Reputation and Cycles*

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Abstract:

We model the interaction between investment firms and a syndicate of investors (i.e., a market) through a dynamic strategic setting. In this setting, the market's assessment of the firms' performance, based on past and current returns, govern the firms' fund-flows incentives. In return, the firms strategically produce excess returns to balance their reputation and payoffs accordingly. Through a unified model we are able to explain much of the well-documented phenomenons of the delegated portfolio-managers problem, including persistent and non-persistent short- and long-term performance. Our model is robust and applies to a wide-range of economic settings where agents are subjected to reputation-based incentives.

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1 Introduction

The phrase 'buy low, sell high' is presumably the timeless, most-simple motto to the world of trading. Easy to explain and easy to understand, but quite difficult to apply since trade is based on predictions, motivated by past results and reputation. Therein lies a problem: would you buy a badly-performing low-reputation product waiting for it to blossom in due time? Some might, others will not. Now consider the portfolio-management industry where past returns and reputation coincide. Would it still be reasonable for an investor to invest through a low-return, potentially losing, investment firm?

Due to the significance of the problem, it is only natural that one of the debatable questions of this field concerns the performance persistence of investment firms, and whether past returns hold some indication to future ones.[1] Recently, Cornell et al. (2017) took one step further in the debate, indicating that past performance might even be inversely correlated with future returns. They showed that returns oscillate in adjacent time periods as top-ranked investment firms, in terms of realized returns, become bottom-ranked firms and vice versa. Similar indications follow from Bessler et al. (2017) portraying limited reversions in realized returns as time progresses (see, e.g., Figure 2 of the relevant study).

In this paper, we follow the quest to settle the persistence-performance problem, particularly targeting the *oscillating-returns* processes. We pursue this goal through a dynamic model where a strategic investment firm interacts with a partially-informed syndicate of investors (i.e., a market). We show that such interaction dictates much of the firm's cyclic performance through a reputation mechanism. Namely, the market's constant evaluation of the firm's returns triggers a strategic counter-reaction, by the firm, to level its performance accordingly.

Our model and analysis spotlight two balancing forces that lead the market-firm interaction and induce the firm's oscillating strategy. On the one hand, the market observes past returns to generate a reputation assessment over the firm's abilities. Reputation assessments are translated into funds-allocation decisions and, due to assets-under-management fees, defines the firm's payoff. These *reputation-based payoffs* are the first key element behind our result. The firm, however, is subjected to convex operating costs. It maintains its reputation by exerting per-period "effort", that translates into returns for the investors. These *decreasing marginal returns* are the second key element, to balance the first, in the dynamic setting. In other words, the returns-based reputation mechanism dictates fund flows, and requires the investment firm to balance current operating costs with future earnings,

[1] On the one hand, papers such as Grinblatt and Titman (1992), Elton et al. (1993), Hendricks et al. (1993), Goetzmann and Ibbotson (1994), Brown and Goetzmann (1995), Carhart (1997), Bollen and Busse (2005), and Busse et al. (2010), indicate that either long- or short-term performance persist, while many others, such as Goyal and Wahal (2008) and Barras et al. (2010), claim otherwise.

thus inducing a cyclic performance.

1.1 Various applications

Though we mainly use the investment-firms terminology, our results and model surpass this setting to a wider scope of scenarios. For example, our set-up also relates to an R&D investment problem, where a firm decides how much to invest (in R&D) in every time period. The decision and costs are immediate, whereas the payoff is accumulated and spread throughout the stages. Any breakthrough, either a technological advantage or superior marketing abilities, would give the firm a long-term edge over the competition. This edge is, essentially, our notion of "reputation". Our use and comparison of different reputation assessments is of essence to changes in, e.g., patent policies.

Another application is attributed to a dynamic advertising problem, where reputation is accumulated in the process of building a franchise. The notion of "reputation" should be interpreted in a straightforward manner, while "effort" relates, for example, to costs of advertising. In addition, our model applies to any trading process involving credit, such that payments are distributed along sequential time periods. On the one hand, the services are instantaneous and, on the other hand, payments are postponed and distributed, evenly or not, over a finite-time horizon. Given such interpretation, "reputation" is translated into direct monetary transfers.

One could even adapt our model to a semi optimal-growth model with post-generational transfers, as a portion of one's wealth is transferred to subsequent generations and so on. By and large, any strategic interaction that combines the two previously-mentioned key components of decreasing marginal-returns and reputation-based payoffs will be closely related to our framework and, therefore, to our conclusions *linking cyclic performance (or business cycles) to incentives.*

1.2 Main results

In practice, the market measures investment firms by their past performance in numerous ways. For example, the New York Times publishes the total returns of various funds according to the previous year, three years, and five years. To capture these alternatives, we consider two distinct mechanisms that represent short- and long-term averaging alike. The first mechanism is the *Transient-Reputation model*, where the market uses the average of the last two-periods returns to evaluate the investment firm's ability. The second mechanism is the *Persistent-Reputation model*, where the market uses a discounted sum of all past returns to generate an evaluation. We present the main results derived from each, separately.

Under the Transient-Reputation model, the firm needs to constantly monitor its last 2-stage average performance (returns). For example, any cost cut at a given stage, immediately limits the firm's payoff at the adjacent stage, since returns of one period are next period's reputation benchmark. This intuition establishes our first main result where, even in a deterministic set-up, the firm's optimal policy dictates that returns oscillate around a certain stable level, while converging towards this level asymptotically. To put it differently, a *cyclic performance is strategically related to incentives*, whenever reputation is based on the average of past results. High-reputation firms have weaker incentives to produce high returns, compared to low-reputation firms, since their benchmark position (of high reputation) entails a lower marginal payoff.

To compare, in the Persistent-Reputation model the market evaluates the firm by a discounted sum of its past returns. The market weighs all past returns while focusing more heavily on recent performance rather than earlier performance. Given decreasing returns to scale, which is the main assumption of Berk and Green (2004), we show that a cyclic performance is plausible. In fact, we prove that the difference between a persistent and a non-persistent performance could be quite mild and vary between time periods.

Nevertheless, the importance of the Persistent-Reputation model follows from wide-perspective analysis, trying to derive policy implications. To be specific, we study how changes in the evaluation process, as more weight is given to recent performance rather than to older one, affects the payoffs of all sides. Non surprisingly, it appears that high-reputation firms profit from a higher weight on past results maintaining their elite status with lower costs, while low-reputation firms benefit from myopic assessments of past returns for the opposite reasons.

On the other hand, we also prove the existence of a *basic tension between incentives and screening.* We show that optimal incentives are reached only in case the market does not value past performance, ignoring past returns completely. To put it differently, a reputation-based screening process cannot produce first-best incentives, as the firm's optimal strategy would rely, to some extent, on past reputation. Therefore, whenever there exists an uncertainty regarding the firms' differential abilities, the market needs to balance between the screening process and optimal incentives. In general, it is quite possible that returns, as a function of the market's evaluation, will take the shape of a *Laffer Curve*: firms' average performance will drop sharply due to poor incentives in case the evaluation is solely based on past performance, and once again will drop sharply due to poor screening, when the evaluation depends completely on future returns, enabling anyone to enter the portfolio-managers market.

1.3 Related literature

In the last two decades, several solutions were proposed to face the performance-persistence problem (to differ from the cyclic performance problem), and the rational model of Berk and Green (2004) contributed one of the key ingredients in that aspect. Berk and Green proposed a non-strategic model, with decreasing returns to scale, where firms that recently outperformed suffer from the positive inflow of funds. Given various assumptions,[2] they showed that much of the non-persistence and other related phenomenons could be attributed to the increasing convex costs, with respect to fund size. Nevertheless, their solution does not account for the cyclic performance situation as emerging from previously-mentioned studies of Cornell et al. (2017) and Bessler et al. (2017). In addition, the motivating conjecture that funds outflow benefit poorly-preforming firms is incompatible with the later work of Coval and Stafford (2007) and Rakowski (2010), suggesting outflows could be just as harmful as inflows.

Dangl et al. (2008) continued the work of Berk and Green (2004) with the introduction of managerial replacements. By differentiating between an investment firm and its manager, they enabled the former to fire the latter in case of poor performance. Therefore, their analysis and main conclusions (supported by the empirical work of Khorana (1996), Chevalier and Ellison (1999), and more recently by Clare et al. (2014)) focus on the interaction between firms and managers, rather than on the firms-market relations. Though the different focal points, the current work follows Dangl et al. (2008) as our "effort" notion strongly relates to their managerial-transfers assumption.

An earlier work of Lynch and Musto (2003) proposed a two-period strategic model to explain the flow-performance relation, specifically targeting recent poor-performance. Though they ignored any diseconomies of scale, their main conclusion is that under-performing funds will adjust their strategies, either through manager replacements or through new investment strategies, while out-performing funds will not. This result carries some intuition to the funds' mean-reverting returns, however their key insight mainly concerns certain fund-flow asymmetries between differently-ranked firms.

Among the various empirical studies of the performance-persistence problem, one can find the work of Pástor et al. (2015) suggesting the existence of decreasing returns to scale at the industry level and, most recently, the work of Bessler et al. (2017) that combines the fund-flow mechanism of Berk and Green (2004) along with Dangl et al. (2008) manager-replacement mechanism. Bessler et al. (2017) show that no single mechanism (among the two) can support the non-persistence phenomenon without the other. In addition, they show that an average monthly gap of 1.81 percentage points between winner and loser funds turns into a -0.22 percentage points difference two years later, only to transform back into a 0.24 percentage points difference four years after the formation year. These findings are important to our work, specifically because it justifies the combination of both mechanisms

[2]Similarly to other studies, they assumed there is no information asymmetry between the market and firms, where all sides maintain rational expectations and a common prior, as an i.i.d. normally-distributed returns-generating process evolves throughout the stages. In addition, their rational-market solution concept requires an infinite-elasticity supply of capital and the ability to eliminate risk by diversifying across funds.

in a unique general model to support an oscillatory performance.

One of the important empirical studies to our work is Amihud and Goyenko (2013). Amihud and Goyenko (2013), building on the work of Carhart (1997), used robust empirical methods to produce a measure of predicting a fund's future performance. They show that the ability to generate excess return over a risk-adjust benchmark is positively associated with funds' expenses, which we denote as *effort*, and negatively associated with the funds' size. These findings are significant to our work, namely because our proposed model aims to explain how these two forces support the key phenomenons of the portfolio-management industry.

1.4 Main contribution

In light of previous studies, we can underline several leading contributions of the current work. First and most importantly, our model accounts for a cyclic performance, while previous models mainly focus on non persistence. Our results indicate that cycles could be attributed to incentives in systems where payoffs are not instantaneous, but distributed throughout the dynamics.

In addition, we bridge between both ends of the empirical research, under a unified model, showing that a persistent and non-persistent performance are not significantly different from a strategic point of view. This result follows from our focus over incentives. Differing from previous work dealing with the portfolio-management problem, we study the incentives of investment firms and their strategic reactions towards them, either through the accumulation of information as in Stoughton (1993) and Admati and Pfleiderer (1997) or through managerial replacements as in Lynch and Musto (2003) and Dangl et al. (2008), among many others.

On a theoretical level, the strategic approach allows us to identify the process and method by which the firm and returns converge to a stable state. That is, we do not limit ourselves to proving the existence of a unique solution, but show how it converges systematically to a stable state, due to the reputation mechanism.

Another contribution is attributed to the inclusion of information frictions between the market and the firms. In our model, the investors form a market that follows simple decision rules to dictate funds flows according to past returns.[3] Thus, there is no need for a common prior, Bayesian updating, or even common knowledge of distributions over returns and abilities.

Lastly, the stochastic elements in our model need not to be normally-distributed, i.i.d., or even ergodic. In contrast, we consider a general Markov Decision Process, and do not limit ourselves to

[3] This non-strategic behavior follows from the market's inability to either easily monitor, or easily interpret, various elements that determine the firms' returns, such as managerial replacements, managers private contracts, implementation of multiple trades, or improved information accumulation by the firms.

Weiner processes. This generality carries some resemblance to the work of Chassang (2013), and opens the door to a broad analysis of the problem from a designer's viewpoint.

1.5 Structure of the paper

The paper is organized as follows. Section 2 concerns the Transient-Reputation model divided into two parts: the deterministic case and the stochastic one. In Section 3 we revert to the Persistent-Reputation model, focusing on the effect that changes in the evaluation process have over incentives. In the same section, we introduce the decreasing-returns-to-scale assumption and study its implications over our model. Final conclusions and remarks are given in Section 4.

2 The Transient-Reputation model

Consider an investment firm interacting with a market or a syndicate of investors[4] in an infinitely-repeated set-up. At every stage, the firm strategically extracts effort that translates into returns for the investors.[5] The market observes the firm's current and past returns and, based on a fixed allocation rule, bestows funds on it to manage. As customary, the firm receives assets-under-management fees. That is, a fixed percentage of the funds it is allocated. The firm's main goal is to maximize its profit given by an infinite discounted sum of the effort-deducted fees. Therefore, the process continues indefinitely as the market observes the newly-realized returns and repeats the funds-allocation process.

This set-up is mainly characterized by two functions: *the return function* and *the market function*, defined as follows. Let $E = [e_{\min}, e_{\max}] \subseteq \mathbb{R}_+$ be a non-empty compact interval denoting the firm's single-period effort choice. For every $e \in E$, the investors' deterministic return $R(e)$ is generated by *the return function* $R : E \to \mathbb{R}_+$. In other words, $R(e)$ is the investors' single-period gross return given the firm's chosen effort level $e \in E$. The investors' pre-determined rule, by which funds are allocated to the firm, is defined by *the market function* $M : R(E) \to \mathbb{R}_+$. The market function's input variable is the firm's average recent performance (returns). Formally, investors average last and current returns to a single factor r such that the firm receives the amount of $M(r)$ to manage in the following period. Due to positive and diminishing marginal returns, both M & R are assumed to be strictly-increasing, strictly-concave, and continuously-differentiable functions.

By the significance of the return and market functions to our model, we wish to dwell on their origin and properties. The extraction of effort (i.e., cost) in the delegated portfolio-management context follows from the need to accumulate information (as in Stoughton (1993), Admati and Pfleiderer (1997),

[4]We use both the investors' and the market's terminologies.

[5]The term 'effort' is used to distinguish between the firm's investment of funds and the firm's operating costs. The firm's effort relates to the latter, and broadly discussed later on.

and Chassang (2013)), as well as the implementation of high-priced multiple trades with significant operating costs. In addition, costs respond to changes in managerial positions and contracts in the spirit of Lynch and Musto (2003) and Dangl et al. (2008) from the theoretical aspect, and Clare et al. (2014), Pástor et al. (2015) and Bessler et al. (2017) from the empirical one. An imperative evidence to the importance of cost-based returns could be found in the previously-surveyed empirical work of Amihud and Goyenko (2013). The same arguments, along with past studies such as Stoughton (1993), also clarify the concavity of the return function. For example, well-performing managers can move to other firms and collect higher wages, thus increasing expenses in (soon to be) top-ranked investment firms. Moreover, one can view the concavity assumption through an auxiliary model where returns are chosen from an interval, while the firm is subjected to an increasing and convex cost function, similarly to the early "cost of adjustment" models (see, e.g., Jorgenson (1963); Lucas (1967); Gould (1968); Treadway (1969); and Mortensen (1973)).

Though the concavity of the return function is well-founded and clear, the market function's concavity is not. In practice, the relation between past performance and fund flows need not be concave. Depending on realized returns, the market function could also be piecewise linear or even convex. For example, Figure 1 in Chevalier and Ellison (1997) describes the flow-performance relations based on a single-year return. The relation alternates between concavity and convexity as performance levels increase.[6] Similar studies, such as Ippolito (1992) and Sirri and Tufano (1998), support this concave-to-convex phenomenon, which was later studied by Lynch and Musto (2003). In addition, Eq. (6) of Berk and Green (2004) indicate that the relation between fund flows and empirically observed returns could be effectively linear (they compare this result to the work of Chevalier and Ellison (1997) in Figure 3 of the same paper). Therefore, we could omit the market function's concavity assumption, as long as the composition of the market and return functions is concave with respect to the firm's operating costs (i.e., effort). In other words, we can restrict ourselves to firms with single-period decreasing marginal payoffs. Nevertheless, for the sake of simplicity, both functions are considered concave henceforth.

The problem begins at stage $t = 1$, with an initial return of $R_0 = R(e_0)$. The firm chooses an effort e_1 to generate a return of $R_1 = R(e_1)$. The market observes both R_0 and R_1, and allocates the firm the amount of $M(R_0 + R_1)$ to manage. Continuing inductively, at every stage $t > 1$ and given past returns $R_0, R_1, \ldots, R_{t-1}$ where $R_{t-1} = R(e_{t-1})$, the firm extracts effort e_t to generate a return

[6] For example, when excess returns range between -5 to $+15$ points, relative to the market return, then expected funds inflow are increasingly concave, whereas between $+15$ to $+20$ points the relation changes to convex, and so on. Chevalier and Ellison (1997) also show that this result depends on the age of the fund, as funds older than 5 years produce a linear relation, with the exception of extremely well-performing firms, producing convex relations. See Figure 2 of the relevant study.

of $R_t = R(e_t)$, and collect the amount of $M\,(R_{t-1} + R_t)$ to manage in the consequent $t+1$ stage.

The firm's payoff in the repeated process is given by a fee-based contract. In simple terms, the contract dictates that the firm collects a fraction of the periodic amount it manages as fees. To simplify the notation and without loss of generality, define the β-discounted firm's payoff[7] by

$$\pi(\underline{e}) = \sum_{t=1}^{\infty} \beta^{t-1}[-e_t + M(R_{t-1} + R_t)], \tag{1}$$

where $\underline{e} = (e_1, e_2, \dots)$ is the firm's infinite-horizon realized actions and $\beta \in (0,1)$. In words, the firm's payoff $\pi(\underline{e})$ is the discounted sum of its per-period payoffs, where at every stage t the firm loses an amount of e_t due to the extraction of effort, and collects a fee of $M(R_{t-1} + R_t)$ based on its returns (and therefore, its effort) in stages t and $t-1$.

A firm's *strategy* σ is a function from all past realized returns (histories) $\bigcup_{t\in\mathbb{N}} R(E)^t$ to the effort set E. A *stationary strategy* σ is a function from the set of single-period returns $R(E)$ to the effort set, or equivalently, a function from the effort set to itself. Given any strategy σ and an initial effort level e_0, denote the firm's payoff by $\pi(e_0|\sigma)$, where all effort levels $\{e_t\}_{t\in\mathbb{N}}$ are determined according to σ. A strategy is considered *optimal* if it solves the optimization problem

$$\pi(e_0|\sigma) = \sup_{\underline{e}\in E^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1}[-e_t + M(R_{t-1} + R_t)].$$

That is, a strategy is optimal if it produces the maximal payoff, denoted $\hat{\pi}(e_0)$, given an initial effort level of e_0. Note that we analyze the interaction of a single firm with a market, rather than the joint interaction of multiple firms. Assuming there are numerous firms in various states, the general properties of the market function and the return capture much of the economic essence of this problem, while abstracting from the technical difficulties of the stochastic game generated by a multiple-firms setting.

The interior-solution property. To simplify the analysis, we require an additional technical assumption stating the the optimal solution is not trivial. Namely, we fix the parameters such that the extreme points $\{e_{\min}, e_{\max}\}$ cannot be the firm's optimal action, independently of R_0. In other words, the firm can only gain from extracting non-minimal effort and cannot gain from the extraction of the maximal effort, thus choosing only interior points of E. One can weaken this assumption by restricting the initial condition to a subset that ensures the non-interior solutions are suboptimal.

[7] The generality of the market function allows us to incorporate all factorial parameters in its definition. For example, we omit the additional parameter to denote the firm's percentage fee of the managed funds.

2.1 Analysis and results - the deterministic case

The payoff function in Eq. (1) presents the basic tension under which the firm operates. On the one hand, the firm receives a reputation-based payoff, given by last and current returns, and prompted through a concave return function. On the other hand, costs are convexly increasing in returns, thus the firm needs to balance its current and future efforts accordingly. For example, an over (under) investment in effort at a single stage, will generate a balancing counter-reaction at the subsequent stage to invest less (more) effort.

This balancing effect motivates the first result of this paper, presented in Theorem 1 that follows. First, it shows that there exists an *absorbing* effort level e^*. Once the absorbing effort level is reached in two consecutive stages, the firm will consistently extract the same level of effort throughout the stages. Second, the theorem proves that the firm balances its performance with respect to the absorbing level, at every two adjacent stages. Namely, in case the initial level is higher (lower) than the absorbing effort, the firm will invest less (more) effort relative to e^*, to level its reputation at the subsequent stage. These alternating effort levels will continue to fluctuate around the absorbing level, while converging to it asymptotically.

Theorem 1. *There exists a unique, stationary and continuous optimal-strategy $\sigma : E \to E$. Given σ, the payoff function $\hat{\pi}(e_0) = \pi(e_0|\sigma)$ is a strictly-concave, and continuously-increasing function of e_0. In addition, if the interior-solution property holds, then:*

- *the optimal strategy σ is strictly decreasing with a single fixed point $e^* \in (e_{\min}, e_{\max})$;*
- *the sequences $\left(\sigma^{2n}(e_0)\right)_{n\in\mathbb{N}}$ and $\left(\sigma^{2n+1}(e_0)\right)_{n\in\mathbb{N}}$ monotonically converge to e^*;*
- *the fixed point e^* is bounded between $\sigma^n(e_0)$ and $\sigma^{n+1}(e_0)$ for every $n \in \mathbb{N}$.*

In other words, Theorem 1 suggests that a cyclic performance, monotonically and systematically converging to equilibrium, is natural when dealing with a firm concerned with performance-based reputation and payoff. This outcome captures two important aspects of the current work. First, depicting a specific path and method of converging to a stable state in a dynamic-optimization problem. Second, linking aggregated reputation, and therefore incentives, to a cyclic-performance phenomenon. The oscillatory movement is consistent with recent studies of Cornell et al. (2017) and Bessler et al. (2017), portraying a short-term mean-reverting performance. In fact, Figure 2 of Bessler et al. (2017) illustrates such an oscillating convergence in practice, with the addition of noisy and decreasing returns to scale, as considered in the following section.

The optimal strategy described in Theorem 1 is, in essence, a combination of two well-documented policies in the economic world. The proposed strategy carries some resemblance to the optimal (S, s)-policies in inventory problems, where an individual agent allows his inventory to fall until is reaches a low level s, only to be imminently and actively increased to a high level S. Such policies were vastly studied in the context of the Pricing Problem (price adjustments and inflation), the Technology-Update Problem, and the Capital Stock Adjustment Problem (see, e.g., Arrow et al. (1951, 1977); Dvoretzky et al. (1952, 1953); Bellman et al. (1955); Bailey (1956); Barro (1972); Sheshinski and Weiss (1977), and Sheshinski and Weiss (1993) for a general survey). Furthermore, our fixed-point convergence relates to the study of global stability in discounted problems, as in Scheinkman (1976); Rockafellar (1976); Cass and Shell (1976); Brock and Scheinkman (1976), among many others. Our analysis produces an advanced combination of these policies, since we not only prove an absorbing possibility, but also depict a systematic method of oscillating convergence towards a stable effort level for any initial condition.

Remark 1. *Due to the technical nature of the analysis and proofs, we postpone them to* Appendix A. *However, we wish to refer the reader to Lemma 1, which studies the properties of the firm's optimal strategy, under the relevant Bellman equation. The generality of this lemma might be of some assistance in similar cases, specifically with the analysis of implicit optimal strategies in dynamic-programming problems.*

2.2 Analysis and results - the stochastic case

The first extension of the Transient-Reputation model concerns the introduction of randomness to the return function. The randomness that we impose need not be i.i.d or even ergodic. Rather, we assume that the return function depends on a randomly-chosen state of the world, dictated by a Markov process, along with prior dependence on the firm's strategic effort. Though its general nature, this extension does not impairment previously-stated results. That is, in this subsection we prove that the conclusions of Theorem 1 still apply, in expectation, under the stochastic extension.

Formally, consider a finite[8] set Ω of states and denote by $P = (P_{ij})_{1 \leqslant i,j \leqslant |\Omega|}$ the transition matrix where P_{ij} is the probability of moving from state i to state j in a single time period. Given the states and transition function, consider a generalization of the return function such that $R : \Omega \times E \to \mathbb{R}_+$ depends on the realized state $\omega \in \Omega$ and on the firm's effort. We assume that the return function

[8] In general, the use of a finite state space could be avoided by taking any countable set or any compact Borel set in $\mathbb{R}$. However, under any compact Borel set, the transition function must hold the Feller property (See Stokey et al. (1989), p. 220) roughly stating that every bounded continuous function is mapped, under the expectation operator and given the transition function, to a bounded continuous function.

maintains its basic properties independently of the realized state. Namely, for every $\omega \in \Omega$, the return function's ω-section $R_\omega : E \to \mathbb{R}_+$ is a strictly-increasing, strictly-concave, continuously-differentiable function. Denote by S the convex hull of the compact set $R(\Omega, E)$ of all possible realized returns.

The stochastic decision problem evolves similarly to the deterministic one. At stage $t = 1$, with an initial state $\omega_0 \in \Omega$ and an initial return of R_0, the firm chooses an effort level $e_1 \in E$. Next, a state ω_1 is realized according to P and ω_0, and the single-stage realized-return is $R_1 = R(\omega_1, e_1)$. Continuing inductively, at every stage $t > 1$ and given a *history* $h_{t-1} = (\omega_0, R_0, \omega_1, R_1, \ldots, \omega_{t-1}, R_{t-1})$ of past realized returns and states, the firm chooses an effort e_t. A state ω_t is realized according to P and ω_{t-1}, and the single-stage return is $R(\omega_t, e_t)$. Therefore, a strategy σ of the firm is a function from the set $\bigcup_{t \in \mathbb{N}} (\Omega \times S)^t$ of all finite histories to E, such that $\sigma(h_{t-1}) = e_t$ is the strategy's realized action at stage t.[9]

Given a strategy σ and initial conditions (ω_0, R_0), the firm's expected β-discounted payoff is

$$\pi(\omega_0, R_0|\sigma) = \mathbf{E}_{\sigma,\omega_0}\left[\sum_{t=1}^{\infty} \beta^{t-1}(-e_t + M(R_{t-1} + R_t))\right], \tag{2}$$

where $\mathbf{E}_{\sigma,\omega_0}[\cdot]$ is the expectation operator with respect to the probability measure induced by the transition probabilities P, the initial state ω_0, and the strategy σ. Note that the strategy is a random variable since it depends on realized states. Thus, the expectation operator also relates to the strategy-induced effort levels throughout the stages.

By the randomness of the process, the firm's realized returns might not accurately follow the same cyclic performance as in Theorem 1. However, in the following Theorem 2 we prove that current returns decrease, in expectation, with respect to previously-realized returns. Moreover, we show that the optimal strategy is a strictly-decreasing function of recently-realized returns. Thus, the oscillating process presented in the deterministic case remains valid, in expectation, under the stochastic one.

Theorem 2. *There exists a unique, stationary and continuous optimal-strategy $\sigma : \Omega \times S \to E$. Given σ and ω_0, the payoff function $\hat{\pi}(\omega_0, R_0) = \pi(\omega_0, R_0|\sigma)$ is a strictly-concave, continuously-increasing function of R_0. In addition, if the interior-solution property holds, then $\boldsymbol{E}_{\sigma,\omega_{t-1}}[R_t|R_{t-1}]$ is a strictly-decreasing function of R_{t-1}, and there exists a return level R^* such that, w.p. 1, returns oscillate around R^* infinitely many times.*

We emphasize that the monotonicity result in the last statement of Theorem 2 relates strictly to previously-realized returns, and not to the different states. Therefore, we fix the state variable ω_{t-1},

[9] Note that the functions M and R are continuous, E is compact, and Ω is finite, therefore measurability requirements, given by Assumptions $9.1' - 9.3'$ in p.$256 - 258$ of Stokey et al. (1989), are met.

and derive monotonicity through changes in the realized return R_{t-1}. Without further assumptions over transitions and states, the expected return in adjacent stages might actually increase in case, e.g., the fixed state variable is significantly worse (in terms of returns) than all other states. Nevertheless, in the proof given in Appendix B, we fix the state variable and show that the optimal strategy is deterministically decreasing in terms of previously-realized returns, thus ensuring the mentioned result. Moreover, this monotonicity supports the general outcome of infinitely many oscillations, as in the deterministic case.

By and large, Theorem 2 coincides with Theorem 1 to support the results of various empirical studies suggesting that firms have a cyclic performance. Our results and model explain such occurrences through a straightforward economic reasoning: when managers' payoffs are reputation dependent, and the accumulation of reputation is increasingly costly, managers' strategies level their performance accordingly.

3 The Persistent-Reputation model

Taking a general perspective on the problem, there are numerous modification to be made in the Transient-Reputation process. One possibility is to condition fund flows on a longer history path with heterogeneous weights. For example, the market function may depend on a discounted sum of all past returns, or on a weighted sum of a fixed number of past returns. Alternatively, one can condition the return function on the amount of managed funds, where an increase in the amount of managed funds carries a negative effect on returns due to the higher price impact.

Though these extensions may seem a matter of mere technicality, they enable us to address the firm-market interaction through a wider scope. Namely, they enable us to study how changes in the market's evaluation process affect, in the long run, the firm's realized returns and its payoff. Therefore, instead of focusing on short-term effects as in Section 2, in this section we inspect long-term effects due to adjustments in the market's fund-flow mechanism.

3.1 Persistent reputation

Starting with the generalization of the market function, consider an optimization problem where the firm's performance, or reputation, is evaluated by a λ-discounted sum of past returns, as $\lambda \in (0,1)$. That is, consider the following optimization problem

$$\hat{\pi}(R_0) = \sup_{\underline{e} \in E^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1}[-e_t + M(\widehat{R_t})],$$

such that $\widehat{R_t} = (1-\lambda)\widehat{R_{t-1}} + \lambda R(e_t)$ and $\widehat{R_0} = R_0$. In words, this optimization problem is similar to the original Transient-Reputation model, other than the exchange of the two-stage returns $R_{t-1} + R_t$ with the discounted sum $(1-\lambda)^t R_0 + \lambda \sum_{n=1}^{t}(1-\lambda)^{t-n} R_n$ of all past returns. Hence, the firm's effective reputation at every stage t is measured by the λ-discounted sum of all past returns. We refer to this problem as the *Persistent-Reputation* model.

The evolution of the Persistent-Reputation set-up is similar to the transient one. At the beginning of every stage t and given a reputation level of $\widehat{R_{t-1}}$, the firm exerts effort e_t to generate a return of $R(e_t)$. The firm's updated reputation is set to $\widehat{R_t} = (1-\lambda)\widehat{R_{t-1}} + \lambda R(e_t)$ and the firm collects a payment of $M(\widehat{R_t})$. The process continues indefinitely.

A comparison of Section 2's results with the following Theorem 3 will certify that many previous results hold under the updated problem, while others change completely. Starting with the similarities, we show that there exists a unique, stationary and continuous optimal-strategy $\sigma : R(E) \to R(E)$, such that the optimal payoff function is a strictly-concave, continuously-increasing function. In addition, we prove that the optimal strategy is strictly decreasing, thus possessing a unique fixed point, as in the previous deterministic set-up.[10] On the other hand, the two models differ in the paths by which the systems converge to a stable return level. In particular, the Persistent-Reputation model generates a monotonic convergence, rather than an oscillating one. We relate to this aspect, among several others, after formally presenting the results of Theorem 3 (proofs are given in Appendix C).

Theorem 3. *In the Persistent-Reputation model, there exists a unique, stationary, and continuous optimal-strategy $\sigma : R(E) \to R(E)$. Given the optimal strategy, the payoff function $\pi(R_0|\sigma)$ is a strictly-concave, continuously-increasing function of R_0. In addition, if the interior-solution property holds, then the optimal strategy is strictly decreasing with a single, interior, fixed point $R^* \in R(E)$, and the sequence $(\widehat{R_t})_{t\in\mathbb{N}}$ of realized discounted performance, generated by σ and R_0, monotonically converges to R^*.*

There are two important aspects that arise from the comparison of Theorem 3 with Theorem 1. First, the monotonic convergence of the discounted performance towards R^* and, second, the optimal strategy's monotonicity. Before relating to each separately, we point out that these results could be reversed when considering decreasing returns-to-scale (see Subsection 3.3).

The monotonic convergence, rather than an oscillating one, follows from the firm's need to minimize costs. The cost of moving from an initial reputation of, e.g., $\widehat{R_0} < R^*$ to $\widehat{R_1} > R^*$ is much higher when the subsequent reputation is a convex combination of current and past returns, rather than just

[10]Note that we now consider strategies as functions from the set of all possible returns to itself, rather than functions from the set of efforts to itself. This alternation is a convenient technical modification of the strategy, that could also apply in the original Transient-Reputation model.

the current return. Therefore, the need to minimize costs guarantees that the adjacent reputation level tends towards the stable effort level, relative to the initial return, but does not cross it. In other words, it is cheaper and therefore more efficient for the firm to monotonically tend towards the stable level, thus discounting costs by extracting more effort in the future, instead of extorting a significant amount of effort in the present.[11]

On the other hand, the reputation's no-crossing statement does not hold when considering actual returns. The optimal strategy's monotonicity suggests that a below stable-level initial return generates an above stable-level return at the subsequent stage, and vice versa. Therefore, we do expect to see an over-performance by a firm, as long as the reputation is below its stable, fixed-point level.

3.2 Long-and short-term memory: Reputation-based screening and the Laffer Curve

In view of the two proposed evaluation mechanisms, one of the most interesting (and potentially most policy-related) question is how much weight should the market put on past returns? This question is not only crucial for the investors' interests, but also critical to the firms, since their optimal strategies and payoffs are directly related to the market's performance evaluation. Therefore, in this section we analyze the ways in which changes in the reputation-aggregation process affect the firm's optimal payoff, as well as the market's returns.

To simplify the analysis, we consider the Persistent-Reputation model, where the evaluation at stage t is given by $\widehat{R_t} = (1-\lambda)\widehat{R_{t-1}} + \lambda R(e_t)$. The advantage of this set-up is its ability to summarize the trade-off between past and current reruns through a single factor, λ. Thus, it facilitates the analysis of changes in the evaluation process and their impact on payoffs and returns. Namely, in Theorem 3 we showed that the firm's performance and reputation converge to a stable level R^*, so we can examine how changes in the market's evaluation, through λ, affect the investors' return, through the steady level R^*. For example, if $\lambda = 0$, the market does not value any future performance, and the steady level becomes R_0 while the firm has no incentive to extract effort. That is, the system remains fixed to the initial condition and the firm produces the minimal feasible effort level. However, if $\lambda = 1$, then past reputation is not taken into account during the market's valuation process, and the firm repeatedly solves the optimization problem $\max_{e \in E}\{-e + M(R(e))\}$.

Before formally stating the results, a few preliminary explanations and notations are needed. For every parameter $\lambda \in [0,1]$, let R^*_λ be the limit return level in the λ-discounted-reputation model

[11] A partially-persistent convergence has some evidence in the empirical literature. For example, Carhart (1997) shows that one-year persistence is possible, while convergence, in the spirit of Theorem 3, is imminent. Figure 2 of Carhart (1997) illustrates such a long-run monotonic convergence.

under the optimal behavior described in Theorem 3. We assume that the firm acts optimally, using the optimal stationary strategy given λ, and returns converge to R^*_λ. In addition, denote the firm's optimal payoff by $\hat{\pi}_\lambda(R_0)$, where R_0 is the initial reputation.

The first result of Theorem 4 relates to the monotonicity of R^*_λ with respect to λ. We prove that the optimal strategy's steady-return level, towards which returns converge, is strictly increasing in λ. To put it another way, returns increase as the market's evaluation is myopic (with respect to past returns), and the reputation depends more heavily on current performance.

On the one hand, this result is quite intuitive from a strategic point of view. When agents cannot rely on past performance, they constantly need to re-justify their abilities in every stage to come, exerting more effort in the process. The accumulated reputation generates a certain *inheritance effect* where the ability to transfer value from one stage to another, leads to less exertion of effort throughout the stages.

On the other hand, the same result also hints to an important economic observation. It implies that the first-best solution, where firms extract the maximal rational effort, is achievable only if the firms do not retain any past reputation. That is, the market's only possibility of extracting the optimal effort from the firms is by ignoring past results completely at any given stage. When such a process comes into play, the market's ability to screen low-level firms is eliminated. Therefore, whenever there exists an uncertainty regarding the firms' differential abilities, the market needs to balance between the screening process (putting more weight on past returns) and optimal incentives (putting more weight on current returns).

In general, it is quite possible that returns, as a function of λ, will take the shape of a *Laffer Curve*. The Laffer Curve, named after economist Arthur Laffer,[12] illustrates the concept of taxable income elasticity. It shows how, in theory, the government's revenue from taxation is eliminated when tax rates are either 0% or 100% (i.e., not collecting any income in the first case, and cutting incentives to produce income, in the other). In our set-up, firms' average performance drops sharply as $\lambda = 0$ due to poor incentives, and once again drops sharply as $\lambda = 1$ due to poor screening, enabling anyone to enter the portfolio-managers market. Hence, the quest to achieve a first-best mechanism ends in an *Incentives-Screening Deadlock*, where one needs to balance between optimal incentives and optimal screening.

The unattainability of the first-best solution should be of no surprise in the delegated portfolio-management context. Both Stoughton (1993) and Admati and Pfleiderer (1997) suggested that the *irrelevance result* poses a critical problem to an optimal incentive scheme.[13] However, their outcome

[12]See report by Laffer (2004).

[13]See Stracca (2006) for a comprehensive survey of this issue.

is based on the existence of some risk-less investment possibility such that the firm can undo the contract's incentive effect, whereas our result is based on the tension between screening and effort-inducing in a reputation-based, decreasing marginal-returns set-up. We believe that the same result could be extended, in future research, beyond the current set-up.

The second part of Theorem 4 concerns the firm's payoff as a function of λ and R_0. These results are best exemplified in Figure 1, showing inverse effects between the discount factor and the initial condition. Namely, fix a discount factor λ_0 and the appropriate limit return level $R^*_{\lambda_0}$. Whenever the initial return level is below $R^*_{\lambda_0}$, the firm's payoff increases if the discount factor is above λ_0. However, whenever the initial reputation is above $R^*_{\lambda_0}$, the firm's payoff increases if the discount factor is fixed below λ_0. Again, the economic intuition is clear. In case the initial reputation is high, the firm will prefer to maintain it as long as possible without extracting additional effort, while a low initial reputation can only harm the expected payoff if past reputation is weighted heavily hereafter.

The combination of these two results generates a somewhat more surprising outcome. It shows that *any discount factor* other than λ is preferable to the firm, given an initial return level of $R_0 = R^*_\lambda$. That is, the discount factor that generates the lowest expected return, given some initial reputation, is the one that imposes the same steady level. As it appears, once returns converge to a stable level, the firm can only profit from either an increased or a decreased discount factor, though the two generate inverse incentives from the market's perspective.

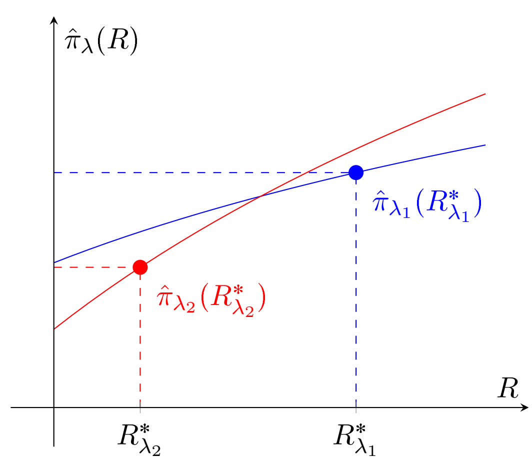

Figure 1: Firm's expected payoff as a function of R_0 and λ, where $\lambda_1 > \lambda_2$. Optimal-payoff functions are convex and continously increasing in the initial reputation.

Theorem 4. *Given the interior-solution property, the steady return level R^*_λ strictly increases as a* *function of λ. Moreover, for every $\lambda_1 \neq \lambda_2$ and given an initial reputation of $R^*_{\lambda_1}$, the firm's payoff is higher under the λ_2-valuation rather than the λ_1-valuation, i.e., $\hat{\pi}_{\lambda_2}\left(R^*_{\lambda_1}\right) > \hat{\pi}_{\lambda_1}\left(R^*_{\lambda_1}\right)$. In addition, If $\lambda_1 > \lambda_2$, then*

- $\hat{\pi}_{\lambda_2}(R) > \hat{\pi}_{\lambda_1}(R)$, *for every* $R \geqslant R^*_{\lambda_1}$;
- $\hat{\pi}_{\lambda_2}(R) < \hat{\pi}_{\lambda_1}(R)$, *for every* $R \leqslant R^*_{\lambda_2}$;
- $\hat{\pi}'_{\lambda_2}(R) > \hat{\pi}'_{\lambda_1}(R)$, *for every* $R \leqslant R^*_{\lambda_1}$.

Theorem 4 is best understood through Figure 1. First, if the market weighs recent performance more heavily (i.e., $\lambda_1 > \lambda_2$), then incentives are sharpened such that: (i) returns converge to a higher level of $R^*_{\lambda_1} > R^*_{\lambda_2}$; (ii) dependence on the initial condition weakens, and the derivative w.r.t. R_0 decreases. Next, in case the initial reputation R_0 is low, the firm would prefer a lower evaluation of past returns (blue line), quickly neglecting past performance, rather than a high evaluation of past returns (red line). The opposite statement holds whenever the initial reputation is high. Moreover, once the initial reputation is a steady state of some λ_i evaluation (i.e., $R_0 = R^*_{\lambda_i}$), then any other λ_{-i} evaluation, *either low or high*, would be strictly preferable to the firm than λ_i.

3.2.1 The stochastic extension

In this subsection we extend the deterministic persistent-reputation model to a stochastic one. By following the same notation and analysis of Section 2.2, we assume that the return function depends on the effort level e and a state $\omega \in \Omega$, chosen according to the transition matrix P. At every stage t, the firm exerts effort e_t given the realized state ω_{t-1} and reputation $\widehat{R_{t-1}}$, where $\widehat{R_t} = (1-\lambda)\widehat{R_{t-1}} + \lambda R(\omega_t, e_t)$ and $\widehat{R_0} = R_0$. Hence, for every strategy σ the firm's expected β-discounted payoff is

$$\pi(\omega_0, R_0|\sigma) = \mathbf{E}_{\sigma,\omega_0}\left[\sum_{t=1}^{\infty} \beta^{t-1}(-e_t + M(\widehat{R_t})\right].$$

The following theorem extends Theorem 3 to the stochastic set-up. The results of Theorem 3 hold, in expectation, similarly to the transient-stochastic extension. Namely, there exist a strictly-concave, continuously-increasing payoff function $\hat{\pi}(\omega_0, R_0)$, and a unique, stationary, and continuous optimal-strategy σ, while the subsequent expected-return decreases as a function of past reputation.

Theorem 5. *There exists a unique, stationary, and continuous optimal-strategy $\sigma : \Omega \times S \to E$. Given σ and ω_0, the payoff function $\hat{\pi}(\omega_0, R_0) = \pi(\omega_0, R_0|\sigma)$ is a strictly-concave, continuously-increasing function of R_0. In addition, if the interior-solution property holds, then $\boldsymbol{E}_{\sigma,\omega_{t-1}}[R_t|\widehat{R_{t-1}}]$ is a strictly-decreasing function of $\widehat{R_{t-1}}$.*

The proof is given in Appendix E.

3.3 Decreasing returns to scale

The rational-market model of Berk and Green (2004) is based on the notion that inwards fund flows decrease expected returns through convex and increasing costs. An increase in managed funds reduce subsequent expected returns as the fund's management requires more information and price impact rises. In this section, we extend the Persistent-Reputation model by conditioning the return function on the amount of managed funds á la Berk and Green (2004). In other words, we adapt our model to the decreasing returns-to-scale assumption to study its affects on previous results.

Formally, consider the Persistent-Reputation set-up of the previous subsection with a return function $R : E \times \mathcal{M} \to \mathbb{R}_+$ where $\mathcal{M} \subset \mathbb{R}_+$ is a non-empty compact interval denoting all possible amounts of managed funds. That is, the return function depends on the effort e and on the amount of managed funds $m \in \mathcal{M}$. For consistency, we assume that $M(R(E, \mathcal{M})) \subseteq \mathcal{M}$. Note that the updated return function, depending on the effort and on the amount of managed funds combines the two important forces suggested by Amihud and Goyenko (2013), to differ from Berk and Green (2004) that do not incorporate effort.

The updated optimization problem evolves as follows. At stage $t = 1$ and given initial return $R_0 \in R(E, \mathcal{M})$, the amount of managed funds is given by $m_0 = M(R_0)$. Then, the firm chooses effort e_1 and receives a payoff of $-e_1 + M(\mathcal{R}_{m_0})$, where $\mathcal{R}_{m_0} = (1-\lambda)R_0 + \lambda R(e_1, m_0)$ is the λ-discounted sum of past returns.[14] In other words, $\mathcal{R}_{m_0}$ is the market's updated evaluation of the firm's performance at the end of stage $t = 1$, thus the amount of managed funds becomes $m_1 = M(\mathcal{R}_{m_0})$. Continuing inductively, at stage $t \geqslant 2$ and given evaluation $\mathcal{R}_{m_{t-2}}$, the firm manages the amount of $m_{t-1} = M(\mathcal{R}_{m_{t-2}})$. The firm chooses an effort level of e_t and receives a payoff of $-e_t + M(\mathcal{R}_{m_{t-1}})$, where $\mathcal{R}_{m_{t-1}} = (1-\lambda)\mathcal{R}_{m_{t-2}} + \lambda R(e_t, m_{t-1})$ is the market's updated evaluation of the firm's performance at the end of the stage. Hence, the new optimization problem is

$$\pi(R_0) = \sup_{\underline{e} \in E^\infty} \sum_{t=1}^{\infty} \beta^{t-1} [-e_t + M(\mathcal{R}_{m_{t-1}})]. \tag{3}$$

Note that the market's evaluation of the firm's performance, $\mathcal{R}_{m_{t-1}}$, is the endogenous state-variable of the optimization problem. The amount of managed funds and the return, at stage t, depend on previously-realized returns only through $\mathcal{R}_{m_{t-1}}$. Therefore, one could follow a similar analysis to the one used in Section 3.1.

[14] We later refer to this notation as the *generalized return function*.

The outcome of this optimization problem depends on the properties of the endogenous state-variable. Due to the price impact of larger funds, it is reasonable to assume (as we do) that the return function is decreasing with respect to the amount of managed funds, while maintaining all previous properties regarding effort. However, these assumptions do not accurately reflect the properties of the state variable $\mathcal{R}_{m_{t-1}}$.

Specifically, consider the *generalized return function* (GRF) defined by $\mathcal{R}(e, r) = (1 - \lambda)r + \lambda R(e, M(r))$. The GRF produces the state variable $\mathcal{R}_{m_{t-1}} = \mathcal{R}(e_t, \mathcal{R}_{m_{t-2}})$, therefore its properties, namely monotonicity and concavity, are essential to our analysis. The previously-discussed properties of the return and market functions ensure that the GRF is continuously-differentiable and strictly concave in both coordinates. Such concavity is consistent with previous studies using convex costs due to the spread of information over multiple trades gathering small activities, or alternatively, a high execution costs leading to a high price impact. Moreover, the return function's monotonicity dictates that the GRF is strictly increasing in effort e, leaving us with the monotonicity assumption with respect to r.

In general, the GRF need not be monotone in r. For example, if $\lambda = 0$, then the GRF is linearly increasing in r. However, taking $\lambda = 1$ and given the decreasing returns-to-scale assumption, the GRF is decreasing in r. Nevertheless, the fact that the return function is decreasing and convex in r asserts that, given a fixed effort level, the GRF has a unique global maximum. Therefore, we can simply assume that the GRF is a single-peak function of r. To conclude, we assume that the return function and the GRF increase in effort and are continuously-differentiable and strictly-concave functions in both arguments, while the return function is strictly decreasing in the amount of managed funds, and the GRF is a single-peak function with respect to r.

The first observation relates to a return function independent of the effort level such that the optimization problem becomes trivial. That is, the t-stage endogenous state variable $\mathcal{R}_{m_{t-1}} = \mathcal{R}(e, \mathcal{R}_{m_{t-2}})$ depends solely on the state variable at stage $t - 1$, whereas e remains fixed. In Observation 1 we assert that the process can converge to a stable state, either cyclically or monotonically, similar to the results of Theorem 1 and Theorem 3. The observation is proven in Appendix F with an extension of the contraction-mapping theorem for concave functions over $\mathbb{R}$.

Observation 1. *If returns are independent of the effort level and the* GRF*'s r-marginal is greater than -1, then the state variable and returns converge to a stable level r^*, either cyclically or monotonically from a certain stage onwards, similarly to the result of* Theorem 3. *If, in addition, the* GRF *is decreasing in r, then it is Lipschitz continuous and contracting. Therefore, the process cyclically converges towards a stable level r^*. That is, the state variable and returns will oscillate around the* *stable level while converging to it, similarly to the result of* Theorem 1.

Observation 1 is motivated by the negative effect of inward fund flows over returns. Positive flows decrease returns through the GRF which, in return, generates a negative flow to repeatedly increase returns and so on. Note that this process is deterministic and non-strategic, and the introduction of a randomness to the returns would increase non-persistent performance, while maintaining the basic results, in expectation. Moreover, the condition over the GRF's derivative is not a necessary one. In the event the GRF's derivative with respect to r equals -1, there exists a possibility of infinitely pivoting between two level, surrounding the stable level, without converging to it. Nevertheless, in many cases the process could still converge, even under a steeper slope assumption.

A significant insight emerging from Observation 1 is the ability to explain, *through a unified model*, the two phenomenons that are at the core of the delegated portfolio-management problem: a persistent versus a non-persistent short-term performance. For example, Carhart (1997) shows that persistence is possible, contrarily to Bessler et al. (2017) depicting a non-persistent convergence. Observation 1 shows that the gap between the two is bridged by the influence of current reputation over the future one. In case the GRF is a decreasing contracting function with respect to past reputation, then the negative effect of past reputation generates a balancing counter-reaction, revealing non-persistent short-term performance. However, in case the GRF is increasing (with respect to past reputation), then a persistent convergence could originate as well.

Contrarily to the degenerated set-up of Observation 1, the second observation relates to effort-dependent returns, with no further assumptions over the GRF. We later point out that additional assumptions could ensure convergence, similarly to Observation 1. The proof is similar to the proofs of Theorem 1 and Theorem 3, hence omitted.

Observation 2. *If returns depend on the effort level, then there exists a unique, stationary, and continuous optimal strategy $\sigma : \mathcal{R}(E, \mathcal{M}) \to E$. Given σ, the optimal payoff is a strictly-concave and continuous function of the initial condition, thus the* GRF*-induced state variable $\mathcal{R}(\sigma(r), r)$ has a fixed point r^*.*

The monotonicity of σ and of the state variable depends on the monotonicity of the GRF with respect to the managed funds, i.e., with respect to the state variable. For example, if the GRF-induced state variable $\mathcal{R}(\sigma(r), r)$ is contracting, or sustains the condition of Observation 1 (concave and marginal decrease is greater than -1), then the system will converge to a stable status. Moreover, the second-order derivatives of the GRF are also imperative for the analysis, since an increase in managed funds can either motivate or discourage the firm from exerting more effort. We leave these assumptions along with further analysis for future research.

4 Concluding remarks

In this paper we presented a model explaining the oscillatory performance issue arising from Cornell et al. (2017) and Bessler et al. (2017). Our reasoning is robust with respect to the governing returns-generating process, and incorporates many information frictions between the managers and the investors. We use these information frictions to generate simple heuristics that subject the firms to a reputation-based payoff mechanism. The payoff mechanism, in return, allows us to extend previous models, such as Berk and Green (2004), to bridge between the two sides of the empirical work in the question of persistent and non-persistent performance.

There are several possible extensions to follow our model and analysis. First, one could model the interaction between firms through performance-based funds-flows, to capture the oscillating performance in a general dynamic market. In light of the theoretical complexity behind such a model (as the dynamic interaction between firms induces a stochastic game), the implementation of numerical analysis is imminent. Second, an extension of our Laffer-Curve idea, associating incentives and screening while taking into account the uncertainty regarding the agent's subjective abilities, is evident. We believe that the main obstacle lays in capturing the same phenomenon in a simple, yet general, static model. In addition, a comprehensive analysis of our model while taking into account the decreasing returns-to-scale assumption, could produce additional insights into the oscillating-performance problem, and towards the market-firms relations, as a whole.

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Appendices

A The deterministic case - analysis and proofs

Eq. (1) produces the following sequential optimization problem

$$\hat{\pi}(e_0) = \sup_{\underline{e} \in E^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} [-e_t + M(R_{t-1} + R_t)], \tag{4}$$

where $\hat{\pi}$ is a optimal payoff given initial effort level $e_0 \in E$. For the simplification of Eq. (4), Bellman (1957) defines the *Principle of Optimality* suggesting the analysis of the corresponding functional equation,

$$\hat{\pi}(e_0) = \sup_{e \in E} \left[-e + M(R_0 + R(e)) + \beta \hat{\pi}(e) \right], \tag{5}$$

where one can focus on a single-period problem instead of an infinite one. Note that the use of $\hat{\pi}$ in Eq. (5) is not trivial.[15] By the properties of E (non-empty, compact, and convex) and the continuity of R and M, it follows that the single-period payoffs are bounded. Thus, we can apply Theorems 4.2, 4.3, and 4.6 from chapter 4 of Stokey et al. (1989) to prove the existence, uniqueness, and continuity of

[15] We also refer to $\hat{\pi}$ as the *value function* of the corresponding problems.

the same solution $\hat{\pi}$ to both Eq. (4) and Eq. (5).In simple terms, we can use the customary approach in discounted dynamic optimization and restrict our analysis to the unique solution of Eq. (5).

Define the correspondence $\sigma : E \to 2^E$ such that

$$\sigma(e_0) = \{e \in E \mid \hat{\pi}(e) = F(e_0, e) + \beta\hat{\pi}(e)\},$$

where $F(e_0, e) = -e + M(R(e_0) + R(e))$ and for every $e_0 \in E$. Since all functions are bounded and continuous, σ is well defined. Moreover, Theorems[16] SLP-4.4, SLP-4.5, and SLP-4.6 prove that σ is a compact-valued, upper hemi-continuous correspondence, that generates the firm's optimal strategy either in the functional equation, Eq. (5), or in the sequential problem, Eq. (4).

In order to show that σ is a single-valued continuous function, we need to show that $F(e_0, e)$ is concave w.r.t. e_0 and e, and strictly concave w.r.t. e_0 (by SLP-4.8). Fix $\delta \in (0, 1)$ and $(e_0, e), (e_0', e') \in E^2$ such that $e_0 \neq e_0'$. By the strict concavity of R if follows that

$$R(\delta e_0 + (1-\delta)e_0') + R(\delta e + (1-\delta)e') > \delta[R(e_0) + R(e)] + (1-\delta)[R(e_0') + R(e')].$$

Since M is monotonic and concave,

$$M(R(\delta e_0 + (1-\delta)e_0') + R(\delta e + (1-\delta)e')) > \delta M(R(e_0) + R(e)) + (1-\delta)M(R(e_0') + R(e')),$$

and

$$F(\delta(e_0, e) + (1-\delta)(e_0', e')) > \delta F(e_0, e) + (1-\delta)F(e_0', e').$$

Thus, σ is a single-valued continuous function and, by SLP-4.8, the value function $\hat{\pi}$ is strictly concave. In addition, the fact that F is strictly increasing in e_0 implies that the value function $\hat{\pi}$ is strictly increasing (see SLP-4.7).

Proving that $\hat{\pi}$ is differentiable using SLP-4.11 requires $\sigma(E)$ to be interior points of E (which holds by the interior-solution property). In such cases, the value function is continuously differentiable, and we can effectively use the envelope theorem: in the FOC of Eq. (5) we plug-in the optimal solution $\sigma(e_0)$ to obtain,

$$M'(R(e_0) + R(\sigma(e_0)))R'(\sigma(e_0)) + \beta\hat{\pi}'(\sigma(e_0)) = 1. \tag{6}$$

Eq. (6) enables us to study the properties of σ. We start with monotonicity. Consider a small increase of e_0 to $e_0 + \varepsilon > e_0$. If $\sigma(e_0 + \varepsilon) \geqslant \sigma(e_0)$, then the LHS of Eq. (6) decrease, violating the equality, since M', R', and $\hat{\pi}'$ are (non-negative) decreasing functions due to the strict concavity of

[16]Hereafter, we refer to Theorems 4.2 – 4.11 in p.71 – 85 of Stokey et al. (1989) as SLP-4.XX. A significant number of these theorems are based on the pioneering work of Blackwell (1965).

M, R, and $\hat{\pi}$. It implies that σ is a strictly-decreasing continuous function from E to E, thus it has a unique fixed point $e^* \in E$ (also an interior point of E) such that

$$M'(R(e^*) + R(e^*))R'(e^*) + \beta\hat{\pi}'(e^*) = 1.$$

The next step of our analysis shows that, for every $e_0 \in E$, the sequences $\left(\sigma^{2n}(e_0)\right)_{n\in\mathbb{N}}$ and $\left(\sigma^{2n+1}(e_0)\right)_{n\in N}$ monotonically converge to e^*, as n tends to infinity. In addition, the uniqueness of e^*, along with the continuity of σ, imply that e^* is bounded between $\sigma^n(e_0)$ and $\sigma^{n+1}(e_0)$ for every n. That is, we show that the repeated use of σ tends monotonically to the fixed point e^*, and $\sigma^n(e_0)$ oscillates around e_0 as a function of n, with an amplitude converging to 0.

Now define $H(x,y) = M'(R(x)+R(y))R'(y)+\beta\hat{\pi}'(y)$ and note that the concavity of R and $\hat{\pi}$ imply that $H(x,y) > H(y,x)$ for every $(y,x) \subseteq E$. In addition, H is continuous and strictly-decreasing in both coordinates. Therefore, H satisfies the conditions of the following Lemma 1.

Lemma 1. *Let $H : E^2 \to \mathbb{R}$ be a continuous function, strictly-decreasing in both coordinates, such that*

$$H(x,y) > H(y,x), \tag{7}$$

for every $(y,x) \subseteq E$. Then,

1. *there exists $c \in \mathbb{R}$ such that for every $x \in E$, the equation $H(x,y) = c$ has a unique solution, $y_c(x)$;*
2. *the function y_c is a continuous, strictly decreasing function with a unique fixed point x_c;*
3. *for every $x \in E$, the sequences $\left(y_c^{2n}(x)\right)_{n\in\mathbb{N}}$ and $\left(y_c^{2n+1}(x)\right)_{n\in\mathbb{N}}$ monotonically tend to x_c as $n \to \infty$;*
4. *the fixed point x_c is bounded between $y_c^n(x)$ and $y_c^{n+1}(x)$ for every $n \in \mathbb{N}$ and every $x \in E$.*

Proof. Ineq. (7) implies that $H(e_{\max}, e_{\min}) > H(e_{\min}, e_{\max})$ and we can fix c between these two values of H. It follows from the strict monotonicity of H that $H(x, e_{\min}) > c > H(x, e_{\max})$ for every $x \in E$. By continuity, for every $x \in E$ there exists a solution $y_c(x)$ for $H(x,y) = c$, and strict monotonicity suggests $y_c(x)$ is unique. In addition, the same two properties of H imply that y_c is continuous and strictly-decreasing. Moreover, y_c is defined from E to E, therefore it has a unique fixed point, denoted x_c. We conclude that $H(x_c, y_c(x_c)) = H(x_c, x_c) = c$.

Fix $x \in E$ such that $x > x_c$. Since $H(x_c, y_c(x_c)) = c$ where H is strictly decreasing, we deduce that $y_c(x) < y_c(x_c) = x_c$. Assume, contrary to the stated lemma, that $y_c^2(x) = x$. Then,

$$c = H(y_c(x), y_c^2(x)) = H(y_c(x), x) < H(x, y_c(x)) = c,$$

where the inequality follows from Ineq. (7). A contradiction. Since $H(y_c(x), x) < c$, we conclude that $x_c < y_c^2(x) < x$, as needed. A similar proof holds for $x > x_c$. Hence, we can now consider the sequences $\left(y_c^{2n}(x)\right)_{n\in\mathbb{N}}$ and $\left(y_c^{2n+1}(x)\right)_{n\in\mathbb{N}}$ bounding x_c. Each of the two sequences tends closer to x_c as n grows. Since both are monotonic and bounded, they converge. Assume, by contradiction, that a sequence, e.g., $\left(y_c^{2n+1}(x)\right)_{n\in\mathbb{N}}$, converges to $x'_c \neq x_c$. Then,

$$x'_c = \lim_{n\to\infty} y_c^{2n+3}(x) = \lim_{n\to\infty} y_c^2\left(y_c^{2n+1}(x)\right) = y_c^2\left(\lim_{n\to\infty} y_c^{2n+1}(x)\right) = y_c^2(x'_c),$$

contradicting the strict monotonicity of the sequences when $x \neq x_c$, and concluding the proof. ■

To conclude the analysis and proof of Theorem 1, consider the previously-defined function $H(x,y) = M'(R(x) + R(y))R'(y) + \beta\hat{\pi}'(y)$, and note that it sustains all the conditions of Lemma 1. Fix $c = 1$ and substitute x and y by e_0 and σ, respectively. The previous analysis, prior to Lemma 1, shows that σ has a unique fixed point e^* given $c = 1$, and Lemma 1 ensures that an iteration of σ cyclically converges to the fixed point e^*. Thus, we conclude the proof of Theorem 1. ■

B The stochastic case - analysis and proof

Similarly to the previous analysis, we transform the optimization problem derived from Eq. (2) to the following Bellman equation

$$\pi(\omega_0, R_0) = \sup_{e\in E} \mathbf{E}_{\omega_0}\left[-e + M(R_0 + R(\tilde{\omega}, e)) + \beta\pi(\tilde{\omega}, R(\tilde{\omega}, e))\right], \tag{8}$$

where the expectation relates to the random variable $\tilde{\omega}$, drawn according to P and ω_0.

Proof of Theorem 2. To prove Theorem 2, we follow Chapter 9 of Stokey et al. (1989), and specifically, Assumptions $9.4 - 9.12$ and $9.16 - 9.17$ along with Exercise 9.7. Formally, Assumption 9.4 follows from the definition of S; Assumption 9.5 follows from the assumptions on Ω and P; Assumptions $9.6, 9.9, 9.11$ and 9.16 hold since E is a fixed convex interval; Assumptions $9.7, 9.8, 9.10$ and 9.12 follow from Assumptions $4.4, 4.5, 4.7$, and 4.9 respectively, mentioned in Appendix A along with the linearity of the expectation operator; and Assumption 9.17 holds by the differentiability of R_ω. In addition, the stage-t return depends solely on the realized action and the realized state at stage $t-1$, thus the condition given in Exercise 9.7-f is met, and the result follows by the interior-solution property. To sum-up, there exists a continuously-increasing, strictly-concave, differentiable payoff function $\hat{\pi}(\omega_0, R_0)$ (all w.r.t. R_0), and there exists a unique, stationary, and continuous optimal-strategy $\sigma : \Omega \times S \to E$.

By the differentiability of the RHS of Eq. (8) w.r.t. e, we use the envelope theorem and plug-in $\sigma(\omega_0, R_0)$ after taking the FOC, to get

$$0 = \mathbf{E}_{\omega_0}\left[-1 + \left(M'(R_0 + R(\tilde{\omega}, e)) + \beta\frac{\partial\pi(\tilde{\omega}, R(\tilde{\omega}, e))}{\partial R(\tilde{\omega}, e)}\right)\frac{\partial R(\tilde{\omega}, e)}{\partial e}\right]_{e=\sigma(\omega_0,R_0)}. \tag{9}$$

The monotonicity and concavity of the return function, the market function, and the payoff function imply that the derivatives on the RHS decrease when either R_0 or $\sigma(\omega_0, R_0)$ increase. Thus, an increase in R_0 must follow a decrease in $\sigma(\omega_0, R_0)$ to maintain Eq. (9). Hence, $\mathbf{E}_{\sigma,\omega_{t-1}}[R_t|R_{t-1}] = \mathbf{E}_{\omega_{t-1}}[R(\tilde{\omega}, \sigma(\omega_{t-1}, R_{t-1})]$, and $\sigma(\omega_{t-1}, R_{t-1})$ decrease w.r.t R_{t-1}.

To prove a cyclic performance, we start with the simple case where the state variable is absorbed, w.p. 1, to some fixed state $\omega \in \Omega$. In such a case, the proof of Theorem 1 holds and a cyclic performance follows. Otherwise, assume w.l.o.g. that the chain is irreducible. For every $\omega \in \Omega$, consider the function $\psi_\omega(r) = \mathbf{E}\left[R(\tilde{\omega}, \sigma(\omega, r))\right]$. By the continuity and monotonicity of σ along with the compactness assumption over $R(E)$, there exists a unique fixed point r_ω such that $\psi_\omega(r_\omega) = r_\omega$. Since the Markov chain is finite and irreducible, we can take the stationary distribution μ and define $R^* = E_\mu[r_{\tilde{\omega}}]$, where the expectation is taken w.r.t. μ.

Fix $\overline{\omega}, \underline{\omega} \in \Omega$ such that $r_{\overline{\omega}} > r_\omega > r_{\underline{\omega}}$, for every $\omega \in \Omega \backslash \{\overline{\omega}, \underline{\omega}\}$. We will show that for every $\varepsilon > 0$, w.p. 1, every trajectory visits the two intervals $(-\infty, r_{\underline{\omega}} + \varepsilon]$ and $[r_{\overline{\omega}} - \varepsilon, \infty)$ infinitely many times, thus oscillating around R^* as needed. The idea behind this statement is that both $\overline{\omega}$ and $\underline{\omega}$ are visited infinitely many times, and whenever the realized return is within $[r_{\underline{\omega}}, r_{\overline{\omega}}]$, then the expected return in the subsequent period is outside $[r_{\underline{\omega}}, r_{\overline{\omega}}]$. Namely, the monotonicity of σ implies that for every state ω, the inequality $\psi_\omega(r) > r_\omega$ holds if and only if $r < r_\omega$. Meaning, a realized return below (above) the fixed point r_ω ensures next-stage's expected return is above (below, resp.) the fixed point. In other words, returns *oscillate in expectation*.

Fix a small $\varepsilon > 0$ such that $R^* \in (r_{\underline{\omega}}+\varepsilon, r_{\overline{\omega}}-\varepsilon)$. The compactness of $R(E)$ along with the oscillation-in-expectation property guarantees that there exists $\delta > 0$ such that $\Pr(R(\tilde{\omega}, \sigma(\overline{\omega}, r)) > r_{\overline{\omega}}) > \delta$ for every $r < r_{\overline{\omega}}$ and, equivalently, $\Pr(R(\tilde{\omega}, \sigma(\underline{\omega}, r)) < r_{\underline{\omega}}) > \delta$ for every $r > r_{\underline{\omega}}$. We will now turn to a proof by contradiction.

Denote $I = [r_{\underline{\omega}} + \varepsilon, r_{\overline{\omega}} - \varepsilon]$ and assume there is a positive probability event $D = \bigcup_{t \in \mathbb{N}} D_t$ where D_t includes all histories such that the realized return from stage t onwards is in I. Since D has positive probability, there exists $D_T \subseteq D$ with positive probability, and a positive-probability finite history h, of length greater than T stages, such that $\Pr(D_T|h) > 1 - \delta$. Now consider all continuations of h. Each continuation h' settles in $\overline{\omega}$ infinitely often. Let $\tau[h']$ be the first stage, after h, where $\overline{\omega}$ is the state variable according to a continuation h'. The construction implies that $R_{\tau[h']} \in I$, and specifically $R_{\tau[h']} < r_{\overline{\omega}}$. By the previous statement, we know that for every $R_{\tau[h']} < r_{\overline{\omega}}$,

$$\Pr\left(R\left(\tilde{\omega}, \sigma\left(\omega_{\tau[h']}, R_{\tau[h']}\right)\right) > r_{\overline{\omega}} |\ h, \omega_{\tau[h']} = \overline{\omega}\right) > \delta.$$

Summing over all stages $\tau(h')$, we get that $\Pr(\overline{D_T}|h) > \delta$, contradicting the initial assumption that $\Pr(D_T|h) > 1 - \delta$ and concluding the proof. ∎

C Proof of Theorem 3

Proof. In this proof we follow the same analysis present in Appendix A. However, to simplify the notation, we use the set $R(E)$ instead of E to denote the firm's actions, and denote the initial return level by $R_0 \in R(E)$. Therefore, the equivalent functional equation to Eq. (5) becomes

$$\hat{\pi}(R_0) = \sup_{r \in R(E)} \left[-R^{-1}(r) + M((1-\lambda)R_0 + \lambda r) + \beta \hat{\pi}((1-\lambda)R_0 + \lambda r) \right], \tag{10}$$

where the firm chooses a return level r, receives a payoff of $-R^{-1}(r) + M((1-\lambda)R_0 + \lambda r)$, and moves on to the next stage with reputation $(1-\lambda)R_0 + \lambda r$. Note that R^{-1} is the inverse function of R, and therefore strictly-increasing, strictly-convex and continuously-differentiable.[17]

By the properties of E, R, and M we can use SLP-4.2, SLP-4.3, and SLP-4.6 (similarly to Theorem 1) to prove the existence, uniqueness, and continuity of $\hat{\pi}$. Re-define the correspondence $\sigma : R(E) \to 2^{R(E)}$ such that

$$\sigma(R_0) = \{ r \in R(E) \mid \hat{\pi}(r) = F(R_0, r) + \beta \hat{\pi}((1-\lambda)R_0 + \lambda r) \},$$

where $F(R_0, r) = -R^{-1}(r) + M((1-\lambda)R_0 + \lambda r)$. Theorems SLP-4.4, SLP-4.5, and SLP-4.6 prove that σ is a compact-valued, upper hemi-continuous correspondence, that generates the firm's optimal strategy.

To show that σ is a single-valued continuous function, we need to prove that $F(R_0, r)$ is concave w.r.t. R_0 and r, and strictly concave w.r.t. R_0 (see SLP-4.8). By the strict convexity of R^{-1} and by the same analysis as in Appendix A, the concavity condition of F holds and σ is a single-valued continuous function, while the value function $\hat{\pi}$ is strictly concave. In addition, the fact that F is strictly increasing in R_0 implies that the value function $\hat{\pi}$ is also strictly increasing (see SLP-4.7).

The interior-solution property and SLP-4.11 prove that the value function is continuously differentiable, and by the envelope theorem we can follow the analysis of Chapter 4 in Stokey et al. (1989), to write down the following FOC of the Bellman equation,

$$0 = -\frac{1}{\lambda R'(R^{-1}(\sigma(R_0)))} + M'((1-\lambda)R_0 + \lambda\sigma(R_0)) + \beta \hat{\pi}'((1-\lambda)R_0 + \lambda\sigma(R_0)),$$

or equivalently,

$$\lambda R'(R^{-1}(\sigma(R_0))) \left[M'((1-\lambda)R_0 + \lambda\sigma(R_0)) + \beta \hat{\pi}'((1-\lambda)R_0 + \lambda\sigma(R_0)) \right] = 1 \tag{11}$$

[17] The fact that subsequent stages are determined by $(1-\lambda)R_0 + \lambda r$ (instead of r) requires some adjustments of the basic model described in Chapter 4 of Stokey et al. (1989). Nevertheless, such adjustments maintain the validity of the results previously used in Appendix A (see, e.g., Chapter 9 of Stokey et al. (1989)). One could consider a similar analysis where the firm chooses the next-stage reputation level, subject to the current position R_0, the return function R, and the discount factor λ. Such analysis in compatible with Chapter 4 of Stokey et al. (1989) and sustains the required assumptions.

Next, consider a small increase of R_0 to $R_0 + \varepsilon > R_0$. If $\sigma(R_0 + \varepsilon) \geqslant \sigma(R_0)$, then the LHS of the last equation decreases (since R, M, and $\hat{\pi}$ are concave), violating the equality. Hence, we proved that σ is a strictly-decreasing continuous function from $R(E)$ to $R(E)$, thus it has a unique, interior, fixed point $R^* \in R(E)$ such that

$$\lambda R'(R^{-1}(R^*)) \left[M'(R^*) + \beta\hat{\pi}'(R^*)\right] = 1.$$

Combining the last two equations (the FOC equality and the last fixed-point equality) yields

$$\lambda R'(R^{-1}(R^*)) \left[M'(R^*) + \beta\hat{\pi}'(R^*)\right] = \lambda R'(R^{-1}(\sigma(R_0))) \left[M'(\hat{R}) + \beta\hat{\pi}'(\hat{R})\right],$$

where $\hat{R} = (1-\lambda)R_0 + \lambda\sigma(R_0)$.

Assume $R_0 < R^*$. The monotonicity of σ implies that $\sigma(R_0) > R^*$, and so $R'(R^{-1}(R^*))) > R'(R^{-1}(\sigma(R_0)))$. Therefore, it follows from the last equation that $M'(\hat{R})+\beta\hat{\pi}'(\hat{R}) > M'(R^*)+\beta\hat{\pi}'(R^*)$, or equivalently $R^* > (1-\lambda)R_0 + \lambda\sigma(R_0) > R_0$. In words, we showed that an initial reputation of $R_0 < R^*$ imposes a return above R^* in the subsequent stage, while maintaining the subsequent reputation level below R^*. By induction, the same result applies in every stage to follow. Symmetrically, one reaches a similar conclusion given $R_0 > R^*$, and we derive that the sequence $(\widehat{R_t})_{n\in\mathbb{N}}$ generated by σ and R_0, monotonically converges to R^*. ■

D Proof of Theorem 4

Proof. To simplify the proof, we use the same notation as in the proof of Theorem 3 where the relevant Bellman equation is given by Eq. (10),

$$\hat{\pi}_\lambda(R_0) = \sup_{r\in R(E)} \left[-R^{-1}(r) + M((1-\lambda)R_0 + \lambda r) + \beta\hat{\pi}((1-\lambda)R_0 + \lambda r)\right],$$

such that R^{-1} is the (strictly increasing and convex) inverse function of R. To use Bellman's principle of optimality and Blackwell's Contraction Mapping Theorem, we need to find a contracting operator from the set of bounded functions to itself. Let B be the set of bounded real-valued functions over $R(E)$. For every λ, define the operator $T_\lambda : B \to B$ such that

$$(T_\lambda f)(R_0) = \max_{r\in R(E)} \left[-R^{-1}(r) + M((1-\lambda)R_0 + \lambda r) + \beta f((1-\lambda)R_0 + \lambda r)\right],$$

for every $R_0 \in R(E)$. This operator, along with the results of chapter 3 of Stokey et al. (1989), was used explicitly to prove Theorem 3, and will be similarly used in the current proof.

The proof is divided into five parts with respect to the different parts of the theorem:

Part I proves $\hat{\pi}_{\lambda_2}\left(R^*_{\lambda_1}\right) \geqslant \hat{\pi}_{\lambda_1}\left(R^*_{\lambda_1}\right)$, for every two discount factors $\lambda_1 \neq \lambda_2$ such that $R_0 = R^*_{\lambda_1}$.

Part II proves $\hat{\pi}_{\lambda_2}(R_0) > \hat{\pi}_{\lambda_1}(R_0)$, for every two discount factors $\lambda_1 > \lambda_2$ such that $R_0 > R^*_{\lambda_1}$.

Part III proves $\hat{\pi}_{\lambda_2}(R_0) < \hat{\pi}_{\lambda_1}(R_0)$, for every two discount factors $\lambda_1 > \lambda_2$ such that $R_0 < R^*_{\lambda_2}$.

Part IV proves R^*_λ is strictly increasing in λ.

Part V proves Part II and Part III for the cases where $R_0 = R^*_{\lambda_1}$ and $R_0 = R^*_{\lambda_2}$, respectively. Applying Parts II, III, and V to any λ and with respect to higher and lower discount factors produces the desired result.

Part I. Since $R_0 = R^*_{\lambda_1}$ is a fixed point of the λ_1-valuation problem, the firm will repeatedly generate a return of $R^*_{\lambda_1}$ and a payoff of $\hat{\pi}_{\lambda_1}(R^*_{\lambda_1})$. Thus, for any function $f \in B$ such that $f(R^*_{\lambda_1}) \geqslant \pi_{\lambda_1}(R^*_{\lambda_1})$, it follows that

$$\begin{aligned}
\left(T_{\lambda_2} f\right)(R^*_{\lambda_1}) &\geqslant -R^{-1}(R^*_{\lambda_1}) + M(R^*_{\lambda_1}) + \beta f(R^*_{\lambda_1}) \\
&\geqslant -R^{-1}(R^*_{\lambda_1}) + M(R^*_{\lambda_1}) + \beta \pi_{\lambda_1}(R^*_{\lambda_1}) \\
&= \hat{\pi}_{\lambda_1}(R^*_{\lambda_1}),
\end{aligned}$$

where the first inequality follows from substituting the optimal r with $R^*_{\lambda_1}$, and the second inequality follows from the assumption over f. By Bellman's principle of optimality and Banach's Contraction Mapping Theorem along with the fact that the set of bounded functions that sustain the condition $f(R^*_{\lambda_1}) \geqslant \pi_{\lambda_1}(R^*_{\lambda_1})$ is closed, it follows that $\hat{\pi}_{\lambda_2}(R^*_{\lambda_1}) \geqslant \hat{\pi}_{\lambda_1}(R^*_{\lambda_1})$, as needed.

Part II. Fix a function $f \in B$ such that $f(r) \geqslant \hat{\pi}_{\lambda_1}(r)$ for every $r > R^*_{\lambda_1}$. By Theorem 3, we know that $R_0 > R^*_{\lambda_1}$ implies $\sigma_{\lambda_1}(R_0) < R^*_{\lambda_1}$, where σ_{λ_1} is the optimal stationary strategy in the λ_1-valuation problem, such that the reputation in the next stage tends towards $R^*_{\lambda_1}$ from above. Hence,

$$\begin{aligned}
\left(T_{\lambda_2} f\right)(R_0) &\geqslant -R^{-1}(\sigma_{\lambda_1}(R_0)) + M((1-\lambda_2)R_0 + \lambda_2\sigma_{\lambda_1}(R_0)) + \beta f((1-\lambda_2)R_0 + \lambda_2\sigma_{\lambda_1}(R_0)) \\
&\geqslant -R^{-1}(\sigma_{\lambda_1}(R_0)) + M((1-\lambda_2)R_0 + \lambda_2\sigma_{\lambda_1}(R_0)) + \beta\hat{\pi}_{\lambda_1}((1-\lambda_2)R_0 + \lambda_2\sigma_{\lambda_1}(R_0)) \\
&> -R^{-1}(\sigma_{\lambda_1}(R_0)) + M((1-\lambda_1)R_0 + \lambda_1\sigma_{\lambda_1}(R_0)) + \beta\hat{\pi}_{\lambda_1}((1-\lambda_1)R_0 + \lambda_1\sigma_{\lambda_1}(R_0)) \\
&= \hat{\pi}_{\lambda_1}(R_0),
\end{aligned}$$

where the first inequality follows from substituting the optimal r with $\sigma_{\lambda_1}(R_0)$, the second inequality follows from the assumption over f, and the third inequality follows from the monotonicity of $\hat{\pi}_\lambda$ (w.r.t. λ) and of M. Since the set of functions f sustaining the required condition is closed, and by the Contraction Mapping Theorem, the result follows.

Part III. Similarly to Part II, fix a function $f \in B$ such that $f(r) \geqslant \hat{\pi}_{\lambda_2}(r)$ for every $r < R^*_{\lambda_2}$. By Theorem 3, we know that $R_0 < R^*_{\lambda_2}$ implies $\sigma_{\lambda_2}(R_0) > R^*_{\lambda_2}$, where σ_{λ_2} is the optimal stationary strategy in the λ_2-valuation problem, such that the reputation in the next stage tends towards $R^*_{\lambda_2}$ from below. Hence,

$$
\begin{aligned}
(T_{\lambda_1} f)(R_0) \;&\geqslant\; -R^{-1}\left(\frac{\lambda_1-\lambda_2}{\lambda_1}R_0+\frac{\lambda_2}{\lambda_1}\sigma_{\lambda_2}(R_0)\right)+M((1-\lambda_2)R_0+\lambda_2\sigma_{\lambda_2}(R_0))\\
&+\;\beta f((1-\lambda_2)R_0+\lambda_2\sigma_{\lambda_2}(R_0))\\
&\geqslant\; -R^{-1}\left(\frac{\lambda_1-\lambda_2}{\lambda_1}R_0+\frac{\lambda_2}{\lambda_1}\sigma_{\lambda_2}(R_0)\right)+M((1-\lambda_2)R_0+\lambda_2\sigma_{\lambda_2}(R_0))\\
&+\;\beta\hat{\pi}_{\lambda_2}((1-\lambda_2)R_0+\lambda_2\sigma_{\lambda_2}(R_0))\\
&>\; -R^{-1}(\sigma_{\lambda_2}(R_0))+M((1-\lambda_2)R_0+\lambda_2\sigma_{\lambda_2}(R_0))+\beta\hat{\pi}_{\lambda_2}((1-\lambda_2)R_0+\lambda_2\sigma_{\lambda_2}(R_0))\\
&=\; \hat{\pi}_{\lambda_2}(R_0),
\end{aligned}
$$

where the first inequality follows from substituting the optimal r with $\frac{\lambda_1-\lambda_2}{\lambda_1}R_0+\frac{\lambda_2}{\lambda_1}\sigma_{\lambda_2}(R_0)$, the second inequality follows from the assumption over f, and the third inequality follows from the monotonicity of R^{-1}. Since the set of functions f sustaining the required condition is closed, and by the Contraction Mapping Theorem, the result follows.

Part IV. Assume that $R^*_{\lambda_1} < R^*_{\lambda_2}$ for $0 < \lambda_2 < \lambda_1 < 1$ (where the result for the end points is trivial). Take $r \in (R^*_{\lambda_1}, R^*_{\lambda_2})$. According to Parts II and III, we get $\hat{\pi}_{\lambda_2}(r) > \hat{\pi}_{\lambda_1}(r) > \hat{\pi}_{\lambda_2}(r)$. A contradiction. Thus, $R^*_{\lambda_1} \geqslant R^*_{\lambda_2}$, for $\lambda_1 > \lambda_2$.

Now assume that $R^*_{\lambda_1} = R^*_{\lambda_2}$ for $\lambda_2 < \lambda_1$. We can take the FOC of the RHS of the stated Bellman equation (similarly to Theorem 3), along with the derivative of $\hat{\pi}_{\lambda_1}(R_0)$ to get the two equations,

$$\lambda\left[M'((1-\lambda)R_0+\lambda\sigma(R_0))+\beta\hat{\pi}'((1-\lambda)R_0+\lambda\sigma(R_0))\right]=\left(R^{-1}\right)'(\sigma(R_0))$$

and

$$\hat{\pi}'(R_0)=(1-\lambda)M'((1-\lambda)R_0+\lambda\sigma(R_0)),$$

where the second equality follows from the envelope theorem. Taking $\lambda=\lambda_1$, $R_0=R^*_{\lambda_1}$, and plugging the second equation into the first yields

$$\lambda_1\left[1+\beta(1-\lambda_1)\right]=\frac{\left(R^{-1}\right)'\left(R^*_{\lambda_1}\right)}{M'\left(R^*_{\lambda_1}\right)}.$$

Since $\beta \in (0,1)$, the LHS is an increasing function of λ_1, subject to $0 \leqslant \lambda_1 \leqslant 1$. Thus, $R^*_{\lambda_1} = R^*_{\lambda_2}$ contradicts the last equality, implying $R^*_{\lambda_1} > R^*_{\lambda_2}$, as needed.

Part V. We only prove the relevant case of Part II where $R_0 = R^*_{\lambda_1}$, while a similar proof holds for $R_0 = R^*_{\lambda_2}$ of Part III. Consider $\lambda_1 > \lambda_2$ and fix $\lambda_3 \in (\lambda_2, \lambda_1)$. According to Part IV, $R^*_{\lambda_1} > R^*_{\lambda_3} > R^*_{\lambda_2}$.

Hence by Part II, $\hat{\pi}_{\lambda_2}(R^*_{\lambda_1}) > \hat{\pi}_{\lambda_3}(R^*_{\lambda_1})$, whereas by Part I, $\hat{\pi}_{\lambda_3}(R^*_{\lambda_1}) \geqslant \hat{\pi}_{\lambda_1}(R^*_{\lambda_1})$, which concludes Part V.

Next, we prove the last statement of the theorem regarding the derivatives. If $R^*_{\lambda_2} \leqslant R \leqslant R^*_{\lambda_1}$, then $\sigma_{\lambda_2}(R) < R < \sigma_{\lambda_1}(R)$, and $(1-\lambda)R + \lambda\sigma_\lambda(R)$ increase with λ. If $R < R^*_{\lambda_2}$, we consider two cases where either $\sigma_{\lambda_1}(R) \leqslant \sigma_{\lambda_2}(R)$ or $\sigma_{\lambda_1}(R) > \sigma_{\lambda_2}(R)$. Assume that $\sigma_{\lambda_1}(R) \leqslant \sigma_{\lambda_2}(R)$. Thus, $\lambda R'(R^{-1}(\sigma_\lambda(R)))$ increase w.r.t. λ, and by Eq. 11 along with the concavity of M and $\hat{\pi}$, it follows that $(1-\lambda)R + \lambda\sigma_\lambda(R)$ increase in λ. Otherwise, $\sigma_{\lambda_1}(R) > \sigma_{\lambda_2}(R) > R$ and, again, we get the same monotonicity of $(1-\lambda)(R) + \lambda\sigma_\lambda(R)$ w.r.t. λ. By the previously-stated equation $\hat{\pi}'_\lambda(R) = (1-\lambda)M'((1-\lambda)R + \lambda\sigma(R))$, along with the concavity of M, it follows that $\hat{\pi}'_{\lambda_2}(R) > \hat{\pi}'_{\lambda_1}(R)$, as stated.

■

E Proof of Theorem 5

Proof. The respective Bellman equation is

$$\pi(\omega_0, R_0) = \sup_{e \in E} \mathbf{E}_{\omega_0}\left[-e + M((1-\lambda)R_0 + \lambda R(\tilde{\omega}, e)) + \beta\pi(\tilde{\omega}, (1-\lambda)R_0 + \lambda R(\tilde{\omega}, e))\right],$$

where the expectation relates to the random variable $\tilde{\omega}$, drawn according to P and ω_0. By the same reasoning as in the proof of Theorem 2, we follow Chapter 9 of Stokey et al. (1989), specifically using Assumptions $9.4-9.12$ and $9.16-9.17$, along with Exercise 9.7. The assumptions and statements hold as in the stochastic transient-reputation model, with the exception of Exercise 9.7-f that does not hold. Specifically, the condition of 9.7-f requires the reputation in the subsequent stage to be independent of the reputation in the previous one. However, in the stochastic persistent-reputation model the dependence is straightforward by $(1-\lambda)R_0 + \lambda R(\tilde{\omega}, e)$. To sum-up, results 9.7-a through 9.7-d ensure the existence of a continuously-increasing, strictly-concave payoff function $\hat{\pi}(\omega_0, R_0)$, all w.r.t. R_0, and ensure the existence of a unique, stationary, and continuous optimal-strategy $\sigma : \Omega \times S \to E$. Thus, we can reformulate the previous Bellman equation as

$$\hat{\pi}(\omega_0, R_0) = -r + \mathbf{E}_{\omega_0}\left[M((1-\lambda)R_0 + \lambda R(\tilde{\omega}, r)) + \beta\hat{\pi}(\tilde{\omega}, (1-\lambda)R_0 + \lambda R(\tilde{\omega}, r))\right],$$

where $r = \sigma(\omega_0, R_0)$.

We wish to prove that r decreases as a function of R_0, while ω_0 is fixed. Following a proof by contradiction, assume there exist $R'_0 > R_0$ such that $r' = \sigma(\omega_0, R'_0) > \sigma(\omega_0, R_0) = r$. We show that a deviation from r to r' is profitable given (ω_0, R_0), contradicting the optimality of r. To simplify the notation, define the functions G by

$$G(x, y) = \mathbf{E}_{\omega_0}\left[M((1-\lambda)x + \lambda R(\tilde{\omega}, y)) + \beta\hat{\pi}(\tilde{\omega}, (1-\lambda)x + \lambda R(\tilde{\omega}, y))\right].$$

Note that G is concave as the sum of two concave functions. Thus,

$$G(R_0', r) - G(R_0, r) > G(R_0', r') - G(R_0, r').$$

In words, the concavity of G implies that an upwards deviation of the x-variable from R_0 to R_0' is more profitable as $y = r$, rather than $y = r' > r$. Therefore,

$$G(R_0, r') - G(R_0, r) > G(R_0', r') - G(R_0', r) > r' - r,$$

where the second inequality follows from the optimality of $r' = \sigma(\omega_0, R_0')$. However, the last inequality suggests that $G(R_0, r') - G(R_0, r) > r' - r$, contradicting the optimality of $r = \sigma(\omega_0, R_0)$ and concluding the proof. ∎

F Extending the contracting-mapping theorem on $\mathbb{R}$.

Lemma 2. *Let $f : [0,1] \to [0,1]$ be a continuously-differentiable*[18] *and strictly-concave function such that $f'(x) > -1$ for every $x \in [0,1]$. Then, for every $x_0 \in [0,1]$ the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, and x^* is a fixed point of f.*

Proof. Assume there are no interior fixed points to f, then either $f(x) < x$ or $f(x) > x$, for every $x \in (0,1)$. Thus, $(f^n(x_0))_{n\in\mathbb{N}}$ is monotonic. Now, suppose there exists $x^* \in (0,1)$ such that $f(x^*) = x^*$. Consider $x \in (0,1)$ where $f(x) \neq x$. By strict-concavity, it follows that $f(x) \geqslant x$ if and only if $x \leqslant x^*$. If $g(x) := \frac{f(x^*)-f(x)}{x^*-x} > 0$, then the numerator and denominator have the same sign, and $g(x) < 1$, bounded by the slope of the diagonal. Otherwise, $g(x) = f'(y) < 0$ for some y between x and x^* (by the Mean-Value Theorem), while $f'(y) > -1$, by assumption. In both cases, $|g(x)| < 1$, thus $|f^n(x_0) - x^*| = |f^n(x_0) - f^n(x^*)|$ decreases. If it tends to 0, the result follows. Otherwise, there are $x_1 \neq x_2$, such that $|x_1 - x^*| = |x_2 - x^*|$ and $f(x_i) = x_{-i}$ for $i = 1, 2$. However, this implies that $|f^n(x_i) - x^*|$ is constant, contradicting the previous statement. ∎

[18] Note that the differentiability condition is not a necessary one, but simplifies the notation.