] σ is said to be a measurable[16] Bayesian ε-equilibrium (ε-MBE). When $\varepsilon = 0$ we will refer simply to an MBE instead of a 0-MBE.

3. Motivation

3.1. **Bayesian Games with a Continuum of States.**

As mentioned in the introduction, the main motivation for the results of this paper is understanding when Bayesian games with everywhere finite (or countable) support over a continuum of states admit or do not admit measurable Bayesian equilibria. Theorem 1 provides the answer, which in the briefest and most telegraphic manner can be reduced to asking whether or not the common knowledge σ-algebra is separable.

It turns out that separability is also a crucial element in resolving other anomalies that have been noted in the literature in games a continuum of states. These are explained in this section in the context of 'three paradoxes' that are all closely related. They all revolve around the question of the distinction between the *ex ante* stage and the interim stage.

The full state space, over which priors are defined, is usually taken to be the *ex ante* stage while the common knowledge component represents the interim stage after each player receives a signal. According to a widely accepted view, in reality there is no chance move that selects a player's type. The true situation the players face is the interim stage after the vector of types has been selected. However, incomplete information requires us to consider the *ex ante* stage in order to understand how the players make their choices in the interim stage, even though it is a fiction and there is no actual distinction between the different stages.

The paradoxes here challenge that view, because in these examples player behaviour is different depending on whether we are in the *ex ante* stage or the interim stage. This is particularly striking in the third example. The

[15]The combination of being Borel measurable and being constant in each knowledge component of Player i is equivalent to requiring that σ^i is $\mathcal{F}^i$-measurable.

[16] It is possible for a game to have Bayesian ε-equilibria that are not measurable as in, for example, Simon (2003). However, for our purposes it will suffice to concentrate on characterising the existence of measurable ε-equilibria, because given a game Γ that admits only non-measurable equilibria it is always possible to create another game Γ' that has no equilibria at all. This is accomplished by adding an additional player k to Γ' who is not in the player set of Γ. The payoffs of players $i \neq k$ in Γ' are defined to be exactly identical to their payoffs in Γ, while k's payoff is given by an integral over the actions of the players $i \neq k$. But if the equilibrium strategies of the players $i \neq k$ are non-measurable, at equilibrium player k cannot even define a payoff, much less an optimal strategy. See Hellman (2012) for an explicit example of such a construction.

concept of 'no acceptable bets' can be extended to 'no trading' (Milgrom and Stokey (1982)); the third example then shows that one can construct knowledge structures in which players can not measurably agree to trade in the *ex ante* stage but will agree to trade in the interim stage.

Taken together, Theorem 1, Theorem 2, and Theorem 3 in this paper show that all three paradoxes essentially disappear when the underlying common knowledge σ-algebra is separable; in fact, these solutions lead to full characterisations of when these pathologies may occur.

3.2. **Paradoxes.** The first two paradoxes, on Bayesian games and common priors in spaces over continuum many states, have been well-known in the literature for about a decade. The third paradox, on no betting, is fairly new.

The "Now You See It, Now You Don't" Bayesian Equilibrium.

Simon (2003) and Hellman (2012) present examples of Bayesian games that have no Bayesian equilibria. In greater detail, let $\mathbf{\Gamma}$ be one of these Bayesian games, with state space Ω. Then there exists no vector of measurable strategies $(\sigma_1, \ldots, \sigma_n)$, one per player, that forms a Bayesian equilibrium.

However, in both cases, one can choose any $\omega_0 \in \Omega$ and consider the common knowledge component of $K^\infty(\omega_0)$, the atom of $\mathcal{F}$ containing ω_0 (as determined by the knowledge structures of the players). Let $\mathbf{\Gamma}|_{K^\infty(\omega_0)}$ be the Bayesian game derived by restricting[17] $\mathbf{\Gamma}$ to the states in $K^\infty(\omega_0)$. Then there *is* a Bayesian equilibrium of $\mathbf{\Gamma}|_{K^\infty(\omega_0)}$, since this component is countable (see Simon (2003)).

The "Now You See It, Now You Don't" Common Prior.

This paradox was first noted in Simon (2000). We present here a slight variation of a version appearing in Lehrer and Samet (2011).

Consider the following type space over a state space Ω, as depicted in Figure 1. Ω is constructed out of four disjoint subsets of $\mathbb{R}^2$, labelled A_j for $j \in \{1, 2, 3, 4\}$:

- $A_1 = \{(x, x+1) \mid -1 \leq x < 0\}$
- $A_2 = \{(x, x) \mid -1 \leq x < 0\}$
- $A_3 = \{(x, x-1) \mid 0 \leq x \leq 1\}$
- $A_4 = \{(x, \psi(x)) \mid 0 \leq x \leq 1\}$, where $\psi(x) = x - c(\text{mod } 1)$ for a fixed irrational c in $(0, 1)$.

The knowledge space is partitionally generated, with Π_1 and Π_2 respectively the partitions of the two players. Player 1 is informed of the first

[17] If one restricts the type functions and payoffs to a set which is common knowledge – that is, in the common-knowledge σ-algebra $\mathcal{F}$ – then the resulting game is well-defined.

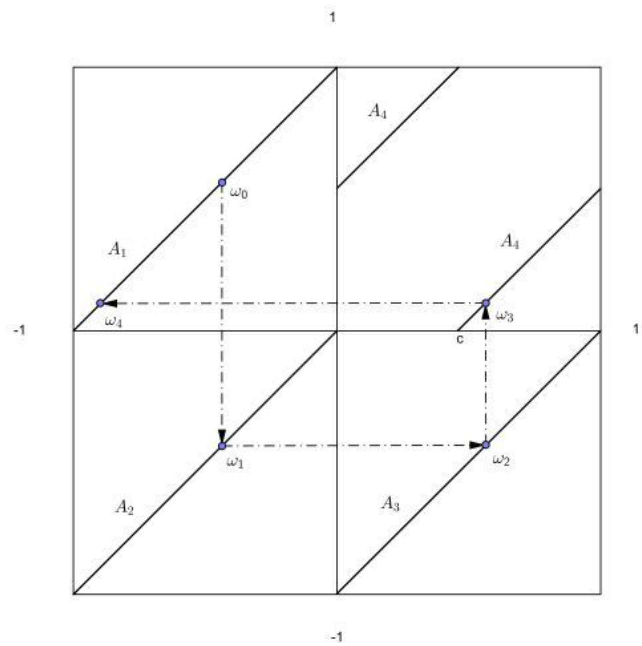

FIGURE 1. The state space consists of the three diagonals A_1, A_2, A_3 and of A_4. The latter is obtained by a rightward shift of the top-right diagonal by an irrational number c.

coordinate of the state and player 2 is informed of the second coordinate. Thus, each element of $\Pi_1(\omega)$ is composed of the two points on the vertical line that contains the state ω. Similarly, $\Pi_2(\omega)$ contains the two points on the horizontal line that includes the state ω.[18]

The posterior $t^i(\omega)$ for each of the two points in $\Pi_i(\omega)$ is $\frac{1}{2}$. Furthermore, let μ be the probability measure $\frac{1}{4}\sum_{j=1}^{4}\psi_j$, where ψ_j is the Lebesgue measure over A_j. Lehrer and Samet (2011) show that measurability conditions are satisfied by the posteriors and that μ is a common prior for (t^1, t^2).

However, although the entire space Ω has a well-defined common prior, if we again concentrate on the common knowledge component $K^\infty(\omega_0)$ of any arbitrary state ω_0 (fixing the posteriors) then there is *no* common prior[19] over $K^\infty(\omega_0)$. The reason for this is that $K^\infty(\omega_0)$ is a doubly infinite

[18] Formally, the knowledge space of Player 1 is

$$\mathcal{F}^1 = \{B \times [0,1] \cap \Omega \mid B \subseteq \mathbb{R} \text{ is Borel }\}$$

and similarly for $\mathcal{F}^2$ with the coordinates switched.

[19] There may, however, be a *common improper prior* over $K^\infty(\omega_0)$. An improper prior allows for the possibility that the total measure it defines over a space diverges.

countable sequence

$$\dots, \omega_{-(k+1)}, \omega_{-k}, \dots, \omega_{-1}, \omega_0, \omega_1, \dots, \omega_k, \omega_{k+1}, \dots \tag{3.1}$$

such that $(\{\omega_k, \omega_{k+1}\}) \subseteq \Pi_1$ for all odd $k \geq 1$, $(\{\omega_k, \omega_{k-1}\}) \subseteq \Pi_1$ for all even $k \leq 0$, $(\{\omega_k, \omega_{k-1}\}) \subseteq \Pi_2$ for all odd $k \geq 0$, and $(\{\omega_k, \omega_{k+1}\}) \subseteq \Pi_2$ for all even $k \leq -1$. Any common prior ν over $K^\infty(\omega_0)$ must satisfy the condition that $\nu(\omega_k) = \nu(\omega_k + 1)$ for all k. Thus all the countably many states in $K^\infty(\omega_0)$ must have the same probability, which is impossible.

The "Now You See It, Now You Don't" Acceptable Bet

For a type space with type spaces $(\Omega, I, (t^i)_{i\in I})$ a ***bet*** is a list of $(f_i)_{i\in I})$ of bounded[20] random variables $f_i : \Omega \to \mathbb{R}$ such that $\sum_{i\in I} f_i(\omega) = 0$ for all $\omega \in \Omega$. An *acceptable bet* is a bet that satisfies the condition that

$$E^i[f_i \mid \omega] := \int_\Omega f_i(s) dt^i(\omega)[s] > 0 \text{ for all } \omega \in \Omega. \tag{3.2}$$

Equation (3.2) states that when a bet is acceptable there is common knowledge everywhere that every player has expectation of positive gain, even though a bet is everywhere zero sum by definition.

Summing the integrals and integrating over the entire space shows that the existence of a common prior implies that there is no acceptable bet (see, for example, a similar argument in Hellman (2014)). By a result in Feinberg (2000) (see also Heifetz (2006)), the converse also holds if Ω is compact and we allow only continuous bets.

As there is a common prior over the entire compact space in the example depicted in Figure 1, there can be no acceptable bet over the entire space. However, one *can* construct acceptable bets on each common knowledge component in this example. Again we concentrate on a particular state ω_0 and the common knowledge component $K^\infty(\omega_0)$ containing it. A variation of a construction from Hellman (2014), using $K^\infty(\omega_0)$ as in (3.1), defines the following function $f : K^\infty(\omega_0) \to \mathbb{R}$:

$$f(\omega_n) = \begin{cases} 1 & \text{if } n = 0 \\ 1 + \sum_{i=1}^{n} \frac{1}{2^i} & \text{if } n > 0 \text{ is even} \\ -(1 + \sum_{i=1}^{n} \frac{1}{2^i}) & \text{if } n > 0 \text{ is odd} \\ \sum_{i=1}^{-n} \frac{1}{2^i} & \text{if } n < 0 \text{ is even} \\ -(\sum_{i=1}^{-n} \frac{1}{2^i}) & \text{if } n < 0 \text{ is odd} \end{cases}$$

It is easy to check that $(f, -f)$ is an acceptable bet over $K^\infty(\omega_0)$, even though there is no globally acceptable bet over the entire space Ω.

[20] We assume boundedness to avoid anomalies; see Feinberg (2000) and Hellman (2014).

4. Results

4.1. Measurable Bayesian Equilibria.

Definition 4. An everywhere finitely supported type space satisfying the condition that its common knowledge σ-algebra is separable will be called a *tame* type space.

A Bayesian game whose underlying type space is tame will be called a *tame Bayesian game*. ♦

Sometimes we will want to specify exactly how a type space fails to be tame:

Definition 5. Let Ω be a standard Borel space, and let $\mathcal{F}$ be a sub-σ-algebra of the Borel σ-algebra that is not separable and which is belief induced. Then a type space whose underlying common knowledge σ-algebra is $\mathcal{F}$ will be called an $\mathcal{F}$*-non-tame* type space. ♦

Theorem 1 extends Harsányi's Theorem, essentially stating that (within the class of type spaces that are everywhere finitely[21] supported) a Bayesian game is guaranteed to have a measurable Bayesian equilibrium if and only if it is tame.

Theorem 1.

I. Let Γ be a tame Bayesian game Then there exists an MBE for Γ.

II. Let Ω be a standard Borel space, and let $\mathcal{F}$ be a belief-induced and non-separable σ-algebra. Then there exists an $\mathcal{F}$-non-tame (positive) type space τ with a common prior and a Bayesian game Γ over τ that does not possess an ε-MBE for small enough $\varepsilon > 0$. In particular, Γ has no MBE.

To prove Theorem 1.I we proceed in three steps. First, we will develop a notion of the space of (not necessarily positive) Bayesian games with countably many states S, player set I and action sets $(A^i)_{i\in I}$, which we will denote by $\mathfrak{B}(S, I, (A^i)_{i\in I})$ (or just $\mathfrak{B}$ for short). Afterwards, we will prove the existence of a Bayesian equilibrium *selection* for this class of games. Then we will show how one can measurably map the games induced on each common knowledge component of a general game into the space of games on countably many states S; the composition of this mapping and the Bayesian equilibrium selection from the second step will give us the required global Bayesian equilibrium.

We can construct such a mapping because the separability, it turns out, allows us measurably to enumerate the elements of each atom, and once we

[21] As shown in Section 5, this can be extended to countably supported.

have this enumeration we can map the game on each atom to its appropriate game in the space $\mathfrak{B}$; when we lack such an enumeration, this cannot be done because we have no canonical way to select the mapping. Details are given in Section 6.5.

For the proof of 1.II, we embed the game given in Hellman (2012) which does not possess ε-MBE into the given structure, using known theorems on embedding countable Borel equivalence relationships into each other.

4.2. **Common Priors over Components.**

Let $\tau = (\Omega, I, (t^i)_{i\in I})$ be a type space with common knowledge σ-algebra $\mathcal{F}$. If $K \in \mathcal{F}$ then $\tau_K := (K, I, (t^i|_K)_{i\in I})$, consisting of the state space K and the type functions restricted to K, is a well-defined type space. This is true in particular if K is an atom of $\mathcal{F}$. Furthermore, if μ is a common prior, then we say that a property holds for *almost every common knowledge component* if the set of components for which it does not hold are all contained in a set of measure zero.

Theorem 2 essentially states that given a type space τ with a common prior, the type space τ_K for any common knowledge component K is guaranteed also to have a common prior if and only if the underlying common knowledge σ-algebra is separable almost everywhere.

Theorem 2. *Let τ be a type space with a common prior μ. The following conditions are equivalent:*

(1) There exists $X \in \mathcal{F}$ with $\mu(X) = 1$ such that $\mathcal{F}|_X$ is separable.
(2) For almost every common knowledge component K, the type space τ_K has a common prior.
(3) There is a proper regular conditional probability t of μ given $\mathcal{F}$ such that for almost every common knowledge component K and each $x \in K$, $t(x)$ is a common prior for τ_K.

Remark 2. In particular, it follows that the common knowledge σ-algebra $\mathcal{F}$ generated in Figure 1 of Section 3 is not separable. This, however, could be seen by more elementary means: the restriction of $\mathcal{F}$ to any one of the sets A_1, A_2, A_3, A_4 is easily seen to be induced by the equivalence relation induced by an irrational rotational of the circle – i.e., $x \to x - c \; mod \; 1$, c being irrational – and this σ-algebra is well-known to be non-separable.

The proof of Theorem 2 can be found in Section 6.4. The main step is to show to that (1) implies (2). The key here is to show that for each player i, if one first takes the regular conditional distribution of μ w.r.t. $\mathcal{F}$ and then from *that* one takes the conditional distribution w.r.t. player i's knowledge structure, one recovers i's original type.

4.3. **No Betting.**

Theorem 3 essentially states that given a type space τ, we are guaranteed that almost all common knowledge components possess no acceptable bets (i.e., there are no acceptable bets at the interim stage) if and only if τ is tame.

The condition guaranteeing consistency in agreeing to trade between the *ex ante* and interim stages thus turns out to be virtually identical to the condition guaranteeing the existence of measurable equilibria in Bayesian games (as in Theorem 1). The basic insight underlying this parallelism comes from the measurable selection, as expressed in Borel transversality, that tameness grants in both cases. In the briefest terms, with measurable selection at hand, a 'local' property associated with each common knowledge component K – such as the existence of an equilibrium in the game restricted to K, or the existence of an acceptable bet at all states in K – can be 'stitched together' by the measurable selection into a corresponding measurable property that is 'global' in the sense of holding at almost all states in Ω. In this way, one avoids situations in which local behaviour differs from globally measurable behaviour.

Theorem 3.

I. *Let τ be a tame type space with a common prior. Then for almost every common knowledge component K there are no acceptable bets on K.*
II. *Let Ω be a standard Borel space, and let $\mathcal{F}$ be a belief-induced and non-separable σ-algebra. Then there exists an $\mathcal{F}$-non-tame (positive) type space τ with a common prior such that on almost every common knowledge component there exists an acceptable bet.*

The proof of Theorem 3.I is given in Section 6.4, as an immediate Corollary of Theorem 2; the proof of Theorem 3.II is given in Section 6.6. Like the proof of Theorem 1.II, the proof Theorem 3.II involves embedding a game in which acceptable bets exist on the common knowledge components – the example of the two-player game from Lehrer and Samet (2011) which appears in Section 3 – into the given structure, and then showing that the acceptable bet for those two players on the image of the embedding can be extended (on each component individually) to an acceptable bet for all players (however many needed to give the desired common knowledge structure) on the entire component.

4.4. **A Condition for Separability.**

Theorem 4. *Suppose Ω is a standard Borel space with metric d, and $\mathcal{F}$ is a sub-σ-algebra of the Borel σ-algebra, and suppose that for each atom A in $\mathcal{F}$, A is countable*[22] *and* $\inf_{x \neq y,\ x,y \in A} d(x,y) > 0$. *Then $\mathcal{F}$ is separable.*

In other words, as long as in each atom the elements 'keep their distance' and do not get 'bunched up', the σ-algebra is separable. The proof appears in Section 6.3. We illustrate with some examples in Section 4.5.

4.5. **Examples of Separability and Non-Separability.**

We present here two examples to illustrate the concept of tameness.

Example 1: The state space is $\Omega = \mathbb{R}$, corresponding to an amount that is chosen via some common prior, which we can assume is a normal distribution. Player 1 is told the absolute value $|\omega|$ of the asset amount but is not told whether it is positive or negative, which can be thought of as not knowing whether the amount he is told is what he is due or what he owes. Player 2 in contrast is told the absolute value of $|\omega + 1|$, that is, not only does he not know whether it is an amount due or owed, the amount is always overestimated by 1 unit. It is easy to verify that the atoms of the common knowledge σ-algebra are of the form $\{\pm x + n \mid n \in \mathbb{Z}\}$.

We claim first that this σ-algebra is separable. As the useful Proposition 4 in the appendices shows, this is equivalent to the existence of a Borel set $B \subseteq \Omega$ which intersects each atom of $\mathcal{F}$ in exactly one point. In our case, simply take $B = [0, \frac{1}{2}]$. Separability also follows immediately from Theorem 4, which in a sense generalises this example.

Theorems 1, 2, and 3 then guarantee that for any type space with these knowledge σ-algebras common priors exist on all components, Bayesian equilibria exist regardless of the payoffs, and there are no agreeable bets on any components. Finding the common priors and Bayesian equilibria, however, can be quite cumbersome, hence the advantage of possessing general existence theorems such as those we present here.

Example 2: To the game in the previous example now add a 3rd player. We will allow ourselves to deviate slightly from the framework presented so far by allowing this player's belief to be supported on a infinite (but countable) number of points; as we explain later in Section 5.1, this more general framework can also be handled by our results.

The 3rd player is informed of some amount but he believes that the actual number may be some integer multiple or divisor of what he is told (for example, he may not be sure if this is the total amount, or the amount per person in a group of unknown size after the amount has been divided).

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[22] The countability of A actually follows from the second condition.

Hence, the atoms of his knowledge are generated by multiplication by a positive integer - and in particular, the atoms of his knowledge are of the form $\{\ldots, \frac{1}{3}x, \frac{1}{2}x, x, 2x, 3x, \ldots\}$. When combined with the uncertainty of Players $1, 2$ – which, as we saw, are generated by addition and subtraction by an integer – we see that in this common knowledge σ-algebra $\mathcal{G}$, x, y are in the same atom iff $x - y$ is rational. It is well known that there cannot exist a set B which intersects each atom in only finitely many points (see for example Chapter 2 of Rudin (1986)).

Hence, this common knowledge structure is not separable, and as a result, common knowledge components may not possess common priors, Bayesian equilibria may not exist for certain priors and payoffs, and for certain type structures – even those induced by a common prior – we may find acceptable bets on components even though there are no globally acceptable bets.

5. Extensions, Variations, and Further Details on Belief Induced Relations

5.1. **Types with Countable Support.**

None of the results of this paper would change if we allow for type spaces with countable support; that is, for each $\omega \in \Omega$ and each Player i, $t^i(\omega)$ is a purely atomic (but not necessarily finitely supported) Borel measure. The proofs all remain largely the same, with only minor alterations. The condition of belief induction also remains unaltered if one allows types with countable support; formally:

Proposition 3. *A σ-algebra $\mathcal{F}$ is belief induced iff there are separable σ-algebras $\mathcal{H}_1, \ldots, \mathcal{H}_m$ with countable atoms such that $\mathcal{F} = \cap_{j=1}^m \mathcal{H}_j$.*

The proof is in Section 6.10.

5.2. **Countable Partitions.**

Consider the following natural model (which is not covered by the results above, as it admits partition elements with continuum support). Let the continuum states be represented by the real numbers in the interval [0,1] and suppose that following receipt of a signal each player gives positive support to a sub-interval of [0,1]. Further assume that players are limited to some finite accuracy in their measurements and therefore the end-points of the sub-intervals in their posteriors are limited to rational numbers. In that case there can only be a countable number of distinct of partition elements in the posteriors of the players.

Bayesian games in which the partition of each player consists of a countable number of elements are guaranteed to have measurable Bayesian equilibria, even when the cardinality of the support of every posterior element is the continuum. This follows from Theorem 1 of Milgrom and Weber

(1985), because the countable cardinality of the partition elements guarantees that the game has absolutely continuous information, as defined in that paper.

5.3. **On Belief Induced Relations.** As we have mentioned, not all σ-algebras with countable atoms are belief induced. Equivalently, using the notion of countable Borel equivalence relationships (Section 6.1.1), there are countable Borel equivalence relationships $\mathcal{E}$ which cannot be written as the transitive closure of finitely many finite Borel equivalence relationships. This can be shown using the concept of the *cost* of a countable Borel equivalence relation $\mathcal{E}$ with an invariant[23] measure μ. We briefly recall this concept; for a more comprehensive treatment, see Kechris and Miller (2004).

A Borel graph G on a standard Borel space Ω is a Borel relation on Ω (i.e., a Borel subset of $\Omega \times \Omega$) that is irreflexive and symmetric. A Borel graph G induces a Borel equivalence relation $\mathcal{E}$ on Ω: $\mathcal{E}$ is the reflexive and transitive closure of G. We say that G is a *graphing* of $\mathcal{E}$. Given such a graph, for each $v \in \Omega$, let $d_G(v) \in \{0, 1, 2, \ldots, \infty\}$ denote the cardinality of the set $\{w \in \Omega \mid (v, w) \in G\}$. Clearly, if $d_G(v)$ is countable for all $v \in \Omega$, then so is the induced equivalence relation $\mathcal{E}$; conversely, if $\mathcal{E}$ is a countable Borel equivalence relation, then it is induced by some Borel graph with vertices of countable degree (this follows easily, e.g., from Proposition 5 of Section 6.1.1.).

The *cost* of a countable Borel equivalence relation $\mathcal{E}$ (with respect to an invariant measure μ) is defined as:

$$C_\mu(\mathcal{E}) := \inf\{\frac{1}{2}\int_\Omega d_G(\omega) d\mu(\omega) \mid G \text{ spans } \mathcal{E}\}$$

A result of Levitt, e.g. (Kechris and Miller, 2004, Ch. 20), is that if T is a Borel transversal for a countable Borel equivalence relation $\mathcal{E}$ with an $\mathcal{E}$-invariant measure μ, then $C_\mu(\mathcal{E}) = \mu(\Omega \backslash T)$; in particular, if μ is finite, then so is $C_\mu(\mathcal{E})$.

Suppose that $\mathcal{E}$ is a countable Borel equivalence relation, and is the equivalence relation generated by $\mathcal{E}_1, \ldots, \mathcal{E}_n$ (that is, the coarsest equivalence relation that each $\mathcal{E}_k$ refines), and suppose that μ is $\mathcal{E}$-invariant. It is then

[23] A (not-necessarily finite) measure is $\mathcal{E}$-invariant if for every Borel bijection $f : \Omega \to \Omega$ satisfying $f(\omega) \sim_{\mathcal{E}} \omega$ for all $\omega \in \mathcal{E}$, it holds that for all Borel $A \subseteq \Omega$, $\mu(f^{-1}(A)) = \mu(A)$.

clearly also $\mathcal{E}_k$ invariant for each $k = 1, \dots, n$, and it's easy to see that[24]

$$C_\mu(\mathcal{E}) \leq \sum_{k=1}^{n} C_\mu(\mathcal{E}_k)$$

Combining this observation with the result of Levitt, and the fact that finite[25] Borel equivalence relations are clearly always smooth[26], we see that if $\mathcal{E}_1, \dots, \mathcal{E}_n$ are finite, $C_\mu(\mathcal{E})$ is finite.

Hence, to show a non-belief induced countable Borel equivalence relation, it suffices to find one with infinite cost w.r.t. some invariant measure $\mathcal{E}$ on it. A result of Gaboriau, e.g. (Kechris and Miller, 2004, Cor 27.10), states that if $\mathcal{E}$ is a countable Borel equivalence relation with finite invariant measure μ, and $\mathcal{T}$ is a Borel tree[27] that is a graphing of $\mathcal{E}$, then $C_\mu(\mathcal{E}) = \frac{1}{2}\int_\Omega d_{\mathcal{T}}(\omega)d\mu(\omega)$.

Now, let F_∞ denote the free (non-abelian) group with countably many generations. This group acts on 2^{F_∞} via $(f(x))(g) = x(f \cdot g)$ for $x \in 2^{F_\infty}$, $f, g \in F_\infty$, and induces a countable Borel equivalence relation by $x \sim y$ iff $\exists g \in F_\infty$ with $g \cdot x = y$. From this, one deduces easily that if $\mu = \prod_{f \in F_\infty}(\frac{1}{2}, \frac{1}{2})$ (which is clearly $\mathcal{E}$-invariant) it holds by Gaboriau's result that $C_\mu(\mathcal{E}) = \infty$.

5.4. **A Further Result: Agreeing to Agree.**

In this section we assume that there are only two players. The following definitions are taken from Lehrer and Samet (2011):

Definition 6. Let E be an event in the state space $(\Omega, \mathcal{B})$ with information structure (Π^1, Π^2) and type functions (t^1, t^2). An agreement on E is an event of the form

$$\{\omega \in \Omega \mid t^1(\omega)(E) = t^2(\omega)(E) = p\}$$

for some $0 < p < 1$. We say that agreeing to agree is possible for E (with μ) if there is a common prior μ for the type functions t^1, t^2 and an agreement A on E such that $\mu(K^\infty(A)) > 0$. ♦

We also define an *ignorance operator* as in Lehrer and Samet (2011):

[24] Note that μ is $\mathcal{E}_k$-invariant for each $k = 1, \dots, n$; hence, for any graphings $G_1, \dots, G_n$ of $\mathcal{E}_1, \dots, \mathcal{E}_n$, respectively, $G = \cup_{k=1}^n G_n$ spans $\mathcal{E}$.

[25] An equivalence relation is called finite if its equivalence classes are all finite.

[26] One can see that finite Borel equivalence relations are always smooth, for example, by taking a Borel ordering $<$ on Ω, and choose the $<$-minimal element in each equivalence class to get a transversal

[27] A Borel tree is a Borel graph with no cycles.

Definition 7. The event that Player i is ignorant of event E is

$$A^i(E) = (\Omega \backslash K^i(E)) \cap (\Omega \backslash K^i(\Omega \backslash E))$$

and $I(E) := I^1(E) \cap I^2(E)$. ♦

In addition, for event F, we define the knowledge operator K_F^i, which is the knowledge operator induced by the partition generated Π^i and $\{F, \Omega \backslash F\}$, and we define the associated higher-order knowledge operators, the common knowledge operator K_F^∞, as well as the operator I_F.

Theorems 1 and 2 from Lehrer and Samet (2011) can be summarized as follows:

Theorem 5. *Assume the state space is countable.*[28] *The following conditions are equivalent for an event E:*

(i) Agreeing to agree is possible for E (with some common prior).
(ii) There exists a non-empty finite event F such that $F \subseteq K^\infty(I(E))$ and $F \subseteq K_F^\infty(I_F(E))$.
(iii) Agreeing to agree is possible for E with a common prior with finite support.

As remarked in Lehrer and Samet (2011), the implication (ii) $\rightarrow$ (i) holds even if the state space is uncountable; but, by example, the converse direction does not hold. We wish to prove the following, answering an open problem raised in Section 5.1 of Lehrer and Samet (2011).

Theorem 6. *Assume that the state space $(\Omega, \mathcal{B})$ is standard Borel. Assume the player's knowledge structure is such that the common knowledge σ-algebra $\mathcal{F}$ is separable. Let μ be a common prior. The following conditions are equivalent for an event E:*

(i) Agreeing to agree is possible for E with μ.
(ii) There exists an event F with $\mu(F) > 0$ such that $F \subseteq K^\infty(I(E))$, $F \subseteq K_F^\infty(I_F(E))$, and such that the intersection of F with any common knowledge component is finite.
(iii) Agreeing to agree is possible for E with a common prior ν, which is absolutely continuous w.r.t. μ, for which there exists a Borel set G, intersecting each common knowledge component in finitely many points, such that $\nu(G) = 1$.

The proof appears in Section 6.9, where the main part of the proof consists of showing that (i) implies (ii). We show that under separability, we can apply Theorem 5 on each common knowledge component, and then select a

[28] Countable sets are automatically endowed with the discrete σ-algebra.

finite subset from each component in a measurable way using measurable-selection theorems.

6. APPENDIX: TOOLS AND PROOFS

6.1. **Mathematical Tools.**

6.1.1. *Countable Borel Equivalence Relationships.* A relation $\mathcal{E}$ on a standard Borel space Ω is said to be *Borel* if it is Borel as a subset of $\Omega \times \Omega$; i.e., if the set $\{(x,y) \in \Omega \mid x\mathcal{E}y\}$ is Borel. A Borel equivalence relation is said to be *countable* if each equivalence class is countable.

If $\mathcal{E}$ is a Borel equivalence relation on a space Ω, then for each $\omega \in \Omega$, $[\omega]_{\mathcal{E}}$ (or just $[\omega]$ when it is clear to which relation we are referring) denotes the equivalence class of ω.

Given a σ-algebra $\mathcal{F}$, there is an *induced equivalence relation*, denoted $\mathcal{E}_{\mathcal{F}}$, defined by

$$[\omega]_{\mathcal{E}_{\mathcal{F}}} := [\omega]_{\mathcal{F}} := \cap_{\omega \in A \in \mathcal{F}} A$$

Given a Borel equivalence relation $\mathcal{E}$ on Ω and a set $T \subseteq \Omega$, the *saturation* $[T]_{\mathcal{E}}$ of T w.r.t. $\mathcal{E}$ is $[T]_{\mathcal{E}} = \cup_{\omega \in T}[\omega]_{\mathcal{E}}$. We will sometimes write $[T]_{\mathcal{F}}$ instead of $[T]_{\mathcal{E}_{\mathcal{F}}}$. Conversely, if $\mathcal{E}$ is a Borel equivalence relation, the induced σ-algebra $\mathcal{F}_{\mathcal{E}}$ is the collection of saturated Borel sets.

In terms that may be more familiar to game theorists used to working with finite atomic partitions as bases for σ-algebras, $[\omega]_{\mathcal{F}}$ is the atom containing ω and for an event T, $[T]_{\mathcal{F}}$, is the union of the atoms intersecting T.

A *transversal* of an equivalence relation is a set that intersects each equivalence class in exactly one point. A Borel equivalence relation $\mathcal{E}$ is said to be *smooth* (or *tame*) if there is a Borel mapping $\psi : \Omega \to X$, where X is some standard Borel space, such that $\psi(x) = \psi(y) \leftrightarrow x \sim y$.

Given a standard Borel space Ω and a sub-σ-algebra $\mathcal{F}$ of the Borel σ-algebra, we let $\Omega/\mathcal{F}$ denote the quotient space whose elements are the equivalence classes induced by $\mathcal{F}$ and the induced σ-algebra consists of precisely the images of the sets in $\mathcal{F}$ under the quotient map.

We will make repeated use of the following proposition:

Proposition 4. *The following conditions are equivalent for a countable Borel equivalence relation $\mathcal{E}_{\mathcal{F}}$ induced*[29] *on Ω by a σ-algebra $\mathcal{F}$:*

(a) $\mathcal{F}$ is separable.
(b) There is a Borel transversal for $\mathcal{E}_{\mathcal{F}}$.

[29] It is not clear whether every sub-σ-algebra of the Borel σ-algebra with countable atoms induces a Borel equivalence relationships.

(c) The quotient space $\Omega/\mathcal{F}$ is standard Borel.
(d) The equivalence relation $\mathcal{E}_{\mathcal{F}}$ is smooth.

Proof. The equivalence (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d) is stated in Propositions 6.3 and 6.4 of Kechris and Miller (2004). If (c) holds and Λ is a countable collection of Borel sets generating the Borel structure on $\Omega/\mathcal{F}$, the collection $\{q^{-1}(U) \mid U \in \Lambda\}$, where $q : \Omega \to \Omega/\mathcal{F}$ is the quotient map, generates $\mathcal{F}$, and hence (a) holds.

Now, suppose (a) holds; let $B_1, B_2, \ldots \in \mathcal{F}$ generate $\mathcal{F}$. The map $p : \Omega \to 2^{\mathbb{N}}$ defined coordinate-wise by $p_n(\omega) = 1_{B_n}(\omega)$ is Borel and satisfies $p(x) = p(y)$ iff $x\mathcal{E}_{\mathcal{F}}y$, and hence $\mathcal{E}_{\mathcal{F}}$ is smooth. $\square$

The following is from Feldman (1977):

Proposition 5. *Let $\mathcal{E}$ be a countable Borel equivalence relation on a standard Borel space Ω. Then there is a countable group G of Borel bijections $\Omega \to \Omega$ such that for each $\omega \in \Omega$, $[\omega]_{\mathcal{E}} = \{g(\omega) \mid g \in G\}$.*

6.1.2. *Preliminaries on Knowledge.* Let τ be a type space with knowledge σ-algebras $(\mathcal{F}^i)_{i\in I}$. For each $i \in I$ and each set $N \subseteq \Omega$, let $K^i(N)$ denote the saturation of N w.r.t. $\mathcal{F}^i$, i.e., $K^i(N) = [N]_{\mathcal{F}^i}$. If $\omega \in \Omega$, write for short $K^i(\omega) = K^i(\{\omega\})$. Since the saturation of countable Borel sets under a Borel equivalence relation is also Borel, we have:

Lemma 6. *If N is Borel, then so is $K^i(N)$.*

For each finite sequence $\hat{i} = (i_1, \ldots, i_k) \in I^* := \cup_{n\geq 0} I^n$ and $N \subseteq \Omega$, let

$$K^{\hat{i}}(N) = K^{i_k}\Big(K^{i_{k-1}}\big(\cdots(K^{i_1}(N))\cdots\big)\Big)$$

and $K^{\hat{i}}(\omega) = K^{\hat{i}}(\{\omega\})$. Then, define

$$K^{\infty}(N) = \cap_{\hat{i}\in I^*} K^{\hat{i}}(N)$$

Lemma 7. *Let Ω be a standard Borel space, let $\mathcal{G}$ be a σ-algebra with countable atoms, let $\mu \in \Delta(\Omega)$ and let t be a proper RCD for μ given $\mathcal{G}$ that satisfies $t(\omega)[\omega] > 0$ for all $\omega \in \Omega$. Let $N \subseteq \Omega$ be a μ-measurable set satisfying $\mu(N) = 0$. Then there is $K \in \mathcal{G}$ with $N \subseteq K$ and $\mu(K) = 0$.*

Proof. For each $n \in \mathbb{N}$, define

$$N_n = \{\omega \in N \mid t(\omega)[\omega] \geq \frac{1}{n}\}$$

Let $K_n = [N_n]_{\mathcal{G}} \in \mathcal{G}$; in other words, $K_n = \cup_{\omega\in N_n}[\omega]_{\mathcal{G}}$. For all $\omega \in \Omega$, if $\omega \notin K_n$ then

$$t(\omega)(K_n) = 0,$$

while if $\omega \in K_n$ then

$$t(\omega)(K_n) = t(\omega)[\omega] \leq n \cdot t(\omega)[\omega] \leq n \cdot t(\omega)(N_n),$$

Therefore

$$\mu(K_n) = \int_\Omega t(\omega)(K_n) d\mu(\omega) \leq n \cdot \int_\Omega t(\omega)(N_n) d\mu(\omega) = n \cdot \mu(N_n) = 0$$

and we can take $K = \cup_{n \in \mathbb{N}} K_n$. □

Corollary 8. *Let τ be a positive type space with a common prior μ. Let $N \subseteq \Omega$ be a μ-measurable set satisfying $\mu(N) = 0$. Then there is $K \in \mathcal{F}$, the common-knowledge σ-algebra, with $N \subseteq K$ and $\mu(K) = 0$.*

To prove Corollary 8, one applies Lemma 7 inductively to show that for each $n \in \mathbb{N}$ and each $\hat{i} \in I^n$, $K^{\hat{i}}(N)$ is Borel. Corollary 8 will often be used implicitly; in many proofs, when useful, we will automatically assume that some null set we are discarding is common knowledge – or, equivalently, and more to the point, that its complement is common knowledge.

Finally, we justify our concentration on positive type spaces. Proposition 9 essentially states that if a space has a common prior (whether or not positive) then under that prior the event containing the states to which any player i assigns zero probability in the posterior is a null event:

Proposition 9. *Let τ be a type space (not necessarily positive) with a common prior μ. Denote, for each $i \in I$,*

$$N^i := \{\omega \in \Omega \mid t^i(\omega)[\omega] = 0\}$$

Then $\mu(N^i) = 0$ for all $i \in I$.

Proof. Recall that each type function is $\mathcal{F}^i$-measurable. Let $x \in N^i$ and $\omega \in \Omega$. If ω is not in the same atom of x then $t^i(x)[\omega] = 0$ since ω is not in the support of $t^i(x)$. Otherwise, since $t^i(x)[x] = 0$ and $t^i(x)[\omega] = t^i(x)[x]$, we again conclude that $t^i(x)[\omega] = 0$. The proposition follows from the definition of an RCD. □

We will also need:

Proposition 10. *Let $\mathcal{F}^1, \ldots, \mathcal{F}^I$ be knowledge σ-algebras with finite atoms on a standard Borel space Ω. Then there exists a positive type space τ with common prior $\nu \in \Delta(\Omega)$. If $\mu \in \Delta(\Omega)$, then ν can be chosen such that $\mu \ll \nu$.*

The proof appears in Section 6.7.

6.1.3. *Some More Descriptive Set Theory Theorems.* The following variants are slight strengthening of the Lusin-Novikov theorem (see, for example, Theorem 18.10 of Kechris (1995)):

Proposition 11. *Given a σ-algebra $\mathcal{F}$ that induces a countable Borel equivalence relation on a standard Borel space Ω:*
I. There exist partial[30] *Borel mappings $f_1, f_2, \ldots$, from Ω to Ω such that for all $\omega \in \Omega$, $[\omega]_{\mathcal{F}} = \bigcup_{\{n | \omega \in dom(f_n)\}} \{f_n(\omega)\}$, $f_n(\omega) \neq f_m(\omega)$ whenever $n \neq m$ and $\omega \in$ dom$(f_n) \cap$ dom(f_m), and if $\omega \notin dom(g_k)$, then $q \notin dom(g_n)$ for all $n > k$.*
II. If $\mathcal{F}$ is separable,[31] *there exist partial Borel mappings $g_1, g_2, \ldots$, from $\Omega/\mathcal{F}$ to Ω such that for all $q \in \Omega/\mathcal{F}$, $q = \bigcup_{\{n | q \in dom(g_n)\}} \{g_n(q)\}$, $g_n(q) \neq g_m(q)$ whenever $n \neq m$ and $\omega \in$ dom$(g_n) \cap$ dom(g_m), and if $q \notin dom(g_k)$, then $q \notin dom(g_n)$ for all $n > k$.*

As a corollary one can show the following well-known result, which we will make repeated implicit use of:

Proposition 12. *Let X, Y be standard Borel spaces, and let $f : X \to Y$ be Borel such that for each $y \in Y$, $f^{-1}(y)$ is at most countable (i.e., the map is countable-to-one). Then for each Borel $B \subseteq X$, $f(B)$ is Borel.*

If $(\Omega, \mathcal{E})$, $(\Lambda, \mathcal{D})$ are standard Borel spaces with Borel equivalence relations $\mathcal{E}$ and $\mathcal{D}$ induced on them, $(\Omega, \mathcal{E})$ is said to *embeddable* into $(\Lambda, \mathcal{D})$ if there is an injective Borel mapping $\psi : \Omega \to \Lambda$ such that for all $\omega, \eta \in \Omega$, $\omega \mathcal{E} \eta \iff \psi(\omega) \mathcal{D} \psi(\eta)$; in this case, we denote $(\Omega, \mathcal{E}) \sqsubset (\Lambda, \mathcal{D})$.

A countable Borel equivalence relation is said to be *hyperfinite* (Dougherty et al. (1994)) if it is induced by the action of a Borel $\mathbb{Z}$-action on Ω; i.e., if there is a bijective[32] Borel mapping $T : \Omega \to \Omega$ such that $x \mathcal{E} y \iff \exists n \in \mathbb{Z}, T^n(x) = y$.

Proposition 13. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be non-smooth countable Borel equivalence relations on standard Borel spaces Ω_1, Ω_2, with $\mathcal{E}_1$ being hyperfinite. Then $(\Omega_1, \mathcal{E}_1) \sqsubset (\Omega_2, \mathcal{E}_2)$.*

Proof. Let $\mathcal{E}_t$ be the tail equivalence relation on $C = 2^{\mathbb{N}}$; i.e., if $S : C \to C$ is defined by $(Sx)_n = x_{n+1}$, then $x \mathcal{E}_t y$ iff $\exists k, m \in \mathbb{N}$ such that $S^k(x) =$

[30] A mapping from a certain domain is called partial if it is defined only on a subset of the domain; it follows that the domain, as in the inverse image of the entire range space, is Borel.

[31] The separability guarantees that $\Omega/\mathcal{F}$ has a standard Borel structure.

[32] If a Borel mapping between standard Borel spaces is injective, a theorem by Kuratowski states that its image is standard Borel and that its inverse is Borel.

$S^m(y)$; $\mathcal{E}_t$ is non-smooth and hyperfinite, see (Dougherty et al., 1994, Sec. 6). By the Glimm-Effros dichotomy for countable Borel equivalence relations, Harrington et al. (1990), since $\mathcal{E}_2$ is not smooth,[33] $(C, \mathcal{E}_t) \sqsubset (\Omega_2, \mathcal{E}_2)$; denote such an embedding by θ. By Theorem 7.1 of Dougherty et al. (1994), any two non-smooth hyperfinite equivalence relations can be embedded into each other; hence $(\Omega_1, \mathcal{E}_1) \sqsubset (C, \mathcal{E}_t)$; denote such an embedding φ. This yields $\psi = \varphi \circ \theta$ as the required embedding. □

6.2. **Embedding of Games.** The following proposition will be of use in proving Theorem 1.II and Theorem 3.II.

Proposition 14. *Let Ω and X be standard Borel spaces, and let $\mathcal{F}$ be a σ-algebra on Ω which is non-smooth and belief induced. Let $\tau_X = (X, J, (t^j_X)_{j\in J})$ be an everywhere finite and positive type space with a common prior μ_X, and assume that its common knowledge equivalence relationship $\mathcal{E}_X$ is hyperfinite. Then there is a Borel embedding $\psi : X \to \Omega$ and a set of players I containing J, with an everywhere finite and positive type space $\tau = (\Omega, I, (t^i)_{i\in I})$, for which:*

- *$\mu := \psi_*(\mu_X)(= \mu_X \circ \psi^{-1})$ is a common prior.*
- *$\mathcal{F}$ is the common knowledge σ-algebra induced by the type space τ.*
- *$t^i \circ \psi(\cdot) = \psi_*(t^i_X(\cdot))$ for each $i \in J$; explicitly, for $\omega, \omega' \in X$, $t^i(\psi(\omega))[\psi(\omega')] = t^i_X(\omega)[\omega']$.*

Note in particular that if $\omega_1 \mathcal{E}_X \omega_2$ then $\psi(\omega_1)\mathcal{E}\psi(\omega_2)$. The bulk of the proof is in:

Lemma 15. *Let $J \subseteq I$ be sets of players, and let $\tilde{\tau} = (\Omega, J, (\tau^j)_{j\in J})$ be a positive type space with knowledge partitions $(\mathcal{F}^j)_{j\in J}$ and common prior μ. Let $(\mathcal{H}^i)_{i\in I\setminus J}$ be σ-algebras with finite atoms. Let $\Phi \subseteq \Omega$ be Borel such that:*

- *$\mu(\Phi) = 1$.*
- *For each $\omega \in \Omega\setminus\Phi$ and each $j \in J$, the knowledge component of Player j containing ω is a singleton.*[34]

Denote $\mathcal{F} = \cap_{i\in J}\mathcal{F}^i \bigcap \cap_{i\in I\setminus J}\mathcal{H}^i$, i.e., $\mathcal{F}$ is generated by $(\mathcal{F}^j)_{j\in J}$ and $(\mathcal{H}^i)_{i\in I\setminus J}$. Then the $(\mathcal{H}^i)_{i\in I\setminus J}$ can be refined[35] *to $(\mathcal{F}^i)_{i\in I\setminus J}$ and the type space $\tilde{\tau}$ can be extended to a positive type space $\tau = (\Omega, I, (\tau^i)_{i\in J})$ with the knowledge*

[33] The Glimm-Effros dichotomy is usually stated for a state space Ω that is Polish; however, a Borel space can always be endowed with a Polish topology inducing the same Borel structure, since all standard Borel spaces are Borel isomorphic.

[34] That is, outside of Φ, the players in J have perfect information.

[35] That is, $\mathcal{H}^i \subseteq \mathcal{F}^i$.

σ-algebras $(\mathcal{F}^i)_{i\in I}$, for which μ is a common prior, such that $\mathcal{F} = \cap_{i\in I}\mathcal{F}^i$, and furthermore for each atom T^i of $\mathcal{F}^i$ for $i \notin J$, $|T^i \cap \Phi| \leq 1$.

The idea is to modify $(\mathcal{H}^i)_{i\in I\setminus J}$ so that any knowledge component for a player $i \in I\setminus J$ intersects Φ in at most one point. Then find some positive type space on $\Omega\setminus\Phi$, which exists by Proposition 10, and 'glue' it together with the type space induced on Φ.

Proof. Fix a player $i \in I\setminus J$. Let Ω^* denote the collection of non-empty finite subsets[36] of Ω , let $\zeta : \Omega^* \to \Omega$ be a Borel selector, which exists by the selection theorem of Kuratowski and Ryll-Nardzewski (1965): $\zeta(F) \in F$ for all $F \in \Omega^*$. Define $\zeta^i : \Omega^* \cap \mathcal{H}^i \to \Omega^*$ by

$$\zeta^i(F) = \begin{cases} F & \text{if } F \cap \Phi = \emptyset \\ F\setminus\Phi \cup \{\zeta(F \cap \Phi)\} & \text{if } F \cap \Phi \neq \emptyset \end{cases}$$

That is, ζ^i leaves the part of an atom of $\mathcal{H}^i$ that is 'outside' of Φ 'intact', while leaving only a single element in Φ. The set $\{F \in \Omega^* \mid F \cap \Phi \neq \emptyset\}$ is Borel, and hence so is $\zeta^i(F)$. Then extend ζ^i to $\mathcal{H}^i$ by

$$\zeta^i(B) = \bigcup \{\zeta^i(T^i) \mid T^i \subseteq B \text{ is an atom of } \mathcal{H}^i\}.$$

$\zeta^i(B)$ is Borel for each $B \in \mathcal{H}^i$.[37] Let

$$\Xi^i = \Phi\setminus\zeta([\Omega\setminus\Phi]_{\mathcal{H}^i})$$

recalling that $[\cdot]_{\mathcal{H}^i}$ is the saturation operator w.r.t. $\mathcal{H}^i$. Finally, define

$$\mathcal{F}^i = \{A \cup \zeta^i(B) \mid B \in \mathcal{H}^i,\ A \subseteq \Xi^i \text{ is Borel}\}$$

i.e., $\mathcal{F}^i$ consists of all the refinements resulting from $\mathcal{H}^i$ and has perfect knowledge on the rest of the space.

For each player $i \notin J$, let

$$\mathcal{G}^i = \{B\setminus\Phi \mid B \in \mathcal{H}^i\} = \{B\setminus\Phi \mid B \in \mathcal{F}^i\}$$

i.e., these are the knowledge σ-algebras induced on $\Omega\setminus\Phi$ by either $\mathcal{H}^i$ or $\mathcal{F}^i$. Let $\eta = (\eta)_{i\in I\setminus J}$ be a positive type space for the knowledge space

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[36] The Borel structure on Ω^* is identified with the Borel structure on $\cup_{n=1}^{\infty}\Omega^n$ under appropriate permutations, or equivalently with the Borel structure induced as a subspace of the space of compact subsets with the Hausdorff topology.

[37] As $\zeta^i(B) = \phi^i(B)$, where ϕ^i is defined by

$$\phi^i(\omega) = \begin{cases} \omega & \text{if } \omega \notin \Phi \\ \zeta([\omega]_{\mathcal{H}^i} \cap \Phi) & \text{if } \omega \in \Phi \end{cases}$$

recalling that $[\cdot]_{\mathcal{H}^i}$ is the saturation operator w.r.t. $\mathcal{H}^i$.

$\{\mathcal{G}^i\}_{i\in I\setminus J}$, which exists by Proposition 10. Finally, for $i \notin J$, define for each $\omega \in \Omega$, where $T^i = [\omega]_{\mathcal{F}^i}$ [38]

$$t^i(\omega)[\omega] = \begin{cases} \eta^i(\omega)[\omega] & \text{if } T^i \cap \Phi = \emptyset \\ \frac{1}{2}\eta^i(\omega)[\omega] & \text{if } T^i \cap \Phi \neq \emptyset \text{ and } T^i\setminus\Phi \neq \emptyset, \omega \notin \Phi \\ \frac{1}{2} & \text{if } T^i \cap \Phi \neq \emptyset \text{ and } T^i\setminus\Phi \neq \emptyset, \omega \in \Phi \\ 1 & \text{if } T^i = \{\omega\} \subseteq \Phi \end{cases}$$

μ remains a common prior for the positive type space $\tau = (\Omega, I, (t^i)_{i\in I})$. $\square$

Now to prove Proposition 14: Letting $\mathcal{E}$ be the equivalence relationship induced by the non-smooth $\mathcal{F}$, and recalling that $\mathcal{E}_X$ is hyperfinite, by Proposition 13 there is an embedding $\psi : (X, \mathcal{E}_X) \sqsubseteq (\Omega, \mathcal{E})$. Denote $\Phi = \psi(X)$. For $i \in J$, set

$$t^i(\omega)[\omega'] = \begin{cases} t^i_X(\psi^{-1}(\omega'))[\psi^{-1}(\omega')] & \text{if } \omega, \omega' \in \Phi \\ 1 & \text{if } \omega = \omega' \notin \Phi \\ 0 & \text{if } \omega \neq \omega' \text{ and } \omega' \notin \Phi \end{cases}$$

I.e., t^i is derived from t^i_X in Φ, and Player i has perfect knowledge outside of Φ. Clearly, $\mu = \psi_*(\mu_X)$ is a common prior for the type space $\tilde{\tau} = (\Omega, J, (t^i)_{i\in J})$, with knowledge spaces

$$\mathcal{F}^i = \{A \cup B \mid \exists C \in \mathcal{F}^i_X s.t. A = \psi(C),\ B \subseteq \Omega\setminus\Phi \text{ Borel}\}$$

I.e., $\mathcal{F}^i$ is induced by $\mathcal{F}^i_X$ on Φ and Player i has perfect knowledge outside of Φ. Since $\mathcal{F}$ is belief induced, there are $(\mathcal{H}^i)_{i\notin I\setminus J}$ such that $\mathcal{F} = \cap_{j\in J}\mathcal{F}^j \bigcap \cap_{i\in I\setminus J}\mathcal{H}^i$. By taking $\Phi = \psi(X) \subseteq \Omega$, we can apply Lemma 15: Refine $(\mathcal{H}^i)_{i\notin I\setminus J}$ to $(\mathcal{F}^i)_{i\notin I\setminus J}$ and extend the type space $(t^i)_{i\in J}$ to the (positive) type space $\tau = (\Omega, I, (t^i)_{i\in I})$ with common prior μ.

6.3. **Proof of Theorem 4.** Let $(V_n)_{n\in\mathbb{N}}$ be a countable basis for Ω, and let G be a countable group which induces the equivalence relation $\mathcal{E} = \mathcal{E}_{\mathcal{F}}$ induced by the σ-algebra $\mathcal{F}$ as in Proposition 5. For each $n \in \mathbb{N}$, define $f_n : \mathbb{N} \to \{0, 1, 2, \ldots, \infty\}$ by

$$f_n(\omega) = |\{V_n \cap [\omega]_{\mathcal{F}}\}|$$

We will show that f_n is Borel: Fix some well-ordering $>$ on G, and define

$$\psi_{g,n}(\omega) = 1_{g^{-1}(V_n)}(\omega) \prod_{g'<g} (1 - 1_{g(\cdot)=g'(\cdot)}(\omega))$$

where 1_A is the indicator function of A; that is, $\psi_g(\omega)$ indicates whether $g(\omega) \in V_n$ and, in addition, that this point $g(\omega)$ has not appeared before for

[38] It suffices to define $t^i(\omega)[\omega]$ for each $\omega \in \Omega$, since we know that the knowledge σ-algebra is $\mathcal{F}^i$.

previous elements of G acting on ω. Hence, $f_n = \sum_{g\in G} \psi_{g,n}$. So for each $n \in \mathbb{N}$, f_n is Borel and our assumption of $\inf_{x\neq y,\ x,y\in A} d(x,y) > 0$ for each atom A of $\mathcal{F}$ is easily seen to imply that for each ω, $E_n(\omega) = \{n \mid f_n(\omega) = 1\}$ is non-empty, and the function $m(\cdot) = \min E_n(\cdot)$ is measurable. Hence, the correspondence

$$\Psi(\omega) = \{g(\omega) \mid g \in G\} \cap V_{m(\omega)} = \cup_{g\in G}(\{g(\omega)\} \cap V_{m(\omega)})$$

has a Borel graph and, by the definition of $m(\cdot)$, satisfies $|\Psi(\omega)| = 1$ for all ω; hence, Ψ is a measurable function. Furthermore, Ψ is constant on each atom, and therefore $\mathcal{F}$ is smooth.

6.4. **Proofs of Theorem 2 and Theorem 3.I.**

Lemma 16. *The mapping $\phi : \Delta(\Omega)\times\Omega \to \mathbb{R}$ given by $\phi : (\nu,\omega) \mapsto \nu(\{\omega\})$ is Borel.*[39]

Proof. Let $C(\Omega)$ be the Banach space of bounded continuous functions on Ω (with supremum norm). Fix some metric on Ω. Then the mapping $\psi : C(\Omega)\times\Delta(\Omega)$ given by $\psi(f,\nu) = \int_\Omega f d\nu$ is easily verified to be continuous. For each $n \in \mathbb{N}, \omega \in \Omega$, let $g_{n,\omega} \in C(\Omega)$ be defined by

$$g_{n,\omega}(x) = \max[0,\ 1 - n \cdot d(x,\omega)]$$

For each n, the mapping $\omega \to g_{n,\omega}$ is continuous, hence so is the mapping $\phi_n(\nu,\omega) := \psi(\nu, g_{n,\omega})$. Finally, observe that by the bounded convergence theorem, $\phi = \lim_{n\to\infty} \phi_n$, the limit being point-wise. ☐

Lemma 17. *The correspondence*

$$\Psi(\omega) = \{\nu \in \Delta_a(\Omega) \mid \nu(K^\infty(\omega)) = 1 \text{ and } \nu|_{K^\infty(\omega)} \text{ is a common prior for } \tau_{K^\infty(\omega)}\}$$

has a Borel graph, and $|\Psi(\omega)| \leq 1$ for all $\omega \in \Omega$; hence, Ψ is a partial Borel function.

Proof. The fact that $|\Psi(\omega)| \leq 1$ (i.e., that on a countable space in which no proper non-empty subset is common knowledge there exists at most one common prior) follows from Proposition 3 of Hellman and Samet (2011).

Let $(f_n)_{n\in\mathbb{N}}$ be as in Proposition 11. Define for each $n \in \mathbb{N}$, $g_n : \Delta_a(\Omega)\times\Omega \to [0,1]$ by:

$$g_n(\nu,\omega) = \begin{cases} \nu(\{f_n(\omega)\}) & \text{if } \omega \in Dom(f_n) \\ 0 & \text{if } \omega \notin Dom(f_n) \end{cases}$$

[39] Recall that $\Delta(\Omega)$ is endowed with the topology of narrow convergence of probability measures.

and for each $m, n \in \mathbb{N}$ and $i \in I$, define

$$D_{n,m} = \Delta_a(\Omega) \times (\text{dom}(f_n) \cap \text{dom}(f_m))$$

and

$$H^i_{n,m} = \{(\nu, \omega) \in D_{n,m} \mid t^i(f_n(\omega))[f_m(\omega)] \cdot \nu(K^i(f_m(\omega))) = \nu(\{f_m(\omega)\})\}$$

Each $D_{n,m}$ and $H_{n,m}$ is Borel - to see this, note that the mapping $\omega \to t^i(\omega)$ is Borel, and

$$\nu(K^i(f_n(\omega))) = \sum_{\omega' \in K^i(f_n(\omega))} \nu(\omega')\})$$

which is measurable by Lemma 16 and by Proposition 11.[40] Finally, it is now immediate that

$$\Psi(\omega) = \{\nu \in \Delta_a(\Omega) \mid \Big(\sum_{n=1}^{\infty} g_n(\nu, \omega) = 1\Big) \\ \wedge_{i \in I} \wedge_{n,m \in \mathbb{N}} \big((\nu, \omega) \notin D_{n,m} \vee (\nu, \omega) \in H^i_{n,m}\big)\}$$

□

Proof of Theorem 2. Clearly, property (3) implies property (2). Suppose (2) holds; then, for Ψ as in Lemma 17, $\Psi(\omega) = 1$ for μ-a.e. $\omega \in \Omega$. Hence, after restricting Ψ to some $X \in \mathcal{F}$ of full μ-measure, the graph of Ψ defines a Borel function $\psi : X \to \Delta_a(\Omega)$, which clearly satisfies $K^\infty(x) = K^\infty(y) \iff \psi(x) = \psi(y)$; hence $\mathcal{F}|_X$ is smooth.

Finally, assume property (1) holds, and assume w.l.o.g., $\Omega = X$. By[41] Theorem 1 of Blackwell and Ryll-Nardzewski (1963), there is a μ-a.e. proper RCD t for μ given $\mathcal{F}$. The claim that t is a common prior on μ-a.e. component follows now from Proposition 23 below of Section 6.7. □

Proof of Theorem 3.I. By Theorem 2, almost every common knowledge component K has a common prior of the restricted type space τ_K. This is sufficient, by Theorem 1.a. in Hellman (2014), to conclude that there can be no acceptable bet over τ_K for such K. □

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40 The latter proposition is used to choose selectors $(f^i_n)_{n \in \mathbb{N}}$ of Player i's knowledge σ-algebra and complete this argument; we do not elaborate.

41 The condition given there for the existence of proper RCD's is easily seen to follow from the existence of a Borel transversal, which, by Proposition 4, follows from separability.

6.5. **Proof of Theorem 1.**

Fix a countable set S and an element $s_0 \in S$. Let $\mathfrak{B}$ denote the collection of all I-tuples $(s^i, g^i)_{i\in I}$ for which $(S, I, (A^i)_{i\in I}, (s^i, g^i)_{i\in I})$ constitutes a Bayesian game, where (s^i) being the types, (g^i) being the payoffs. $\mathfrak{B}$ is endowed with the topology of point-wise convergence[42]: $(s^i_\alpha, g^i_\alpha)_{i\in I} = \Upsilon_\alpha \to \Upsilon = (s^i, g^i)_{i\in I}$ in $\mathfrak{B}$ if for every player $i \in I$, every $\omega \in S$, and every pure action profile $a \in \prod_{i\in I} A^i$ in $g^i_\alpha(\omega, a) \to g^i(\omega, a)$ and $s^i_\alpha(\omega) \to s^i(\omega)$:

Proposition 18. *$\mathfrak{B}$ is homeomorphic to a Borel subset $\Xi := \big((S\times[0,1])^* \times \mathbb{R}^{\prod_{i\in I} A^i}\big)^{S\times I}$ and hence is standard Borel (where for a set A, $A^* = \cup_{n=0}^\infty A^n$ with each A^n being both closed and open).*

The simple intuition is that for each player and state pair $(\omega, p) \in S \times I$, we need to specify both an element in $(S \times [0,1])^*$ – a finite list of states that are in the same element of the knowledge partition as ω, and the probabilities themselves to these states – as well as an element of $\mathbb{R}^{\prod_{i\in I} A^i}$, which specifies what payoff that player will receive as a result of each possible action profile.

Although we will not need it, the proof shows this mapping can be chosen to be natural up to a choice of a well-ordering on S. Henceforth, we will identify $\mathfrak{B}$ with some such fixed subset of Ξ.

Proof. Write $\mathfrak{B} = \prod_{i\in I}(\mathfrak{B}^i_s \times \mathfrak{B}^i_g)$, where $\mathfrak{B}^i_s$ (resp. $\mathfrak{B}^i_g$) denotes the projection of $\mathfrak{B}$ to the space of types (resp. payoffs) for Player i, with the induced topologies. It's enough to show that $\mathfrak{B}^i_s$ is homeomorphic to a Borel subset of $\big((S \times [0,1])^*\big)^S$ and that $\mathfrak{B}^i_g$ is homeomorphic to Borel subseteq of $\mathbb{R}^{S\times\prod_{i\in I} A^i}$.

The latter claim is trivial once one notices that for any countable set C, the set of bounded functions in $\mathbb{R}^C$ is Borel, as it can be written

$$\cup_{n\in\mathbb{N}} \cap_{c\in C} \{a \in \mathbb{R}^C \mid |a_c| \le n\}.$$

and that the Tychonoff topology is indeed the required topology of point-wise convergence. We turn to the former claim. Fix some well-ordering $<$ on S. As mentioned above, the intuition describing the map from $\mathfrak{B}^i_s$ to $\big((S \times [0,1])^*\big)^S$ is the following: for each $\omega \in S$, the player has to specify the finite list of states he believes he could be in and the weight each one receives. Finite lists of states are ordered by $>$. Hence, the image of $\mathfrak{B}^i_s$ under such a map is given by the subset of Ξ defined by three conditions: Being supported on finite sets, they have total mass of unity, and they are

[42] We define the topology in terms of nets.

constant on the set they are supported on. Mathematically:

$$\cap_{\omega\in S} \cup_{F\subseteq S,\ |F|<\infty} \cap_{x\notin F} \{s^i \in \left((S\times[0,1])^*\right)^S \mid s^i(\omega)(x) = 0\}$$
$$\bigcap \cap_{\omega\in S}\{s^i \in \left((S\times[0,1])^*\right)^S \mid \sum_{x\in S} s^i(\omega)[x] = 1\}$$
$$\bigcap \cap_{\omega,\eta,\zeta\in S}\{s^i \in \left((S\times[0,1])^*\right)^S \mid s^i(\omega)[\eta] > 0 \to s^i(\omega)(\zeta) = s^i(\eta)[\zeta]\}$$

and, again the topology is the topology of point-wise convergence. □

The space Σ^i of strategies for Player i on a countable space is clearly a compact subspace of $(\Delta(A^i))^S$, hence the space of strategy profiles $\Sigma = \prod_{i\in I}\Sigma^i$ is a compact space.

Proposition 19. *The Bayesian equilibrium correspondence $BE : \mathfrak{B} \to \Sigma$ has a Borel graph and takes on compact non-empty values.*

Proof. The fact that every Bayesian game with a countable state space has at least one Bayesian equilibrium follows from standard fixed point arguments; see, e.g., Simon (2003). The fact that the set of Bayesian equilibrium is compact also follows by standard arguments. To show that the graph G of the BE correspondence is Borel, note that

$$G = \{((s^i, g^i)_{i\in I}, \sigma) \in \mathfrak{B}\times\Sigma \mid \forall\omega\in S, \forall i\in I, \forall x\in\Delta_{\mathbb{Q}}(A^i),$$
$$\sum_{v\in S} 1_{v\in K^i(\omega)}\cdot g^i(v,\sigma(w))s^i(\omega)[v] \geq \sum_{v\in S} 1_{v\in K^i(\omega)}\cdot g^i(v,x,\sigma^{-p}(w))s^i(\omega)[v]\}$$

where for a finite set A, $\Delta_{\mathbb{Q}}(A)$ denotes the probability distributions on a A which give rational weights to all points. □

The following corollary then results from Proposition 19 and the selection theorem of Kuratowski and Ryll-Nardzewski (1965) (see also Himmelberg (1975)):

Corollary 20. *There exists a Borel mapping $\psi : \mathfrak{B} \to \Sigma$ such that for all $\Lambda \in \mathfrak{B}$, $\psi(\Lambda)$ is a Bayesian equilibrium of Λ.*

Given two Bayesian games

$$(S, I, (A^i)_{i\in I}, (s^i_S)_{i\in I}, (g^i_S)_{i\in I})$$

and

$$(T, I, (A^i)_{i\in I}, (s^i_T)_{i\in I}, (g^i_T)_{i\in I})$$

with countable state spaces and the same player and action sets, an *isomorphism* from S to T is a bijective mapping $\phi : S \to T$ such that:

- For all $\omega \in S$ and pure action profile x, $g_S(\omega, x) = g_T(\phi(\omega), x)$.
- For all $\omega, \eta \in S$ and $i \in I$, $s^i_S(\omega)[\eta] = s^i_T(\phi(\omega))[\phi(\eta)]$.

Proposition 21. *Let $\Gamma = (\Omega, I, (A^i)_{i\in I}, (t^i_S)_{i\in I}, (r^i_S)_{i\in I})$ be a Bayesian game such that the common knowledge σ-algebra $\mathcal{F}$ is separable and aperiodic,*[43] *and let $\mathfrak{B} = \mathfrak{B}(S, I, (A^i)_{i\in I})$ be the set of Bayesian games with countable state space S with the same player and action space as Γ. Then $\Omega/\mathcal{E}$ is standard Borel and there is a Borel map $\Phi : \Omega \to S$ which is $\mathcal{F}$-measurable and a Borel map $\Lambda : \Omega/\mathcal{F} \to \mathfrak{B}$ such that for each $\omega \in \Omega$, if we denote*

$$\Gamma_\omega = (K^\infty(\omega), I, (A^i)_{i\in I}, (t^i|_{K^\infty(\omega)})_{i\in I}, (r^i|_{K^\infty(\omega)})_{i\in I})$$

then $\Theta|_{K^\infty(\omega)}$ is an isomorphism of Γ_ω to $\Lambda(K^\infty(\omega))$.

Proof. Let $\zeta_1, \zeta_2, \dots$ be an enumeration of S, and let $g_1, g_2, \dots$ be as in Proposition 11 w.r.t. $\mathcal{F}$, which exist by separability of $\mathcal{F}$. Define $\Phi : \Omega \to S$ by $\Phi(\omega) = \zeta_{n(\omega)}$, where $n(\omega)$ is the unique n such that $g_n(K^\infty(\omega)) = \omega$. We can then define $\Lambda(q) = (g^i_q, s^i_q)_{i\in I}$ by

$$g^i_q(\Phi(\omega), x) = r^i(\omega, x)$$

and

$$s^i_q(\Phi(\omega))[\Phi(\eta)] = t^i(\omega)[\eta]$$

It is straightforward to check that Φ and Λ so defined satisfy the requirements. □

Proof of Theorem 1.I. For simplicity, take the case that the common knowledge equivalence relation is aperiodic. Otherwise, partition the space into the common knowledge components of each size, and on each use a modified version of Proposition 21 with S being of a fixed countable or finite size.

Let $\psi : \mathfrak{B} \to (\prod_{i\in I} \Delta(A^i))$ be a Bayesian equilibrium selection as in Corollary 20. Let Φ, Λ be as in Proposition 21 for some countable set S. For each $\omega \in \Omega$, define

$$\sigma(\omega) = \psi(\Lambda(K^\infty(\omega)))(\Phi(\omega))$$

Such σ is then an MBE. □

Proof of Theorem 1.II. Let $C = 2^{\mathbb{N}}$ denote the Cantor space and let $\mathcal{E}_t$ be the tail equivalence relation; i.e., if $S : C \to C$ is defined by $(Sx)_n = x_{n+1}$, then by $x\mathcal{E}_t y$ iff $\exists k, m \geq 0, S^k(x) = S^m(y)$. This is a countable Borel equivalence relation which is non-smooth and hyperfinite, see (Dougherty et al., 1994, Sec. 6). Now, let $X = \{-1, 1\} \times C$, and define $S_X : X \to X$

[43] An equivalence relation is aperiodic if each equivalence class is infinite. We therefore say that a σ-algebra is aperiodic if each atom is infinite.

by $S_X(x_0, x_1, x_2, \ldots) = (-x_0, S(x_1, x_2, \ldots))$. Let $\mathcal{E}_X$ be the equivalence relation on X given by

$$\mathcal{E}_X = \{(x, y) \mid \exists k, m \geq 0, S_X^k(x) = S_X^m(y)\}$$

This relationship is hyperfinite: Let $Y = (\{-1, 1\} \times 2)^{\mathbb{N}}$, and identify the element $(x_0, x_1, x_2, \ldots) \in X$ with the element $(x_0 \cdot x_1, -x_0 \cdot x_2, x_0 \cdot x_3, -x_0 \cdot x_4, \ldots) \in Y$; let $\iota : X \to Y$ denote this embedding and let S_Y denote the shift in Y. Then we have $\iota \circ S_X = S_Y \circ \iota$, and S_Y induces a countable Borel equivalence relationship $\mathcal{E}_Y$ like above, $x \mathcal{E}_Y y$ iff $\exists k, m \geq 0, S_Y^k(x) = S_Y^m(y)$. As above, $\mathcal{E}_Y$ is hyperfinite, and hence so is its restriction to the $\mathcal{E}_Y$-saturated subspace $\iota(X)$.

Let $\Gamma_X = (X, \{1, 2\}, \{L, R\} \times \{L, R\}, t_X^1, t_X^2, r_X^1, r_X^2)$ be the two-player game presented in Hellman (2012) with state space X as above which does not possess an ε-MBE for small enough ε; fix such an $\varepsilon > 0$; the common knowledge equivalence relation of that game is indeed $\mathcal{E}_X$. Let $\psi : X \to \Omega$ denote an embedding as in Proposition 14, with the induced positive type space $\tau = (\Omega, I, (t^i)_{i \in I})$ on Ω and induced common prior $\mu = \psi_*(\mu_X)$. Define the payoffs:

$$r^j(\omega, x) = \begin{cases} r_X^j(\psi^{-1}(\omega, x)) & \text{if } j = 1, 2 \text{ and } \omega \in \psi(X) \\ 0 & \text{otherwise} \end{cases}$$

By the properties of Γ_X listed above and for ε chosen above, the game

$$\Gamma = (\Omega, \{L, R\}^I, I, t^1, t^2, t^3, \ldots, t^I, r^1, r^2, r^3, \ldots, r^I)$$

which has common prior μ, does not possess an ε-MBE. □

6.6. **Proof of Theorem 3.II.**

The proof of Theorem 3.I appears above in Section 6.4. To prove Theorem 3.II, we first note that it holds on the example Γ_X given in Section 3, as explained there, and in fact using only two players. The type space there is $(X, \{1, 2\}, (t_X^1, t_X^2))$ with knowledge spaces $(\mathcal{F}_X^1, \mathcal{F}_X^2)$. The common knowledge equivalence relation $\mathcal{E}_X$ is easily seen to be hyperfinite, as it is clearly induced by a $\mathbb{Z}$-action. Let $\psi : X \to \Omega$ denote an embedding as in Proposition 14, with the induced positive type space $\tau = (\Omega, I, (t^i)_{i \in I})$ on Ω and induced common prior $\mu = \psi_*(\mu_X)$.

A modified definition (compare with Eq. (3.2) of Section 3) of an acceptable bet which is helpful is that of a *Dutch book*: In this case, we require that $f_i : \Omega \to \mathbb{R}$ are bounded and measurable such that $\sum_{i \in I} f_i < 0$ and $E^i[f_i \mid \cdot] > 0$ at each point for each player. More generally, if $L \subseteq K$, we will say that $(f_i)_{i\in I}$ is a Dutch book in L if these inequalities hold throughout L. It is easy to show that the existence of an acceptable bet (on a countable space, or any subset of it) is equivalent to the existence of a Dutch book.

Let K be a common knowledge component in Γ such that $\psi^{-1}(K)$ is one of those components in Γ_X on which there is an acceptable bet (f_1, f_2) (with $f_2 = -f_1$). By assumption, this is true for μ-almost every component K, since it is true for μ_X-almost every component in X. Hence, we also have a Dutch book for these players $\{1,2\}$ on that subset, and we need to show that there is a Dutch book on the *entire* space K for *all* players:

Proposition 22. *Let $\Gamma_K = (K, I, (t^i)_{i\in I})$ be a countable positive*[44] *type space such that K does not strictly contain any non-empty common knowledge component, let $J \subseteq I$ and $L \subseteq K$, and suppose there is a Dutch book $(g_j)_{j\in J}$ for the players in J on L. Then there is a Dutch book for all the players in I on all of K.*

Note that the fact that L is not common knowledge for all players is not relevant; in fact, we do not even make use of the fact that it is common knowledge for the players in J. Also note that the assumptions imply that $E^i[g_i \mid \cdot] > 0$ for all $i \in Q$ on all K.

Proof. First, we observe that there exists such a Dutch book for all players in I on all of L: if we define $(g_i)_{i\in I\setminus J}$ to be positive but small enough in L and vanishing outside of L, then $(g_i)_{i\in I}$ is a Dutch book in L and we still have $E^i[g_i \mid \cdot] > 0$ for all $i \in I$ on all L.

From here we fix some $M > \sup_{i\in I,\omega\in L} |g_i(\omega)|$. Now, we proceed inductively and keep enlarging L: Let $\omega_0 \in K\setminus L$ and $i_0 \in I$ be such that $K^{i_0}(\omega_0) \cap L \neq \emptyset$; if there are no such ω_0, i_0, then we are done by the assumption that there are no subsets which are common knowledge. Fix some $\omega_1 \in K^{i_0}(\omega_0) \cap L$. By assumption, $E^{i_0}[g_i \mid \omega_0] = E^{i_0}[g_i \mid \omega_1] > 0$. Let

$$\gamma = \frac{1}{2}\min\left[-\sum_{i\in I} g_i(\omega_1),\ t^{i_0}(\omega_0)[\omega_0]\cdot\left(M - \sup_{i\in I,\omega\in L}|g_i(\omega)|\right)\right]$$

$\gamma > 0$, since $(g_i)_{i\in I}$ is a Dutch bet in L. Hence, define $(g'_i)_{i\in I}$ by:

$$g'^i(\omega) = \begin{cases} g^i(\omega) & \text{if } i \neq i_0, \omega \neq \omega_0 \text{ or } i = i_0, \omega \neq \omega_0, \omega_1 \\ g^i(\omega) + \gamma & \text{if } i = i_0, \omega = \omega_1 \\ g^i(\omega) - \gamma \cdot \frac{t^{i_0}(\omega_0)[\omega_1]}{t^{i_0}(\omega_0)[\omega_0]} & \text{if } i = i_0, \omega = \omega_0 \\ g^i(\omega) + \frac{\gamma}{|I|} \cdot \frac{t^{i_0}(\omega_0)[\omega_1]}{t^{i_0}(\omega_0)[\omega_0]} & \text{if } i \neq i_0, \omega = \omega_0 \end{cases}$$

[44] We could modify the proof to include the non-positive case, but it would be unnecessary and also somewhat more cumbersome.

It is then easy to check that $(g'^i)_{i\in I}$ is a Dutch book on $L \cup \{\omega_0\}$ which still satisfies $E^i[g'_i \mid \cdot] > 0$ for all $i \in I$ on all K (note that $E^{i_0}[g_{i_0} \mid \omega_0] = E^{i_0}[g'_{i_0} \mid \omega_0]$, and all others players' payoffs have decreased nowhere and have increased at ω_0). Furthermore, we still have $M > \sup_{i\in I, \omega\in L} |g'_i(\omega)|$.

Now repeat the procedure with $L \cup \{\omega_0\}$ replacing L; the resulting Dutch book from this process will also be bounded by M. It's easy to see that the values of $(g^i)_{i\in I}$ are only altered finitely many times at each point $\omega \in K$ during this inductive construction, hence the process converges point-wise and the strong inequalities required of the Dutch book hold in the limiting construction. □

6.7. **Proposition for Proof of Theorem 2.**

Proposition 23. *Let $\mathcal{E}, \mathcal{E}'$ be smooth countable Borel equivalence relations on a standard Borel space Ω, with $\mathcal{E}'$ refining $\mathcal{E}$ (that is, $\mathcal{E}' \subseteq \mathcal{E}$) let μ be a regular Borel probability measure on Ω, and let t, t' be proper RCD's of μ w.r.t. the σ-algebras $\mathcal{F}, \mathcal{F}'$ induced by $\mathcal{E}, \mathcal{E}'$, respectively. Then for μ-a.e. $\omega \in \Omega$ and $\mathcal{E}'$-equivalence class C' with $\omega \in C'$,*

$$t(\omega)(\cdot \mid C') = t'(\omega)(\cdot) \tag{6.1}$$

Proof. It suffices to show that for μ-a.e. $\omega \in \Omega$ and each $\mathcal{E}'$-equivalence class C' such that $\omega \in C'$,

$$t(\omega)(\{\omega\} \mid C') = t'(\omega)[\omega]$$

Indeed, this suffices since both t, t' are constant in each $\mathcal{E}'$-equivalence class, and both sides of Equation (6.1) vanish for sets supported outside of C'. Since $\omega \in C'$, this is equivalent to showing that for μ-a.e. $\omega \in \Omega$ and such C',

$$t'(\omega)[\omega] \cdot t(\omega)(C') = t(\omega)[\omega] \tag{6.2}$$

Note that since $\mathcal{E}, \mathcal{E}'$ are smooth, the induced quotient spaces $\Omega/\mathcal{F}, \Omega/\mathcal{F}'$ are standard Borel by Proposition 4 and μ induces measures on these quotient spaces. Throughout this proof, it will be convenient to view t, t' as functions on $\Omega/\mathcal{F}, \Omega/\mathcal{F}'$ - i.e., to view the RCD's as a function of the equivalence class, not of its elements.

Lemma 24. *For any bounded real-valued random variable X on (Ω, μ),*

$$\int_\Omega X(\omega) d\mu(\omega) = \int_{\Omega/\mathcal{F}} \left(\sum_{\omega\in C} X(\omega) \cdot t(C)[\omega] \right) d\mu(C) \tag{6.3}$$

Proof. It suffices to verify Equation (6.3) in the case $X = 1_A$, A being Borel, and then to use an approximation argument. In this case, the left-hand side of Equation (6.3) is just $\mu(A)$, while the other side is

$$\int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} 1_A(\omega) \cdot t(C)[\omega] \right) d\mu(C) = \int_{\Omega/\mathcal{F}} t(C)(A \cap C) d\mu(C)$$
$$= \int_{\Omega/\mathcal{F}} t(C)(A) d\mu(C)$$

In general, for an $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ – which induces a measurable function $f : \Omega/\mathcal{F} \to \mathbb{R}$ – we have

$$\int_{\Omega/\mathcal{F}} f(C) d\mu(C) = \int_{\Omega} f(\omega) d\mu(\omega)$$

(again, one checks it first for simple $\mathcal{F}$-measurable functions) and in particular for $f(\cdot) = t(\cdot)(A)$. Hence,

$$\int_{\Omega/\mathcal{F}} t(C)(A) d\mu(C) = \int_{\Omega} t(\omega)(A) d\mu(\omega) = \mu(A)$$

as required. □

Now, note that on $\Omega/\mathcal{F}'$ there is the equivalence relation $\mathcal{E}^*$ induced by $\mathcal{E}$; that is, two elements of $\Omega/\mathcal{F}'$ are $\mathcal{E}^*$ equivalent if they are subsets of the same equivalence class of $\mathcal{E}$. $\mathcal{E}^*$ is easily seen to be Borel and smooth as well; denote its induced σ-algebra on $\Omega/\mathcal{F}'$ as $\mathcal{F}^*$. Let t^* denote the proper RCD of μ (as a measure on $\Omega/\mathcal{F}$) w.r.t $\mathcal{F}^*$, which exists by[45] Theorem 1 of Blackwell and Ryll-Nardzewski (1963).

Lemma 25. *For μ-a.e. $C \in \Omega/\mathcal{F}$ and each $\mathcal{E}'$-equivalence class $C' \subseteq C$,*

$$t^*(C)[C'] = t(C)(C')$$

Proof. For any bounded real-valued random variable X on $(\Omega/\mathcal{F}', \mu)$ (by abuse of notation, we let X also denote the induced $\mathcal{F}'$-measurable random variable defined on Ω), by repeated use of Lemma 24,

$$\int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} X(\omega) \cdot t(C)[\omega] \right) d\mu(C) = \int_{\Omega} X(\omega) d\mu(\omega) = \int_{\Omega/\mathcal{F}'} X(C') d\mu(C')$$
$$= \int_{\Omega/\mathcal{F}} \left(\sum_{C' \in C} X(C') \cdot t^*(C)[C'] \right) d\mu(C)$$

[45] See explanation and footnote when this result is used in the proof of Theorem 2.

where the sum over $C' \in C$ is taken over $\mathcal{E}'$-equivalence classes. (The middle equality follows by definition for indicator functions.) However, for μ-a.e. $\omega \in \Omega$,

$$\sum_{\omega \in C} X(\omega) \cdot t(C)[\omega] = \sum_{C' \in C} X(C') \cdot t(C)(C')$$

Hence,

$$\int_{\Omega/\mathcal{F}} \left(\sum_{C' \in C} X(C') \cdot t(C)(C') \right) d\mu(C) = \int_{\Omega/\mathcal{F}} \left(\sum_{C' \in C} X(C') \cdot t^*(C)[C'] \right) d\mu(C)$$

and this holds for any bounded real-valued random variable X. □

We now complete the proof. For any bounded real-valued random variable X on Ω, by Lemma 24 (applied first to the equivalence relation $\mathcal{E}'$ on Ω, and then to the equivalence relation $\mathcal{E}^*$ on $\Omega/\mathcal{F}'$), and by Lemma 25,

$$\begin{aligned}
\int_{\Omega} X(\omega) d\mu(\omega) &= \int_{\Omega/\mathcal{F}'} \left(\sum_{\omega \in C'} X(\omega) \cdot t'(C')[\omega] \right) d\mu(C') \\
&= \int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C'} X(\omega) \cdot t'(C')[\omega] \right) \sum_{C' \subseteq C} t^*(C)[C'] d\mu(C) \\
&= \int_{\Omega/\mathcal{F}} \left(\sum_{C' \subseteq C} \sum_{\omega \in C'} X(\omega) \cdot t(C)(C') \cdot t'(C')[\omega] \right) d\mu(C) \\
&= \int_{\Omega/\mathcal{F}} \left(\sum_{\omega \in C} X(\omega) \cdot t(C)([\omega]_{\mathcal{E}'}) \cdot t'([\omega]_{\mathcal{E}'})[\omega] \right) d\mu(C)
\end{aligned}$$

Comparing this to Equation (6.3), we see that for μ-a.e. $\omega \in \Omega$,

$$t([\omega]_{\mathcal{E}})([\omega]_{\mathcal{E}'}) \cdot t'([\omega]_{\mathcal{E}'})[\omega] = t([\omega]_{\mathcal{E}})[\omega]$$

or, denoting $C' = [\omega]_{\mathcal{E}'}$ and recalling $t([\omega]_{\mathcal{E}}) = t(\omega)$, and similarly for $\mathcal{E}', t'$, we deduce Equation (6.2). □

6.8. **Proof of Proposition 10.** Let G^i be a countable group which generates the knowledge structure $\mathcal{F}^i$, as in Proposition 5 of Section 6.1.1. Let G be the countable group generated by $G^1, \dots, G^I$. Let $(\alpha_g)_{g \in G}$ be some collection of positive real numbers which sum to unity, fix some probability measure μ on Ω, and set

$$\nu = \sum_{g \in G} \alpha_g g_*(\mu)$$

where $g_*(\mu) = \mu \circ g^{-1}$, let τ^i be a proper RCD of ν w.r.t. to $\mathcal{F}^i$ for each $i \in I$. Note that $\mu \ll \nu$. It follows from Proposition 9 of Section 6.1.2 that the set N of $\omega \in \Omega$ for each $t^i(\omega)[\omega] = 0$ for some $i \in I$ is of ν-measure 0. We contend that $\nu(K^\infty(N)) = 0$: Once we have this, one can redefine the types in an arbitrary measurable and positive way on $K^\infty(N)$ (while still assuring that they generate $\mathcal{F}^1, \ldots, \mathcal{F}^n$ on $K^\infty(N)$, of course; for example, for $\omega \in K^\infty(N)$ and $i \in I$, let $t^i(\omega)$ be uniform on $K^i(\omega)$); ν remains a common prior for the altered types.

In fact, we contend that for each $g \in G$, g sends ν-null sets to ν-null sets; this will suffice, since the group G generates the common knowledge equivalence relation, i.e., $K^\infty(N) = \cup_{g \in G} g(N)$.

So fix $g_0 \in G$, and let N be any null set. Observe that

$$\nu(N) = \sum_{g \in G} \alpha_g g_*(\mu)(N)$$

and hence $g_*(\mu)(N) = 0$ for all $g \in G$; hence,

$$\begin{aligned} \nu(g(N)) &= \sum_{g \in G} \alpha_g g_*(\mu)(g_0(N)) \\ &= \sum_{g \in G} \alpha_g (g \cdot g_0^{-1})_*(\mu)(N) = \sum_{g \in G} \alpha_{g \cdot g_0} (g)_*(\mu)(N) = 0. \end{aligned}$$

□

6.9. **Proof of Theorem 6.** Regardless of whether $\mathcal{F}$ is separable or not, by taking $\nu(\cdot) = \mu(\cdot \mid F)$, we see that (ii) implies (iii); also, that (iii) implies (i) is immediate.

To prove that (i) implies (ii), we rely on the countable case. Let A be an agreement on E, $\mu(K^\infty(A)) > 0$. For each common knowledge component C, let $I(\cdot, C)$, $K^\infty(\cdot, C)$ and, for each $H \subseteq C$ let $I(\cdot, C), K^i(\cdot, C), K^\infty(\cdot, C)$, $I_H(\cdot, C), K^i_H(\cdot, C), K^\infty_H(\cdot, C)$ denote the versions of the operators $I(\cdot), K^i(\cdot), K^\infty(\cdot)$, $I_H(\cdot), K^i_H(\cdot), K^\infty_H(\cdot)$ restricted to C. Note that if $\mathfrak{C}$ is a collection of common knowledge components and $A, H \subseteq \cup_{C \in \mathfrak{C}} C$ are any sets, then

$$K^\infty(A) = \cup_{C \in \mathfrak{C}} K^\infty(A \cap C, C)$$

and similarly for K^∞ and I, while

$$K^\infty_H(A) = \cup_{C \in \mathfrak{C}} K^\infty_{H \cap C}(A \cap C, C)$$

and similarly for K^∞_H and I_H. (Intuitively, these operators apply independently on each common knowledge component.) Define a correspondence from the standard Borel space $\Omega/\mathcal{F}$ to the standard Borel space[46] Z of non-empty finite subsets of Ω

$$\Theta(C) = \{F \in Z \mid F \subseteq K^\infty(I(E \cap C, C)) \text{ and } F \subseteq K_F^\infty(I_F(E \cap C, C), C)\}$$

Θ is then seen to have a Borel graph. By Theorem 5, $\Theta(C) \neq 0$ iff agreeing is possible for $E \cap C$. We note that since the common knowledge σ-algebra is separable, there is by Theorem 2 a mapping ρ from $\Omega/\mathcal{F}$ to probability distributions, assigning to μ-a.e. common knowledge component a common prior on it. We contend that there is $\mathfrak{C} \subseteq \Omega/\mathcal{F}$ satisfying $\mu(\mathfrak{C}) > 0$ (where μ also denotes the measure induced on $\Omega/\mathcal{F}$) such that, for all $C \in \mathfrak{C}$, agreeing to agree is possible for $E \cap C$ with $\rho(C)$. Indeed, since $K^\infty(A) \cap C = K^\infty(A \cap C)$ for $C \in \Omega/\mathcal{F}$, we have

$$0 < \mu(K^\infty(A)) = \int_\Omega \rho([\omega]_\mathcal{F})(K^\infty(A \cap [\omega]_\mathcal{F}))d\mu(\omega) = \int_{\Omega/\mathcal{F}} \rho(C)(K^\infty(A \cap C))d\mu(C)$$

Hence, taking

$$\mathfrak{C} = \{C \in \Omega/\mathcal{F} \mid \rho(C)(K^\infty(A \cap C)) > 0\}$$

we have $\mu(\mathfrak{C}) > 0$. By the Aumann selection theorem (e.g., Himmelberg (1975)), up to the need discard a set of measure zero, there is a Borel mapping $\theta : \mathfrak{C} \to Z$ such that $\theta(C) \in \Theta(C)$ for all $C \in \mathfrak{C}$. Since this map is clearly injective ($\theta(C) \cap \theta(C') = \emptyset$ in fact if $C \neq C'$) its image is Borel,[47] and hence it's easy to see that so is $F = \cup_{C \in \mathfrak{C}} \theta(C)$. This F is the required set, since

$$F = \cup_{C \in \mathfrak{C}} \theta(C) \subseteq \cup_{C \in \mathfrak{C}} K^\infty(I(E \cap C, C)) = K^\infty(I(E))$$

and similarly

$$F = \cup_{C \in \mathfrak{C}} \theta(C) \subseteq \cup_{C \in \mathfrak{C}} K_{F \cap C}^\infty(I_{F \cap C}(E \cap C, C)) = K_F^\infty(I_F(E))$$

6.10. **Proof of Proposition 3.** An equivalent formulation is that a countable Borel equivalence relation $\mathcal{E}$ is belief induced iff there are smooth countable Borel equivalence relations $\mathcal{E}_1, \dots, \mathcal{E}_n$ which generate $\mathcal{E}$. Since clearly the countable Borel equivalence relation generated by a single player's type is smooth, it suffices to show that if $\mathcal{E}$ is a smooth countable Borel equivalence relation, then there are Borel equivalence relations $\mathcal{E}_1, \mathcal{E}_2$ with finite equivalence classes which generate $\mathcal{E}$. Let $g_1, g_2, \dots$ correspond to $\mathcal{F}$, the σ-algebra induced by $\mathcal{E}$, as in Proposition 11. For convenience, we write $\{g_m(\omega), g_n(\omega)\}$ even if ω is not in one or both of the domains of g_m, g_n; in

[46] The Borel structure on Ω^* is identified with the Borel structure on $\cup_{n=1}^\infty \Omega^n$ under appropriate permutations, or equivalently with the Borel structure induced as a subspace of the space of compact subsets with the Haussdorf topology.

[47] This follows from Kuratowski's theorem.

these cases, this set is either empty (if in neither domain) or consists of a single element (if belonging to one domain). Then set,

$$\mathcal{E}_1 = \{(x, y) \in \Omega \times \Omega \mid \exists q \in \Omega/\mathcal{F}, k \in \mathbb{N}, (x, y) \subseteq \{g_{2k-1}(q), g_{2k}(q)\}\}$$

$$\mathcal{E}_2 = \{(x, y) \in \Omega \times \Omega \mid \exists q \in \Omega/\mathcal{F}, k \in \mathbb{N}, (x, y) \subseteq \{g_{2k}(q), g_{2k+1}(q)\}\}$$

It is easy to see that the equivalence classes of $\mathcal{E}_1, \mathcal{E}_2$ are all of size at most 2, and that $\mathcal{E}$ is generated by $\mathcal{E}_1, \mathcal{E}_2$.

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