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Abstract

We show that the equilibrium existence result for markets with adverse selection of Azevedo and Gottlieb [2017] does not generically extend to settings with unbounded consumer types. We provide conditions under which equilibria do exists in these environments in terms of equilibria of associated truncated economies. We illustrate the role of our conditions by explicitly building the equilibrium in a non-trivial insurance market, and show that it features unbounded prices.

1 Introduction

In this note, we consider the notion of equilibrium in markets with adverse selection developed by Azevedo and Gottlieb [2017] (henceforth AG). We show that an AG-equilibrium need not exist in settings when consumer types (in particular, consumer riskiness or cost) has an unbounded distribution. We also provide class of settings where risk is unbounded but an AG equilibrium nonetheless exists. Furtheremore, in specific cases, we build that equilibrium explicitly and we show that it features unbounded prices.

We begin by recalling the model and notion of equilibrium used in AG. A consumers type is a vector $\theta \in \Theta$, where Θ is a Polish space with measure P.[1] This type can describe, for instance, each individuals risk, risk aversion, wealth, etc. A contract is a vector $x \in X$, where X is also Polish. For instance, contracts might be characterized

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[1] A Polish space is a complete, separable metrizable space.

by deductibles, co-insurance rates, etc. Price is denoted by $p \in \mathbb{R}_+$. We consider Borel-measurable price functions $p : X \to \mathbb{R}_+$ where $p(x)$ is the price of alternative x. Individual utility is a continuous function $u(\theta, x, p)$ (sometimes denoted $u_\theta(p, x)$), strictly decreasing in prices where $u : \Theta \times X \times \mathbb{R} \to \mathbb{R}$. Cost is also a continuous function $c(\theta, x) \geq 0$, where $c : \Theta \times X \to \mathbb{R}_+$. Crucially, cost depends on type θ which creates the possibility for adverse selection. At various points in the paper we may add additional structure or assumptions, but the continuity of $u(\cdot,\cdot,\cdot)$ and $c(\cdot,\cdot)$ are implicitely assumed throughout.

When extending the model presented in AG to unbounded settings, to avoid certain anomologies we assume that for each compact set $K \subseteq X$, $\int_\Theta \sup_{x \in K} c(x, \theta) dP(\theta) < \infty$ (i.e., if all agents choose alternatives in the compact set K, the cost will be finite). A distributions α on $\Theta \times X$ with marginal P on Θ, such that $\int_{\Theta \times X} c(\theta, x) d\alpha < \infty$, is known as an allocations. $\alpha(\{\theta, x\})$ can be thought of as the density of types θ purchasing contract x under α.

Definition 1. A pair (p, α), consisting of a price function p and an allocation α is a weak equilibrium of an economy $\mathcal{E} = [\Theta, X, P]$ if, for α-a.e. $(\theta, x) \in \Theta \times X$, consumers maximize their utility and each contract breaks even. Formally,

$$\sup_{x' \in X} u(\theta, p(x'), x') = u(\theta, p(x), x) \text{ and}$$

$$p(x) = E_\alpha[c(\theta, x) \mid x], \qquad \text{for } \alpha - a.e.(\theta, x) \in \Theta \times X \tag{1}$$

The conditional expectation is well-defined, as c is α-integrable. AG consider settings where Θ, X are compact. In these environments, there typically exists a large multiplicity of weak equilibria, which motivates AG's focus on equilibria.

Definition 2. A pair (p, α) is an equilibrium of $\mathcal{E} = [\Theta, X, P]$ if there exists:

- A Polish space $\overline{X}$ such that X is dense in $\overline{X}$.[2]

[2] Formally, X embeds to a dense subset of $\overline{X}$, but we disregard such technicalities for the sake of brevity at no cost to the generality. In AG, X is assumed to be compact, and hence $\overline{X}$ must coincide with X.

- A sequence $(\overline{X}^n)_{n\in N}$ of finite subsets of $\overline{X}$ which converge to $\overline{X}$ in the sense of Haussdorf.[3]
- A sequence $(\eta^n)_{n\in N}$ of measures, with η_n supported on $\overline{X}^n$, strictly positive on $\overline{X}^n$, and $\eta^n(\overline{X}^n) \to 0$.
- A sequence of pairs $(p^n, \alpha^n)_{n\in\mathbb{N}}$, such that:
  - (p^n, α^n) is a weak equilibrium of the economy $[\Theta \cup \overline{X}^n, \overline{X}^n, P + \eta^n]$, where type $x \in \overline{X}^n$ has zero cost and prefers x to any other alternative (regardless of price); the types $x \in \overline{X}^n$ are known as behavioral types.
  - $\alpha^n \to \alpha$ weakly.[4]
  - Whenever $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ with $x_n \in \overline{X}^n$, then $p^n(x^n) \to p(x)$.

In this case, we say that $(\overline{X}, (\overline{X}^n), (p^n, \alpha^n), (\eta^n))$ witnesses that (p, α) is an equilibrium. We note that we allow the perturbed economies to include alternatives which are not in X but can only be approximated by contracts in X - i.e., in $\overline{X}$. This is natural when the equilibrium may include unbounded prices. The price, say, of full insurance coverage may be infinite when infinitely costly types exist, but when considering a only subset of them with bounded costs, the riskiest types may desire purchasing full insurance. We note that an equilibrium is also an equilibrium; see Proposition 8.[5]

The main result of AG is to show that an equilibrium exists under the following addition conditions: Firstly, X, Θ are compact metric spaces; and secondly, u obeys a form of Lipschitz-ness in X, uniformly over types; when utility is of the form $u(\theta, x, p) = v(\theta, x) - p$, this amounts precisely to uniform Lipshitz-ness in X.

Much of this paper focuses on the role of the compactness of Θ and X in equilibrium existence. We pay particular attention to the empirically relevant case of economies where cost $c(x, \theta)$ is unbounded above. Indeed, consumer costs are often approximated by unbounded distributions (see, for instance, Veiga and Weyl [2016], Cohen and Einav [2007], Einav et al. [2010, 2012] and, in fact, the calibration in AG). We will

[3] I.e., for each $x \in \overline{X}$, there is $(x_n)_{n\in\mathbb{N}}$ converging to x with $x_n \in \overline{X}^n$ for each $n \in \mathbb{N}$.

[4] i.e., for each $f : \Theta \times X \to \mathbb{R}$ continuous and bounded, $\int f d\alpha^n \to \int f d\alpha$.

[5] It is in the proof of Proposition 8 that the assumption that for each compact set $K \subseteq X$, $\int_\Theta \sup_{x\in K} c(x, \theta) dP(\theta) < \infty$, is used.

show that, in these settings, the AG equilibrium generically does not exist. Then, we provide conditions under which equilibria exist for unbounded economies. We illustrate these conditions by explicitly building the equilibrium for an insurance market with unbounded cost.

Large parts of the paper will further restrict attention to consumers with CARA utility facing wealth shocks (at times, for further concreteness, we will consider the shocks to be Gaussianly distributed). Moreover, we consider insurance contracts that cover a fixed share of the individual's cost, so certainty equivalents are linear. We begin by showing that, when risk is unbounded, an AG equilibrium (if it exists) necessarily features unbounded prices. We then provide an example of a setting where each individual's risk aversion is bounded above by a linear function of the individual's risk, and show that an AG equilibrium does not exist in this setting. We then establish some tools for constructing, or showing the existence of, an equilibrium of an unbounded economy as a limit of equilibria of truncated bounded economics. Finally, using these tools, we show that an equilibrium does exist if risk is unbounded but risk aversion increases sufficiently fast with risk.

2 Utility and Cost Functions

Fome of our results, we consider utility functions which are defined on $[0,1]\times\mathbb{R}_+$, where the contract (x,p) can be understood to denote insurance coverage of an x-fraction of any loss (1 being full insurance, 0 being no insurance). We assume a continuous map μ on the type space $\mu : \Omega \to \mathbb{R}_{++}$ assigning to a type θ his riskiness μ_θ. The cost function is given by $c(\theta,x) = \mu_\theta \cdot x$. In this framework, we add two assumptions to those already given in Section 1:

Assumption #1: $u_\theta(p,x) = g_\theta(x) - p$ for some continuous strictly concave twice differentiable $g_\theta : [0,1] \to \mathbb{R}$ with $\frac{\partial g_\theta}{\partial x} > 0$, $\frac{\partial^2 g}{\partial x^2} < 0$.

We denote, for $\theta \in \Omega$, the marginal willingness to pay (which, under Assumption #1, is independent of price):

$$w_\theta(x) = \frac{\partial u_\theta}{\partial x}(x,p) > 0.$$

Assumption #2: At full insurance ($x = 1$), the marginal willingness to pay equals riskiness. That is, $w_\theta(1) = \mu_\theta$ for all $\theta \in \Omega$.

To see how such utility functions arise, generalizing Rothschild and Stiglitz [1976], suppose the utility of type $\theta \in \Omega$ is given by

$$v_\theta(x, p) = E[U_\theta(w_\theta - (1-x)Z_\theta - p)] \tag{2}$$

where U_θ is a CARA utility function with constant coefficient of absolute risk aversion a_θ, $U_\theta(c) = -e^{-a_\theta c}$, w_θ is θ's unitial wealth, and Z_θ type θ's loss, a random variable for which $E[Z_\theta] = \mu_\theta$. This model was adopted in Veiga and Weyl [2016] and Levy and Veiga [2017]. Hence $U_\theta(x, p) = -e^{a_0 p} \cdot e^{-a_\theta(w_\theta - (1-x)Z_\theta)}$, so defining the smooth strictly monotonic transformation $u_\theta = -\frac{1}{a_\theta}\ln(-v_\theta)$ of utility gives

$$u_\theta(x, p) = -\frac{1}{a_\theta} \cdot \ln(E[e^{-a_\theta(w_\theta - (1-x)Z_\theta)}]) - p \tag{3}$$

It follows that near full insurance, $x \approx 1$:[6]

$$\begin{aligned} u_\theta(x,p) &\approx -\frac{1}{a_\theta} \ln E[e^{-a_\theta w_\theta}(1 + a_\theta(1-x)Z_\theta)] - p \\ &\approx -\frac{1}{a_\theta} \ln(e^{-a_\theta w_\theta}(1 + a_\theta(1-x)E[Z_\theta])) - p \approx w_\theta - p - (1-x)E[Z_\theta] \end{aligned}$$

and hence $w_\theta(1) = E[Z_\theta] = \mu_\theta$.

To illuminate further, in some specific examplse we follow Veiga and Weyl [2016] and Levy and Veiga [2017] further, which assumed $Z_\theta \sim \mathcal{N}(\mu_\theta, \sigma_\theta^2)$, so $\ln(E[Z_\theta]) = \mu_\theta + \frac{1}{2}\sigma_\theta^2$. Denoting $\nu_\theta = a_\theta \sigma_\theta^2$, it follows from (3) and properties of the normal distribution that the utility - which is just the certainty equivalent of (x, p) for type θ - is

$$u_\theta(x, p) = x\mu_\theta + \frac{1}{2}\left(1 - (1-x)^2\right)\nu_\theta - p \tag{4}$$

and marginal willingness to pay for additional insurance is

$$w_\theta(x) := \partial u_\theta / \partial x = \mu_\theta + (1-x)\nu_\theta. \tag{5}$$

[6]We suffice for less-than-rigorous arguments here only for the intuition.

More generally, we will denote for each $\delta > 0$,

$$\overline{\nu}_\delta(\theta) := \max\{-\frac{\partial w_\theta}{\partial x}(x) \mid x \in [1-\delta, 1]\} = \max\{-\frac{\partial^2 u_\theta}{\partial x^2}(x) \mid x \in [1-\delta, 1]\} \tag{6}$$

$$\underline{\nu}_\delta(\theta) := \min\{-\frac{\partial w_\theta}{\partial x}(x) \mid x \in [1-\delta, 1]\} = \min\{-\frac{\partial^2 u_\theta}{\partial x^2}(x) \mid x \in [1-\delta, 1]\} \tag{7}$$

Under the CARA-Normal case presented above, $\overline{\nu}_\delta(\theta) = \underline{\nu}_\delta(\theta) = \nu$, regardless of δ.

Assumption (*): There is $\overline{\delta} > 0$ such that $\overline{\nu}_{\overline{\delta}}$ can be bounded as a function of μ; formally, for each $\overline{\mu}$ s.t. $P(\mu(\theta) \leq \overline{\mu}) > 0$, there is $\overline{\rho} = \overline{\rho}_{\overline{\delta}}(\overline{\mu})$ s.t. $P(\overline{\nu}_{\overline{\delta}}(\theta) \leq \overline{\rho} \mid \mu(\theta) \leq \overline{\mu}) = 1$.

WLOG we assume that $\overline{\rho}_{\overline{\delta}}(\cdot)$ is non-decreasing and right-continuous.[7]

Since the cost to a firm when type θ purchases coverage x is $c(\theta, x) = \mu_\theta x$, the break-even condition (1) becomes

$$p(x) = x \cdot E_\alpha[\mu \mid x],\ \alpha - a.s. \tag{8}$$

We will use repeatedly the fact that:

Lemma 1. If$1 \geq x_2 \geq x_1 \geq \delta \geq 0$, then for a type $\theta \in \Theta$.

$$u_\theta(p_2, x_2) \geq u_\theta(p_1, x_1) \Longrightarrow \frac{p_2 - p_1}{x_2 - x_1} \leq \mu_\theta + \overline{\nu}_\delta(\theta) \cdot (1 - \frac{x_1 + x_2}{2}) \tag{9}$$

while

$$u_\theta(p_2, x_2) \leq u_\theta(p_1, x_1) \Longrightarrow \frac{p_2 - p_1}{x_2 - x_1} \geq \mu_\theta + \underline{\nu}_\delta(\theta) \cdot (1 - \frac{x_1 + x_2}{2}) \tag{10}$$

Proof. As in Assumption #1, write a decomposition $u_\theta(x, p) = g_\theta(x) - p$. Since

[7] If $\overline{\rho}_{\overline{\delta}}(\cdot)$ is not monotonic, it can just be replaced with its 'monotonic closure', $x \to \sup_{0 \leq y \leq x} \overline{\rho}_{\overline{\delta}}(y)$. If it is not right-continuous, it can be replaced with $x \to \lim_{y \to x^+} \overline{\nu}(y)$. Both of these operations will preserve the linear growth, which we will assume in Section 4.

$w_\theta(1) = \frac{\partial g_\theta}{\partial x}(1) = \mu_\theta$,

$$\frac{\partial g_\theta}{\partial x}(s) = \mu_\theta - \int_s^1 \frac{\partial^2 g_\theta}{\partial^2 x} dx$$

Now, for any $0 \leq x_1 < x_2 \leq 1$,

$$u_\theta(x_{2,}p_2) - u_\theta(x_1, p_1) = p_1 - p_2 + \int_{x_1}^{x_2} \frac{\partial g_\theta}{\partial x} dx = p_1 - p_2 + \mu_\theta(x_2 - x_1) + \int_{x_1}^{x_2} \int_s^1 (-\frac{\partial^2 g_\theta}{\partial^2 x}) dx$$

Hence, denoting

$$\Delta(x_1, x_2) = \int_{x_1}^{x_2} \int_s^1 1 dx = \frac{1}{2}(1 - x_1)^2 - \frac{1}{2}(1 - x_2)^2 = (x_2 - x_1)(1 - \frac{x_1 + x_2}{2})$$

we see that if $x_2 > x_1 \geq \delta$,

$$p_1 - p_2 + \mu_\theta(x_2 - x_1) + \Delta(x_1, x_2) \cdot \underline{\nu}_\delta(\theta) \leq u_\theta(x_{2,}p_2) - u_\theta(x_1, p_1) \leq p_1 - p_2 + \mu_\theta(x_2 - x_1) + \Delta(x_1, x_2) \cdot \overline{\nu}_\delta(\theta)$$

Dividing by $x_2 - x_1$gives the lemma. □

A few remarks are in order. An equilibrium price function can be shown to be continuous (see Lemma 5 below).[8] Moreover, $p(x)$ must be (weakly) increasing. Heuristically, suppose, instead that, $x_2 > x_1$ and $p(x_2) < p(x_1)$. Then, no consumer would choose (p_1, x_1) or even be indifferent. Then, in equilibrium, we must have $p(x_1) = 0$ but then $p(x_2) < 0$ is a contradiction. This argument is formalized in Lemma 8.

3 Unbounded Risks Imply Unbounded Prices

We begin by showing that, when riskiness μ is unbounded (and our other assumptions hold), an AG-equilibrium (if it exists) must feature unbounded prices $p(x)$. This result is interesting in itself since it suggests that, for instance, a numerical calibration that necessarily uses a bounded distribution to approximate an unbounded one, can produce an equilibrium price function which looks significantly different from the price function in the unbounded the economy of interest.

We will also assume that the projection of P to riskinesses, $P_\mu := P \circ \mu^{-1}$, is not

[8] The continuity (and in fact Lipshitz-ness) was proven in AG under the stronger conditions there.

compactly supported.[9] Equivalently, μ is not essentially bounded w.r.t. to P; i.e., for all $M > 0$, $P(\{\theta \mid \mu(\theta) \geq M\}) > 0$.

We take the set of contracts X to be any Borel[10] subset of $[0, 1]$ for which full insurance is a limit point.[11] Notice that full insurance ($x = 1$) a priori may or may not be an option in X and, moreover, X need not be compact.

Due to the unboundedness of μ and the form of w_θ in (5), this model does not satisfy AG's Assumptions. In particular, Θ is not compact, X need not be compact and u is not Lipschitz in x uniformly in θ.

We will first prove that, under these conditions, any AG-equilibrium (if it exists) features an unbounded price. The full proof is in appendix B. We proceed in the following five steps. First, we establish bounds on the slope of $p(x)$ for weak equilibria. Second, we show these bounds also hold for equilibria. Third, we show that full insurance cannot be an atom of an equilibrium (unless it is concentrated on a single type). Fourth, we show that smaller and smaller neighborhoods of full insurance attracts arbitrarily large risks. Finally, we show that the price cannot be bounded in equilibrium.

Proposition 1. If the distribution $P_\mu = P \circ \mu^{-1}$ is not compactly supported, and Assumption (*) holds, any AG equilibrium features unbounded price $p(\cdot)$.

Proof. See Appendix B. □

4 Non-Existence of Equilibrium: An Example

Now we assume that, some for some $\delta, C, D > 0$, P-a.s. it holds that $\overline{\nu}_\delta(\theta) \leq C\mu(\theta) + D$, where v_δ was defined in (8). That is, ν_δ is function of at most linear growth in μ. This includes the case of consumers homogeneous in risk aversion in the CARA-normal model ($\nu \equiv \nu_0 > 0$ a.s.). This assumption implies Assumption (*), and hence, as we

[9] Recall that the support of a measure is the smallest closed set which has null complement.

[10] We require this to be able to define a Borel measure on X. We allow X to be fairly general - e.g., countable. As the example of Section 7 hints, a countable discrete set is often appropriate. An obvious initial objection to a non-compact space of contracts is that agents may not have a best contract, even when prices are continuous. However, when utilities extend continuously to $[0, 1] \times \mathbb{R}_+$ and $\lim_{x \to 1} p(x) = \infty$, agents do attain their maximal utility contract.

[11] I.e., there is a sequence (x_n) in X with $x_n \to 1$ but $\forall n, x_n < 1$.

have already shown, that p must be unbounded in any AG-equilibrium. We will now show that, under these conditions, if (p,α) is an equilibrium, that p is bounded: a contradiction to Proposition 1.

Proposition 2. If the distribution P is s.t. μ is not essentially bounded and, P-a.s., $\nu_\delta \le C\mu + D$ for some $\delta, C, D > 0$, then there exists no AG equilibrium.

Proof. We deal here with the case where X includes the left-neighborhood $(\delta, 1)$ of full insurance, and the price function is absolutely continuous. The general case is handled in Section C. Denote its derivative by $p'(x)$. Consider any $x \ge \delta$. Lemma 8 of the Appendix - which gives bounds on the slope of the equilibrium prices - implies

$$p'(x) \le \frac{p(x)}{x} + (1-x)\overline{\nu}_\delta(\frac{p(x)}{x}) \le \frac{p(x)}{x} + (1-x)(C\frac{p(x)}{x} + D) \le (1+C)\frac{p(x)}{x} + D \le Bp(x) + D.$$

where we have denoted $B = \frac{1+C}{\delta}$. This will imply that

$$(e^{-Bx}p(x))' \le e^{-Bx} \cdot D$$

which implies, for $x \ge \delta$,

$$e^{-Bx}p(x) \le e^{-B\delta}p(\delta) - \frac{D}{B}[e^{-Bx} - e^{-B\delta}].$$

Therefore $p(x)$ is bounded, which results in a contradiction: no AG equilibrium exists.

□

5 Unbounded Equilibrium via Bounded Approximations

In Section 7 below, we will present a class of economies with unbounded riskiness in which we have an equilibrium (with unbounded prices, as Proposition 1 implies); in fact, we will also focus on a sub-class in which we will construct the equilibria explicitely. Here, we present a result which shows that, under appropriate conditions, the equilibrium of an unbounded economy exists and is the limit (in the appropriate sense) of a sequence of equilibria of 'bounded approximation' economies which approximate the unbounded economy of interest.

We are ultimately interested in an unbounded economy $\mathcal{E} = [\Theta, X, P]$, with Θ, X Polish and locally compact, but Θ not compact. For example, Proposition 1 implies that in the set up considered there, equilibrium can only exist if $p(x)$is unbounded as full insurance is approached ($x \to 1$), hence we allow price to be unbounded by taking the contract space to be $X = [0, 1)$.

We will approximate this economy by a sequence approximating economies indexed by the subscript n. Each economy n has a bounded type and contract spaces. We consider a sequence of compact subsets $\Theta_1 \subseteq \Theta_2 \subseteq \cdots \subseteq \Theta$ with $\cup_n \Theta_n = \Theta$. We will take the contract space of the approximating economies to be $\bar{X}$, a compactification of X.[12] For instance, if $X = [0, 1)$, then we may take $\bar{X} = [0, 1]$. We will also assume that $c(\cdot, \cdot)$ and $u(\cdot, \cdot, \cdot)$ extend continuously from $X \times \Theta$ and $X \times \Theta \times \mathbb{R}_+$ to $\overline{X} \times \Theta$ and $\overline{X} \times \Theta \times \mathbb{R}_+$, respectively, s.t. $\int_\Theta \max_{x \in \overline{X}} c(\cdot, \theta) dP < \infty$. That is, even if each agent chooses the costliest available option, total cost is finite. (For example, in the framework of Section 2, the utilities and costs are defined and continuous on $\Theta \times [0, 1] \times \mathbb{R}_+$ and $\Theta \times [0, 1]$, respectively, and $\int_\Theta \max_{x \in \overline{X}} c(\cdot, \theta) dP = \int_\Theta \mu(\theta) dP$, which is finite by assumption.[13]

Our goal is to build the equilibrium of the unbounded economy $\mathcal{E}$. To do this, we consider the equilibria of the sequence of bounded approximating economies $\mathcal{E}_n$, where $\Theta_n \to \Theta$. However, to construct the equilibrium of each $\mathcal{E}_n$, we must then consider the sequence of weak equilibria of the perturbation economies $\mathcal{E}_n^k$, per Definition 2.

Proposition 3. For each n, let (p_n, α_n) be an AG-equilibrium of the restricted economy $[\Theta_n, \overline{X}, P(\cdot \mid \Theta_n)]$, such that:

- There is a function $p : X \to \mathbb{R}_+$ s.t. $p_n \to p$ uniformly on compact subsets of X.
- There is a distribution α on $\Theta \times X \subseteq \Theta \times \overline{X}$ s.t. $\alpha_n \to \alpha$ weakly.

Then (p, α) is an equilibrium of the economy $[\Theta, X, P]$.

Proof. See Appendix D. □

[12]The compactification of a bounded subset of $\mathbb{R}^N$ is its closure (the union of that set with all its limit points).

[13]Formally, for any $x \in X \subseteq [0, 1]$ with $x \neq 0$, and $x \cdot \int_\Theta \mu(\theta) dP = \int_\Theta c(x, \theta) dP < \infty$by assumption.

As we will see, it is often not tractable to know wheat the candidate price function or allocation in equilibrium of the limiting economy will be. Hence, we have the follow useful variant of Proposition 3; recall that a collection of real-valued functions $\mathcal{F}$ on a metric space (X, d) is point-wise bounded if $\forall x \in X$, $\sup_{f \in \mathcal{F}} |f(x)| < \infty$, and $\mathcal{F}$ is equi-continuous if for each $\varepsilon > 0$ and each $x \in X$, there is $\delta > 0$ such that if $y \in X$ with $d(x, y) < \delta$, then $|f(y) - f(x)| < \varepsilon$ for all $f \in \mathcal{F}$.

Proposition 4. Assume that for every two alternatives $x, y \in X$, price $p \geq 0$, and type $\theta \in \Theta$, there is price q high enough s.t. $u(\theta, x, p) > u(\theta, y, q)$.[14] For each n, let (p_n, α_n) be an equilibrium of the restricted economy $[\Theta_n, \overline{X}, P(\cdot \mid \Theta_n)]$, such that:

1. The collection $(p_n)_{n=1}^{\infty}$ is point-wise bounded and equicontinuous in X.
2. For every $M \in \mathbb{R}$, there is a compact subset K of X, s.t. $\inf_{x \notin K} p_n(x) \geq M$ for all n large enough.

Then there exists an equilibrium (p, α) of the economy$[\Theta, X, P]$, which is a limit of a subsequence of the equilibria $(p_n, \alpha_n)_{n=1}^{\infty}$ in the sense of Proposition 3.

6 Equilibrium Properties

Before we present a class of economies in which equilibria exist, we need some properties of equilibria. We will actually only apply these results to bounded economies (used to approximate unbounded economies), but the properties may be of interested more broadly.

Suppose, as in Section 2, utility functions which are defined on $[0, 1] \times \mathbb{R}_+$. Suppose a continuous map μ on the type space $\mu : \Omega \to \mathbb{R}_{++}$ assigning to a type θ his riskiness μ_θ. The cost function is given by $c(\theta, x) = \mu_\theta \cdot x$. We make two assumptions:

Assumption #1': For each type θ, $u_\theta(p, x)$ is continuously differentiable, with $\frac{\partial u_\theta}{\partial x} > 0$, $\frac{\partial u_\theta}{\partial p} < 0$.

We denote, for $\theta \in \Omega$, the marginal willingness to pay (which, unlike Section 2,

[14] This is satisfied automatically if u is of the form $u(x, \theta, p) = v(\theta, x) - p$, as in Section 2.

may not depend on price):

$$w_\theta(x) = -\frac{\partial u_\theta}{\partial x}(x,p)/\frac{\partial u_\theta}{\partial p}(x,p) > 0.$$

Assumption #2': The marginal willingness is at least riskiness - that is, $w_\theta(x,p) \geq \mu_\theta$ for all $\theta \in \Omega$ and all $p \in \mathbb{R}_+$ - with strictly inequality $w_\theta(x,p) > \mu_\theta$ for $x < 1$.

Assumptions #1 and #2 imply of Section 2 imply Assumptions #1' and #2', but the latter pair is clearly weaker.

We also introduce the following assumption:

Assumption WIR (Willingness Increasing in Risk): For each $x \in [0,1]$, $p \geq 0$, it holds that $w_{\theta_2}(x,p) \geq w_{\theta_1}(x,p)$ iff $\mu(\theta_2) \geq \mu(\theta_1)$.

The condition that the marginal willingness to pay for insurance is strictly increasing in riskiness was already used in Riley [1979]. Note that in particular under Assumption WIR, utility is completely determined by riskiness: two types with the same riskiness have the same marginal willingnesses to pay.

Let $X \subseteq [0,1]$ with $\sup X = 1$ be the alternative space.

Proposition 5. Assume Assumptions #1', #2', hold, and Assumption WIR holds. Then any AG equilibrium (p,α) of the economy $[\Theta, P, X]$ satisfies the following properties:

1. Types with higher riskiness purchase strictly higher levels of insurance (except possibly at 0 insurance), and types of same riskiness purchase the same level: Formally, it holds α-a.s. that for each pair $(\theta_2, x_2), (\theta_1, x_1)$ for which $x_1, x_2 \neq 0$, $\mu(\theta_2) > \mu(\theta_1)$ iff $x_2 > x_1$.

2. If P_μ denotes the induced measure on riskiness ($P_\mu = P \circ \mu^{-1}$), there is a continuous mapping $\sigma : supp(P_\mu) \to [0,1]$ s.t. $\alpha\big\{(\theta,x) \mid x = \sigma(\mu(\theta))\big\} = 1$, such that σ is strictly monotonic on $supp(P_\mu)\backslash\sigma^{-1}(\{0\})$.

3. Contracts are actuarily fair: $P-a.s.$, $p(\sigma(\theta)) = \mu(\theta) \cdot \sigma(\theta)$.

4. If $x_0 \in X$ and $L \geq w_\theta(x,p)$ for any $x \leq x_0$, $p \leq p(x_0)$, and a.e. θ which chooses

coverage up to x_0,[15] then $p(\cdot)$ is L-Lipshitz in $[0, x_0] \cap X$.

5. If Θ is compact and $\frac{\partial u_\theta}{\partial x}, \frac{\partial u_\theta}{\partial p}$ are continuous in all variables - type as well as contracts - then full insurance is in the support of the equilibrium; formally, if α_X denotes the projection of α to $X \subseteq [0,1]$, then $1 \in supp(\alpha_X)$.

6. If $\mu_1 < \mu_2$ are atoms of P_μ but $P_\mu(\mu_1, \mu_2) = 0$ - i.e., there are α-a.s. no types with riskiness between μ_1, μ_2- for any type $\theta \in \Omega$ with $\mu(\theta) = \mu_2$ is indifferent between $(\sigma(\mu_2), p(\sigma(\mu_2))$ and $(\sigma(\mu_1), p(\sigma(\mu_1))$.

Note that since the type space and alternative spaces are compact, an equilibrium exists by Azevedo and Gottlieb [2017] if the additional uniform Lipshitz conditions given there hold.

We remark that when $0 \in X$, it is not clear if 0 can be chosen with positive measure in an AG equilibrium (and, hence, it is not clear if σ is strictly increasing on the entire $supp(P_\mu)$.

7 Equilibrium with Unbounded Types: A Class

Section 4 showed that, when riskiness μ is unbounded, an AG equilibrium need not exist. However, Section 5 provided conditions under which an unbounded economy can indeed have an AG equilibrium. To illustrate the role of those conditions, in this section we construct a (non-trivial) example of a setting where μ is unbounded and an AG equilibrium nonetheless exists.

We assume utilities satisfying Assumptions #1 and #2 given in Section 2 on contract space $[0, 1]$. Truncalted We will first construct equilibria for bounded truncated economies, as described in Section 5. We show that, under the assumptions we make, the prices of the bounded economies converge to the unbounded equilibrium price function which we will construct for the unbounded economy. An application of Proposition 3 will then complete the proof.

Throughout, we will assume - in addition, as said, to Assumptions #1 and #2 given

[15] Formally, for every θ in a set Θ' such that $\alpha((\Theta' \times X)\Delta(\Theta \times [0, x_0]))$.

in Section 2 - the Assumption WIR - willingness increasing in riskiness - introduced in Section 6. We also assume:

Assumption LL (Local-Lipshitz): On each compact subset Θ_0 of Θ, $u(\cdot,\cdot,\cdot):\Theta\times[0,1]\times\mathbb{R}_+$ is Lipshitz in coverage (the middle variable), uniform over all types in Θ_0; i.e., $w_\theta \le L$ for some L and all $\theta\in\Theta_0$.[16]

Recall that

$$\underline{\nu}_\delta(\theta) := \min\{-\frac{\partial w_\theta}{\partial x}(x) \mid x\in[1-\delta,1]\} \tag{11}$$

Since under Assumption (WIR), types with the same riskiness have the same utility function, we may view $\underline{\nu}_\delta$ as a function of riskiness, defined on the support of $P_\mu := P\circ\mu^{-1}$. We wish to prove:

Theorem 1. Suppose Assumptions (*), WIR and LL hold. Suppose for some $\delta>0$,

$$\int_0^\infty \frac{1}{\underline{\nu}_\delta(\mu)} dF(\mu) < \infty \tag{12}$$

where F is the culmulative distribution of $P_\mu = P\circ\mu^{-1}$, and the integral is the Lebesgue-Steijles integral.[17] Denote $X=[0,1)$. Then the economy $[\Theta, X, P]$ possesses an equilibrium.

Two cases of an interest are the absolutely continuous case - i.e., when the culmulative distribution of μ is absolutely continuoues with density f - in which case (12) can be written

$$\int_0^\infty \frac{1}{\underline{\nu}_\delta(\mu)} d\mu < \infty$$

and the completely discrete case - the distribution P on Θ of types, when projected to riskinesses, is concentrated on a countable set $\mu_1<\mu_2<\cdots$ with $\mu_n\to\infty$,[18] in which case (12) can be written

$$\sum_{n=1}^\infty \frac{\mu_{n+1}-\mu_n}{\underline{\nu}_\delta(\mu_{n+1})} d\mu < \infty \tag{13}$$

[16] By Assumption #1, utility is 1-Lipshitz in price anyway.

[17] For μ not in the support of F, we may define $\underline{\nu} = \liminf_{\mu'\ge\mu} \underline{\mu}_\delta(\mu')$, the infimum taken over elements in the support of F.

[18] The requirement that the elements in the support be an unbounded sequence is stricter than merely requiring the distribution of riskinesses to be purely atomic.

Referring back to the model in which utility functions are CARA and shocks in wealth are given by a log-normal distribution, $\frac{\partial w_\theta}{\partial x} = -\nu_\theta$, and in particular $\underline{\nu}_\delta$ independent of δ. In this case, for example, the conditions will be satisfies if P-a.s., $\nu_\theta \sim C \cdot (\mu_\theta)^{r+1}$ for some $C > 0$ and some $r > 1$.

Notice that (12) cannot be the case when $\nu \leq C\mu + D$ as it was in our non-existence example of Section 4.[19]

To begin with, fix a sequence of compact subets $\Theta_1 \subseteq \Theta_2 \subseteq \Theta_3 \subseteq \cdots \Theta$ with $\Theta = \cup \Theta_n$. Let M^n be the essential supremum of μ w.r.t. $P(\cdot \mid \Theta_n)$. Then $M^1 \leq M^2 \leq M^3 \leq \cdots$ with $M^n \to \infty$, $M^n \in supp(P_\mu)$ for each $n \in \mathbb{N}$. Let $P^n = P(\cdot \mid \Theta_n)$, and $\overline{X} = [0,1]$. By the results of AG (and Assumption LL), the economy $[\Theta^n, \overline{X}, P^n]$ possesses an equilibrium; fix one such equilibrium (α^n, p^n) for each economy. By Proposition 5 implies that in each of these equilibria, there is a strictly increasing function $\sigma^n : supp(P^n_\mu) \to [0,1]$ such that type with riskiness $\mu \in supp(P^n_\mu)$ purchases coverage $\sigma^n(u)$, with $\sigma^n(M^n) = 1$.

We need two claims:

Lemma 2. For each $\mu \in supp(P_\mu)$, $\limsup_{n\to\infty} \sigma^n(\mu) < 1$.

Proof. For each θ_2, θ_1 with $\mu_2 = \mu(\theta_2) > \mu_1 = \mu(\theta_1)$ and each $u \in [0,1]$, let $\phi(\mu_1, \mu_2, u)$ denote the unique $v < u$ s.t. type θ_2 is indifferent between contracts $(\mu_2 \cdot u, u)$ and $(\mu_1 \cdot v, v)$; such unique v clearly exists as $w_{\theta_2}(x) > \mu_2 > \mu_1$ for all $x \in (0,1)$, ϕ is clearly continuous on $\mathbb{R}_{++} \times \mathbb{R}_{++} \times [0,1]$, and $\phi(\mu_1, \mu_2, u) < u$ for $u \in (0,1]$. Suppose $\mu_1 \in supp(P_\mu)$ with $\lim_{n\to\infty} \sigma^{k_n}(\mu_1) = 1$ for some indices (k_n). Fix some $\mu_2 > \mu_1$ with $\mu_2 \in supp(P_\mu)$, and hence w.l.o.g., $\mu_2 \in supp(P^{k_n}_\mu)$ for all $n \in \mathbb{N}$. For each k_n, $\sigma^{k_n}(\mu_1) \leq \phi(\mu_1, \mu_2, \sigma^{k_n}(\mu_2))$, as otherwise type θ with $\mu(\theta) = \mu_1$ would instead of choose coverage $\sigma^{k_n}(\mu_2)$. Then $\lim_{n\to\infty} \sigma^{k_n}(\mu_2) = 1$ as each σ^{k_n} is monotonically increasing. Hence,

$$1 = \lim_{n\to\infty} \sigma^{k_n}(\mu_1) \leq \lim_{n\to\infty} \phi(\mu_1, \mu_2, \sigma^{k_n}(\mu_2)) = \phi(\mu_1, \mu_2, \lim_{n\to\infty} \sigma^{k_n}(\mu_2) = 1) = u(\mu_1, \mu_2, 1) < 1$$

[19]For simplicity, observe the purely discrete case. Suppose (13) holds. Then denoting $\nu_n = \overline{\nu}_\delta(\mu_n) \geq \underline{\nu}_\delta(\mu_n)$, so $\sum_{n=1}^{\infty} \frac{\mu_{n+1} - \mu_n}{\nu_{n+1}} < \infty$, and summation by part shows $\sum_n \mu_n (\frac{1}{\nu_n} - \frac{1}{\nu_{n+1}}) = \sum_n \frac{\mu_n}{\nu_n}(1 - \frac{\nu_n}{\nu_{n+1}}) < \infty$. But in Section 4., there are C, D s.t. $\nu_n \leq C \cdot \mu_n + D$, so $\liminf_{n\to\infty} \frac{\mu_n}{\nu_n} < \frac{1}{C}$, so we must have $\sum_n (1 - \frac{\nu_n}{\nu_{n+1}}) < \infty$, which requires that $\prod_{j=1}^{\infty} \frac{\nu_n}{\nu_{n+1}} > 0$, but $\lim_{n\to\infty} \prod_{j=1}^{n} \frac{\nu_n}{\nu_{n+1}} = \frac{\nu_1}{\nu_{n+1}} \to 0$, a contradiction.

a contradiction. □

Lemma (2) does not rely on our assumption (12). Lemma 3 however, crucially, does:

Lemma 3. For each $\eta > 0$, there is $M > 0$ such that if $\mu > M$ and $n \in \mathbb{N}$ is such that $\mu \in supp(P^n_\mu)$, then $\sigma^n(\mu) > 1 - \eta$. Must prove!

As a result, the onditions of 4 hold. As remarked there, Assumption #1 implies clearly that for every two alternatives $x, y \in X$, price $p \geq 0$, and type $\theta \in \Theta$, there is price q high enough s.t. $u(\theta, x, p) > u(\theta, y, q)$. As for the required properties of the equilibria $(p_n, \alpha_n)_{n=1}^{\infty}$:

1. First we show that if $[0, m] \subseteq [0, 1)$, then (p_n) is point-wise (in fact, uniformly) bounded and equicontinuous on $[0, m]$: Choose some $\theta_0 \in \Theta$ and m' such that, denoting $\mu_0 = \mu(\theta_0)$ such that $\sigma^n(\mu_0) > m$ whenever $\mu_0 \in supp(P^n_\mu)$. . Such θ_0 exists by Lemma 3. Then for all such n and all $x \leq m$, $p_n(x) \leq p_n(m) \leq \mu_0 \cdot m$, so we have the boundedness in $[0, m]$. Denote $w \equiv w_{\theta_0}$. Then for α-a.e. type θ that choose coverage in $[0, m]$, $\sigma(\mu(\theta)) \leq \sigma(\mu_0) = \sigma(\mu(\theta_0))$ so by Assumption WIR, $w_\theta \leq w_{\theta_0}$, and each type has Lipshitz utility.
2. Furtheremore, $\lim_{x \to 1} p(x) = \infty$ by 1.

Hence, an equilibrium (p, α) of the economy $[X, \Theta, P]$ exists.

71 Equilibrium with Unbounded Types: A Sub-Class

To get a feel for the construction, will expand on a particular case, : Utilities for contracts are derived from CARA utility for money with shocks that distributed normally, as discussed in Section 2. Hence, a type θ is completely characterized by a pair of parameters (μ_θ, ν_θ), the utility is given by (4).

For simplicity, let $\Theta = (\mu_n, \nu_n)_{n=1}^{\infty}$ where $\mu_1 < \mu_2 < \cdots$, $\nu_1 < \nu_2 < \cdots$, and $\lim_{n \to \infty} \mu_n = \infty$, and $w_n(x) := w_{\mu_n, \nu_n}(x) = \mu_n + (1 - x)\nu_n$. Assumption (*), LL and WIR clearly hold, and $\underline{\nu}_\delta(\mu_n, \mu_n) \equiv \nu_n$ regardless of $\delta > 0$. Assume (13). We will construct a (fully separating) equilibrium, in which consumer n of type (μ_n, ν_n)

purchases contract (p_n, x_n), where $(p_n)_{n=1}^{\infty}$, $(x_n)_{n=1}^{\infty}$ are strictly increasing with $p_n \to \infty$, $\lim_{n\to\infty} x_n = 1$. Moreover, we will have $p_n = x_n \mu_n$ so that p_n is the actuarily fair price for covering agent of riskiness μ_n at insurance level x_n. Finally, it will be the case that type μ_{n+1} is indifferent between contracts (p_{n+1}, x_{n+1}) and (p_n, x_n).

The intuitive reason for assuming (13) is as follows: The equilibrium price $p(x)$ will be the upper envelope of the consumer indifference curves. As mentioned above, the slope of these indifference curves, at the contract chosen by each type (μ, ν) is given by $w_\theta(x) = \mu_\theta + (1-x)\nu_\theta$, (5). However, we must have $\lim_{x\to 1} p(x) = \infty$ by (1). Therefore, we need ν to increase sufficiently fast with μ so as to allow both an unbounded price function as $x \to 1$ and for insurance prices to be actuarially fair at the purchased contracts. This condition ensures that that "$p_n \to p$ uniformly on compact subsets of X" as required by Proposition 3.

For each $n \in \mathbb{N}$ and each $u \in [0,1]$, let $\phi_n(u)$ denote the unique $v < u$ s.t. type (μ_{n+1}, ν_{n+1}) is indifferent between contracts $(\mu_{n+1} \cdot u, u)$ and $(\mu_n \cdot v, v)$; such unique v clearly exists as $w_{(\mu_{n+1},\nu_{n+1})}(x) > \mu_{n+1} > \mu_n$ for all $x \in (0,1)$. ϕ is clearly continuous on $[0,1]$, and $\phi_n(u) < u$ for $u \in (0,1]$. Furthermore, ϕ_n is strictly monotonically increasing, and in particular $\phi_n(u) = 0$ iff $u = 0$.[20]

We would like to have $x_n = \phi_n(x_{n+1})$, while preserving the incentive compatibility required by (??). To this end, denote for each pair of integers $k \le n$,

$$x_k^n := \phi_k(\phi_{k+2}(\cdots(\phi_{n-1}(1))\cdots)).$$

Notice that $x_n^n = 1$ and $x_k^n = \phi_k(x_{k+1}^n)$. By the strict monotonicity of the ϕ_i, we

[20]For each $u \in [0,1]$, $\phi_n(u)$ is the unique $v \in [0,1)$ s.t.

$$u - v = t_n \cdot v \cdot \left[1 - \frac{u+v}{2}\right]^{-1}.$$

where $t_n = \frac{\mu_{n+1}-\mu_n}{\nu_{n+1}} > 0$. Denoting $w = (1-u)^2, z = (1-v)^2$ (or $u = 1-\sqrt{w}), v = 1-\sqrt{z}$), this is equivalent (for u, v) to

$$\sqrt{z} - \sqrt{w} = 2t_n \frac{1-\sqrt{z}}{\sqrt{z}+\sqrt{w}}.$$

which can then be written as

$$w = z - 2t_n(1-\sqrt{z}) = z + 2t_n\sqrt{z} - 2t_n$$

and the right-hand side is strictly monotonic (and continuous) in z.

see that for each k, $(x_k^n)_{n=1}^\infty$ is a strictly decreasing sequence and, since it is bounded by below by 0, it converges; denote this limit by $x_k \in [0,1)$. Then, by the continuity of ϕ_{k+1}, we have $x_k = \phi_k(x_{k+1})$, and in particular as long as $x_k > 0$, we have $x_k < x_{k+1}$.

Lemma 4. For each $k \in \mathbb{N}$, $x_k > 0$.

Proof. Clearly suffices to show $x_1 > 0$. Applying Lemma 3 shows that for some N, $x_N > 0$. But $x_1 = \phi_1(x_2) = \cdots = \phi_1(\phi_2(\cdots(\phi_{N-1}(x_N))\cdots))$, and for each $u > 0$ and each $k \in \mathbb{N}$, $\phi_k(u) > 0$. □

Denote $p_k^n = \mu_k \cdot x_k^n$, $p_k = \mu_k \cdot x_k$. We now proceed to construct the equilibrium price function, for the truncated economies, where economy $\mathcal{E}_n$ is a truncation consisting of the first n types $(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)$. For each type k in economy $\mathcal{E}_n$, let $g_k^n : [0,1] \to \mathbb{R}$ be the indifference curve of type k through the contract (p_k^n, x_k^n), obtained from (9). For $x \leq x_k^n$, $g_k^n(x) \leq \mu_n x_k^n$, and for $x \in (0,1)$, the derivative is given by $\frac{dg_k^n}{dx}(x) \equiv w_n(\cdot, x)$, where w_n is the marginal willingness to pay of type (μ_n, ν_n), given in (5). Define $p^n : [0,1) \to \mathbb{R}_+$ by

$$p^n(x) = g_k^n(x) \text{ if } x \in \left[x_{k-1}^n, x_k^n\right]$$

where $x_0^n = min[x \mid g_0^n(x) \geq 0]$, and for convenience, set $x_{-1}^n = 0$ and $g_0^n = 0$. Since $g_k^n(x_k^n) = g_{k+1}^n(x_k^n) = p_k^n$, $p^n(\cdot)$ is well-defined and continuous. We also note, $p^n = \max_{k \leq n} g_k^n$.

Similarly define for each k, $g_k : [0,1] \to \mathbb{R}$ as the indifference curve of μ_k through (p_k, x_k), and piece them together to a single function $p : [0,1) \to \mathbb{R}$ in the same manner:

$$p(x) = g_k(x) \text{ if } x \in [x_{k-1}, x_k]$$

where $x_0 = min[x \mid g_0(x) \geq 0]$, and for convenience, set $x_{-1} = 0$ and $g_0 = 0$. Again $p(\cdot)$ is well-defined and continuous, and $p = \max_{k \in \mathbb{N}} g_k$. Notice that the price $p\,(x)$ is defined by the upper envelope of the indifference curves of buyers, where buyers (μ_n, ν_n) chooses contract (x_n, p_n). (Note that p involves 'infinitely many pieces' and hence is not defined at full insurance.) Proposition 5 gives:

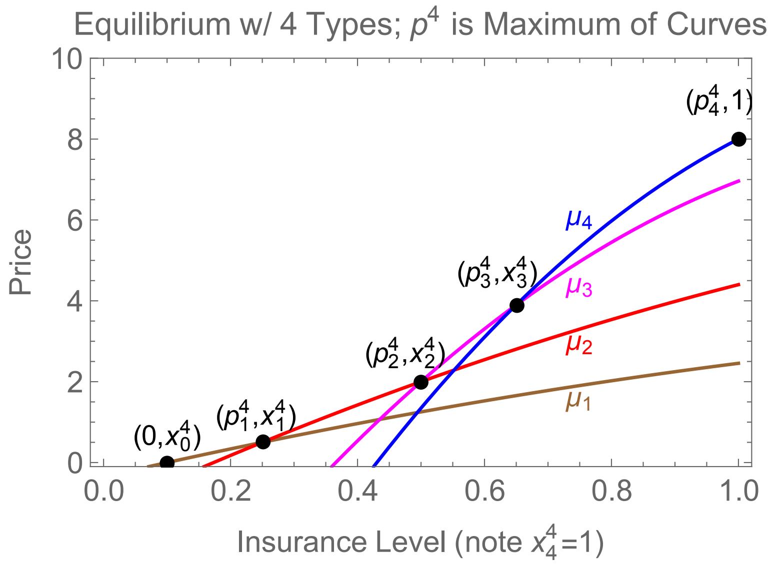

Figure 1: Construction of Price Function for Bounded Economy w/ 4 Types. For all contracts $x \in \left[x_{n-1}^4, x_n^4\right]$, prices are given by $p^4(x) = g_n^4(x)$ where $g_n^4(x)$ is the indifference curve of type n (who chooses x_n).

Proposition 6. Fix $N \in \mathbb{N}$. Denote $\Theta_N = \{(\mu_1, \nu_1), \ldots, (\mu_N, \nu_N)\}$. In the truncated economy $[\Theta_N, [0,1], P(\cdot \mid \Theta_N)]$, the unique equilibrium is given by prices $p^N(\cdot)$ and distribution α^N concentrated on $(\mu_k, \nu_k, x_k^N)_{k \leq N}$, with $\alpha^N(\{(\mu_k, \nu_k, x_k^N)\}) = Q_k := \frac{P(\mu_k)}{P(\Theta_n)}$.

The equilibrium of the truncated economy is depicted in Figure 1. A proof that the described pair (p^N, α^N) is an equilibrium could also be given using the arguments in Section F (which are applied there to the unbounded economy, but with minor modifications apply to the bounded economy as well).

Define a distribution α on $\Theta \times [0,1)$ by

$$\alpha(\{(\mu_n, x_n)\}) = P_n$$

i.e., type n purchases contract (p_n, x_n) and there is a mass P_n of such types. We then obtain the following proposition.

Proposition 7. $p_n \to p$ uniformly on compact subsets of $[0,1)$, and $\alpha^n \to \alpha$ weakly.

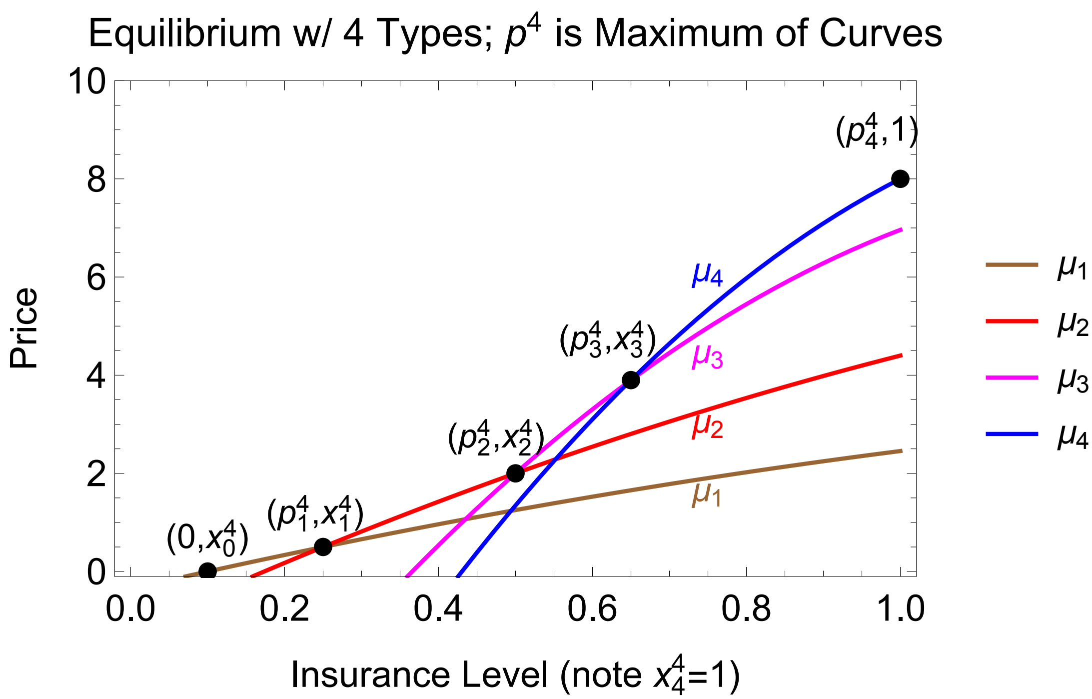

Figure 2: Construction of Price Function for Unbounded Economy. For all contracts $x \in [x_{n-1}, x_n]$, prices are given by $p(x) = g_n(x)$ where $g_n(x)$ is the indifference curve of type n (who chooses x_n). Notice that we show only $x \in \left(0, \frac{3}{4}\right)$. $p(x)$ is only well defined on $[0,1)$ since $\lim_{x \to 1} p(x)$ diverges.

The function $p(\cdot)$ is illustrated in Figure 2.

In light of Proposition 3, we obtain that (p, α) is an equilibrium. In Section F, we provide an alternative, and more direct proof that (p, α) is an equilibrium.[21]

Finally, we remark that if $\sum_{j=1}^{\infty} \frac{\mu_{n+1} - \mu_n}{\nu_{n+1}} = \infty$ (as it must be under the conditions of Section 4), then we would have $x_j = 0$ for all $j \in \mathbb{N}$ so the construction above would fail. In that case, we would find that, for all $x > 0$, $p_n(x) \underset{n \to \infty}{\to} \infty$.

8 Conclusion

We have shown that, without the assumption of bounded risk, the equilibrium described in AG need not exist. However, we have also constructed a non-trivial example

[21] We do not know if (p, α) is the only equilibrium; there is no guarantee that the only equilibria of the economy are those which are approximated by equilibria of truncated economies.

of an environment where risk is unbounded and a (fully separating) AG equilibrium nonetheless exists.

A Properties of Equilibria

The continuity (and in fact Lipshitz-ness) of prices was proven in AG (for the bounded environments they consider). Here, we prove the continuity of prices in generic unbounded settings. Note that this proof does not make use of the CARA-Gaussian framework used elsewhere in the paper.[22]

Lemma 5. If (p, α) is an AG-equilibrium, then p is continuous.

Proof. Suppose $x_n \to x$ in X, and let $(p^n, \alpha^n)_{n=1}^{\infty}$ be the approximating sequence of weak equilibria with alternatives $(\overline{X}_n)_{n=1}^{\infty}$. By passing to a subsequence of (p^n, α^n), we may assume that for each n, there is $y_n \in \overline{X}_n$ such that

$$|y_n - x_n| < \frac{1}{n} \text{ and } |p^n(y_n) - p(x_n)| < \frac{1}{n}$$

Hence, $y_n \to x$. Therefore, by the second inequality and the choice of $(p^n, \alpha^n)_{n=1}^{\infty}$, we have

$$\lim_{n\to\infty} p^n(x_n) = \lim_{n\to\infty} p^n(y_n) = p(x).$$

□

Similar, AG shows under their weaker assumptions that equilibria are, in particular, weak-equilibrium. Also in our case:

Proposition 8. An equilibrium is also a weak equilibrium.

The following lemma, stated in greater generality than needed, may be in independent interest.

Lemma 6. Let X be a locally compact separable metric space, (X_n) a sequence of finite subsets, $p : X \to \mathbb{R}$ continuous and for each $n \in \mathbb{N}$, $p_n : X_n \to \mathbb{R}$, s.t. if (x_n) is

[22] In fact, the continuity of prices in the particular case of the utility functions discussed in this paper, as introduced in Section 2, follow from Lemma 8.

a sequence in X with $x_n \to x \in X$ s.t. $x_n \in X_n$ for each $n \in \mathbb{N}$, then $p_n(x_n) \to p(x)$. Then there are extensions of the p_n to continuous functions $\tilde{p}_n : X \to \mathbb{R}_+$ s.t. $\tilde{p}_n \to p$ uniformly on compact sets. In particular, if X is compact, then $\forall \varepsilon > 0$, there is $N \in \mathbb{N}$, s.t. $\forall n > N$ and$\forall x \in X_n$, $|p_n(x) - p(x)| < \varepsilon$.

The latter conclusion, for the case of compact X, although follows from the first part of the lemma's statement, actually already follows from the first step in the proof.

Proof. Let $(K_j)_{j=1}^{\infty}$ be an increasing sequence of compact sets with $X = \cup_j K_j$ and $K_j \subseteq K_{j+1}^{\circ}$; such exists as X is locally compact and separable metric. Fix $J \in \mathbb{N}$: We contend that $\forall \varepsilon > 0$, there is $N \in \mathbb{N}$, s.t. $\forall n > N$ and$\forall x \in X_n \cap K_J$, $|p_n(x) - p(x)| < \varepsilon$. Indeed, if not, there is $\varepsilon > 0$, a sequence $n_1 < n_2 < \cdots$ of indices, a sequence (x_j) with $x_j \in X_{n_j} \cap K_J$, $|p_{n_j}(x_j) - p(x_j)| \geq \varepsilon$, and such that (x_j) converges; denote the limit $x \in K_J$. Hence, $p_{n_j}(x_j) \to p(x)$ by assumption. Since p is continuous by Lemma 5, $p(x_j) \to p(x)$. Together, these give a contradiction.

Hence, define $q_n : X_n \to \mathbb{R}$ by $q_n = p_n - p$. Denote $Y_n = X_n \cap K_n$, $\varepsilon_n = \max_{x \in Y_n} |q_n|$. By the last paragraph, $\varepsilon_n \to 0$. The Tietze extension theorem implies, for each $n \in \mathbb{N}$, the existence of a continuous extension $\tilde{q}_n$ of q_n to X satisfying $\varepsilon_n = \max_{K_n} |\tilde{q}_n|$. (Formally, first extend the restriction of q_n to Y_n to a function $\tilde{q}_n$ on K_n satisfying $\varepsilon_n = \max_{K_n} |\tilde{q}_n|$ via Tietze's theorem, and then extend it to a function on X agreeing with q_n on X_n in an arbitrary continuous way, again via Tietze's theorem.) Defining $\tilde{p}_n = \tilde{q}_n + p$ for each $n \in \mathbb{N}$ give the required extensions, since for any compact subset $K \subseteq X$, there is J s.t. for all $j > J$, $K \subseteq K_j$. □

Now, the proof of Proposition 8 follows along lines similar to the corresponding Proposition in AG, with much additional care taken.

Proof. For any continuous function $f : X \to \mathbb{R}$ with compact support, since the $\tilde{p}_n$ are uniformly bounded on compact sets (p is continuous and $\tilde{p}_n \to p$ uniformly on compact sets), and since $\tilde{p}^n(x) = p^n(x) = E_{\alpha^n}[c \mid x]$ for all $x \in supp(\alpha^n)$,

$$\int_{\Theta \times X} f \cdot p \cdot d\alpha = \lim_{n \to \infty} \int_{\Theta \times X} f \cdot p \cdot d\alpha^n = \lim_{n \to \infty} \int_{\Theta \times X} f \cdot \tilde{p}^n \cdot d\alpha^n = \lim_{n \to \infty} \int_{\Theta \times X} f(x) \cdot c(x, \theta) d\alpha^n(x, \theta)$$

Now denoting by $K \subseteq X$ the compact support of f, we know that for each $\varepsilon > 0$, there is compactly support $g_\varepsilon : \Omega \to [0,1]$ s.t. $\int_\Theta (1 - g_\varepsilon(\theta)) \sup_{x\in K} c(x,\theta) dP(\theta) < \varepsilon$ (e.g.,) Hence

$$\int_{\Theta\times X} f(x) \cdot c(x,\theta) d\alpha^n(x,\theta) = \int_{\Theta\times X} f(x) g_\varepsilon(\theta) \cdot c(x,\theta) d\alpha^n(x,\theta) + \int_{\Theta\times X} f(x)(1-g_\varepsilon(\theta)) \cdot c(x,\theta) d\alpha^n(x,\theta)$$

$$\int_{\Theta\times X} f(x) \cdot c(x,\theta) d\alpha(x,\theta) = \int_{\Theta\times X} f(x) g_\varepsilon(\theta) \cdot c(x,\theta) d\alpha(x,\theta) + \int_{\Theta\times X} f(x)(1-g_\varepsilon(\theta)) \cdot c(x,\theta) d\alpha(x,\theta)$$

Now, $lim_{n\to\infty} \int_{\Theta\times X} f(x) g_\varepsilon(\theta) \cdot c(x,\theta) d\alpha^n(x,\theta) = \int_{\Theta\times X} f(x) g_\varepsilon(\theta) \cdot c(x,\theta) d\alpha(x,\theta)$, and the errors terms are at most $\varepsilon \cdot \sup |f|$, and $\varepsilon > 0$ was arbitrary. Hence,

$$\int_{\Theta\times X} f \cdot p \cdot d\alpha = \int_{\Omega\times X} f(x) \cdot c(x,\theta) d\alpha(x,\theta)$$

and this was for any $f : X \to \mathbb{R}$ compactly supported. Hence, $p(x) = E_\alpha[c(x,\theta) \mid x]$ α-a.s.

Now, let $\phi : \mathbb{R} \to \mathbb{R}$ be a strictly monotonically increasing continuous funding with bounded range, e.g., $\phi(x) = arctan(x)$ or $\phi(x) = \frac{x}{1+|x|}$. Since α_n is a weak equilibrium, it holds

$$u(\theta, \tilde{p}_n(x), x) = \sup_{x'\in X_n} u(\theta, \tilde{p_n}(x'), x'), \text{for } \alpha_n - a.e.(\theta, x) \in \Theta \times X_n$$

Hence, it is also true that, denoting $v = \phi \circ u$

$$v(\theta, \tilde{p}_n(x), x) = \sup_{x'\in X_n} v(\theta, \tilde{p_n}(x'), x'), \text{for } \alpha_n - a.e.(\theta, x) \in \Theta \times X_n$$

Let α' be a 'deviation to α' - i.e., a measure on $\Theta \times X$ whose projection to Θ is P, and letting (α'_n) be a sequence of measures on $(\Theta \cup X) \times X$, with α'_n supported on $(\Theta \cup X_n) \times X_n$ and $\alpha'_n \to \alpha$ weakly, we have since (p_n, α_n) is a weak equilibrium,

$$\int_{\Theta\times X} v(\theta, \tilde{p}_n(x), x) d\alpha_n \geq \int_{\Theta\times X} v(\theta, \tilde{p}_n(x), x) d\alpha'_n$$

Since $\alpha_n \to \alpha$, $\alpha'_n \to \alpha'$, so the families (α_n) and (α'_n) are tight, and v is bounded. Hence, for each $\varepsilon > 0$, there is $\zeta_\varepsilon : \Theta \times X \to [0,1]$ continuous and compactly supported, such that

$$|\int_{\Omega\times X}(1-\zeta_\varepsilon(x,\theta))v(\theta,\tilde{p}_n(x),x)d\beta| < \varepsilon, \text{ for } \beta = \alpha_n, \alpha, \alpha'_n, \alpha', n \in \mathbb{N}$$

Since $\tilde{p}_n \to p$ uniformly on the support of ζ_ε,

$$\int_{\Theta\times X}\zeta_\varepsilon(x,\theta)v(\theta,\tilde{p}_n(x),x)d\alpha_n \to \int_{\Theta\times X}\zeta_\varepsilon(x,\theta)v(\theta,p(x),x)d\alpha$$

and

$$\int_{\Theta\times X}\zeta_\varepsilon(x,\theta)v(\theta,\tilde{p}_n(x),x)d\alpha'_n \to \int_{\Theta\times X}\zeta_\varepsilon(x,\theta)v(\theta,p(x),x)d\alpha'$$

Since this was for any comapctly supported ζ_ε, it follows that

$$\int_{\Theta\times X}v(\theta,p(x),x)d\alpha \geq \int_{\Theta\times X}v(\theta,p(x),x)d\alpha'$$

Since this was for any measure α' on $\Theta \times X$ whose projection to Θ is P,

$$v(\theta,p(x),x) = \sup_{x'\in X} v(\theta,p(x'),x'), \text{for } \alpha - a.e.(\theta,x) \in \Theta \times X$$

and therefore

$$u(\theta,p(x),x) = \sup_{x'\in X} u(\theta,p(x'),x'), \text{for } \alpha - a.e.(\theta,x) \in \Theta \times X$$

□

B Unboundedness of Prices: Proof of Prop. 1

In this section prove Proposition 1 : We show that, when μ is unbounded and Assumption (*) holds, any AG equilibrium (if it exists), must have $p(\cdot)$ unbounded.

Fix $\overline{\delta} > 0$ for which Assumption (*) holds. We introduce the following notation.

Suppose that $p : X \to \mathbb{R}_+$ and the distribution α on $\Theta \times X$ constitute an AG-equilibrium. Let the marginal of α on Θ be P, while the marginal of α on X we denote by α_X. Consider any $x \in supp(\alpha_X)$. We define,[23]

$$\psi^+(x) = \lim_{\delta \to 0^+} \left[\sup \left\{ \mu \mid \alpha\big(\{\theta \mid \mu_\theta \geq \mu\} \times (x-\delta, x+\delta)\big) > 0 \right\} \right] \tag{14}$$

$$\psi^-(x) = \lim_{\delta \to 0^+} \left[\inf \left\{ \mu \mid \alpha\big(\{\theta \mid \mu_\theta \leq \mu\} \times (x-\delta, x+\delta)\big) > 0 \right\} \right]. \tag{15}$$

Intuitively, $\psi^+(x)$ captures the largest value of μ which purchases x under α, and $\psi^-(x)$ as the lowest such value of μ.

Notice also that, for every $x \in supp(\alpha_X)$

$$limsup_{y \to x} \psi^+(y) \leq \psi^+(x), \qquad liminf_{y \to x} \psi^-(y) \geq \psi^-(x) \tag{16}$$

where the limits are taken along $supp(\alpha_X)$. These hold with equality when x is not an atom of $supp(\alpha_X)$.

We will proceed in five steps, as described in Section 3. Our first auxiliary results establishes bounds on prices in a weak equilibrium.

Lemma 7. Assume (p, α) is a weak equilibrium s..t. p is continuous on $supp\,(\alpha_X)$.[24] Let $0 < \overline{\delta} < x_1 < x_2$ be two points in $supp\,(\alpha_X)$. Then,

$$\begin{aligned}
\frac{p(x_1)}{x_1} &\leq \psi^+(x_1) \leq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \\
&\leq \psi^-(x_2) + (1 - \frac{x_1 + x_2}{2})\overline{\rho}_{\overline{\delta}}(\psi^-(x_2)) \leq \frac{p(x_2)}{x_2} + (1 - \frac{x_1 + x_2}{2})\overline{\rho}_{\overline{\delta}}(\frac{p(x_2)}{x_2})
\end{aligned}$$

In particular, $\psi^+(x)$ must be finite for each $x \in supp(\alpha_X)$ with $x < 1$.

Proof. (8) requires that for α_X-a.e. $x > 0$ in $supp\,(\alpha_X)$, $\psi^-(x) \leq \frac{p(x)}{x} \leq \psi^+(x)$. This,

[23] Observe that, if α' is the marginal of α on the variables (μ, x) - i.e., $\alpha' = P \circ (\mu, id)^{-1}$ - and $x \to \alpha'(\cdot \mid x)$ is a decomposition of α' conditional on x, then for α-a.e. $x \in X$, $\psi^+(x)$ is a supremum of the support of $\alpha'(\cdot \mid x)$. Similarly, $\psi^-(x)$ is an infimum of this support. The limits exist as the terms they are taken over are monotonic.

[24] In particular, this is true if $supp\,(\alpha_X)$ is finite.

together with the monotonicity of $\overline{\nu}$, implies the first and last inequalities.

To show the third inequality, notice that for any alternative which is chosen instead of x_1 is revealed preferred to x_1 (at these prices), we have

$$\alpha(\{u(\theta, p(x), x) \geq u(\theta, p(x_1), x_1)\}) = 1$$

In particular, by the definition of ψ^-, there is a sequence $y_n \to x_2$ in $supp(\alpha_X)$ and types (θ_n) with $\mu(\theta_n) \to \psi^-(x_2)$ (if x_2 is an atom of α_X, take $y_n \equiv x_2$), such that for all n, $u(\theta_n, p(y_n), y_n) \geq u(\theta_n, p_1, x_1)$. Recall that $\overline{\rho}_{\overline{\delta}}(\cdot)$ is monotonically increasing and right-continuous. Hence also $(\overline{\rho}_{\overline{\delta}}(\theta_n))$ satisfies$\limsup_{n\to\infty} \overline{\nu}_{\overline{\delta}}(\theta_n) \leq \overline{\rho}_{\overline{\delta}}(\psi^-(x_2))$. An application of (9), the fact that $liminf_{y\to x_2}\psi^-(y) = \psi^-(x_2)$ if x_2 is not an atom of α_X, the right-continuity of $\overline{\rho}_{\overline{\delta}}$ and the continuity of p on $supp(\alpha_X)$ completes the proof.

The second inequality follows similarly, by also using the trivial minimal bound 0 for ν. □

The second auxiliary result shows the outer-most bounds hold for equilibria globally (not just on the support of α_X).

Lemma 8. Assume (p, α) is an equilibrium. If $0 < \overline{\delta} < x_1 < x_2$ with $p(x_2) > 0$ (and x_1, x_2 not necessarily in the support of α_X), then

$$\frac{p(x_1)}{x_1} \leq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \leq \frac{p(x_2)}{x_2} + (1 - \frac{x_1 + x_2}{2})\overline{\rho}_{\overline{\delta}}\left(\frac{p(x_2)}{x_2}\right)$$

In particular, p is non-decreasing.

Proof. If $p(x_1) = 0$, the first inequality is trivial, so assume $p(x_1) > 0$. Since $p(x)$ is part of an equilibrium, there are finite subsets $\overline{X}^n \subseteq X$, prices $p_n : \overline{X}^n \to \mathbb{R}_+$ and associated distributions α_n on $(\Theta \cup \overline{X}^n) \times \overline{X}^n$ as described in Section 1. Let $y_n \to x_1$ and $z_n \to x_2$ with $y_n, z_n \in \overline{X}^n$. Since each (p_n, α_n) is a weak equilibrium whose projection to X is finitely supported, it follows from Lemma 7 that, at each n,

$$\frac{p_n(y_n)}{y_n} \leq \frac{p_n(z_n) - p(y_n)}{z_n - y_n} \leq \frac{p_n(z_n)}{z_n} + \left(1 - \frac{y_n + z_n}{2}\right)\overline{\rho}_{\overline{\delta}}\left(\frac{p(z_n)}{z_n}\right)$$

Taking $n \to \infty$, and recalling that $\overline{\rho}_{\overline{\delta}}$ is monotonically increasing and right-continuous, completes the proof. □

We now proceed to show that full insurance ($x = 1$) cannot be an atom of α_X if μ is not essentially bounded w.r.t. P. Of course, this proposition is only relevant when $1 \in X$.

Lemma 9. If $x = 1$ is an atom of α_X, it must be that $\alpha(\cdot \mid \{x = 1\})$ is concentrated on one riskiness, i.e., there must be $\tilde{\mu} \in \mathbb{R}_+$ s.t. $\alpha(\mu(\theta) = \tilde{\mu} \mid x = 1) = 1$.

Proof. Suppose, by way of contradiction, that $x = 1$ is an atom of α_X but not concentrated on a single riskiness. Then, (8) holding requires that there are some types buying $x = 1$ who are less risky than the average buyers of that contract, with the price of $x = 1$ being determine by the these average buyers. Define $\mu^* = \psi^-(1)$ and $p^* := p(1)$; then $\mu^* < E_\alpha[\mu \mid x = 1] = p(1) = p^*$.

Lemma 8 implies that, for any $x \geq \delta^*$,

$$\frac{p(x)}{x} \leq \frac{p^* - p(x)}{1 - x} \leq \mu^* + \frac{1}{2}\overline{\rho}_{\overline{\delta}}(\mu^*)(1 - x).$$

In turn, this implies

$$p^* x \geq p(x) \geq p^* - \mu^*(1 - x) - \frac{1}{2}\overline{\rho}_{\overline{\delta}}(\mu^*)(1 - x)^2.$$

It then follows that

$$\frac{1}{2}\overline{\rho}_{\overline{\delta}}(\mu^*)(1 - x) \geq p^* - \mu^*.$$

However, this last condition cannot hold for x close enough to 1 since $p^* > \mu^*$, a contradiction. □

Figure B illustrates the proof of Lemma 9.

Our next auxiliary results shows that contracts arbitrarily close to full insurance attract types with arbitrarily large riskiness μ.

Lemma 10. $\sup_{x<1} \psi^+(x) = \infty$

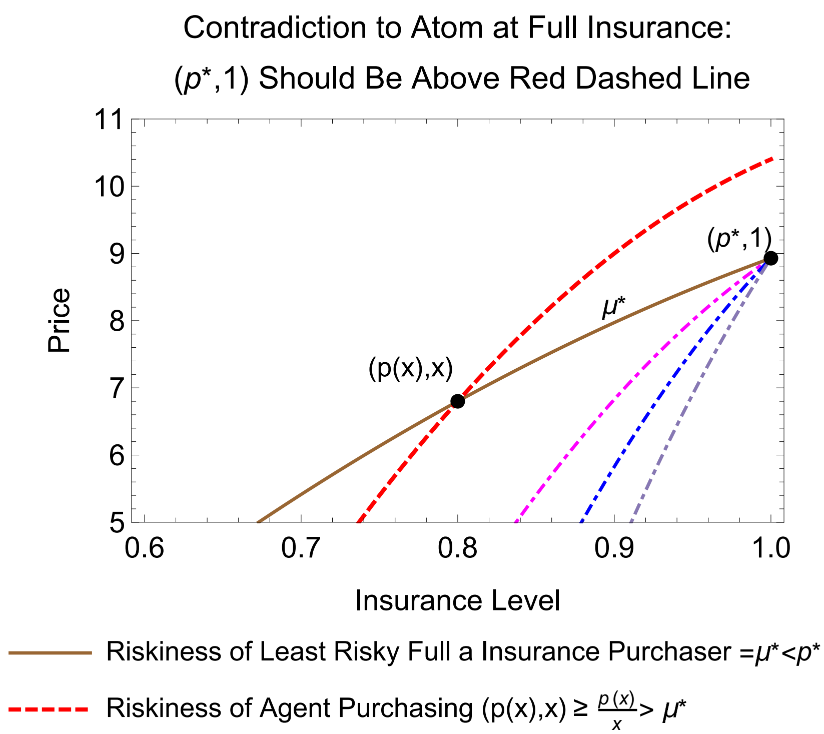

Figure 3: Proof of Proposition 9.

Proof. For $0 < z < 1$, $g(z) = \sup_{x \leq z}(\psi^+(x))$ is such that the projection of $\alpha(\cdot \mid x \leq z)$ to riskiness μ is supported on $[0, g(z)]$. Since full insurance cannot be an atom with unbounded support of riskiness of α, the projection of $\alpha(\cdot \mid [0,1))$ to riskiness μ cannot be compactly supported, so we must have $\lim_{x \to 1^-} g(x) = \infty$.

□

With these results in hand, we are now finally able to prove our desired result, Proposition 1.

Proof. Suppose not. Denote $p^* = \lim_{x \to 1} p(x) < \infty$. Fix some $0 < \overline{\delta} \leq \delta < 1$. Recall that $p^* = p(1)$. Then, from Lemma 8 and the facts that p and $\overline{\rho}_{\overline{\delta}}$ are non-decreasing, for all $\delta \leq x_1 < x_2$ in X,

$$\psi^+(x_1) \leq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \leq \frac{p(x_2)}{x_2} + \overline{\nu}(\frac{p(x_2)}{x_2}) \leq M := \frac{p^*}{\delta} + \overline{\rho}_{\overline{\delta}}(\frac{p^*}{\delta}),$$

while for $x_1 \leq \delta$ in X,

$$\psi^+(x_1) \leq \lim_{x_2 \to 1^-} \frac{p(x_2) - p(x_1)}{x_2 - x_1} = \frac{p^* - p(x_1)}{1 - x_1} \leq \frac{p^*}{1 - \delta}$$

Hence, $\sup_{x<1} \psi^+(x) \leq \max[M, \frac{p^*}{1-\delta}]$, contradicting Lemma 10.

□

C Proof of Proposition 2

We prove Proposition 2 in the general case.

Proof. Let $x_n \to 1$ strictly monotonically in X. WLOG, $x_1 > \overline{\delta}$. Denote $B = \frac{1+C}{x_1}$ and $p_n = p(x_n)$. Then,

$$\frac{p_{n+1} - p_n}{x_{n+1} - x_n} \leq \frac{p_{n+1}}{x_{n+1}} + (1 - x_n)\overline{\rho}_{\overline{\delta}}(\frac{p_{n+1}}{x_{n+1}}) \leq Bp_{n+1} + D$$

and therefore, if furthere w.l.o.g. we assume $1 - x_1 < \frac{1}{B}$, we have,

$$p_{n+1} \leq \frac{p_n + D(x_{n+1} - x_n)}{1 - B(x_{n+1} - x_n)}.$$

Now denote $\delta_n = \max[D, B](x_{n+1} - x_n)$ and assume WLOG $\delta_n < 1$ for all n.

Then, we have

$$p_{n+1} \leq \frac{p_n + \delta_n}{1 - \delta_n}$$

Inductively, it follows that

$$p_n \leq (p_1 + \sum_{j<n} \delta_j) \cdot \prod_{j<n} \frac{1}{1 - \delta_j}$$

Then, $\sum \delta_n = \sum \max[D, B](x_{n+1} - x_n) = \max[D, B](1 - x_1) < \infty$ which implies $\prod_{j<\infty} \frac{1}{1-\delta_j} < \infty$. This, together with the monotonicity of p, shows that

$$\sup_{x \in X} p(x) = \lim_{n \to \infty} p_n < \infty.$$

□

D Proofs of From Section 5

We prove Proposition 3: Briefly, each of the equilibria (p_n, α_n) is itself a limit, in the appropriate sense, of weak equilibria of perturbed economies $\mathcal{E}_n^k$ with contract-spaces, behavioral types, and weak equilibria, indexed by the subscript n and superscript k. The weak equilibria which witness that (p, α) is an equilibria will be constructed from these by an approrpriate diagnol argument.

Proof. For each n, let $(\overline{X}_n), (Y_n^k)_{k\in\mathbb{N}}, (\zeta_n^k)_{k\in\mathbb{N}}, (q_n^k, \beta_n^k)_{k\in\mathbb{N}}$ be sequences of Polish spaces X is dense in, finite sets of alternatives, of behavioral types, and of weak equilibria which witness that (p_n, α_n) is an equilibrium of the restricted economy n. (Note that Y_n^k will in general include points of $\overline{X}$ which are not in X.) Then for each $n \in \mathbb{N}$, $\overline{X}_n \subseteq \overline{X}$.[25] ($q_n^k$ refers to the price function while β_n^k refers to the distribution over types and contracts.) Let $(Z_j)_{j\in\mathbb{N}}$ be a sequence of compact subsets of X with $X = \cup_j Z_j$,

[25] Formally, for each $n \in \mathbb{N}$ there is an embedding $\phi_n : \overline{X}_n \to \overline{X}$, which is identity on X.

and for each $j \in \mathbb{N}$, $Z_j \subseteq Z_{j+1}^{\circ}$; such exists as X is locally compact and seperable Note that since each $p_n(\cdot)$ is continuous by Lemma 5, X is locally compact, and $p_n \to p$ uniformly on compact sets, it follows that p is continuous. Hence, by passing to a subsequence of (p_n, α_n), we may assume that:

- For all $n \in \mathbb{N}$ and all $x \in Z_n$, $|p_n(x) - p(x)| < \frac{1}{n}$..

W.lo.g., since each Y_n^n is finite and $p_n \to p$ on uniformly on compact sets in X, we may assume by Lemma 6 there there are indices $(k_n)_n$ such that (after passing possibly to a sequence of (p_n, α_n)):

- For all $n \in \mathbb{N}$ and all $x \in Y_{k_n}^n$, $|q_{k_n}^n(x) - p_n(x)| < \frac{1}{n}$.
- For all $n \in \mathbb{N}$, $d(\beta_{k_n}^n, \alpha^n) < \frac{1}{n}$, where $d(\cdot, \cdot)$ is a metric for the weak topology.

Now, denote $\gamma_n = \beta_{k_n}^n$, $r_n = q_{k_n}^n$, $\overline{W}_n = Y_{k_b}^n$. Then γ_n is concentrate on $(\Theta \times \overline{W}_n) \times \overline{W}_n$, $\gamma_n \to \alpha$, and for all $n \in \mathbb{N}$, $|r_n(x) - p(x)| < \frac{2}{n}$ for all $x \in \overline{W}_n \cap Z_n$. We contend that for each $x \in X$, each sequence (x_n) in X with $x_n \to x$ and $x_n \in \overline{W}_n$ for each $n \in \mathbb{N}$, $r_n(x_n) \to p(x)$. Indeed, since $x \in X$ there is N s.t. for all $n \geq N$, $x_n \in int(Z_n)$; hence $x_n \in int(Z_n) \cap \overline{W}_n$, so $|r_n(x_n) - p(x_n)| < \frac{2}{n}$, and since p is continuous at x, so $p(x_n) \to p(x)$.

□

Before proving Proposition 4, we present a useful generalization of the Arzela-Ascoli theorem. This generalization is found, e.g., Thm 17, Ch 7, of Kelley, "General Topology". The key generalization in this version visa-a-vis more classical statements is the requirement that X be only locally compact (rather than compact), and requires the functions f to be point-wise bounded (rather than uniformly bounded).

Theorem 2. Let X be a locally compact metric space. Given a sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions on X, equicontinuous and point-wise bounded, there is a continuous function $f : X \to \mathbb{R}$ and a subsequence of $(f_n)_{n=1}^{\infty}$ converging to f uniformly on compact sets.

We now prove Proposition 4:

Proof. The existence of the limit function $p(\cdot)$ with the required first property of $p(\cdot)$ in Proposition 3 follows from Theorem 2. Furthermore, $(\alpha_n)_{n=1}^{\infty}$ (or any of its subsequences) is tight (since $\alpha_n(\Theta_k \times \overline{X}) = P(\Theta_k)$ for all $n, k \in \mathbb{N}$) and hence it w.l.o.g. (passing to a subsequence) converges weakly to some measure α on $\Theta \times \overline{X}$. We now need to show that $\alpha(\Theta \times (\overline{X} \backslash X)) = 0$.

Suppose not, set $B := \alpha(\Theta \times \overline{X} \backslash X) > 0$. Fix some $y_0 \in X$, and fix some $D > \sup_n p_n(y_o)$. (By assumption, such $D < \infty$ exists.) We note that for each $\theta \in \Theta$, and each alternative $y \in X$, there is $q \geq 0$ s.t. $u(y_0,\theta, D) > u(y, \theta, q)$; by possibly decreasing D slightly, the continuity of utility in $\overline{X} \times \Theta \times \mathbb{R}_+$ shows that this statement is true for all $y \in \overline{X}$; and finally a standard continuity argument shows that q may be chosen independent of $y \in X$ (only dependent on θ); i.e., $\cap_{M>0}\{\theta \mid \exists y \in \overline{X} \text{ s.t } u(y_0, \theta, D) \leq u(y, \theta, M)\} = \emptyset$. Fix M s.t. $P\big(\{\theta \mid \exists y \in \overline{X} \text{ s.t } u(y_0, \theta, D) \leq u(y, \theta, M)\}\big) < \frac{1}{2}B$. By assumption, there is a neighborhood V of $\overline{X} \backslash X$ such that for all n large enough and all $y \in V$, $p_n(y) > M$. Therefore, $\alpha_n(\Theta \times V) \leq \frac{1}{2}B$ for all n large enough. By Portmanteau theorem, however, since V is open

$$\frac{1}{2}B \geq \liminf \alpha_n(\Theta \times V) \geq \alpha(\Theta \times V) \geq \alpha(\Theta \times \overline{X} \backslash X) = B > 0,$$

a contradiction. □

E Proof of Proposition 5

Here, we prove Proposition 5. Let (α, p) be an AG equilibrium. We break the proofs into steps:

Lemma 11. It holds α-a.s. that for each pair $(\theta_2, x_2), (\theta_1, x_1)$, $\mu(\theta_2) > \mu(\theta_1)$ implies $x_2 \geq x_1$. This is also true if $X' \subseteq X$ is finite and α' is a weak equilibrium of the economy $[\Omega, P, X']$.

Proof. Suppose not - then there are open subsets U, V of $\Omega \times [0, 1]$ with $\alpha_X(U) > 0$, $\alpha_X(V) > 0$, and such that for each $(\theta_1, x_1) \in U, (\theta_2, x_2) \in V$, $x_1 > x_2$ but $\mu_1 = \mu(\theta_1) < \mu_2 = \mu(\theta_2)$. Fix such a pair. Then $q(x_1)$ must be above the indifference curve of type θ_2 through $(x_2, p(x_2))$, and $p(x_2)$ must be above the indifference curve of type

θ_1 through $(x_1, p(x_1))$. Since the latter indifference curve is strictly flatter, this is impossible. The same logic holds for weak equilibrium on a finite set of contracts. $\square$

Corollary 1. If for some $x \in (0,1)$ and $\mu_0 \in \mathbb{R}_+$, $\alpha\big((\theta, y) \mid \mu(\theta) \geq \mu_0, y < x\big) > 0$,[26] then $p(x) \geq \mu_0 \cdot x$. Similarly, if $\alpha\big((\theta, y) \mid \mu(\theta) \leq \mu_0, y > x\big) > 0$,[27] $p(x) \leq \mu_0 \cdot x$.

Proof. We prove the first statement. Fix $y < x$ with $\alpha\big((\theta, y) \mid \mu(\theta) \geq \mu_0, y < x\big) > 0$. Let $(X_n), (\eta_n), (p_n, \alpha_n)$ be a sequence of weak equilibria[28] which witness that (α, p) is an equilibrium. In particular, using the continuity of μ and the fact that $\alpha_n \to \alpha$ weakly, for each $\varepsilon > 0$ there is N, such that for $n > N$ there is $x_n \in X_n \cap (z, x)$ s.t. $\alpha_n((\theta, y) \mid \mu(\theta) > \mu_0 - \varepsilon, y = x_n) > 0$. This implies by the previous lemma, that for all $t \in X_n$ with $x_n < t$ under α_n, that only types θ with $\mu(\theta) \geq \mu_0 - \varepsilon$ choose t. Taking $n \to \infty$, we see that $p(x) \geq (\mu_0 - \varepsilon) \cdot x$, and this was for any $\varepsilon > 0$. $\square$

Denote ψ^+, ψ^- be defined as in (14) and (15) on the support of the projection α_X of α to $X = [0,1]$. It follows from Lemma 11 that ψ^+, ψ^- are monotinically non-decreasing, and in fact if $x < y$, then $\psi^+(x) < \psi^-(y)$. Also, clearly,

$$z \cdot \psi^-(z) \leq z \cdot E_\alpha[\mu \mid z] = p(z) \leq z \cdot \psi^+(z), \ \forall \text{z} \in \text{supp}(\alpha_X) \tag{17}$$

Lemma 12. It holds α-a.s. that for each pair $(\theta_2, x_2), (\theta_1, x_1)$ with $x_2, x_1 \neq 0$, $\mu(\theta_2) > \mu(\theta_1)$ implies $x_2 > x_1$.

Proof. We already know by the previous lemma that

$$\alpha\big\{(\theta_1, x_1), (\theta_2, x_2) \mid x_2 \geq x_1, \mu(\theta_2) > \mu(\theta_1)\big\} = 1$$

Suppose by way of contradiction,

$$\alpha\big\{(\theta_1, x_1), (\theta_2, x_2) \mid x_1 = x_2, \mu(\theta_2) > \mu(\theta_1)\big\} > 0$$

[26]I.e., with positive probability under α an agent with riskiness at least μ_0 purchases a contract with coverage less than x.
[27]I.e., with positive probability under α an agent with riskiness at most μ_0 purchases a contract with coverage more than x.
[28]X is already compact.

By the Lebesgue density theorem[29] it follows that there is $x \in (0,1]$, and $\mu_1 < \mu_2$ such that every neighborhood of x has positive α-measure of both: types of riskiness at most μ_1, and types of riskiness at least μ_2.

If $0 < x < 1$, the corollary shows that this implies $p(y) \leq \mu_1 \cdot y$ for $y < x$, and $p(z) \geq \mu_2 \cdot z$ for $z > x$, contradicting the continuity of p.

If $x = 1$, it follows in the same way that if $y < x$ then $q(y) \leq \mu_1$ for $y < 1$. Hence, by p's continuity, $p(1) \leq \mu_1$, whenever μ_1 is such that any neighborhood of full insurance has positive α-measure of types $\theta$with $\mu(\theta) < \mu_1$. It also follows that for $\varepsilon > 0$ small enough, $\alpha(\theta, x \mid \mu(\theta) > \mu_2 - \varepsilon, x = 1) > 0$. These two contradict $E_\alpha[c \mid x = 1] = p(1)$. □ Complete details.

Lemma 13. $\psi^- = \psi^+$ and are strictly increasing in $supp(\alpha_X)\backslash\{0\}$.

Proof. Suppose $\psi^-(z) < \psi^+(z)$ for some $z \in supp(\alpha_X)$ with $z > 0$. Lemma 12 implies that z is not an atom of α_X; therefore, (16) hold with equility:

$$limsup_{y \to x} \psi^+(y) = \psi(x), \qquad liminf_{y \to x} \psi^-(y) = \psi^-(x) \tag{18}$$

with the limits being taken along $supp(\alpha_X)$. The fact that $\psi^-(x) \leq \psi^+(x) < \psi^-(y) \leq \psi^+(y)$ whenever $x < y$ in $supp(\alpha_X)$ imply that $limsup_{y \to x} \psi^+(y) = limsup_{y \to x} \psi^-(y)$ and $liminf_{y \to x} \psi^+(y) = liminf_{y \to x} \psi^-(y)$. □

Lemma 14. $\alpha(\{\theta, x \mid p(x) \neq \mu(\theta) \cdot x\}) = 0$.

Proof. Suppose the conclusion does not hold: Then w.l.o.g., there are $c, \delta \geq 0$ and an open set $U = V \times (a,b) \subseteq \Theta \times [0,1]$ with $\alpha(U) > 0$ such that for $(\theta, x) \in U$, $p(x) > c + \delta > c - \delta > \mu(\theta) \cdot x$ (the case $p(x) < c - \delta < c + \delta < \mu(\theta) \cdot x$ is handled similarly).

Let $x_0 \in (a,b)$ be density point of both α_X and of $\alpha_X(\cdot \mid W)$, where W is the projection of U to X. Hence, using the continuity of p and the definition of ψ^-, $\psi^-(x_0) \cdot x_0 \leq c - \delta < c + \delta \leq p(x_0)$. By (17), $p(x_0) \leq \psi^+(x_0)$. But by the previous lemma, $\psi^-(x_0) = \psi^+(x_0)$, a contradiction. □

[29] Applied to the measure induced by α on $\mathbb{R}_+ \times [0,1]$ induced by the map $(\theta, x) \to (\mu(\theta), x)$.

In particular it follows that agents purchase in equilibrium contracts a.s. only at actuarily fair prices (at 0 in any case costs are 0).

Corollary 2. Hence, there is a mapping $\sigma : supp(P_\mu) \to [0,1]$, strictly increasing on $supp(P_\mu)\backslash\sigma^{-1}(\{0\})$, s.t. $\alpha\big\{(\theta,x) \mid x = \sigma(\mu(\theta))\big\} = 1$.

Proof. Denote $\psi = \psi^+ = \psi^-$ in $supp(\alpha_X)\backslash\{0\}$. Let $W \subseteq \Theta$ such that $\alpha(\theta \in W \mid x > 0) = 1$, i.e., those types which choose positive coverage. Indeed, let $\sigma = \psi^{-1}$. This is well-defined P_μ-a.e. on $supp((P_\mu(\cdot \mid W))$, and by the previous results in strictly monotonic. Extend $\sigma$to $supp(P_\mu(\cdot \mid \Theta\backslash W))$ by 0; by the previous results, this is well-defined (α-a.s., any types θ s.t. $\mu(\theta) \in supp((P_\mu(\cdot \mid W)) \cap supp(P_\mu(\cdot \mid \Theta\backslash W))$ choose 0 coverage, i.e., are not in W.) □

Lemma 15. If Θ is compact and the derivatives of utility are continuous in all arguments, then the supremum of the support of α_X is full insurance.

Proof. If $x^* < 1$ is the supremum of the support of α_X, clearly $x^* > 0$ and $p(x^*) > 0$. Denote $\overline{\mu} = \frac{p(x^*)}{x^*}$. Since σ is strictly monotonic on $supp(P_\mu)\backslash\sigma^{-1}(0)$, it follows that $\overline{\mu}$ is an essential bound of μ w.r.t. P_μ; hence $p(x) \leq \overline{\mu}\cdot x$ for all $x \in X$, and $p(x^*) = \overline{\mu}\cdot x^*$. Let $\overline{\theta}$ be such that $\mu(\overline{\theta}) = \overline{\mu}$. Such exists as μ is continuous and $\Theta$is compact, and furtheremore, such $\overline{\theta}$can be chosen in the support of P. Since $w_{\overline{\theta}}(x,p) > \overline{\mu}$ for all $x \in (0,1)$ and all $p \in \mathbb{R}_+$, $\overline{\theta}$ strictly prefers $(p(1),1)$ to $(p(x^*),x^*)$.

The most intuitive case to negative is if $\overline{\mu}$ is an atom of P_μ this gives a contradiction, as this holds for all types with riskiness $\overline{\mu}$.

Assume $\overline{\mu}$ is not an atom; we need to refine the argument: By the assumed continuity, there is some neighborhood W of $\overline{\theta}$ and some $\delta > 0$ such that if $\sigma(\mu(\theta)) > x^* - \delta$ and $\theta \in W$, then θ also strictly prefers $(p(1),1)$ to $(p(\sigma(\mu(\theta))), \sigma(\mu(\theta)))$. Since σ is strictly increasing on $supp(P_\mu)\backslash\sigma^{-1}(\{0\})$, and $\overline{\mu}$ is not an atom of P_μ, we see that for some $\mu' < \overline{\mu}$, $\mu(\theta) > \mu'$ implies $\sigma(\mu(\theta)) > x^* - \delta$. But since μ is continuous, we may w.l.o.g. assume that for all $\theta \in W$, $\mu(\theta) > \mu'$; since $\overline{\theta}$ is in the support of $\overline{\mu}$, $P(W) > 0$, a contradiction, as every type is W chooses full insurance. □

The Lipshitz-type property of equilibrium on subsets follows along the lines of Part 3 of Proposition 1 of Azevedo and Gottlieb [2017], so we omit the proof; essentially, the restriction of the economy to those types that choose coverage up to x_0 satisfies the framework and Lipshitz-ness conditions of that paper.

Now, to prove the final property: If $\mu_1 < \mu_2$ are atoms in the support of P_μ but $P_\mu(\mu_1, \mu_2) = 0$, then for any type $\theta \in \Omega$ with $\mu(\theta) = \mu_2$ (such exists as μ_1, μ_2 are atoms of P_μ) and any such θ is indifferent between $(\sigma(\mu_2), p(\sigma(\mu_2))$ and $(\sigma(\mu_1), p(\sigma(\mu_1))$. Indeed, denote $x_1 = \sigma(\mu_1) < x_2 = \sigma(\mu_2)$. Let x' be such that if $p' = \mu_1 \cdot x'$ then for θ_2 with $\mu(\theta_2) = \mu_2$ (such exists by the continuity of μ and the compactness of Θ), $u_{\theta_2}(p(\sigma(\mu_2)), \sigma(\mu_2)) = u_{\theta_2}(p', x')$. Clearly we must have $\sigma(\mu_1) \leq x'$, as otherwise $(p(\sigma(\mu_1))), \sigma(\mu_1))$ would be strictly preferable by θ to $(\sigma(\mu_2), p(\sigma(\mu_2))$. If $\sigma(\mu_1) < x'$, then $u_{\theta_2}(p(\sigma(\mu_2)), \sigma(\mu_2)) < u_{\theta_2}(p(x'), x')$, as $w_{\theta_2} > \mu_1$. Let (p^n, α^n) be a sequence of weak equilibria converging to (p, α), with α^n supported on $\overline{X}_n$. By Lemma 11, and disregarding finitely many of the sequence, we may assume that for all n, x with $x \in \overline{X}_n$ and $\sigma(\mu_1) < x < x'$, α^n-a.s. only agents with riskiness $\mu_1$or μ_2 choose x. Hence, we must have a sequence (x^n) converging to $\sigma(\mu_1)$ from above s.t. $\alpha^n(\mu(\theta) = \mu_2 \mid x^n) > 0$ and $x^n \in \overline{X}_n$; since otherwise there would be a neighborhood $(\sigma(\mu_1), z)$ for some z s.t. $p(x) \leq \mu_1 \cdot x$ in this neighborhood, but then some such x would be preferable to types choosing μ_1 then $\sigma(\mu_1)$. However, since $u_\theta(p(\theta), \sigma(\theta)) = u_\theta(p', x')$ for θ s.t. $\mu(\theta) = \mu_2$, and $(p^n(x_n), x_n) \to (p(\sigma(\mu_1)), \sigma(\mu_1))$, we have $u_{\theta_2}(p(\sigma(\mu_2)), \sigma(\mu_2)) < u_{\theta_2}(p^n(x_n), x_n)$ for large enough n, contradicting the fact that (p^n, α^n) is a weak equilibrium of $[\Theta \cup \overline{X}^n, P, \overline{X}^n]$.

F Direct Proof of Equilibrium

The following is a direct construction of the equilibrium described in Section 7. As remarked there, a slight modification of this construction could also be used to show that the price and allocation described in Proposition 6 is indeed an equilibrium.

Proof. Since utilities are quasi-linear and $p > 0$ on $(0, 1)$,[30] it is enough to approximate (p, α), in the same manner described in Section 1, but on $X' = (x_0, 1)$ instead of $[0, 1)$ (as $p \equiv 0$ in $(0, x_0)$) and with η_n not necessarily strictly positive on the behavioral

[30] $p(0) = 0$, but $\forall n, \overline{X}^n \subseteq (0, 1)$ in our construction to follow, so p is positive on $\overline{X}^n$.

types $\overline{X}^n$; afterwards the weight of the behavioral types could be increased slightly to be strictly positive in such a way that the price goes down by the same amount for each alternative in $\overline{X}^n$.

We will also index the sequence of economies by n. For each n, let $\overline{X}^n$ be the set

$$\overline{X}^n = \{x_{ij} \mid i = 1, \ldots, n, j = 1, \ldots, n\} \cup \{x_1, \ldots, x_n\}.$$

The contracts $x_1, \ldots, x_n$ are obtained from (??) above. That is, economy n has only the first n contracts $x_1, \ldots, x_n$. Moreover, to each contract i are associated n behavioral types $x_{i1}, \ldots, x_{in}$ are distributed (e.g., evenly) strictly between x_{i-1} and x_i. (Recall that x_0 is the right-most point s.t. $p(x_0) = 0$, i.e., where type 1 is indifferent between $(0, x_0)$ and (p_1, x_1).) The mass of agents at each x_{ij} (which de denote η_n) is defined below.

As in AG, the behavioral agents in $\overline{X}^n$ have riskiness $\mu = 0$, i.e., zero cost. We set prices $p_n \equiv p$ for contracts on $\overline{X}^n$. Moreover, we set the distribution of the weak equilibrium (α_n) such that

$$\alpha_n(\{\mu_i, x_i\}) = P_i \left[1 - \frac{1}{n}\right], \forall i = 1, \ldots, n$$

$$\alpha_n(\{\mu_{i+1}, x_{ij}\}) = P_{i+1}\frac{1}{n^2}, \forall i, j = 1, \ldots, n$$

That is, of the original mass P_i of "regular" types μ_i, all but a $\frac{1}{n}$-fraction choose x_i, while the rest evenly spread themselves between the contracts $x_{i,1}, \ldots, x_{i,n}$, such that the mass of type μ_i in each of these contracts is a share $\frac{1}{n^2}$ of the total mass P_i. Recall that $x_{i-1} < x_{i1} < \ldots < x_{i,n} < x_i$ and moreover $p(x)$ is defined so that types μ_i are indifferent between all these contracts.

We also construct the distribution α_n such that, all types $k > n$ (each with mass P_k) purchase the highest coverage available (x_n):

$$\alpha_n(\{\mu_k, x_n\}) = P_k, \qquad \forall k > n.$$

Since μ_n, ν_n increasing, this maximizes their utility when contracts x_k for $k > n$ are

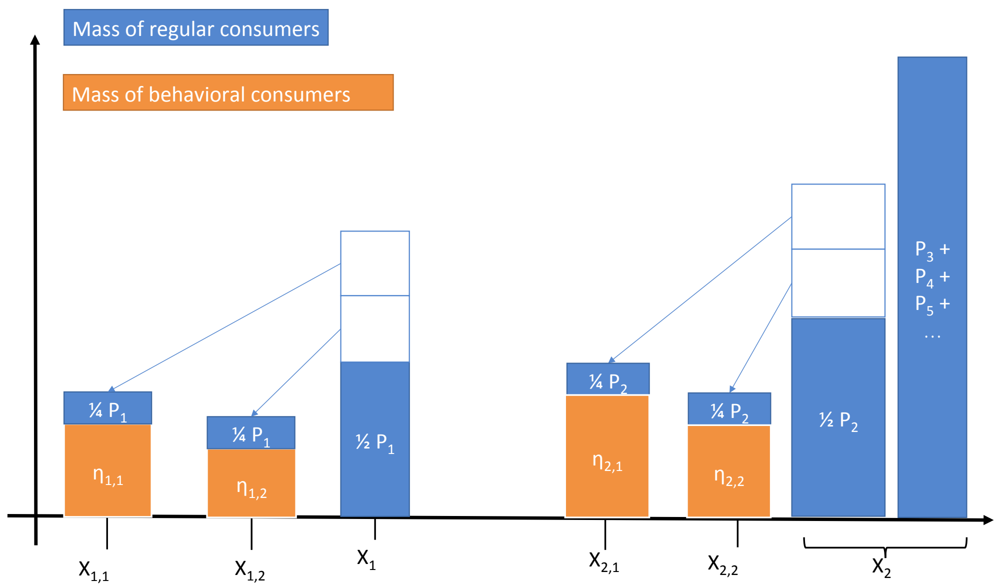

Figure 4: Illustration of α_n used in Section F. Specifically, the figure illustrates the second perturbation, α_3.

not available.

This construction of α_n is illustrated by Figure F.

We then define $\alpha_n(\{(x_{ij}, x_{ij})\}) = \eta_n(\{x_{ij}\})$ for all i, j to be the mass of behavioral types who purchase contract x_{ij} (which, recall, will also be purchased by some mass of types μ_i). We define $\eta_n(x_{ij})$ to satisfy

$$g_i(x_{ij}) = x_{ij}\mu_i \frac{P_i \cdot \frac{1}{n^2}}{P_i \cdot \frac{1}{n^2} + \eta_n(x_{ij})} \leq 1.$$

This will imply that each contracts x_{ij} breaks even:

$$E_{\alpha_n}[\mu \cdot x \mid x_{ij}] = x_{ij}\mu_{i+1} \cdot \frac{\alpha_n(\mu_i, x_{ij})}{\alpha_n(\mu_i, x_{ij}) + \eta_n(x_{i,j})} + 0 = x_{ij}\mu_{i+1} \cdot \frac{P_i \cdot \frac{1}{n^2}}{P_i \cdot \frac{1}{n^2} + \eta_n(x_{i,j})} = g_i(x_{ij}) = p(x_{ij}) \tag{19}$$

Moreover, since $x_{i-1} \leq x_{ij} \leq x_i$, we also have

$$\frac{g_i(x_{i-1})}{x_i\mu_i} \leq \frac{P_i \cdot \frac{1}{n^2}}{P_i \cdot \frac{1}{n^2} + \eta_n(x_{ij})}$$

and therefore, as $n \to \infty$, we have $\sup_{x_{ij}} \eta_n(x_{ij}) \to 0$.

We also assume that, in economy n, there are no behavioral types purchasing contracts x_i for $i \leq n-1$:

$$\eta_n(x_i) = 0,\ i = 1, \dots, n-1$$

Regarding the top contract x_n, the mass of behavioral types $\eta_n(x_n)$ is defined such that

$$\mu_n = \frac{p(x_n)}{x_n} = E_{\alpha_n}[\mu \mid x_n] = \frac{\mu_n P_n \frac{n-1}{n} + \sum_{j>n} \mu_j P_j}{P_n \frac{n-1}{n} + \sum_{j>n} P_j + \eta_n(x_n)}$$

i.e., $\eta_n(x_n)$ is chosen such that although the riskiest agents all choose the top contract, its price nonetheless satisfies $\mu_n = \frac{p(x_n)}{x_n}$. The fact that $\sum_n P_n\mu_n < \infty$ implies $\lim_{n\to\infty} \eta_n(x_n) = 0$.

In this way, for each i, the break even condition (8) $E_{\alpha_n}[\mu x \mid x_i] = p_n(x_i) = p(x_i)$ holds for each $i = 1, \dots, n$ in (p_n, α_n); indeed, for each $i = 1, \dots, n-1$, only types μ_i purchase x_i, while for $i = n$ this results from our definition of $p_n(x_n) = p(x_n)$ and by (19).

We claim that the sequence (p_n, α_n) demonstrates that (p, α) is an equilibrium. Clearly, $p_n \equiv p$ on $\overline{X}^n$ and $\overline{X}^n \to X' = [x_0, 1)$ in the sense of Haussdorf.

Moreover, $\alpha_n \to \alpha$ weakly: Notice α_n is concentrated on the set of types $\{(\mu_k, \nu_k)\}_{k\in\mathbb{N}}$ and the behavioural types, with $\alpha_n(\mu_k, \nu_k) = \alpha(\mu_k, \nu_k) = P_k$. Define $\mathbb{I}_{k,m} = 1\{k = m\}$ be an indicator function. Then,

$$\alpha_n(\mu_k, x_m) = \mathbb{I}_{k,m} \cdot P_m \left[1 - \frac{1}{n}\right] \to \mathbb{I}_{k,m} \cdot P_m = \alpha(\mu_k, x_m)$$

and for each $m \in \mathbb{N}$,

$$\alpha_n(\{x \in (x_{m-1}, x_m)\}) = P_m \frac{1}{n} \to 0 = \alpha(\{x \in (x_{m-1}, x_m)\}).$$

Hence for each $\delta < 1$, $\alpha_n(\cdot \mid \{x \leq \delta\}) \to \alpha(\cdot \mid \{x \leq \delta\})$ converges in total variation norm. This implies that $\alpha_n \to \alpha$ weakly.

Furthermore, α_n-a.s. the original agents $\{(\mu_n, \nu_n)\}_{n=1}^{\infty}$ are utility maximizing: agents of type $i \leq n$ are utility maximizing since they either choose the same option x_i in $\overline{X}^n \subseteq X = [0, 1)$, at the same price $p_n(x_i) = p(x_i)$, as they do when they can choose any alternative in X, or they choose an alternative $x_{i,1}, \ldots, x_{i,n}$ which delivers the same utility as x_i at prices $p_n \equiv p$. Agents of type $k > n$ are utility maximizing since their willingness to pay for x is higher than that of type n, who (weakly) prefers the contract $x_n = \max\left[\overline{X}^n\right]$ to any other alternative in $\overline{X}^n$ at prices $p_n \equiv p$.

Therefore, each (p_n, α_n) is a weak equilibrium, so (p, α) is an equilibrium. □

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